Nonminimal coupling for the gravitational and electromagnetic fields: Black hole solutions and solitons

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Using a Lagrangian formalism, a three-parameter nonminimal Einstein-Maxwell theory is established. The three parameters q_1, q_2, and q_3 characterize the cross-terms in the Lagrangian, between the Maxwell field and terms linear in the Ricci scalar, Ricci tensor, and Riemann tensor, respectively. Static spherically symmetric equations are set up, and the three parameters are interrelated and chosen so that effectively the system reduces to a one parameter only, q. Specific black hole and other type of one-parameter solutions are studied. First, as a preparation, the Reissner-Nordström solution, with $q_1 = q_2 = q_3 = 0$, is displayed. Then, we search for solutions in which the electric field is regular everywhere as well as asymptotically Coulombian, and the metric potentials are regular at the center as well as asymptotically flat. In this context, the one-parameter model with $q_1 \equiv -q$, $q_2 = 2q$, $q_3 = -q$, called the Gauss-Bonnet model, is analyzed in detail. The study is done through the solution of the Abel equation (the key equation), and the dynamical system associated with the model. There is extra focus on an exact solution of the model and its critical properties. Finally, an exactly integrable one-parameter model, with $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$, is considered also in detail. A special submodel, in which the Fibonacci number appears naturally, of this one-parameter model is shown, and the corresponding exact solution is presented. Interestingly enough, it is a soliton of the theory, the Fibonacci soliton, without horizons and with a mild conical singularity at the center.

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I. INTRODUCTION

A natural and very general extension of the Einstein-Maxwell Lagrangian yielding a general system of equations for a nonminimal coupling between the gravitational and electromagnetic fields, with nonlinear terms, was set up and studied in [1]. Within this general theory, a special theory, worthy of discussion, arises when one restricts the general Lagrangian to a Lagrangian that is Einstein-Hilbert in the gravity term, quadratic in the Maxwell tensor, and the couplings between the electromagnetism and the metric are linear in the curvature terms. The motivations for setting up such a theory are phenomenological, see, e.g., [2,3] for reviews and references. This theory has three coupling constants q_1, q_2, and q_3, which characterize the cross-terms in the Lagrangian between the Maxwell field F_{ij} and terms linear in the Ricci scalar R, Ricci tensor R_{ik}, and Riemann tensor R_{ikmn}, respectively. The coupling constants q_1, q_2, and q_3 have units of area, and are *a priori* free parameters, which can acquire specific values in certain effective field theories. More specifically, the action functional of the nonminimal theory linear in the curvature is

$$S = \int d^4x\sqrt{-g}\mathcal{L}, \tag{1}$$

where g is the determinant of the spacetime metric g_{ik}, and the Lagrangian of the theory is

$$\mathcal{L} = \frac{1}{4\pi}\left(\frac{R}{\kappa} + \frac{1}{2}F_{mn}F^{mn} + \frac{1}{2}\chi^{ikmn}F_{ik}F_{mn}\right). \tag{2}$$

Here $\kappa = 2G$, G being the gravitational constant and we are putting the velocity of light c equal to one, $F_{ik} = \partial_i A_k - \partial_k A_i$ is the Maxwell tensor, with A_k being the electromagnetic vector potential, and Latin indexes are spacetime indexes, running from 0 to 3. The tensor χ^{ikmn} is the nonminimal susceptibility tensor given by

$$\chi^{ikmn} \equiv \frac{q_1 R}{2}(g^{im}g^{kn} - g^{in}g^{km}) + \frac{q_2}{2}(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) + q_3 R^{ikmn}, \tag{3}$$

where q_1, q_2, and q_3 are the mentioned phenomenological parameters. The action and Lagrangian (1)–(3) describe thus a three-parameter class of models, nonminimally

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coupled, and linear in the curvature [1–3]. Lagrangians of this type have been used and studied by several authors.

The first and important example of a calculation of the three couplings was based on one-loop corrections to quantum electrodynamics in curved spacetime, a direct and nonphenomenological approach considered by Drummond and Hathrell [4]. This model is effectively one-parameter since the coupling constants are connected by the relations $q_1 \equiv -5q$, $q_2 = 13q$, $q_3 = -2q$. The positive parameter q appears naturally in the theory, and is constructed by using the fine structure constant α, and the Compton wavelength of the electron λ_e, $q \equiv \frac{\alpha\lambda_e^2}{180\pi}$. In these models it is useful to define a radius r_q, an effective radius related to the nonminimal interaction, through $r_q = \sqrt{2|q|}$. Thus, the corresponding effective radius for the nonminimal interaction in this case, is the Drummond-Hathrell radius $r_{q\mathrm{DH}}$, given by $r_{q\mathrm{DH}} \equiv \lambda_e\sqrt{\frac{\alpha}{90\pi}}$. In [5] one also finds a quantum electrodynamics motivation for the use of generalized Einstein-Maxwell equations.

Phenomenological models, i.e., models based on external considerations to obtain the couplings, or parameters, q_1, q_2, and q_3, have also been considered. Prasanna [6,7] wanting to understand how the strong equivalence principle can be weakly violated in the context of a nonminimal modification of Maxwell electrodynamics, has shown that $q_1 = q_2 = 0$, $q_3 \equiv -q$, q a free parameter, is a good phenomenological model. Another type of requirement, one with mathematical and physical motivations, is to impose that the differential equations forming the nonminimal Einstein-Maxwell system are of second order (see, e.g., [8,9]). For instance, in [9], by imposing a Kaluza-Klein reduction to four dimensions from a Gauss-Bonnet model in five dimensions, thus guaranteeing second order equations for the electric field potential A_i, and metric g_{ik}, it was discussed a model in which $q_1 + q_2 + q_3 = 0$ and $2q_1 + q_2 = 0$, i.e., with $q_1 \equiv -q$, $q_2 = 2q$ and $q_3 = -q$. So the extra nonminimal term is a kind of Gauss-Bonnet term, and the model is called the Gauss-Bonnet model. Yet another type of requirement, this time purely mathematical, was suggested in [1]. The idea is connected with the symmetries of the nonminimal susceptibility tensor χ^{ikmn} [see Eq. (3)]. For instance, one can recover the relations $q_1 + q_2 + q_3 = 0$ and $2q_1 + q_2 = 0$, used in [9], by the ansatz that the nonminimal susceptibility tensor χ_{ikmn} is proportional to the double dual Riemann tensor ${}^*R^*_{ikmn}$, i.e., $\chi_{ikmn} = \gamma {}^*R^*_{ikmn}$, for some γ (see, [1] for details and motivations). Analogously, one can use the Weyl tensor $\mathcal{C}_{ikmn}$ in the relation $\chi_{ikmn} = \omega\mathcal{C}_{ikmn}$, for some ω, or the difference $R_{ikmn} - \mathcal{C}_{ikmn}$ instead of ${}^*R^*_{ikmn}$, to introduce some new linear relations between q_1, q_2, q_3, namely $3q_1 + q_2 = 0$ and $q_2 + q_3 = 0$. Yet another type of requirement is to choose the parameters so that one obtains exact solutions. As we will see this will lead to a model with $q_1 + q_2 + q_3 = 0$ and $q_3 = 0$, i.e., $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$. Since this model is integrable we call it the integrable model. A subcase of this has additional interest and is called the Fibonacci soliton.

Up to now we have a theory defined through Eqs. (1)–(3), with each chosen set of values for the parameters q_1, q_2, and q_3, giving a model. We have seen that the reduction from three-parameter models to one-parameter models, specified by the one-parameter q and the relations between q_1, q_2, and q_3, happens in several instances, either through direct calculation, as in [4,5], or through phenomenological and other considerations, as in [6–9] or [1] and here. This certainly simplifies the analysis, and we will consider this one-parameter type of models, in which q_1, q_2, and q_3, have a specified relation to the parameter q. For all these models, one can pick an effective radius $r_q \equiv \sqrt{2|q|}$, as in the Drummon-Hathrell case, which gives the range of the nonminimal interaction between the gravitational and electric fields. Of course, r_q can be set to zero, in the case the world is pure Einstein-Maxwell, or otherwise can have a given specified value. The radius $r_{q\mathrm{DH}}$ defined above is a candidate but in principle not the unique choice. Thus, possible estimations of the parameter q, and so of r_q, from, for instance, astrophysical observations, are undoubtedly of interest (see, e.g., [7]).

Now, after choosing a model, specified by q and by the relations between q_1, q_2, and q_3, it is important to study exact solutions. Exact solutions of the equations of nonminimal electrodynamics in nonlinear gravitational wave backgrounds were obtained in [10–14], and a nonminimal Bianchi-I cosmological solution was discussed in [15] in this context. Here we want to study charged black hole and other charged solutions of nonminimal models. The Reissner-Nordström solution is a standard solution in pure Einstein-Maxwell theory, with two horizons, an event and a Cauchy horizon, and a timelike singularity at the center (see, e.g., [16]). Paradigmatic charged black hole solutions also appear in the framework of Einstein theory minimally coupled to nonlinear electromagnetic fields, as well as other matter fields. Such solutions were found by Bardeen and others [17–21] and the main feature is that they are regular, without singularities inside the horizon. Within quartic gravity nonsingular charged black hole solutions have also been found [22]. Since the nonminimal theory we are considering possesses new degrees of freedom, namely, the phenomenological parameter q and its relations to q_1, q_2, and q_3, we believe that these allow us to introduce new aspects to the problem of finding black hole and other solutions of each chosen model. Three aspects can be mentioned. First, one wants to have a gauge in order to compare the new solutions. Thus, we study the Reissner-Nordström solution, the trivial solution in this context, where $q_1 = q_2 = q_3 \equiv q$ with $q = 0$, in order to understand the novel features, such as causal and singularity structure, of the new solutions. Second, one should try to search for models exactly or quasiexactly soluble. This requirement will take us, among the models cited above and the many other possible models, to two interesting models. They are, the Gauss-Bonnet type model where $q_1 \equiv -q$, $q_2 = 2q$ and $q_3 = -q$, and the integrable model where $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$. In both models we perform a detailed analysis. Third, we vary q within each of the two nontrivial models and in trying to consider nonminimal extensions of the Reissner-Nordström solution, we search for features that are similar or distinct from the two paradigmatic solutions, the Reissner-Nordström solution itself and the Bardeen solutions. In our search for charged black hole solutions in nonminimal models we work in Schwarzschild coordinates and impose certain requirements. The first requirement is connected with the electric field $E(r)$. We demand it is a regular function on the interval $0 \leq r < \infty$, the origin being also regular (i.e., the value $E(0)$ is finite), and for large values of r the electric field is Coulombian, $E(r) \rightarrow \frac{Q}{r^2}$, where Q is the electric charge. The second requirement is concerned with the metric functions g_{ik}. These should take finite values at the center ($g_{ik}(0) \neq \infty$). Horizons at $r \neq 0$ are not excluded, and far away the solutions should be asymptotically flat. Upon these conditions and solving the nonminimal equations in certain cases we will find electric charged solutions with one horizon only, thus causally distinct from the Reissner-Nordström. However, like Reissner-Nordström, the solutions have a singularity at the center, although here the singularity is spacelike instead, as the Schwarzschild case, but conical, thus much milder. We have also found in one model a gravitational charged soliton, without horizons, where the fields are well behaved, apart from a mild conical singularity at the center. Although this and other solutions with horizons are almost regular at the center, we have not obtained strictly nonsingular black hole of the type found by Bardeen and others [17–22]. The difference is based on two aspects. First, we consider spherically symmetric static solutions with $g_{00}(r)g_{rr}(r) \neq 1$, in contrast to the Schwarzschild, Reissner-Nordström, and minimal regular Bardeen solutions. Second, we assume that the values $g_{00}(0)$ and $g_{rr}(0)$ are finite, but can differ from one. Moreover, we admit, that $g_{00}(0)$ can, in principle, be equal to zero. This means that the scalar curvature invariants for such a metric can take infinite values at the center, and the solution of the nonminimal Einstein equations is not regular at the center in this general sense. However, in many cases the singularity is a conical one, so much milder than the nasty ones of Schwarzschild and Reissner-Nordström. In summary, we find charged black hole solutions different in horizon structure from the Reissner-Nordström but similar to Bardeen black holes, and although singular, in certain cases can be considered quasiregular conical singularities, in-between Schwarzschild and Reissner-Nordström types of singularities and the no singularities of Bardeen.

