Entanglement entropy in $d + 1$ $SU(N)$ gauge theory

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We consider the entanglement entropy for a subsystem in $d + 1$ dimensional $SU(N)$ lattice gauge theory. The $1 + 1$ gauge theory is treated exactly and shows trivial behavior. Gauge theories in higher dimensions are treated within Migdal-Kadanoff approximation. We consider the gauge theory in the confinement phase. We demonstrate the existence of a nonanalytical change from the short distance to long distance form in the entanglement entropy in such systems ($d > 2$) reminiscent of phase transition. The transition is manifested in nontrivial change in the renormalization group flow of character expansion coefficients defining the partition function.

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I. INTRODUCTION

Interest in the study of entanglement entropy has a relatively long history in quantum field theory. The early motivation was due to the connection with black hole physics. General properties of the entanglement entropy, such as its dependence only on the surface, were demonstrated for a system of oscillators and massless noninteracting scalar field theory [1]. Later the entanglement entropy was studied in gravity duals of confining large N_c gauge theories [2] using the AdS/CFT approach of [3]. In this work the d dimensional space was divided into two complementary regions A and $\bar{A}$ by two imaginary $d - 1$ dimensional hypersurfaces placed a distance l apart along one of the space directions:

$$A = \mathbb{R}^{d-1} \times \mathbb{I}_l, \qquad \bar{A} = \mathbb{R}^{d-1} \times (\mathbb{R} - \mathbb{I}_l), \tag{1}$$

where $\mathbb{I}_l$ is a line segment of length l. The authors studied the entanglement entropy as a function of l and found that it exhibits a nonanalytical change in behavior at $l = l_c^*$ reminiscent of a phase transition [4].

In the present work we aim to prove that this is a general scenario for $SU(N)$ gauge theories (at arbitrary N) at temperatures corresponding to the confinement phase. We consider a $d + 1$ dimensional gauge theory at finite temperature T. The zero temperature system is recovered as the limit $T \to 0$. We consider the same geometry of the entangled region as in [2], see Eq. (1).

If we are given the density matrix for such a system, we can integrate out all degrees of freedom associated with region $\bar{A}$. The resulting density matrix can be used to construct the entanglement entropy

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho, \qquad S_A = -\mathrm{Tr}_A \rho_A \log \rho_A, \tag{2}$$

which is the entropy as seen by an observer with no access to the degrees of freedom in $\bar{A}$.

We will use a method of gluing replicas of the system under consideration into a multisheet Riemann surface, which was used in an extensive treatment of 2D CFT in [6,7]. We consider n replicas of a system after the trace over region $\bar{A}$ has been taken (for this the boundaries in time direction of this region were identified) [8]. Each of these replicas is glued to another along the boundary of region A normal to the time direction. The first replica's upper boundary (coordinate $t = 1/T$) is glued to the lower boundary (coordinate $t = 0$) of the second replica and so on. The upper boundary of the last nth replica is glued to

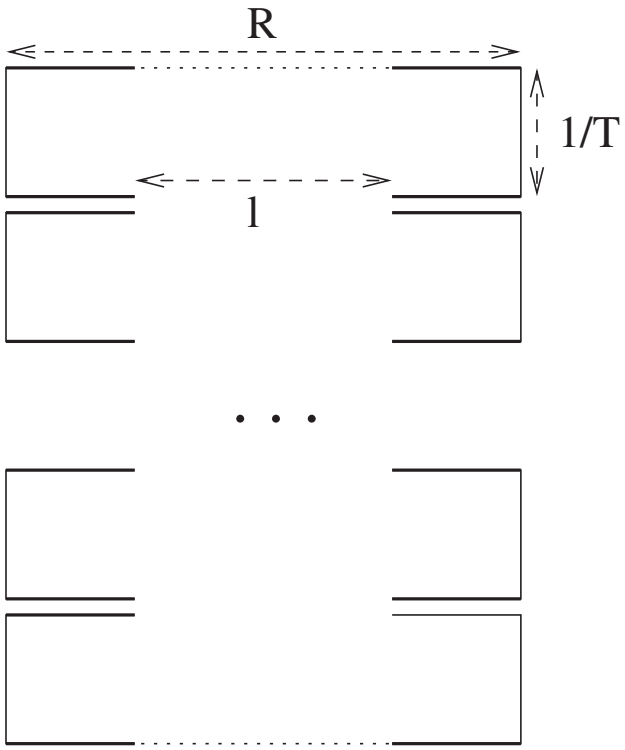

FIG. 1. Z_n for $1 + 1$ dimensional gauge theory.

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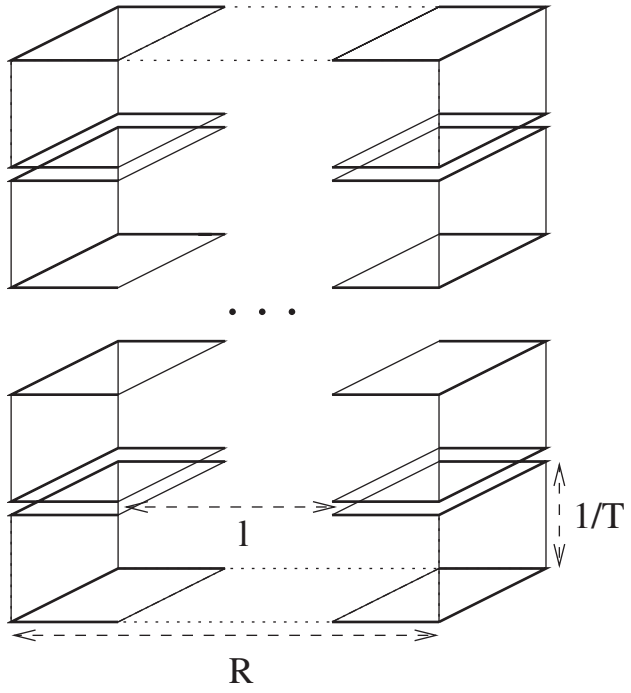

FIG. 2. Z_n for $2+1$ dimensional theory.

the lowest boundary of the first replica, thus closing the system. For illustration of such gluing in $1+1$ and $2+1$ dimensional theories, see Figs. 1 and 2. One can observe that in such a system

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_n(A)}{Z^n}, \tag{3}$$

where Z_n is the partition function of the glued system and Z is the standard partition function of the original system ($Z = Z_1$).

This approach allows one to construct the entanglement entropy

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_n(A)}{Z^n}. \tag{4}$$

II. $SU(N)$ GAUGE THEORY IN $d+1$ DIMENSIONS

The partition function for $SU(N)$ lattice gauge theory is

$$Z = \int\prod_l dU_l \prod_p e^{-S_p}, \tag{5}$$

where the action is $S_p \equiv S(U_p) = -\beta/(2N)\mathrm{Tr}U_p + \mathrm{H.c.}$, $\beta = 2N/g^2$ is the lattice inverse coupling, and the plaquette variable is the ordered product of gauge fields which live on the links constituting the plaquette $U_p = \prod_{l\in\partial p} U_l$. The gauge invariant action is a class function and therefore it can be expanded in group characters

$$e^{-S_p} = \sum_r F_r d_r \chi_r(U_p) \equiv F_0\Big(1 + \sum_{r\neq 0} c_r d_r \chi_r(U_p)\Big), \tag{6}$$

where the first sum runs over all irreducible representations, while the second sum excludes the trivial $r = 0$ representation. For general $SU(N)$ group r is a set of indices; d_r is the dimension of the representation, $c_r = F_r/F_0 < 1$ and F_r are the coefficients of expansion

$$F_r = \int dU e^{-S(U)}\frac{1}{d_r}\chi_r^*(U). \tag{7}$$

A. $d = 1$ gauge theory

The 2-dimensional $SU(N)$ gauge theory is exactly solvable, see [9] for an overview and large N treatment of zero temperature $U(N)$ gauge theory. It is possible to treat the zero temperature gauge theory in this study in an analogous fashion; however, we would like to consider a more general case of a gauge theory at finite temperature T and therefore adopt a different approach. Finite temperature gauge theory in 2 dimensions normally is formulated on a $\mathbb{R}\times\mathbb{S}_1$ surface periodic in time direction with period $1/T$. For practical reasons we consider a finite system in space direction. The corresponding discretized theory is formulated on a $N_r \times N_t$ lattice, with space-time cutoff a and $aN_t = 1/T$ and $aN_r = R$.

At this point it is instructive to consider how integration on the surface is performed. Consider an elementary part of a surface bounded by a single loop [10]. Its contribution to the resulting partition function [11] is

$$f(\{a\};\partial A) \equiv 1 + \sum_{i\neq 0} d_i a_i \chi_i(\partial A), \tag{8}$$

where ∂A is a product of link variables along the surface perimeter and the function is defined when all a_i coefficients are specified. The expression for a junction of two surface elements A and B with a common boundary $A\cap B$ is

$$\begin{aligned} f(\{c\};\partial(A\cup B)) &= \int d(A\cap B) f(\{a\};\partial A) f(\{b\};\partial B) \\ &= 1 + \sum_{i\neq 0} d_i c_i \chi_i(\partial(A\cup B)), \\ c_i &= a_i b_i. \end{aligned} \tag{9}$$

The integration over the common boundary $U = A\cap B$ is performed using the character property:

$$\int dU \chi_r(VU)\chi_s(U^\dagger W) = \frac{1}{d_r}\delta_{r,s}\chi_r(VW). \tag{10}$$

In other words the junction of the surfaces in the space of character coefficients is represented by an ordinary product.

For any 2-dimensional surface we can expand the partition function (5) in characters according to (6) and then

integrate all the internal plaquettes using (9). The resulting expression for the partition function is

$$Z = \int \prod_{l\in\partial A} dU_l \sum_r F_r^A d_r \chi_r(U_{\partial A}), \tag{11}$$

where $A = N_r N_t$ is the area of the total surface in plaquette units (number of tiling plaquettes) and ∂A is the contour enclosing the surface.

The multisheet n-replica partition function Z_n for 2D model is shown in Fig. 1. Remember that the links of time boundary of region $\bar{A}$ for each replica (bold lines) are identified and so are the links from region A that form the time boundary of Z_n (dotted lines). Being subjected to the same treatment as Z, the partition function of the glued system Z_n will result in the same expression (11), but with corresponding surface area $A_n = nA = nN_rN_t$ and perimeter ∂A_n.

To perform the perimeter integration, first we choose free boundary condition (b.c.) in the spatial direction. The invariance of the group integration (Hurwitz/Haar measure) allows one to manipulate the link variables in the perimeter integral, so that the final integration is performed over a single plaquette perimeter (cf. Gross-Witten one plaquette integral) in both ∂A and ∂A_n integrations. Specifically we absorb the spacelike links that separate different replicas in Z_n (time boundary of region $\bar{A}$) so that the system becomes identical to a simple plaquette with free spatial b.c.