This paper is organized as follows. In Sec. II, in particular, in subsection II A, using the Lagrangian formalism of the Introduction, we establish a three-parameter nonminimal Einstein-Maxwell model. In Sec. II B, we set up static equations for studying black holes and reduce the three-parameter model to a one-parameter model. In Sec. III we study specific spherically symmetric one-parameter solutions. In Sec. III A we define the basic quantities and basic variables. In Sec. III B we display the Reissner-Nordström solution as a preparation, where $q_1 = q_2 = q_3 \equiv q$ with $q = 0$. In subsection III C we analyze in detail a one-parameter model, the Gauss-Bonnet model, with $q_1 \equiv -q$, $q_2 = 2q$, $q_3 = -q$ (i.e., $q_1 + q_2 + q_3 = 0$ and $2q_1 + q_2 = 0$), using the known solution of the Abel equation, the key equation of the model, and the dynamical system associated with this model. We display the charged black hole solutions, and we focus on a specific exact solution and its critical properties. In subsection III D we consider in detail an exactly integrable one-parameter model, the integrable model, with $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$ (i.e., $q_1 + q_2 + q_3 = 0$ and $q_3 = 0$). We examine a special submodel, the Fibonacci soliton, of this one-parameter model and present the corresponding exact solution. In Sec. III E we present, by means of a table, the summary of the results of the models studied. In Sec. IV we conclude.

II. NONMINIMAL COUPLING, LINEAR IN THE CURVATURE, BETWEEN GRAVITY AND ELECTROMAGNETISM: EQUATIONS AND REDUCTION FROM THREE PARAMETERS TO ONE PARAMETER FOR STATIC SPHERICALLY SYMMETRIC SYSTEMS

A. The three-parameter model: General equations

The variation of the Lagrangian (1)–(3) with respect to the metric yields (see, [1] for details),

$$R_{ik} - \frac{1}{2}R g_{ik} = \kappa[T^{(0)}_{ik} + q_1 T^{(1)}_{ik} + q_2 T^{(2)}_{ik} + q_3 T^{(3)}_{ik}]. \quad (4)$$

The energy-momentum tensor of the pure electromagnetic field $T^{(0)}_{ik}$ is

$$T^{(0)}_{ik} = \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{im} F_k{}^m. \quad (5)$$

The definitions for the other three parts of the stress-energy tensor, $T^{(1)}_{ik}$, $T^{(2)}_{ik}$, and $T^{(3)}_{ik}$, are

$$T^{(1)}_{ik} = R T^{(0)}_{ik} - \frac{1}{2} R_{ik} F_{mn} F^{mn} - \frac{1}{2} g_{ik} \nabla^l \nabla_l (F_{mn} F^{mn}) + \frac{1}{2} \nabla_i \nabla_k (F_{mn} F^{mn}), \quad (6)$$

$$\begin{aligned} T^{(2)}_{ik} = & -\frac{1}{2} g_{ik} [\nabla_m \nabla_l (F^{mn} F^l{}_n) - R_{lm} F^{mn} F^l{}_n] \\ & - F^{ln} (R_{il} F_{kn} + R_{kl} F_{in}) - R^{mn} F_{im} F_{kn} \\ & - \frac{1}{2} \nabla^l \nabla_l (F_{in} F_k{}^n) + \frac{1}{2} \nabla_l [\nabla_i (F_{kn} F^{ln}) \\ & + \nabla_k (F_{in} F^{ln})], \end{aligned} \quad (7)$$

$$T^{(3)}_{ik} = \tfrac{1}{4}g_{ik}R^{mnls}F_{mn}F_{ls} - \tfrac{3}{4}F^{ls}(F_i{}^nR_{knls} + F_k{}^nR_{inls}) - \tfrac{1}{2}\nabla_m\nabla_n(F_i{}^nF_k{}^m + F_k{}^nF_i{}^m). \quad (8)$$

In addition, the nonminimal electrodynamics associated with the Lagrangian (1)–(3) obeys the equation

$$\nabla_k[F^{ik} + \chi^{ikmn}F_{mn}] = 0, \qquad \nabla_kF^{*ik} = 0, \quad (9)$$

where F_{mn} is the Maxwell tensor and F^*_{kl} is dual to it. Consider now these master equations in the case of a static spherically symmetric spacetime.

B. Static spherically symmetric nonminimally coupled fields. Reduced three-parameter system of equations: One-parameter models

1. Preliminaries

Using Schwarzschild coordinates, the line element for a static spherically symmetric system can be put in the form

$$ds^2 = B(r)c^2dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (10)$$

where B and A are metric potentials that depend on the radial coordinate r only. This form of the line element is useful as when B and $1/A$ are simultaneously zero it signals the presence of an event horizon. Assume also that the electromagnetic field inherits the static and spherical symmetries. Then the potential four-vector of the electric field A_i has the form

$$A_i = A_0(r)\delta^0_i. \quad (11)$$

From (11) the Maxwell tensor is equal to $F_{ik} = A_0'(r)\times(\delta^r_i\delta^0_k - \delta^0_i\delta^r_k)$, where a prime denotes the derivative with respect to r. To characterize the electric field, it is useful to introduce a new scalar quantity $E(r)$ as $E^2(r) \equiv -\frac{1}{2}F_{mn}F^{mn}$. Then the electric field squared is $E^2(r) = \frac{1}{AB}(A_0')^2$ from which one obtains in turn $F_{r0} = -(AB)^{1/2}E(r)$. Since the expressions $1/A$ and $\sqrt{AB}$ enter frequently in the master equations, it is sometimes convenient to use the functions $N(r)$ and $\sigma(r)$ defined as $N(r) \equiv \frac{1}{A(r)}$ and $\sigma(r) \equiv \sqrt{A(r)B(r)}$. In summary, the functions

$$E^2(r) \equiv \frac{1}{\sigma^2}(A_0')^2, \quad (12)$$

and

$$N(r) \equiv \frac{1}{A(r)}, \qquad \sigma(r) \equiv \sqrt{A(r)B(r)}, \quad (13)$$

are alternatives to the functions $A_0(r)$, $A(r)$, and $B(r)$.

2. Key equation for the Maxwell field and its solution

The Maxwell equations (9) with (11) give only one nontrivial equation, namely,

$$[r^2E(r)(1 + 2\chi^{0r}{}_{0r}(r))]' = 0, \quad (14)$$

which can be integrated immediately to give

$$E(r)\left\{r^2\left[1 + (q_1+q_2+q_3)\left(N'' + 3N'\frac{\sigma'}{\sigma} + 2N\frac{\sigma''}{\sigma}\right)\right] + 2r(2q_1+q_2)\left(N' + N\frac{\sigma'}{\sigma}\right) + 2q_1(N-1)\right\} = Q, \quad (15)$$

where Q is a constant, to be associated with the central electrical charge of the solution. This equation gives the electric field of a central charge, corrected by the radial component of the dielectric permeability tensor, $1 + 2\chi^{0r}{}_{0r}(r)$. This component describes the vacuum screening effect on the central charge, due to the interaction of the vacuum with curvature, analogously to the screening of a charge by a nonhomogeneous medium in a spherical cavity. Supposing the spacetime to be asymptotically flat, i.e., $R^i{}_{klm}(\infty) = 0$, one can see that (15) yields asymptotically the Coulomb law $E \to Q/r^2$, and the constant Q indeed coincides with the total electric charge of the object.

3. Key equations for the gravitational field

The equations for the gravitational field (4) with (5)–(8) and the metric potentials (10) redefined as in (13) can be rewritten as a pair of equations for $N(r)$ and $\sigma(r)$, respectively,

$$\frac{[r(1-N)]'}{\kappa r^2} = -(E^2)''N(q_1+q_2+q_3) + (E^2)'\left[-\frac{1}{2}(q_1+q_2+q_3)\left(N' + \frac{8N}{r}\right) + \frac{N}{r}(2q_1+q_2)\right] + E^2\left[\frac{1}{2} + (q_1+q_2+q_3) \times\left(N'' + 3N'\frac{\sigma'}{\sigma} + 2N\frac{\sigma''}{\sigma} - \frac{N'}{r} - 2\frac{N}{r^2}\right) + (2q_1+q_2)\left(2\frac{N'}{r} + 2\frac{N}{r}\frac{\sigma'}{\sigma} + \frac{N}{r^2}\right) + q_1\frac{(N-1)}{r^2}\right], \quad (16)$$

$$\frac{2\sigma'}{\kappa r\sigma} = -(E^2)''(q_1+q_2+q_3) + (E^2)'\left[(q_1+q_2+q_3)\left(\frac{\sigma'}{\sigma} - \frac{4}{r}\right) + (2q_1+q_2)\frac{2}{r}\right] + E^2\left[(q_1+q_2+q_3)\frac{2\sigma'}{r\sigma} - \frac{2q_3}{r^2}\right]. \quad (17)$$

The first equation can be reduced to an equation for $E(r)$ and $N(r)$, by extracting the term $\frac{\sigma'}{\sigma}$ from the second one. The second equation contains the unknown functions $E(r)$ and $\sigma(r)$ only. Thus, Eqs. (15)–(17) form the key system of equations for the nonminimal Einstein-Maxwell model of a static spherically symmetric object. It is a system of three ordinary differential equations of second order for the three unknown functions, $E(r)$, $N(r)$, and $\sigma(r)$. The form of equations is not canonical. In principle, the electric field $E(r)$ can be extracted explicitly from (15) as a function of N'', N', N, $\frac{\sigma''}{\sigma}$, $\frac{\sigma'}{\sigma}$, and r. Inserting such $E(r)$ into the Eqs. (16) and (17), we obtain equations for $N(r)$ and $\sigma(r)$ of fourth order in their derivatives.