Because of periodicity in time direction the ordered contour product of gauge fields generally has the form

$$U_{\partial A} = U_{0,\hat{1}} V_{1,\hat{0}} U_{0,\hat{1}}^\dagger V_{2,\hat{0}}^\dagger. \tag{12}$$

Here $U_{n,\hat{i}}$ denotes the gauge field at coordinate n in $\hat{i} = 0, 1$ direction, where $\hat{0}$ is chosen to be the time direction.

We use another property of character integration

$$\int dU_{0,\hat{1}} \chi_r(U_{0,\hat{1}} V_{1,\hat{0}} U_{0,\hat{1}}^\dagger V_{2,\hat{0}}^\dagger) = \frac{1}{d_r} \chi_r(V_{1,\hat{0}}) \chi_r(V_{2,\hat{0}}^\dagger). \tag{13}$$

The integral over the remaining two gauge variables decouples and has support only for the trivial representation $\chi_0 = 1$. This leads to a simple result

$$Z = F_0^A. \tag{14}$$

The ratio of the partition functions is unity and the entanglement entropy is zero.

Next we consider a lattice periodic in the spatial direction which effectively mimics infinite spatial extent. The perimeter integral for Z now is

$$\int dV \int dU \chi_r(UVU^\dagger V^\dagger) = \int dV \frac{1}{d_r} \chi_r(V) \chi_r(V^\dagger) = \frac{1}{d_r}, \tag{15}$$

where in the last part we used the character orthonormality property. The partition function becomes

$$Z = \sum_r F_r^A. \tag{16}$$

It is easy to check that the Z_n perimeter integral results in

$$\int dU_1 \dots dU_n \frac{1}{d_r} \frac{\chi_r(U_1)\dots\chi_r(U_n)}{d_r^{n-1}} \frac{\chi_r(U_1^\dagger)\dots\chi_r(U_n\dagger)}{d_r^{n-1}} = \frac{1}{d_r^{2n-1}}. \tag{17}$$

Note that this expression for $n = 1$ correctly reproduces the result of perimeter integration for Z, cf. (15). The partition function ratio is

$$\frac{Z_n}{Z^n} = \frac{\sum_r F_r^{nA}/d_r^{2n-2}}{(\sum_r F_r^A)^n} = \frac{1 + \sum_{r\neq 0} c_r^{nA}/d_r^{2(n-1)}}{(1 + \sum_{r\neq 0} c_r^A)^n}. \tag{18}$$

The entanglement entropy then is

$$S_A = -\frac{\partial}{\partial n} \frac{Z_n}{Z^n}\bigg|_{n=1} = \log\Big(1 + \sum_{r\neq 0} c_r^A\Big) - \frac{\sum_{r\neq 0} c_r^A \log c_r^A/d_r^2}{1 + \sum_{r\neq 0} c_r^A}. \tag{19}$$

Note that the series of character expansion coefficients is vanishing ($1 > c_r > c_s|$ if $d_s > d_r$) in such a way that the sums in (18) are converging even for the smallest surface $A = 1$. One then can choose the surface area large enough to guarantee that (19) is finite. We observe that this expression is l-independent [12] and valid for $l > 0$. The entanglement entropy expression (19) is universal in the sense that it does not depend on the initial lattice cutoff and is dependent only on the physical dimensions of the system. This is due to the fact that after a number of iterations the coefficients are attracted to the renormalization group (RG) trajectory independently of the starting point. It is interesting that at $l = 0$ Z_n factors into n copies of Z so that ratio $Z_n/Z^n = 1$ and $S_A = 0$. This is reminiscent of the 2-dimensional theory end-point phase transition at temperature $T = 0$.

If the surface area is very large, one can truncate the series to obtain a manageable expression. This is in fact similar to the strong coupling limit treatment. Using strong coupling expansion in evaluation of F_r, we obtain an approximate expression for the entanglement entropy (19). In general, one can compute F_r term by term to any desired order. For our purposes, however, it is enough to keep the first two lowest order terms, which give the coefficients for the trivial $r = 0$ and fundamental $r = 1$ representations

$$F_r \approx \int dU \Big(1 + \frac{\beta}{2N}[\chi_1(U) + \text{H.c.}]\Big) \frac{1}{d_r} \chi_r^*(U). \tag{20}$$

Thus $F_0 = 1$ and $c_1 = F_1 = \beta/(2N^2)$ for $N > 2$ [note that characters of $SU(2)$ group are self-conjugate and therefore $c_1 = \beta/N^2$]. The entropy becomes

$$S_A = \log\left(1 + \left(\frac{\beta}{2N^2}\right)^A\right) - \frac{(\frac{\beta}{2N^2})^A \log((\frac{\beta}{2N^2})^A/N^2)}{1 + (\frac{\beta}{2N^2})^A}$$
$$\approx \left(\frac{\beta}{2N^2}\right)^A \left(1 - \log\left(\left(\frac{\beta}{2N^2}\right)^A/N^2\right)\right). \quad (21)$$

Simplifications can be also achieved in the large N limit. The expressions for the first two representations F_r [integrals (7)] are readily available [9]. In the Gross-Witten paper notation $F_0 = z$ and $c_1 = \omega$

$$F_1 = \omega z = F_0 \times \begin{cases} 1/\lambda, & \lambda \geq 2 \\ 1 - \lambda/4, & \lambda \leq 2, \end{cases} \quad (22)$$

where $\lambda = g^2 N$ is the 't Hooft coupling.