4. General features and notes

Below we focus on models admitting solutions to Eqs. (15)–(17), such that they can be represented by a series expansion regular at $r = 0$, i.e.,

$$\mathcal{A}(r \to 0) = \mathcal{A}(0) + \mathcal{A}'(0)r + \tfrac{1}{2}\mathcal{A}''(0)r^2 + \ldots \quad (18)$$

where $\mathcal{A}(r)$ symbolizes generically the functions $E(r)$, $N(r)$, and $\sigma(r)$. Our purpose is to find solutions satisfying three conditions: First, the electric field $E(r)$ should be a continuous function regular at $r = 0$ ($E(0) \neq \infty$) and also should be of Coulombian form at $r \to \infty$. Second, the metric functions $N(r)$ and $\sigma^2(r)N(r)$ should be regular at $r = 0$. Third, in terms of the functions $A(r)$ and $B(r)$ the asymptotic flatness requires that $B''(\infty) = A''(\infty) = B'(\infty)A'(\infty) = 0$, and $A(\infty) = \text{const}$, $B(\infty) = \text{const}$. So, essentially, one can put, $\sigma(\infty) = 1$ and $N(\infty) = 1$.

Note that the regularity of the functions $E(r)$, $N(r)$, and $\sigma(r)$ at $r = 0$ does not guarantee that the solution of the Einstein-Maxwell model is characterized by regular curvature invariants. For instance, when $N(0)$ is finite but $N(0) \neq 1$, the model displays a conical singularity and the scalar invariants of the curvature tend to infinity as $r \to 0$. In considering solutions such that the fields are finite we try to be as close as possible to Bardeen's idea of having black hole solutions without singularities, by finding a regular $E(r)$ and putting the metric coefficients in the form $N(r) = 1 - 2Mr^2(r^2 + r_0^2)^{-(3/2)}$ and $\sigma(r) = 1$, for some r_0 [17]. As we will see it will turn out that this is not achieved, since although the electric and metric potentials are regular, the black hole solutions found here are singular at the center, where the curvature invariants blow up. Notwithstanding, these solutions are very interesting. Using the ansatz (18) and the Eqs. (15)–(17) one can couple the values $E(0)$, $N(0)$, $\sigma(0)$, and q_1, q_2, q_3. The relations are different for the cases $\sigma(0) = 0$ and $\sigma(0) \neq 0$, which we now analyze.

(i) $\sigma(0) = 0$.—When $\sigma(0) = 0$, but $\sigma'(0) \neq 0$, one obtains from the system (15)–(17) that $(\frac{\sigma'}{\sigma})(r \to 0) \to \frac{1}{r}$, and the decomposition (18) is valid at $r \to 0$, when the following conditions are satisfied,

$$E(0)\frac{3q_1q_2 + q_2^2 + 2q_1q_3}{3q_1 + q_2 - q_3} = Q, \quad E^2(0)(q_1 + q_2) - 1 = 0, \quad N(0)[2(3q_1 + q_2 - q_3)] = 2q_1 + q_2, \quad (19)$$

for generic q_1, q_2, and q_3. There are two specific cases. When $q_1 + q_2 + q_3 = 0$, but both $2q_1 + q_2 \neq 0$ and $q_1 + q_2 \neq 0$, then $Q = \frac{1}{2}E(0)(q_2 - q_1)$ and $N(0) = \frac{1}{4}$. When $q_1 + q_2 + q_3 = 0$ and $2q_1 + q_2 = 0$, simultaneously, then $\kappa q_1 E^2(0) = -1$, providing q_1 is negative, and $N(0)$ is fixed by the relation $N(0) = 1 + \frac{Q}{2q_1E(0)} \neq 1$. As in the case $\sigma(0) \neq 0$, see below, here the Ricci scalar $R(r)$ takes an infinite value at $r = 0$.

(ii) $\sigma(0) \neq 0$.—When all three functions, $E(r)$, $N(r)$, and $\sigma(r)$ are regular at $r = 0$, and $\sigma(r)$, appearing in the denominator of Eqs. (15)–(17), does not vanish at $r = 0$, one obtains from the system (15)–(17) the following set of equations

$$E(0)2q_1[N(0) - 1] = Q, \qquad E^2(0)2q_3 = 0, \qquad N(0)[1 + \kappa E^2(0)(q_1 - q_2 - 2q_3)] = 1 + q_1\kappa E^2(0), \quad (20)$$

for generic q_1, q_2, and q_3. Since the charge of the object, Q, is considered to be nonvanishing, one obtains immediately from the first equation of the set that $E(0) \neq 0$ and $N(0) \neq 1$. Thus, we infer,

$$q_1 = \frac{Q}{2E(0)[N(0) - 1]}, \qquad q_2 = \frac{2[N(0) - 1] + \kappa E(0)Q}{2N(0)\kappa E^2(0)}, \qquad q_3 = 0. \quad (21)$$

The relations (20) give that the curvature invariants are infinite in the center $r = 0$. For instance, when $N(r)$ and $\sigma(r)$ are regular in the center and $\sigma(0) \neq 0$, then the Ricci scalar

$$R(r) = 2N\frac{\sigma''}{\sigma} + N'' + 3N'\frac{\sigma'}{\sigma} + \frac{4}{r}\left(N' + N\frac{\sigma'}{\sigma}\right) + \frac{2}{r^2}(N - 1) \quad (22)$$

tends to infinity at $r \to 0$, since $N(0) \neq 1$, as well as generally $N'(0) \neq -N(0)\frac{\sigma'}{\sigma}(0)$.

5. The order of differential equations and the choice of the parameters: One-parameter models

Now we want to analyze the simplest cases of the system of equations (15)–(17). One sees that there is an immediate simplification when $q_1 + q_2 + q_3 = 0$, since second order derivatives and products of first order derivatives disappear from the equations. In such a case the system (15)–(17) reduces to

$$E(r)\left\{r^2 + 2r(2q_1 + q_2)\left(N' + N\frac{\sigma'}{\sigma}\right) + 2q_1(N - 1)\right\} = Q, \quad (23)$$

$$\frac{r\sigma'}{\kappa\sigma} = r(2q_1 + q_2)(E^2)' - q_3E^2, \quad (24)$$

$$\frac{[r(1-N)]'}{\kappa} = r(E^2)'(2q_1+q_2)N + E^2\left[\frac{r^2}{2} - q_1 + 2r(2q_1+q_2)\left(N' + N\frac{\sigma'}{\sigma}\right) + N(3q_1+q_2)\right]. \tag{25}$$

Moreover, the three-dimensional matrix, composed of the coefficients in front of the first derivatives $(E^2)'$, N' and σ', has rank two. This means that for $q_1 + q_2 + q_3 = 0$ the system (23)–(25) can be reduced to one algebraic equation and two differential equations of the first order. The corresponding algebraic equation is

$$\kappa(2q_1+q_2)(r^2-2q_1)E^3 - 2\kappa(2q_1+q_2)QE^2 + [r^2 + 2q_3(N-1)]E - Q = 0, \tag{26}$$

which in turn links two functions, $E(r)$ and $N(r)$. Now from Eq. (26), one can consider three subcases, that emerge from the case $q_1 + q_2 + q_3 = 0$. First we consider briefly the trivial case in this context, $q_1 = q_2 = q_3 \equiv q$, $q = 0$, i.e., the Reissner-Nordström limit, see subsection III B. Then we consider two interesting nontrivial cases: first, $q_3 \neq 0$, second, $q_3 = 0$, When $q_3 \neq 0$ it is easy to express $N(r)$ in terms of $E(r)$. In the subsection III C we consider a specific model in this class, characterized by the supplementary condition $q_1 \equiv -q$, $q_2 = 2q$, $q_3 = -q$ (i.e., $q_1 + q_2 + q_3 = 0$, $2q_1 + q_2 = 0$). This is the Gauss-Bonnet type model, which has been considered as an important model in [8,9], and for which we present an extended analysis with relevant new details. For the second case $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$ (i.e., $q_1 + q_2 + q_3 = 0$, $q_3 = 0$), $E(r)$ decouples from $N(r)$ and we deal with a cubic equation for the determination of the electric field. We will consider such a model, the integrable model, in subsection III D.

III. SOLUTIONS OF THE REDUCED THREE-PARAMETER MODEL: SOLUTIONS OF ONE-PARAMETER MODELS

A. Basic quantities and variables

One should first define three quantities, r_M, r_Q, E_Q, as follows

$$r_M \equiv 2GM, \qquad r_Q \equiv \sqrt{G}|Q|, \qquad E_Q \equiv \frac{Q}{r_Q^2}. \tag{27}$$

Now, the models we are going to discuss here are essentially one parametric, with q_1, q_2, and q_3 being a multiple of some parameter q. It is useful to introduce first a quantity r_q, given through

$$r_q = \sqrt{2|q|}, \quad \text{and} \quad 2q = \pm r_q^2, \tag{28}$$

with r_q being a radius. From r_M, r_Q, and r_q, one can then construct two independent dimensionless quantities, namely

$$a \equiv \frac{2q}{r_Q^2} = \pm\frac{r_q^2}{r_Q^2}, \tag{29}$$

and

$$K \equiv \frac{r_M}{r_Q}. \tag{30}$$

The a quantity gives the deviation from the standard Reissner-Nordström case, and K fixes the ratio between the total mass to the total charge of the object.

In addition, for what follows below, it is useful to write the equations of motion by defining two dimensionless variables, a normalized radius x and a normalized electric field $Z(x)$, defined as follows

$$x = \frac{r}{r_Q}, \qquad Z(x) = \frac{E(r)}{E_Q}. \tag{31}$$

Also, the function $N(x)$ can be defined in terms of another useful function $y(x)$, i.e.,

$$N(x) = 1 - \frac{y(x)}{x}. \tag{32}$$

The physical interpretation of $y(x)$ is connected with the effective mass of the object, i.e., $M(r)$, as we will see below. With these quantities defined we now discuss the Reissner-Nordström limit and the two new models.

B. The Reissner-Nordström limit: $q_1 = q_2 = q_3 \equiv q$, with $q = 0$

When the nonminimal parameters q_1, q_2, and q_3 are set to zero, i.e.,

$$q_1 = q_2 = q_3 \equiv q, \qquad q = 0, \tag{33}$$

and so $a = 0$ as well, we can integrate immediately Eqs. (15)–(17). In terms of the above functions, the system of key equations can be rewritten as

$$x^2 Z - 1 = 0, \tag{34}$$

$$xN'(x) + N(x) = 1 - 2Z + x^2Z^2, \tag{35}$$

$$\frac{\sigma'(x)}{\sigma} = 0. \tag{36}$$

The solutions to these equations are

$$Z(x) = \frac{1}{x^2}, \tag{37}$$

$$N(x) = 1 - \frac{K - 1/x}{x}, \tag{38}$$

$$\sigma(x) = 1. \tag{39}$$

The function $y(x)$ in (32) is here given by

$$y(x) = K - \frac{1}{x}, \tag{40}$$

where K is defined in (30), and one can see through Eq. (35) that y obeys the equation

$$\frac{dy}{dx} = \frac{1}{x^2}. \tag{41}$$

Of course one can transform to the original fields $E(r)$, $A(r)$, and $B(r)$, giving

$$E(r) = \frac{Q}{r^2}, \tag{42}$$

$$\frac{1}{A(r)} = 1 - \frac{2GM(r)}{r}, \tag{43}$$

$$B(r) = 1 - \frac{2GM(r)}{r}, \tag{44}$$

which are the usual Reissner-Nordström functions. Note, then, that the physical interpretation of $y(x)$ given in (40) is connected with the effective mass of the object, i.e., $M(r)$, which in turn is given by the definition

$$N(r) = \frac{1}{A(r)} = 1 - \frac{2GM(r)}{r} = 1 - \frac{y(x)}{x}. \tag{45}$$

Thus, $y(x)$ is related to the dimensionless effective mass, since $y(x) = \frac{r_M}{r_Q}\frac{M(r)}{M}$, with $M(r) \equiv M - \frac{Q^2}{2r}$, M being the asymptotic mass of the object.