Again for very large surface area it is reasonable to assume that the terms in this series are rapidly vanishing. Therefore the entanglement entropy becomes

$$S_A \approx \omega^A \left(1 - \log\frac{\omega^A}{N^2}\right). \quad (23)$$

Note that for the strong coupling $\omega = 1/\lambda = \beta/(2N^2)$, and the expression for S_A is equal to the strong coupling expansion derived earlier (we can interchange strong coupling and large N). It is no surprise that the entanglement entropy is sensitive to the 2D Gross-Witten phase transition and is different for strong and weak coupling phases.

B. $d \geq 2$ gauge theory

Next we consider the $d + 1$ dimensional theory, with $d \geq 2$. This is a nontrivial theory which cannot be solved exactly. We employ the Migdal-Kadanoff (MK) [13–15] decimation procedure to solve this theory approximately. For illustrative purposes we concentrate on $2 + 1$ theory, the generalization to higher dimensions is straightforward.

In general, for a finite temperature system one has to use anisotropic lattice. The time and space direction bond moving can be performed independently, cf. λ- and ρ-transformations for finite temperature gauge theory [16–18]. To simplify the treatment we consider a vanishing temperature system in a symmetric box.

The standard MK decimation procedure (λ-transformation) moves the internal plaquettes to the hypersurfaces which constitute the elementary cells of the resulting coarse lattice

$$e^{-S_p(U)} = \left[\sum_r F_r^A d_r \chi_r(U)\right]^{\zeta^{1-b}},$$
$$F_r = \int dU e^{-\zeta^b S_p(U)} \frac{1}{d_r} \chi_r^*(U), \quad (24)$$

where the choice $b = 0$ corresponds to Migdal, while $b = 1$ to Kadanoff prescription. Here $\zeta = \lambda^{d-2}$ is the factor by which we strengthen the interaction on the resulting coarse lattice in order to compensate for missing internal plaquettes, $A = \lambda^2$ is the surface of the new elementary plaquette in units of fine lattice plaquettes (number of tiling fine plaquettes), and λ is the scaling factor of the RG transformation and is equal to the number of plaquettes (internal and from the surface) moved to the surface from each of $d - 2$ directions.

It is known that the Kadanoff procedure results in the overcompensation of the strength of the coupling thus resulting in the upper bound for partition function, on the other hand leaving the coupling on the surface unchanged while dropping internal interactions $\zeta = 1$ results in the lower bound on the partition function [19]

$$Z(\zeta = 1) \leq Z \leq Z(\zeta = \lambda^{d-2}). \quad (25)$$

This relation relies on translation invariance and therefore does not hold for Z_n; however, one may expect it to hold approximately. As a result a generalization of the MK procedure which preserves the partition function may be possible to construct. Here, however, we use the standard MK decimation.

After each step of decimation iteration the partition function decomposes into the product of the coarse lattice partition function and the integrated out bulk part, which (after m step iteration) is

$$\prod_{j=0}^{m} F_0(j)^{|\Lambda|/\lambda^{jd}}. \quad (26)$$

Significant simplification can be achieved if we carry out decimations for Z_n and Z in exactly the same manner. As a result of equal volumes, the bulk contributions in Z_n and Z^n are identical and cancel out in their ratio at each step.

We start with a symmetric $d + 1$ dimensional decimation [λ-transformation (24)] in Z and Z_n, see Fig. 2. Note that there are periodicity conditions in the t-direction for each $\bar{A}$ part of n-replicas (bold links) and for the links of time boundary of the glued system (Z_n) belonging to A (dotted links). The decimation should be altered when the lattice spacing becomes equal to l (the smallest scale in the problem). At this point the l-like plaquettes (directed along l) inside the slab of thickness l (extending through all n replicas) have to be treated differently. These plaquettes can be decimated in the remaining directions, very much like timelike plaquettes in the finite temperature gauge theory treatment. Such transformations are normally referred as ρ-transformations, for them the decimation prescription (24) is modified:

$$e^{-S_{p;l}(U)} = \left[\sum_r F_r^\lambda d_r \chi_r(U)\right]^{\zeta^{1-b}},$$
$$F_r = \int dU e^{-\zeta^b S_{p;l}(U)} \frac{1}{d_r} \chi_r^*(U). \quad (27)$$

We still can move plaquettes in the $d - 2$ direction but the tiling is done with λ plaquettes. All the other plaquettes are unaffected by this change and are decimated according to the standard (λ-transformation) procedure.

In Appendix A we consider a gauge theory formulated in a box R^3. This system will be the building block for construction of expressions for Z_n and Z.

Let us assume that the imaginary surfaces that cut out the part for which we compute the entanglement entropy belong to $x-t$ planes and are a distance l apart in the y direction (note that we consider $2+1$).

We begin with Z. The surface with normal along x consists of 3 pieces after decimation is stopped. There is exactly the same contribution from the surface with normal $-x$. At the center there is a boundary of the slab c_x^s and two pieces which complement it, we refer to their joint as $\bar{c}_x^s$. The combined contribution is $c_x = c_x^s \bar{c}_x^s$ and should be substituted into the corresponding equation from the Appendix. We note here that technically the complement to the slab (more precisely two complementary volumes) is not symmetric, therefore the recursion at some point has to be switched from λ to ρ; however, we consider the scale $R \gg l$ and therefore we can always take R large enough so that the corresponding coefficients are in the strong coupling limit and no transition from the RG flow to the infrared fixed points can occur in these bulks.

The 2 surfaces with normals $\pm y$ each contribute c_y. There is only one group of surfaces (similar to c_x) with normal $-t$ (our convention) with contribution $c_t = c_t^s \bar{c}_t^s$. Therefore from (A2)

$$Z = 1 + \sum_{i\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2 + \sum_{i,j\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2 d_j c_{t,j}^s \bar{c}_{t,j}^s D_{ij}^i. \tag{28}$$

After many successive steps of decimation iteration, the only remaining degrees of freedom are defined on the surface of the system. In case of Z_n we also have $n-1$ l-like plaquettes inside the bulk ($c_{t,j}^s$). At this point we can move these l-like plaquettes in Z_n to the bottom surface. This decimation step has no counterpart in the denominator and therefore the bulk term ($\tilde{F}_0$) of this last decimation procedure does not cancel. This decimation step is achieved only with moving the internal plaquettes along the time direction onto a single surface plaquette; there is no integration of the tiling plaquettes for this procedure, therefore the new coefficients for the resulting surface plaquette after Kadanoff-type moving are

$$\tilde{F}^{\,s}_{t,j} = \int dU \left(1 + \sum_{i\neq 0} d_i c_{t,i}^s \chi_i(U)\right)^n \frac{1}{d_j} \chi_j(U^\dagger) \tag{29}$$

and

$$\tilde{c}^{\,s}_{t,j} = \frac{\tilde{F}_{t,j}^s}{\tilde{F}_{t,0}^s}. \tag{30}$$

Next we assume that the boundary between region $\bar{A}$, which has spacelike links with coordinates $t=0$ and $t=1/T$ identified, and region A, which has no such constraint is defined in such a way that the end links (directed along x) of the cut in Z_n belong to region A. Since there is no periodicity requirement for these links we can integrate them out. As a result the internal $n-1$ timelike surface terms of $\bar{A}$ have support only at the trivial representation and therefore do not contribute to the partition function. There is still, however, a contribution from the first replica timelike surface (bottom) of Z_n.

After simple considerations one can convince oneself that the surface integral in Z_n is similar (in $2+1$ dimensional theory up to factor $1/d_i^{4(n-1)}$) to the surface integral of a $nN_t \times N_r^2$ cube [20]. The side surface coefficients are modified to account for gluing n replicas, while the bottom surface coefficient involves a term computed according to (30) and is $\bar{c}_{t,i}^s \tilde{c}_{t,i}^s$. The partition function becomes

$$\begin{aligned} Z_n &\equiv \tilde{F}_{t,0}^s f_n \\ &= \tilde{F}_{t,0}^s \Bigg(1 + \sum_{i\neq 0} \frac{1}{d_i^{4(n-1)}} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^{2n} \\ &\quad \times \Bigg[1 + \sum_{j\neq 0} d_j \bar{c}_{t,j}^s \tilde{c}_{t,j}^s D_{ij}^i\Bigg]\Bigg). \end{aligned} \tag{31}$$

The ratio of the partition functions including the bulk term is

$$\frac{Z_n}{Z^n} = \tilde{F}_{t,0}^s \frac{1 + \sum_{i\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^{2n}/d_i^{4(n-1)} [1 + \sum_{j\neq 0} d_j \bar{c}_{t,j}^s \tilde{c}_{t,j}^s D_{ij}^i]}{(1 + \sum_{i\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2 [1 + \sum_{j\neq 0} d_j c_{t,j}^s \bar{c}_{t,j}^s D_{ij}^i])^n}. \tag{32}$$

In order to obtain a higher dimensional expression for this ratio, one needs to adjust accordingly the sides contribution and the contribution from the surface integration.

The entanglement entropy is

$$S_A = -\dot{\tilde{F}}_{t,0}^s + \log Z - \frac{\dot{f}_n}{Z}, \tag{33}$$

where the dot stands for $\dot{X} = \frac{\partial}{\partial n} X|_{n=1}$. Note that

$$\begin{aligned} \dot{f}_n &= \sum_{i\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2 \log \frac{(c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2}{d_i^4} \Bigg(1 + \sum_{j\neq 0} d_j \tilde{c}_{t,j} D_{ij}^i\Bigg) \\ &\quad + \sum_{i\neq 0} (c_{x,i}^s \bar{c}_{x,i}^s c_{y,i})^2 \sum_{j\neq 0} d_j \bar{c}_{t,j}^s \dot{\tilde{c}}_{t,j}^s D_{ij}^i. \end{aligned} \tag{34}$$

In order to manipulate these expressions we will need the following derivatives:

$$\dot{\tilde{c}}^{\,s}_{t,j} = \dot{\tilde{F}}_{t,j}^s - c_{t,j}^s \dot{\tilde{F}}_{t,0}^s \tag{35}$$

$$\begin{aligned} \dot{\tilde{F}}_{t,j}^s &= \int dU \Bigg[1 + \sum_{i\neq 0} d_i c_{t,i}^s \chi_i(U)\Bigg] \log\Bigg(1 + \sum_{i\neq 0} d_i c_{t,i}^s \chi_i(U)\Bigg) \\ &\quad \times \frac{1}{d_j} \chi_j(U^\dagger). \end{aligned} \tag{36}$$