We study in this context the usual static spacetimes with $a = 0$, i.e., the Schwarzschild and Reissner-Nordström spacetimes, which are special solutions of the full system of equations. These two solutions are heavily singular, both the metric and the Kretschmann scalar diverge at $r = 0$. So these solutions are outside the spirit of the solutions we want to find. They do not serve as models. However, they are of interest to set the nomenclature, and to have a gauge with which we can compare the solutions we find in the two models studied below. Note that for these solutions $\sigma^2(x) = A(r)B(r) = 1$. In these cases the problem of searching for horizons is equivalent to finding their radial position r_h through the solutions of $B(r) = \frac{1}{A(r)} \equiv 1 - \frac{2GM(r)}{r} = 0$, where $M(r)$ is the effective mass. In terms of dimensionless mass $y(x)$, see Eq. (45), or Eq. (40), this condition can be written as $y(x) = x$. For the Schwarzschild metric $y(x) = K$, where $K = r_M/r_Q$ is defined in (30). Then, one obtains only one horizon at $x = x_h = K$, which is just the Schwarzschild radius, $r = r_M$. For the Reissner-Nordström metric one has $y(x) = K - \frac{1}{x}$, and the equation $K - \frac{1}{x} = x$ gives three different cases: (i) $K > 2$ (i.e., $r_M > 2r_Q$, or $GM^2 > Q^2$ in the standard notation): there are two solutions $x_h = \frac{K}{2} + \sqrt{\frac{K^2}{4} - 1}$, and $x_h = \frac{K}{2} - \sqrt{\frac{K^2}{4} - 1}$, corresponding to the outer and inner horizons, respectively, of a usual Reissner-Nordström black hole. (ii) $K = 2$ (i.e., $r_M = 2r_Q$, or $GM^2 = Q^2$): there is one horizon only, given by $x_h = 1$, corresponding to an extremal black hole. (iii) $K < 2$ (i.e., $r_M < 2r_Q$, or $GM^2 < Q^2$): there is no solution to the equation $y(x) = x$. The object is a naked singularity.

C. The Gauss-Bonnet model: $q_1 \equiv -q$, $q_2 = 2q$, $q_3 = -q$ (i.e., $q_1 + q_2 + q_3 = 0$ and $2q_1 + q_2 = 0$)

1. Preliminaries

Consider now, in Eqs. (1)–(3), the following specific one-parameter model

$$q_1 \equiv -q, \qquad q_2 = 2q, \qquad q_3 = -q \quad (\text{i.e., } q_1 + q_2 + q_3 = 0, 2q_1 + q_2 = 0), \tag{46}$$

for some parameter q. In this model the susceptibility tensor is proportional to the double-dual Riemann tensor and is divergence-free [1], i.e., $\chi_{ikmn} = q\,{}^*R^*_{ikmn}$, and $\nabla_n \chi^{ikmn} = 0$. Moreover, the coupled Einstein and electromagnetic equations are second order in the derivatives, which is the reason why this model is called a Gauss-Bonnet model. Gauss-Bonnet gravity in five and higher dimensions has the property that its equations are of second order, Lovelock gravity being a generalization of it. Actually, one can show that the model specified by (46) comes from Kaluza-Klein reduction to four dimensions from five a dimensional Gauss-Bonnet theory, i.e., Einstein gravity plus a Gauss-Bonnet term [9]. Using (46), Eqs. (23)–(25) convert, respectively, into

$$E(r)\{r^2 + 2q(1 - N)\} = Q, \tag{47}$$

$$r\frac{\sigma'}{\sigma} = \kappa q E^2, \tag{48}$$

$$[r(1 - N)]' = \tfrac{1}{2}\kappa E^2[r^2 + 2q(1 - N)]. \tag{49}$$

After appropriate redefinitions these equations agree with the ones discussed in [9]. The correspondingly modified algebraic equation (26) coincides with (47). We now consider two ways of analyzing this system of equations: first, we use a power series expansion, and second, we apply the formalism of dynamical systems.

2. Abel equation and its solution

(I) The Abel equation.—Using from Eq. (47) that

$$N(r) = 1 + \frac{1}{2q}\left[r^2 - \frac{Q}{E(r)}\right], \tag{50}$$

one obtains the Abel equation (see, e.g., [23]) for $E(r)$ from (49),

$$rE'(r) = E - 3\frac{r^2}{Q}E^2 - \kappa q E^3. \tag{51}$$

Similarly, eliminating $E(r)$ from Eq. (47) one can transform Eq. (49) into the Abel equation for a function $\Theta(r)$, here defined as

$$\Theta(r) \equiv 1 - N(r). \tag{52}$$

Thus, Eq. (49) is given by

$$[r\Theta'(r) + \Theta(r)][r^2 + 2q\Theta(r)] = \frac{\kappa Q^2}{2}. \tag{53}$$

Clearly, the values $E(0)$ and $N(0)$ are related through $N(0) = 1 - \frac{Q}{2qE(0)}$. Searching for solutions $N(r)$ regular at $r = 0$, we have to consider $E(0)$ to be nonvanishing, $E(0) \neq 0$. In such a case Eq. (51) yields $\kappa q E^2(0) = 1$. This means that q has to be positive in these solutions. Thus, although we analyze all cases, we tend to focus in models with $a \equiv \frac{2q}{r_Q^2} > 0$. Finally, a possible solution, regular at $r = 0$, should be characterized by $\sigma(0) = 0$. It is convenient to use the auxiliary quantities $r_q \equiv \sqrt{2|q|}$, r_Q, and a used before [see Eqs. (28) and (29)], to write

$$E(0) = \frac{1}{\sqrt{\kappa q}} = \frac{Q}{r_Q r_q} = \frac{E_Q}{\sqrt{a}},$$
$$N(0) = 1 - \frac{Q}{2}\sqrt{\frac{\kappa}{q}} = 1 - \frac{r_Q}{r_q} = 1 - \frac{1}{\sqrt{a}}. \tag{54}$$

Using Eqs. (47)–(49), plus the asymptotic conditions $\sigma(\infty) = 1$, $N(\infty) = 1$, as well as the condition that the electric field is asymptotically Coulombian $E \to Q/r^2$, one can obtain the following formula for the asymptotic mass M,

$$r_M = 2GM = \lim_{r\to\infty}\{\sigma^2[r(1-N) - \tfrac{1}{2}r\kappa QE + 2N\kappa qE^2]\} = \lim_{r\to\infty} r(1-N). \tag{55}$$

(II) The solution of the Abel equation for small r.—In the vicinity of the point $r = 0$ the solutions for $E(r)$, $N(r)$ and $\sigma(r)$ are assumed to have a polynomial form of the type given in Eq. (18). The decomposition of a regular solution $E(r)$ with nonvanishing $E(0)$ is

$$E(r) \to \frac{E_Q}{\sqrt{|a|}}\left[1 - \frac{3}{\sqrt{|a|}}\left(\frac{r}{r_Q}\right)^2 + \ldots\right], \tag{56}$$

and the corresponding $N(r)$ with finite $N(0)$ is given by

$$N(r) \to 1 - \frac{1}{\sqrt{|a|}} + \frac{1}{4a^2}\left(\frac{r}{r_Q}\right)^2 + \ldots. \tag{57}$$

For this solution the effective mass $M(r)$ becomes equal to zero at $r = 0$. Moreover, $N(0) = 0$, when electric and nonminimal radii coincide, i.e., $r_Q = r_q$, $a = 1$.

(III) Power series expansion with respect to $\frac{r_Q}{r}$.—The decomposition of the electric field yields

$$E(r) = \frac{Q}{r^2}\left[1 - a\frac{r_M}{r_Q}\left(\frac{r_Q}{r}\right)^3 - \sum_{n=5}^{\infty} n b_n \left(\frac{r_Q}{r}\right)^{n-1}\right], \tag{58}$$

where the b_n are defined below. Infinity is a regular point for $N(r)$, thus, taking into account (55) one obtains the following decomposition of $N(r)$

$$N(r) = \frac{1}{A(r)} = 1 - \frac{r_M}{r} + \frac{r_Q^2}{r^2} - a\frac{r_M}{4r_Q}\left(\frac{r_Q}{r}\right)^5 - \sum_{n=5}^{\infty} b_n\left(\frac{r_Q}{r}\right)^{n+1}, \tag{59}$$

where again, the b_n are given below. The function $\sigma(r) = \sqrt{A(r)B(r)}$ is equal to one in the Schwarzschild and the Reissner-Nordström cases, but not in general. When $q \neq 0$ the logarithm of this function can be represented by the decomposition

$$\ln\sigma(r) = -\frac{a}{4}\left(\frac{r_Q}{r}\right)^4\left[1 - \sum_{n=4}^{\infty}\frac{8nb_n}{n+3}\left(\frac{r_Q}{r}\right)^{n-1} + \sum_{n=4}^{\infty}\sum_{m=4}^{\infty}\frac{4nm}{n+m+2}b_n b_m\left(\frac{r_Q}{r}\right)^{n+m-2}\right]. \tag{60}$$

The b_n coefficients can be taken from Eq. (59), and starting from b_5 can be found by the recurrence formula

$$b_{n+3} = -a\sum_{m=0}^{n-1}\left(\frac{n-m}{n+3}\right)b_m b_{n-m}, \tag{61}$$

with

$$b_0 = \frac{r_M}{r_Q}, \qquad b_1 = -1, \qquad b_2 = b_3 = 0,$$
$$b_4 = a\frac{r_M}{4r_Q}, \qquad b_5 = -\frac{a}{5}, \qquad b_6 = 0, \tag{62}$$
$$b_7 = -a^2\frac{r_M^2}{7r_Q^2}, \qquad b_8 = a^2\frac{9r_M}{32r_Q}, \ldots.$$

The decompositions (58)–(60) are regular at $r = \infty$ and absolutely converge in the interval $r > r_Q H(a)$, where $H(a) \equiv \lim_{n\to\infty}|\frac{b_{n+1}}{b_n}|$. Note that the terms b_0 and b_1 are the Schwarzschild and Reissner-Nordström terms, respectively, and that the b_n for $n \geq 2$ are the terms that give the post-Reissner-Nordström behavior. Note also that for $q = 0$, i.e., the Reissner-Nordström case (or the Schwarzschild case when, further, $Q = 0$), the function σ in (60) is equal to one, as it should. For $r \to \infty$, Eqs. (58)–(60) give useful asymptotic formulas, showing that, for $E(r)$ and $N(r)$, the first post-Reissner-Nordström terms are of fifth order in $(\frac{r_Q}{r})$, and that the decomposition for $\log\sigma$ starts with a term of fourth order. When necessary, one should convert from N and σ to A and B. Numerical calculations, see Figs. 1–3, confirm that the corresponding curves tend to the corresponding horizontal asymptotes, when r goes to infinity, i.e., $E(r)$ tends to zero, and $\frac{1}{A(r)}$ and $B(r)$ tend to 1. It follows from (58) that, for positive q, $q > 0$, the curvature coupling effects on the electric field are analogous to a dielectric medium, since the asymptotic electric field effectively decreases. For negative q, $q < 0$, there are no solutions with $N(r)$ regular at $r = 0$, and, since we are mostly interested in regular or quasiregular solutions, we do not fully discuss this case.