The expression for the entanglement entropy can be evaluated if the system flows towards the IR fixed point. This is the strong coupling limit for $c^s_{t,j}$ therefore we can expand logarithms and simplify the expression:

$$\tilde{F}^s_{t,j\neq 0} = c^s_{t,j} + \sum_{i,i'\neq 0} \frac{d_i d_{i'}}{d_j} c^s_{t,i} c^s_{t,i'} D^j_{ii'} = c^s_{t,j} + O(c^2)$$

$$\tilde{F}^s_{t,0} = \sum_{i,j\neq 0} c^s_{t,i} c^s_{t,j} d_i d_j \int dU \chi_i(U)\chi_j(U) = O(c^2).$$

The leading term in the entropy is

$$S_A \approx -(c^s_{x,1}\bar{c}^s_{x,1}c_{y,1})^2 \log(c^s_{x,1}\bar{c}^s_{x,1}c_{y,1})^2. \quad (37)$$

Note that the dependence on l is encoded in the value of $c^s_{x,1}$.

C. Analyzing the RG flow

Now recall that our choice of temperature makes the box symmetric and $c^s_{t,i} = c^s_{x,i} = c^s_i$. The resulting expression for the entanglement entropy (33) is a very complicated function of c^s_i. Note that this is a general feature valid for higher dimensional theories as well. The dependence on l enters through the value of these coefficients. Essentially l regulates the moment when λ-transformation is switched to ρ-transformation, which in turn sets the initial value for the $c^s_i(m_0)$ iteration under ρ-transformations thus defining where the theory will flow before reaching the boundary.

Next we analyze the RG flow of $SU(2)$ gauge theory for $c^s_i(m)$ as a function of number of iterations m under Migdal recursion (27) and depending on the starting point. In Fig. 3 we plot the projection of the flow (for a $3+1$ dimensional theory) from the infinite dimensional space of character coefficients onto the fundamental-adjoint $c_{1/2} - c_1$ plane. We consider $\lambda = 1.1$ and 2 values and observe a significant dependence on the choice of the scaling factor. In what follows we will use the former value, since it is known to reproduce the $SO(3)$ critical coupling value [21]. This value was also used to extract an approximately correct phase diagram for the mixed action fundamental-adjoint $SU(2)$ gauge theory [22].

One can clearly observe that depending on the starting value the flow will go to either of the two fixed points—the infrared trivial fixed point or nontrivial UV fixed point. This is a clear indication of a transition. The starting point for the system is set by the λ-transformations and depends on the value of l. Generally at the starting point the action is a single plaquette action but with an infinite number of couplings for terms in all irreducible representations. In the numerical simulation that gives Fig. 3, we simplify this situation by considering a starting action in the Wilsonian (only fundamental representation) form on the $N_t = 1$ lattice, noting that this should not affect the observed picture of existence of transition.

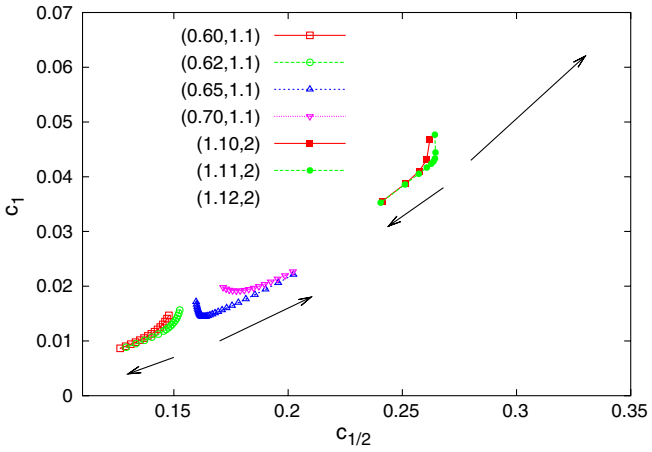

FIG. 3 (color online). Migdal decimation flow for $3+1$ dimensional $SU(2)$ gauge theory. Projection to $c^s_{1/2}$ and c^s_1; (β, λ) are indicated.

The lattice inverse coupling value $\beta^*_c \in (0.62, 0.65)$ where the transition in the flow occurs should be compared to $N_t = 1$ gauge theory finite temperature phase transition $\beta_c \approx 0.86$ [23]. This allows one to relate the scale l^*_c of the entanglement entropy transition to the finite temperature phase transition scale $l_c = 1/T_c$. For this we use the standard 1-loop scaling relationship

$$a(\beta)\Lambda_L = \left(\frac{\beta}{2Nb_0}\right)^{b1/2b_0^2} \exp\left(-\frac{\beta}{4Nb_0}\right), \quad (38)$$

where $b_0 = 11/24\pi^2$ and $b_1/2b_0^2 = 51/121$. Substituting the couplings we obtain

$$l^*_c/l_c \in (1.56, 1.66). \quad (39)$$

III. DISCUSSION OF THE RESULTS

In this paper we studied the entanglement entropy in $d+1$ $SU(N)$ gauge theory. We use the multireplica trick to relate the entanglement entropy to a simple ratio of partition functions. The $d = 1$ theory is solved exactly. Free spatial b.c. lead to trivially zero entanglement entropy. Periodic spatial b.c. show nonzero universal value independent of the size l of the entangled region. As the entangled region is removed the entropy becomes zero, showing in this manner behavior similar to the end-point phase transition of $1+1$ dimensional theories.

Using MK decimation, we approximately computed the ratio of partition functions and entanglement entropy in $d \geq 2$ dimensional theories. A note of caution should be made regarding our choice to carry out the decimation for Z_n and Z in the same way. This allows us to significantly simplify the computational procedure. The nonanalyticity in the RG flow observed for Z_n, however, is also induced in Z by this choice. This should not be a problem if one is interested only in the location of the transition.