3. The dynamical system associated with the model

We now study this model, specified through Eq. (46), using a dynamical system analysis.

I. First analysis: The plots and numerics

(A) Key dynamic equation.—In order to find the regular solutions $E(r)$, $N(r)$, and $\sigma(r)$ in the whole interval $0 < r < \infty$ let us transform the master equations (47)–(49) to the independent variable $x \equiv \frac{r}{r_Q}$, a dimensionless radius, and to the unknown dimensionless function $y(x)$ given in Eqs. (38) and (40), i.e.,

$$y(x) \equiv x[1 - N(x)]. \tag{63}$$

The physical interpretation of $y(x)$ is connected with the so-called effective mass of the object, $M(r)$, see Eq. (45). Putting these definitions into Eq. (49), we obtain the following key equation

$$\frac{dy(x)}{dx} = \frac{x}{x^3 + ay(x)}. \tag{64}$$

This equation is indeed a key one, since using its solution $y(x)$ we can represent explicitly the electric field by

$$E(x) = E_Q Z(x), \quad \text{with } Z(x) \equiv \frac{1}{x^2 + ay(x)/x}, \tag{65}$$

the metric function $N(x)$ by Eq. (45), and $\sigma(x)$ by the integral form

$$\ln\sigma(x) = \ln\sqrt{A(x)B(x)} = a\int_{\infty}^{x} \frac{dx'}{x'} Z^2(x'). \tag{66}$$

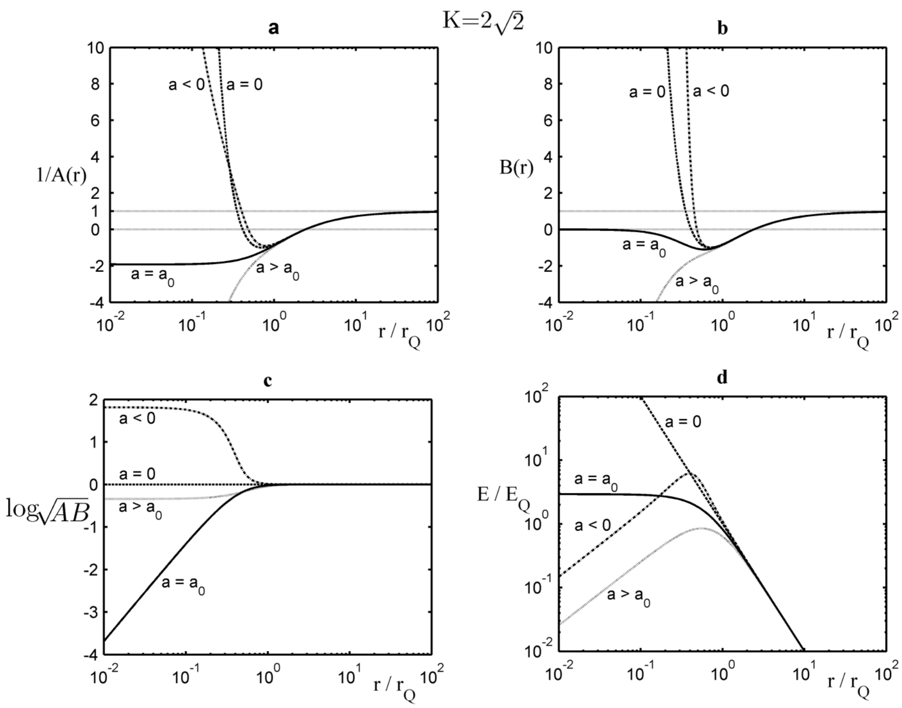

FIG. 1. Nonminimal solution of the Gauss-Bonnet model, with $q_1 = -q$, $q_2 = 2q$, and $q_3 = -q$, of gravitational electrically charged objects characterized by $a = 2q/r_Q^2$, for $K = 2\sqrt{2}$ ($K > 2$). Plots (a), (b), (c), and (d) depict the metric potentials $1/A(r)$ and $B(r)$, the function $\log\sqrt{A(r)B(r)}$, and the electric field $E(r)/E_Q$, respectively, as functions of $x = \frac{r}{r_Q}$, for solutions with different values of the nonminimal quantity a. The Reissner-Nordström black hole has $a = 0$, the curves of which are clearly shown in the plots. In this case the curves for $1/A(r)$ and $B(r)$ have two zeros representing the inner and outer horizons, and for $r \to \infty$ they go to one, while the curve $E(r)/E_Q$ tends to zero, respectively, and when $r \to 0$ these curves tend to $-\infty$ (the electric field is in a logarithmic scale). For $a < 0$ and $0 < a < a_0$ the black holes behave quite similarly as the case $a = 0$, with two horizons, and the function $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a = a_0$ the curve for $1/A(r)$ tends to a finite negative value when $r \to 0$, and takes the value zero only once. On the other hand $B(r)$ has two zeros one at the same point as $1/A(r)$, the event horizon, and the other at $r = 0$, signaling the presence of a singularity there. The function $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a > a_0$ the curves $\frac{1}{A(r)}$ and $B(r)$ have one zero, and thus one horizon only, at the same r, and then tend to infinity as $r \to 0$, an analogous behavior to the Schwarzschild black hole. The function E/E_Q tends to finite values as $r \to 0$. See text for more details.

Moreover, A follows from $A = 1/N$ and B from $B = \sigma^2/A$, i.e.,

$$B(x) = \left[1 - \frac{y(x)}{x}\right] \exp\left\{2a \int_\infty^x \frac{dx'}{x'} Z^2(x')\right\}. \tag{67}$$

Note that the function $y(x)$ is also a function of the quantity a, so in general should be written as $y(x, a)$.

(B) Three typical cases.—In this problem there are two independent dimensionless quantities, constructed from r_M, r_Q, and r_q, namely a and K, see Eqs. (29) and (30). Note that K, besides fixing the ratio between the total mass and the charge of the object, gives the value of the dimensionless mass $y(x)$ at $r = \infty$, since $y(\infty) = K$. Taking into account the quantity K, and in conformity with the Reissner-Nordström solution, let us distinguish three different situations, (i) $K > 2$, (ii) $K = 2$, (iii) $K < 2$, within each situation the quantity a can vary from zero to infinity. Figures 1–3 display typical cases in each situation, and Fig. 4 shows the behavior of $y(x)$. In slightly more detail: (i) $K > 2$ (i.e., $r_M > 2r_Q$): For $K > 2$, we use $K = 2\sqrt{2}$ as a typical value for the numerical analysis, see the plots in Fig. 1, [see Fig. 1(a) for $\frac{1}{A(r)}$, Fig. 1(b) for $B(r)$, Fig. 1(c) for $\ln\sqrt{A(r)B(r)}$, and Fig. 1(d) for $E(r)$). When $a = 0$ this case gives the usual Reissner-Nordström black hole with two horizons. For other a's there are also black holes, some with different properties. (ii) $K = 2$ (i.e., $r_M = 2r_Q$): For $K = 2$ see the plots in Fig. 2, [see Fig. 2(a) for $\frac{1}{A(r)}$, Fig. 2(b) for $B(r)$, Fig. 2(c) for $\ln\sqrt{A(r)B(r)}$, and Fig. 2(d) for $E(r)$). When $a = 0$ this case gives the extreme Reissner-Nordström black hole with one horizon. For other a's there are also interesting solutions with black holes. (iii) $K < 2$ (i.e., $r_M < 2r_Q$): For $K < 2$, we use $K = 1$ as a typical value for the numerical analysis, see the plots in Fig. 3, [see Fig. 3(a) for $\frac{1}{A(r)}$, Fig. 3(b) for $B(r)$, Fig. 3(c) for $\ln\sqrt{A(r)B(r)}$, and Fig. 3(d) for $E(r)$]. When $a = 0$ this case gives a Reissner-Nordström naked singularity, a solution without horizons. For other a's there are also solutions.

(C) Scaling of the key equation.—The key equation (64) remains invariant after the following scale transformations:

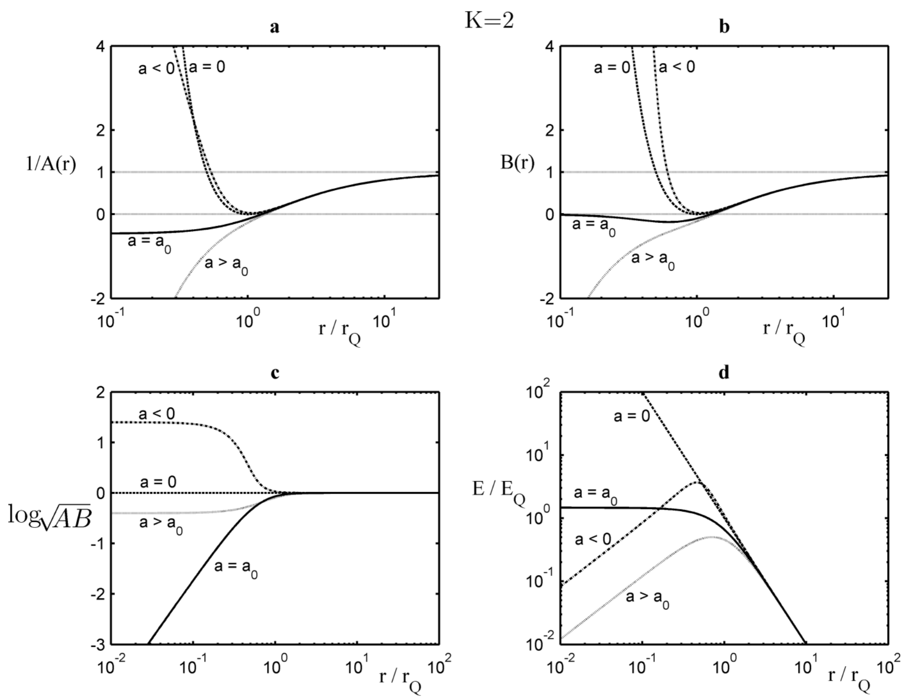

FIG. 2. Nonminimal solution of the Gauss-Bonnet model, with $q_1 = -q$, $q_2 = 2q$, and $q_3 = -q$, of gravitational electrically charged objects characterized by $a = 2q/r_Q^2$, for $K = 2$. Plots (a), (b), (c), and (d) depict the metric potentials $1/A(r)$ and $B(r)$, the function $\log\sqrt{A(r)B(r)}$, and the electric field $E(r)/E_Q$, respectively, as functions of $x = \frac{r}{r_Q}$, for solutions with different values of the nonminimal quantity a. The extremal Reissner-Nordström black hole has $a = 0$, the curves of which are clearly shown in the plots. In this case the curves for $1/A(r)$ and $B(r)$ have one double zero representing an extremal horizon, and for $r \to \infty$ they go to one, while the curve $E(r)/E_Q$ tends to zero, respectively, and when $r \to 0$ these curves tend to $-\infty$ (the electric field is in a logarithmic scale). For $a < 0$ and $0 < a < a_0$ the black holes behave quite similarly as the case $a = 0$, with one horizon, and the function $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a = a_0$ the curve for $1/A(r)$ tends to a finite negative value when $r \to 0$, and takes the value zero only once. $B(r)$ has two zeros one at the same point as $1/A(r)$, signaling there is only one horizon, and the other at $r = 0$, signaling the presence of a singularity there. The function $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a > a_0$ the curves $\frac{1}{A(r)}$ and $B(r)$ have at the same r, one zero, and thus one horizon only, and then tend to infinity as $r \to 0$, an analogous behavior to the Schwarzschild black hole. The function $E(r)/E_Q$ tends to finite values as $r \to 0$. See text for more details.

$$x \to \frac{1}{K}x, \quad y \to Ky, \quad a \to \frac{1}{K^4}a. \tag{68}$$

Thus, the critical values of a that one may eventually encounter when $K = 1$, are also critical values that one can easily find for arbitrary K using the formula $a_0(K) = \frac{1}{K^4}a_0(1)$.

II. Second analysis: Critical properties of the family of the solutions

(A) About the mass function $y(x, a)$ when $x = 0$, $y(0, a)$: the critical value of the quantity a, a_0.—We now write explicitly that y is a function of both x and a, $y = y(x, a)$ since this is important to our analysis. Plots of this dimensionless mass function $y(x, a)$ are displayed in Fig. 4. In Fig. 4(a), $y(x, a)$ is shown for several values of a, and in Fig. 4(b), a plot for $y(0, a)$ as a function of a is shown. A simple qualitative analysis shows that the mass function $y(x, a)$ at the central point $x = 0$, i.e. $y(0, a)$, as a function of the quantity a, has to possess a zero. Indeed, when $a = 0$, $y(x, 0) = K - \frac{1}{x}$, (where, recall, $K \equiv \frac{r_M}{r_Q}$), corresponding thus to the Reissner-Nordström solution. When $a = \infty$, $y(x, \infty) = K$, corresponding thus to the Schwarzschild solution. Note as well that when $a = -\infty$, $y(x, -\infty) =$ constant. In addition, when $x = \infty$, $y(\infty, a) = K$, a condition at infinity that holds for arbitrary a. In order to prove our assertion, that $y(0, a)$ as a function of a possesses a zero, consider then $y(0, a)$ as a function of the quantity a in the interval $0 < a < \infty$. One can see that $y(0, 0) = -\infty$ and $y(0, \infty) = K > 0$. Supposing that $y(0, a)$ is continuous in such an interval, one can conclude that there exists at least one specific value of the a quantity for which $y(0, a_0) = 0$, where a_0 is the value of a for which $y(0, a_0) = 0$. Figure 4(b) shows that, for $K = 1$ the zero of the function $y(0, a) = 0$ happens, when $a \equiv a_0 \simeq 7.49$. But the most interesting fact is that the curve $y = y(0, a)$ displays a discontinuity in the first derivative with respect to a just at $a = a_0$. One can see explicitly a finite jump of the derivative at this point, $a = a_0$. Nevertheless, the function $y = y(0, a)$ itself is continuous at this point. That is why a_0 is a critical value of the quantity a. For other Ks the

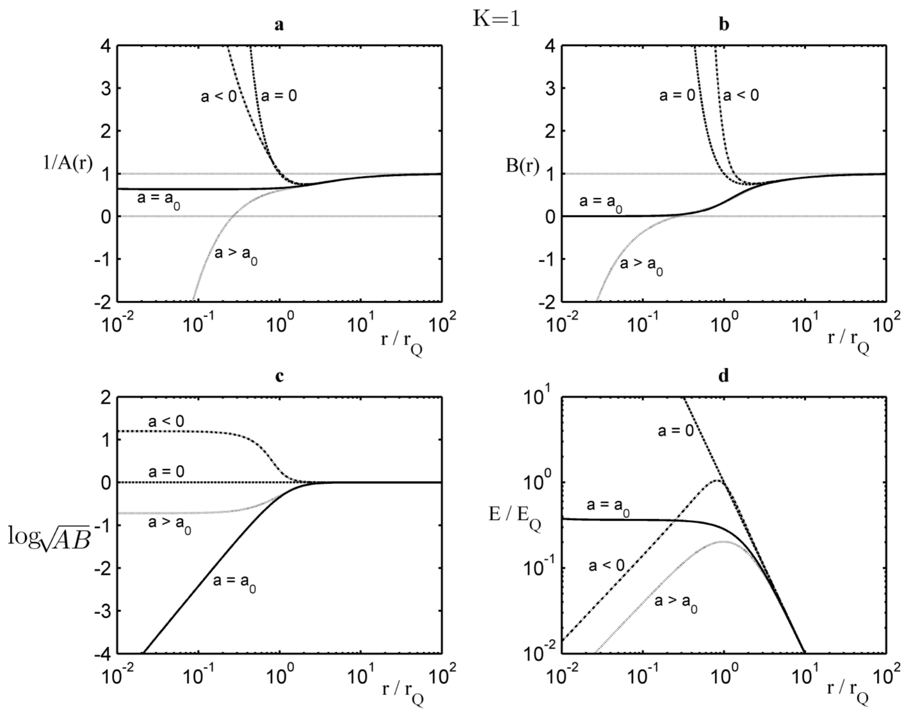

FIG. 3. Nonminimal solution of the Gauss-Bonnet model, with $q_1 = -q$, $q_2 = 2q$, and $q_3 = -q$, of gravitational electrically charged objects characterized by $a = 2q/r_Q^2$, for $K = 1$ ($K < 2$). Plots (a), (b), (c), and (d) depict the metric potentials $1/A(r)$ and $B(r)$, the function $\log\sqrt{A(r)B(r)}$, and the electric field $E(r)/E_Q$, respectively, as functions of $x = \frac{r}{r_Q}$, for solutions with different values of the nonminimal quantity a. The Reissner-Nordström naked singularity has $a = 0$, the curves of which are clearly shown in the plots. In this case the curves for $1/A(r)$ and $B(r)$ have no zeros, and for $r \to \infty$ they go to one, while the curve $E(r)/E_Q$ tends to zero, respectively, and when $r \to 0$ these curves tend to $-\infty$ (the electric field is in a logarithmic scale). For $a < 0$ and $0 < a < a_0$ the naked singularity behaves quite similarly as the case $a = 0$, and the function and $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a = a_0$ the curve for $1/A(r)$ tends to a finite positive value as $r \to 0$, while $B(r)$ has a zero at $r = 0$, signaling the presence of a singularity there. The function $E(r)/E_Q$ tends to finite values as $r \to 0$. For $a > a_0$ the curves $\frac{1}{A(r)}$ and $B(r)$ have one zero at the same r, and thus one horizon only, and then tend to infinity as $r \to 0$, an analogous behavior to the Schwarzschild black hole. Thus, by tuning the nonminimal quantity a one can turn a Reissner-Nordström naked singularity, which has $a = 0$, into a black hole, when $a > a_0$. The function $E(r)/E_Q$ tends to finite values as $r \to 0$. See text for more details.

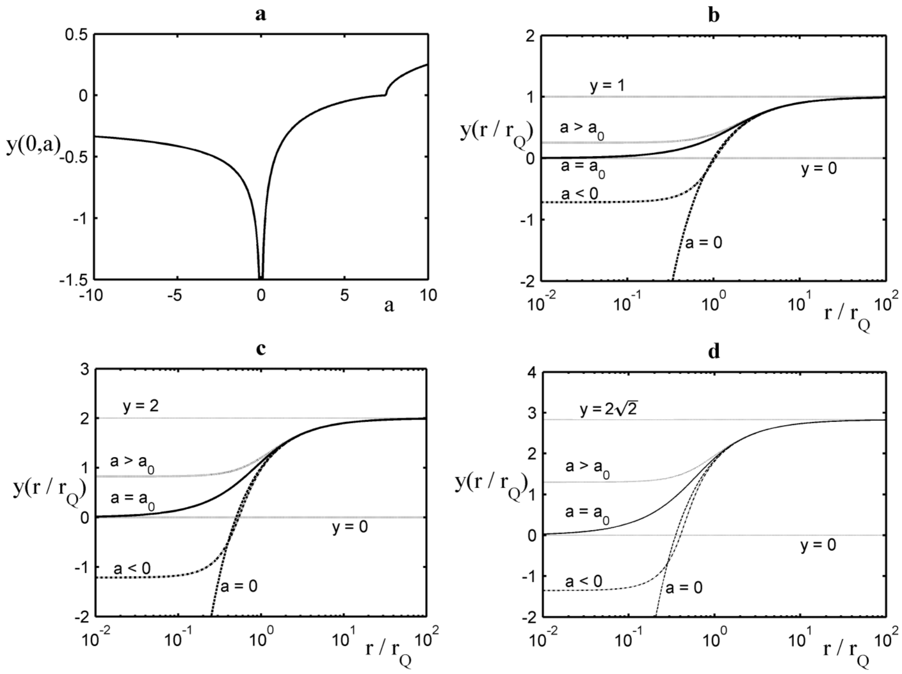

FIG. 4. Reduced mass profiles $y(0, a)$ and $y(x, a)$ for the nonminimal Gauss-Bonnet model, with $q_1 = -q$, $q_2 = 2q$, and $q_3 = -q$, of gravitational electrically charged objects characterized by $a = 2q/r_Q^2$. Plots of the curves $y(0, a)$ and $y(x, a)$ are shown, with $x = r/r_Q$. For $a_0 < a$ (with $a_0 > 0$) the characteristic curve $y = -x^3/a$ is not intercepted by the function $y(x, a)$. For $a = a_0$ the intersection takes place at $x = 0$. For $0 < a < a_0$ the point of crossing floats along the characteristic curve, and the integral curve has two branches. Finally, for $a < 0$ there is no intersection of $y(x, a)$ with the characteristic curve. In the plot $y(0, a)$ it is shown explicitly the existence of a point obeying $y(0, a_0) = 0$ with positive a_0.

critical values can be found using the scaling properties (68), yielding $a_0(K) = \frac{7.49}{K^4}$. For instance $a_0(2) = 0.468$ and, for the typical case we study, $a_0(2\sqrt{2}) = 0.117$. For negative a, $y(0, a) \to 0$ asymptotically when $a \to -\infty$.