In the case of $3+1$ $SU(2)$ gauge theory, we demonstrated that there is a nonanalytical change in the RG flow for coefficients of character expansions which define the entanglement entropy. We find that the length scale of this transition is $l_c^*/l_c \in (1.56, 1.66)$. Unfortunately the systematic error due to the use of the MK approximation is not easily tractable. It is interesting that in the large N_c case it was shown [2] that $l_c^*/l_c = 2$.

It is important to note that the MK procedure does not find a transition in the RG flow for $2+1$ dimensional theories. This transition is only observed for $d+1$ theories with $d > 2$. Most likely this is an artifact of the MK procedure and $d = 2$ theory exhibits a transition similar to higher dimensional theories. The MK decimation is known to miss the order of phase transition while correctly identifying its location. It is conceivable that in $d = 2$ theory a proper transition can be seen by the MK procedure as a crossover. This observation is supported by the fact that we indeed observe an interesting qualitative change in the flow around $\beta = 3.2$, see Fig. 4. For values of lattice inverse coupling below this value, the flow is directed immediately toward the IR critical point (monotonously decreasing series of c_i, $\forall i$), while for larger values of β the flow is directed from the IR fixed point for a few steps of iteration then switching to the flow toward the IR fixed point (initial increase of $c_{1/2}$ followed by monotonous decrease). We illustrate such a scenario for weak coupling regime $\beta = 4.0$ in the inlet of Fig. 4. It is interesting that formally the MK ρ-transformation in $d+1$ theory with compact direction can be effectively viewed as λ-transformation with an effective RG scaling parameter $\sqrt{\lambda}$ in $2d-2$ dimensions, see [16–18]. Thus $d > 2$ theories are related to the zero temperature theories above the critical dimension 4 (have bulk phase transition), while $d = 2$ is related to 4-dimensional zero temperature theory.

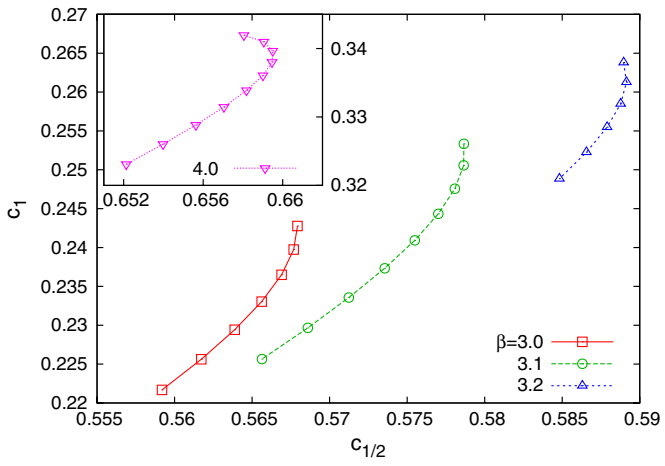

FIG. 4 (color online). Migdal decimation flow for $2+1$ dimensional $SU(2)$ gauge theory. Projection to $c^s_{1/2}$ and c^s_1; $\lambda = 1.1$, $\beta = 3.0$, 3.1, 3.2, and 4.0 (inlet).

Similar results hold for $SU(3)$ and other N_c groups. Therefore our claim is that the transition in the entanglement entropy is observed for any number of colors N_c and the critical scale l_c^* where the transition takes place most likely is N_c dependent and asymptotically reaches 2 as $N_c \to \infty$.

We note that the finite temperature phase transition studies of $SU(2)$ and $SU(3)$ gauge theory within MK formalism [16–18] relied on the same analysis of the RG flow. It is important to emphasize that the periodic boundary conditions in the time direction do not play any role in such studies. One has to impose the periodicity on the $N_t = 1$ system after λ-transformations are switched to ρ-transformations [16]. This will result in an effective lower dimensional spin system which exhibits a phase transition for $d \geq 2$.

The study of the entanglement entropy effectively is transformed into an MK analysis of a gauge system defined with one compact direction and no periodicity imposed. Possibly Monte Carlo simulations of such systems can define the location of the transition more accurately. However, this would be still a crude approximation since the MK treatment results in a well-defined boundary, which is in reality rather soft. Therefore direct numerical computation of the entanglement entropy should be preferred.

It is also interesting to relate our results to studies of the vortex free-energy order parameter [24], which provides a complete characterization of the possible phases of gauge theory. For $SU(2)$ it was found [25] that when the transverse size of the lattice is around 0.7 fm there is a sharp crossover in the vortex free energy. This crossover has an obvious physical interpretation: the lattice size has to be large enough to accommodate sufficient spreading of the vortex flux ("fat" vortex) to enter the regime of exponential free-energy lowering by further spreading, i.e. the confining or color magnetic mass-gap creation regime. Assuming that $\sqrt{\sigma} = 420$ MeV (σ is the string tension), we get in this theory $1/T_c = 0.681$ fm. Therefore the transition in the vortex free energy happens approximately at $1/T_c$ scale.

Using this observation, we suggest that the transition in the entanglement entropy happens when the size of the entangled region is large enough to accommodate a fat vortex. The difference in the geometry should account on small difference of the scales when such transition occurs.