(B) The critical points of the associated autonomous two-dimensional dynamical system.—The key equation (64) can be put as an autonomous dynamical system

$$\dot{y} = x, \qquad \dot{x} = x^3 + ay, \tag{69}$$

where $\dot{} \equiv \frac{d}{d\tau}$ and τ is an auxiliary parameter. In Eq. (69) there is one critical point at

$$(x, y) = (0, 0). \tag{70}$$

In the vicinity of this critical point the variables x and y are connected by the relation $x^2 - ay^2 = \text{constant}$, which means this point is a saddle point when a is positive, and a center when a is negative. If $a > 0$ there are two separatrices $y = \pm \frac{x}{\sqrt{a}}$. The equation for $N(r) = \frac{1}{A(r)}$ can also be written as dynamical,

$$\dot{N} = (1 - N)r^2 + 2q(1 - N)^2 - r_Q^2, \qquad \dot{r} = r[r^2 + 2q(1 - N)], \tag{71}$$

which is much more complicated than Eq. (69) for $y(x)$. Nevertheless, if q is positive (i.e., a is positive), one can find the critical points immediately,

$$(r, N) = \left(0, 1 \pm \frac{r_Q}{r_q}\right). \tag{72}$$

In order to present the integral curves for the total interval of the auxiliary parameter τ, we resort to numerical calculations. The results are presented in Figs. 1–3. It is clear that for the critical $a = a_0$ the curve for $N(r) = \frac{1}{A(r)}$ tends to one when $r \to \infty$, and takes a finite value $\frac{1}{A(0)} = 1 - \frac{1}{\sqrt{a_0}}$ at the center of the object, $r = 0$. This critical curve is a separatrix between the curves having $a > a_0$ and those having $a < a_0$. The same type of behavior happens with the curves for $B(r)$. One also has that for $a = a_0$ there exists a unique integral curve for $y(x)$ with asymptotic value given by $y(\infty) = \frac{r_M}{r_Q}$, and for $r \to 0$ one has $y(0, a_0) = 0$. Again, this curve behaves as a separatrix. Similar reasoning goes to the curve $E(r)$. In other words, the critical point $N = 1 - \frac{r_Q}{r_q}$, given in Eq. (72), is just a saddle point, corresponding to the critical value a_0, obtained numerically, and it is the final point of the unique integral curve, which coincides with the separatrix of the function $y = +\frac{x}{\sqrt{a_0}}$ at small x. When $q < 0$ and, thus, a is negative, there are no regular or quasiregular solutions to the master equations for the whole interval $0 \le r < \infty$.

(C) Vertical asymptotes and the electric barrier.—From Eq. (64) one sees that, when $y(x, a) = -\frac{x^3}{a}$, the derivative

$y'(x, a)$ becomes infinite and vertical asymptotes appear in the graph of $y(x)$ versus x. At the point $x = x^*$, for which $y(x^*, a) = -\frac{x^{*3}}{a}$, the electric field (65) becomes infinite, and so these vertical asymptotes can be interpreted in terms of an electric barrier.

When $a < 0$, Figs. 1(d), 2(d), and 3(d) show that for all three values of K the curve for the electric field has a form of a finite barrier. Indeed for $r \to \infty$ the electric field $E(r)$ tends to zero, then at some radius it reaches a maximum, the barrier height, and finally goes to a finite positive value when $r \to 0$. A charged test particle with sufficiently high energy can overcome this barrier, be trapped in the potential well and then oscillate inside. When $a = 0$, more specifically, when $a \to 0_-$, the barrier height increases and tends towards the center $r \to 0$. For $a = 0$ [see again Figs. 1(d), 2(d), and 3(d)] one obtains the standard, Reissner-Nordström, behavior, $E(r) = \frac{Q}{r^2}$. This means that the electric barrier has become infinite, taking its maximum value (an infinite value) at the center $r = 0$. In other words, the vertical asymptote for the electric field appears at $r = 0$. Finally, when $a > 0$ this vertical asymptote and the position of the infinite electric barrier shift from the center towards positive r values, then stop at some value of the quantity a, drift back again to smaller values of r and, finally, the infinite electric barrier disappears at $a \geq a_0$, (with $a_0 > 0$). When $a = a_0$ the curve tends to the horizontal asymptote at $r \to 0$, there is yet no trap. When $a > a_0$, one can see a finite electric barrier with the corresponding traps, for all values of K.

Since the integrals in (66) and (67) diverge when $E(r)$ is discontinuous, then in searching for solutions with regular functions $E(r)$, $\frac{1}{A(r)}$, and $B(r)$, we should reject all the cases where vertical asymptotes appear. For $a \geq a_0$, numerical calculations show that vertical asymptotes do not appear. So we will mainly consider solutions for this range of the quantity a, i.e., $a \geq a_0$.

(D) Horizons.—To analyze the $a \geq a_0$ case, with a nonsingular electric field and possible nonsingular metric potentials $1/A$ and B, we have studied previously, for comparison, the usual static spacetimes with $a = 0$, i.e., the Schwarzschild and Reissner-Nordström spacetimes, which are special solutions of Eq. (10). One has for these that $A(r)B(r) = 1$. For $a \geq a_0$ one sees from (60) that $A(r)B(r) \neq 1$, in contrast to the $a = 0$ case. Nevertheless, as in the $a = 0$ case, horizons are still given by the condition $\frac{1}{A(r)} = 0$, or $y(x) = x$. We have analyzed this numerically. For $a > a_0$ the results are the following: (i) $K > 2$ (i.e., $r_M > 2r_Q$): A typical case is $K = 2\sqrt{2}$, see Fig. 1. One finds that for $a > a_0 = 0.117$, there is only one horizon. (ii) $K = 2$ (i.e., $r_M = 2r_Q$): For $K = 2$, see Fig. 2. One finds that for $a > a_0 = 0.468$, there is one horizon also. (iii) $K < 2$ (i.e., $r_M < 2r_Q$): A typical case is $K = 1$, see Fig. 3. One finds that for $a > a_0 = 7.49$, there is only one horizon also.

Note that Figs. 1–3 show that when $a > a_0$, for all formal possibilities ($K > 2$, $K = 2$, $K < 2$) the curves $1/A$ tend monotonically to minus infinity and cross the line $1/A = 0$ only once. Thus, for arbitrary $a > a_0$ the plots of $1/A$ are continuous, irregular at the center and characterized by one horizon. These solutions have thus an analogous behavior to the Schwarzschild black hole. Moreover, by tuning the nonminimal quantity a one can turn a Reissner-Nordström naked singularity, with $K < 2$ and $a = 0$, into a black hole, when $a > a_0$. Now, since the $a = a_0$ is a very special case, we discuss it in particular.

(E) The solution with $a = a_0$.—In the framework of the model under discussion, i.e., when $q_1 = -q$, $q_2 = 2q$, and $q_3 = -q$, all the three functions, $E(r)$, $\frac{1}{A(r)}$, and $B(r)$, are regular in the interval $0 \leq r < \infty$ if and only if $a = a_0(K) = \frac{7.49}{K^4}$ (recall $a \equiv \frac{2q}{r_Q^2}$). This means that we deal with a one-parameter family of exact solutions, the arbitrary quantity being $K = \frac{r_M}{r_Q}$, and all the curvature coupling constants, q_1, q_2 and q_3, being expressed explicitly via K. The critical q corresponding to a_0 is thus, with the help of Eqs. (27)–(30), given by, $q_{a_0} = 7.49Q^6/32GM^4$. We now consider these solutions in more detail: (i) There are two different solutions, corresponding to the separatrices $y(x) = \pm \frac{x}{\sqrt{a_0}}$ at small x. The physical solution, the solution that gives the appropriate limit when $x \to \infty$ and has no jumps on the derivative of the characteristic curve $ay(x) = -x^3$, is $y(x) = +\frac{x}{\sqrt{a_0}}$. The plots of $\frac{1}{A(r)}$, $B(r)$ and $E(r)$ are displayed in Figs. 1–3. (ii) The solutions, which we discuss, are characterized by finite values at the center, and $E(0) \neq 0$, $A(0) \neq 0$, and $N(0) \neq 0$, but $\sigma(0) = 0$ and $B(0) = 0$. (iii) For this model $\frac{1}{A(r)} \neq B(r)$, and there are two distinct critical radii; first, the radius for which $\frac{1}{A(r)} = N(r) = 0$; second, the radius for which $B(r) = 0$ which signals an infinite redshift surface, and an event horizon in the case of static spacetimes, such as the ones we are treating here. (iv) For $\frac{1}{A(r)} = 0$, one finds that such a solution exists when $r_q < r_Q$, i.e., $a_0 < 1$; this is a necessary condition [see, e.g., (57)]. Moreover, within this case it is possible to have such a zero when $K^4 > 7.49$, i.e., $K > 1.65$ or $r_M > 1.65r_Q$. For $r_q > r_Q$, one has, $\frac{1}{A(0)} = 1 - \frac{r_Q}{r_q}$ is positive and without zeros. (v) For the infinite redshift surface and event horizon, $B(r) = 0$, one finds that when $a = a_0$, for the critical quantity, the function $B(r)$ is zero both at $r = 0$ and at the radius for which $\frac{1}{A(r)} = 0$. (vi) The Kretschmann scalar diverges at $r = 0$, so although the metric functions are regular, spacetime is not, time stops.

(F) Remark.—Some important aspects derived from qualitative and numerical analyses of this nonminimal model, with $q_1 = -q$, $q_2 = 2q$ and $q_3 = -q$, can be found in [9]. Our results are in thorough concordance with this initial analysis. We have supplemented those aspects on several grounds, of which we stress briefly three novel details obtained here: (i) We have found a complete

converging decomposition of the solution of the Abel equation based on the recurrence formula (61). This gives us not only the asymptotic decompositions for $r \to \infty$, but also the possibility to link the limiting formulas for $r \to 0$ and $r \to \infty$ [see (58)–(60)]. (ii) We have formulated and discussed the problem of the infinite electric barrier, associated with the vertical asymptote appearing when $x^3 + ay(x, a) = 0$, thus completing physically and mathematically the analysis given in [9]. (iii) We have found, first, the critical value of the nonminimal parameter q, namely, $q_{a_0} = 7.49Q^6/32GM^4$, second, the scaling law of the critical parameter q_{a_0} for different values of the asymptotic quantity $K = r_M/r_Q$, namely, $q_{a_0}(K) = 7.49/K^4$, third, the significance of the choice for K, namely, $K > 2$, $K = 2$, and $K < 2$, in the qualitative analysis.