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APPENDIX: 2 + 1 DIMENSIONAL GAUGE THEORY IN A BOX

In this Appendix we consider a $2+1$ dimensional $SU(N)$ gauge theory formulated in a symmetric box R^3 at temperature $T = 1/R$, which corresponds to confined phase temperatures for sufficiently large R. The theory is formulated on a lattice with the UV cutoff a, with periodic boundary condition in the t direction and free boundary condition in spatial directions. The MK decimations (24) with scale factor λ are performed iteratively N times ($\lambda^N = \hat{R} \equiv R/a$). At this point all degrees of freedom are "pushed" to the boundary and the resulting lattice spacing becomes equal to R, see Fig. 5. Here we are not interested in the bulk contribution. There are 8 independent gauge degrees of freedom which live on links. We use normals to the cube faces, which are directed outside to identify plaquettes. The partition function has contributions from 4 plaquettes with normals along spatial directions and one plaquette (due to periodicity) from t-direction

$$f(\{c_z\}; \partial A) \equiv 1 + \sum_{i\neq 0} d_i c_{z;i} \chi_i(\partial A_z), \tag{A1}$$

where $z = \pm x, \pm y, t$ marks the plaquettes. The character coefficients $c_{z;i}$ can be obtained numerically and are the result of the RG flow in infinite dimensional coupling space. The symmetry of the box implies $c_{x;i} = c_{y,i} = c_{t,i}$. For a general nonsymmetric box, one has to consider a series of ρ-transformations, resulting in an anisotropic lattice with all coefficients $c_{z,i}$ different.

Because of the free spatial boundary condition, we can further integrate out three timelike links (thin lines in Fig. 5). By doing this we join the surfaces according to (9), with the resulting surface term $f(\{c_{xy,i}\}; U^\dagger V U V^\dagger)$ and $c_{xy,i} = c_{x,i}^2 c_{y,i}^2$. The partition function is

$$\begin{aligned} Z &= \int dU dV f(\{c_{xy,i}\}; U^\dagger V U V^\dagger) f(\{c_{t,i}\}; V) \\ &= 1 + \sum_{i\neq 0} c_{xy,i} + \sum_{i,j\neq 0} c_{xy,i} d_j c_{t,j} D^i_{ij}, \end{aligned} \tag{A2}$$

where

$$D^k_{ij} = \int dV \chi_k(V^\dagger) \chi_i(V) \chi_j(V) \tag{A3}$$

we recognize as the coefficients of the Clebsch-Gordan series $\mathcal{D}^{(i)} \times \mathcal{D}^{(j)} = \sum_k D^k_{ij} \mathcal{D}^{(k)}$ for the Kronecker product of irreducible representations. Using Gaunt's formula

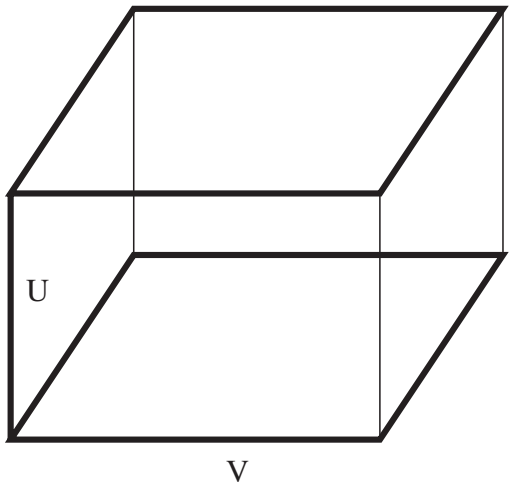

FIG. 5. $2+1$ dimensional symmetric box.

$$\begin{aligned} &|G|^{-1} \int_G \mathcal{D}^{(j_1)}(R^{-1})_{n_1 m_1} \mathcal{D}^{(j_2)}(R)_{n_2 m_2} \mathcal{D}^{(j_3)}(R)_{n_3 m_3} dR \\ &= \begin{pmatrix} j_1 \\ n_1 \mu \end{pmatrix} \begin{pmatrix} j_1 \\ \nu m_1 \end{pmatrix}^* \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu & n_2 & n_3 \end{pmatrix}^* \begin{pmatrix} j_1 & j_2 & j_3 \\ \nu & m_2 & m_3 \end{pmatrix}, \end{aligned} \tag{A4}$$

where $|G|$ is the volume of the group space, we can express D^k_{ij} through the Wigner coefficients (1-j and 3-j symbols) for general group [26,27]

$$D^k_{ij} = \begin{pmatrix} k \\ n_1 \mu \end{pmatrix} \begin{pmatrix} k \\ \nu n_1 \end{pmatrix}^* \begin{pmatrix} k & i & j \\ \mu & n_2 & n_3 \end{pmatrix}^* \begin{pmatrix} k & i & j \\ \nu & n_2 & n_3 \end{pmatrix}. \tag{A5}$$

Coefficients D^r_{rs} can be easily evaluated for the $SU(2)$ group, using the Clebsch-Gordan equation

$$\chi_i \chi_j = \sum_{k=|i-j|}^{i+j} \chi_k. \tag{A6}$$

Thus the integral becomes

$$D^i_{ij} = \int dV \sum_{k=0}^{2i} \chi_k(V) \chi_j(V) = H_1(2i - j), \tag{A7}$$

where $H_1(x)$ is the Heaviside step function [$H_1(0) = 1$]. Therefore

$$Z_{SU(2)} = 1 + \sum_{i\neq 0} c_{xy,i} + \sum_{i,j\neq 0; j\leq 2i} c_{xy,i} d_j c_{t,j}. \tag{A8}$$

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