D. The integrable model: $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$ (i.e., $q_1 + q_2 + q_3 = 0$ and $q_3 = 0$)

1. Preliminaries

Now, we consider the model with

$$q_1 \equiv -q, \qquad q_2 = q, \qquad q_3 = 0 \qquad \text{(i.e., } q_1 + q_2 + q_3 = 0, q_3 = 0), \tag{73}$$

which, as we have seen, has the property that $E(r)$ decouples from $N(r)$ and we deal with a cubic equation for the determination of the electric field. Since the basic feature of the model is that it is integrable, we call it the integrable model. For such a model the susceptibility tensor χ^{ikmn} can be written in terms of the Einstein tensor $G^{ik} \equiv R^{ik} - \frac{1}{2}Rg^{ik}$ as follows, $\chi^{ikmn} = \frac{q}{2}[G^{im}g^{kn} - G^{in}g^{km} + G^{kn}g^{im} - G^{km}g^{in}]$, where we have put $q_1 = -q_2 \equiv -q$. Thus, the model becomes one-parametric and we can introduce the dimensionful quantities defined above, r_Q, E_Q, r_q, and the dimensionless quantities a and K. We assume that E_Q inherits the sign of the charge Q and the quantity a can be positive or negative depending on the sign of q. Then, we introduce the two dimensionless variables, the normalized radius x and the normalized electric field $Z(x)$, defined in (31). In terms of these, the system of key equations can be rewritten as

$$a(x^2 + a)Z^3 - 2aZ^2 - x^2Z + 1 = 0, \tag{74}$$

$$xN'(x) + N[1 - ax(Z^2)'(x)] = 1 - 2Z + (x^2 + a)Z^2, \tag{75}$$

$$\frac{\sigma'(x)}{\sigma} = -a(Z^2)'(x). \tag{76}$$

Clearly, Eq. (74) for $Z(x)$ is the key equation for finding $N(x)$ and $\sigma(x)$ from (75) and (76), respectively. If instead of $N(x)$ one uses $y(x)$ then Eq. (75) gives an equation for $\frac{dy}{dx}$ of the type of Eq. (41) or Eq. (64), but more complicated, which for this analysis is not very illuminating. For this model it is better to start analyzing the electric field $E(r)$, or its redefinition $Z(x)$.

2. Electric field

Consider now Eq. (74) in detail. Equation (74) is a one-parameter algebraic equation of third order for the dimensionless electric field. Below we denote its solution as $Z(x, a)$. The function $Z(x, a)$ can be generally presented by the well-known Cardano formula, nevertheless we prefer to analyze qualitatively this one-parameter family of solutions. Depending on the value of the quantity a the solution $Z(x, a)$ can possess one or three real branches. The corresponding plots are presented in Fig. 5. When $a = 0$, one obtains, as it should, the Coulombian solution $Z(x, 0) = \frac{1}{x^2}$. The curves displaying $Z(x, a)$ for nonvanishing values of the quantity a are more sophisticated.

(i) $a > 0$.—When a is positive, the functions $Z(x, a)$ take finite values for all values of the quantity a, see the curves a, b, c, d in Fig. 5. The initial values $Z(0, a)$ satisfy the cubic equation $a^2Z^3(0, a) - 2aZ^2(0, a) + 1 = 0$. There is only one real solution of this cubic equation for $a > 0$, if the discriminant $\mathcal{D} = \frac{1}{108a^5}(27a - 32)$ is positive, i.e., when $\frac{32}{27} < a < \infty$, see box a of Fig. 5 for details. This plot displays three real branches of the solution $Z(x, a)$, nevertheless, only one of them is defined on the whole interval $0 \le x < \infty$. Two other branches are defined for $x \ge x_{\rm min}(a)$ only and contact at the point $x = x_{\rm min}(a)$. Only one branch tends to the horizontal asymptote $Z = 0$ at $x \to \infty$. When $0 < a \le \frac{32}{27}$, the discriminant $\mathcal{D}$ is negative or equal to zero, which guarantees that there are three real starting points, $Z_1(0, a)$, $Z_2(0, a)$, $Z_3(0, a)$ for the three corresponding branches of the solution $Z(x, a)$, see the curves in boxes b, c, d of Fig. 5. Nevertheless, when $1 < a \le \frac{32}{27}$, two branches of the solution $Z(x, a)$ are not continuous, only the third being defined on the whole interval $0 \le x < \infty$, see box b of Fig. 5. When $0 < a \le 1$ all three branches are continuous and are defined on the whole interval $0 \le x < \infty$, one of them is asymptotically Coulombian. There are three horizontal lines $Z = -\frac{1}{\sqrt{|a|}}$, $Z = 0$ and $Z = +\frac{1}{\sqrt{|a|}}$, which yield distinct ranges for the functions $Z_1(x, a)$, $Z_2(x, a)$, and $Z_3(x, a)$. Clearly, the curve of Coulombian type is in between the separatrices $Z = 0$ and $Z = +\frac{1}{\sqrt{|a|}}$. The model with critical a, call it a_0 again, is the one that has $a = 1$, i.e., $a \equiv a_0 = 1$. This model can be solved analytically, and we consider this case in detail below. Finally, when a tends to zero remaining positive, the starting points $Z_1(0, a)$ tend to minus infinity, and $Z_2(0, a)$, $Z_3(0, a)$ grow infinitely. Clearly, at $a \to 0_+$ the branch $Z_2(x, a)$ is the only branch that remains visible at finite values of x, and is Coulombian.

(ii) $a = 0$.—When $a = 0$, one obtains the Coulombian solution $Z(x, 0) = \frac{1}{x^2}$, see the curve in box e of Fig. 5.

(iii) $a < 0$.—When a is negative, the coefficient $(x^2 + a)$ in the first term of Eq. (74) vanishes at $x = \sqrt{|a|}$ and the line $x = \sqrt{|a|}$ is the vertical asymptote of the graph $Z(x, a)$, see the example of the curve for $a = -1$ in box f of Fig. 5. When a tends to zero remaining negative, the vertical asymptote shifts towards the line $x = 0$, and the solution $Z(x, a \to 0_-)$ converts, finally, into the Coulombian solution. At large values of x the plots of the function $Z(x, a < 0)$ tend to the Coulombian curve $Z(x, 0)$ for all values of the quantity a. Thus, for $a \leq 0$ the solutions $Z(x, a)$ are not regular at all in the range $0 \leq x < \infty$.

3. Metric functions

To study the gravitational part of the solution, we observe that the solution of (75) for N can be presented in quadratures

$$N(x, a) = 1 - \frac{1}{x} e^{aZ^2(x,a)} \left\{ K - \int_\infty^x d\xi e^{-aZ^2(\xi,a)} \times \left[(\xi^2 + a) Z^2(\xi, a) - 2Z(\xi, a) + 2a\xi Z(\xi, a) \frac{d}{d\xi} Z(\xi, a) \right] \right\}, \quad (77)$$

where $Z(x, a)$ is supposed to be already found. The constant of integration K can clearly be related to the asymptotic mass of the object M. When $x \to \infty$ and $Z(x \to \infty, a) \to \frac{1}{x^2}$, Eq. (77) yields that $K = \frac{r_M}{r_Q}$, as defined in (30). On the other hand, searching for a solution $N(x, a)$, which is finite at $x = 0$, and taking into account that $Z(0, a)$ is finite, we should require that

$$K = \int_\infty^0 d\xi e^{-aZ^2(\xi,a)} \left[(\xi^2 + a) Z^2(\xi, a) - 2Z(\xi, a) + 2a\xi Z(\xi, a) \frac{d}{d\xi} Z(\xi, a) \right]. \quad (78)$$

In this case the formula (77) transforms into

$$N(x, a) = \frac{1}{x} e^{aZ^2(x,a)} \int_0^x d\xi e^{-aZ^2(\xi,a)} [(\xi^2 + a) Z^2(\xi, a) - 2Z(\xi, a) + 1], \quad (79)$$

providing

$$N(0, a) = 1 + aZ^2(0, a) - 2Z(0, a). \quad (80)$$

Since the electric field at the center satisfies the condition

$$1 - 2aZ^2(0, a) + a^2 Z^3(0, a) = 0, \quad (81)$$

[see (74)], then $N(0, a)$ can be rewritten as $N(0, a) = 1 - \frac{1}{aZ(0,a)}$ in accordance with the first relation from (20). Thus, for the family of solutions with regular functions $N(r, a)$ and $E(r, a)$ the quantity K is a function of the value $Z(0, a)$, i.e., depends on the quantity a according to the formula (78), $K = K(a)$.

Also, from (76) for σ one finds

$$\sigma(x, a) = \sigma(\infty, a) \exp\{-a[Z^2(x, a) - Z^2(\infty, a)]\}, \quad (82)$$

with $Z(x, a)$ being found from (74). For the asymptotically Coulombian branch we have to set $Z(\infty, a) = 0$ and thus $\sigma(\infty, a) = 1$. Then, the function $1/A$ is taken from $1/A = N$ and the function B is taken from A and σ, $B = \sigma^2/A$.

In the Reissner-Nordström solution and in the previous discussed model, i.e., the Gauss-Bonnet model (see Secs. III B and III C, respectively), we divided the solutions according to $K > 2$, $K = 2$, and $K < 2$. Here it is no more convenient to make such a division. The reason is that K is not a free quantity here, rather $K = K(a)$. Since it is not a free quantity, we cannot classify the models with respect to it, we can only calculate this quantity (numerically) after solving the problem as a whole. In the Gauss-Bonnet model of Sec. III C we could classify through K because the equations for $Z(x)$ and $N(x)$ cannot be decoupled, thus, $Z(0, a)$ and $N(0, a)$ are connected. This means that we can choose K as a convenient quantity for the classification, the dependent quantity being $Z(0, a)$. Here, in this model, the solution for $Z(x)$ satisfies a decoupled equation, the latter does not depend on $N(x)$. Thus solving the decoupled equation for $Z(x, a)$, we can classify the quantity $Z(0, a)$ as the independent one. Then, we obtain $N(x, a)$ and, as we see, $N(0, a)$ depends on $E(0, a)$, an so K is a dependent quantity, $K = K(a)$. An explicit example where $K(a)$ is calculated is given below for the case $a \equiv a_0 = 1$. So, as when discussing the electric field, we again divide the analysis into three cases, here with subcases.

(i) $a > 0$.—We should further divide into three subcases.

$\infty > a > 1$.—The electric field is discontinuous or irregular at the center. Since in this work we focus on regular electric fields everywhere, although interesting, we do not discuss these models here.

$a = 1$, i.e., $a \equiv a_0 = 1$.—We plot in Fig. 6, boxes (a), (b), (c), (d), the functions $1/A$, B, $\log\sqrt{AB}$, and E/E_Q, respectively. From the figure it is clear that $a = 1$ is characterized by the absence of horizons. For $a \equiv a_0 = 1$ the solution is regular, or better, quasiregular, and is a soliton of the theory, the Fibonacci soliton. Because of its interest, the case $a \equiv a_0 = 1$ will be solved next explicitly.

$1 > a > 0$.—This case is interesting. In this case we plot in Fig. 6, boxes (a), (b), (c), (d), the functions $1/A$, B, $\log\sqrt{AB}$, and E/E_Q, respectively, for two positive values of a within this range, namely, $a = 0.999$, 0.750. From the figure it is clear that positive a in this range gives one horizon, and the solution possesses a quasiregular center. Since the singularity at the center is a conical one, these black hole solutions can be considered as quasiregular solutions, and thus are of great interest. Note that extremal black holes have only one zero, which in turn is a double zero. So from the figure above, the nonminimal black holes

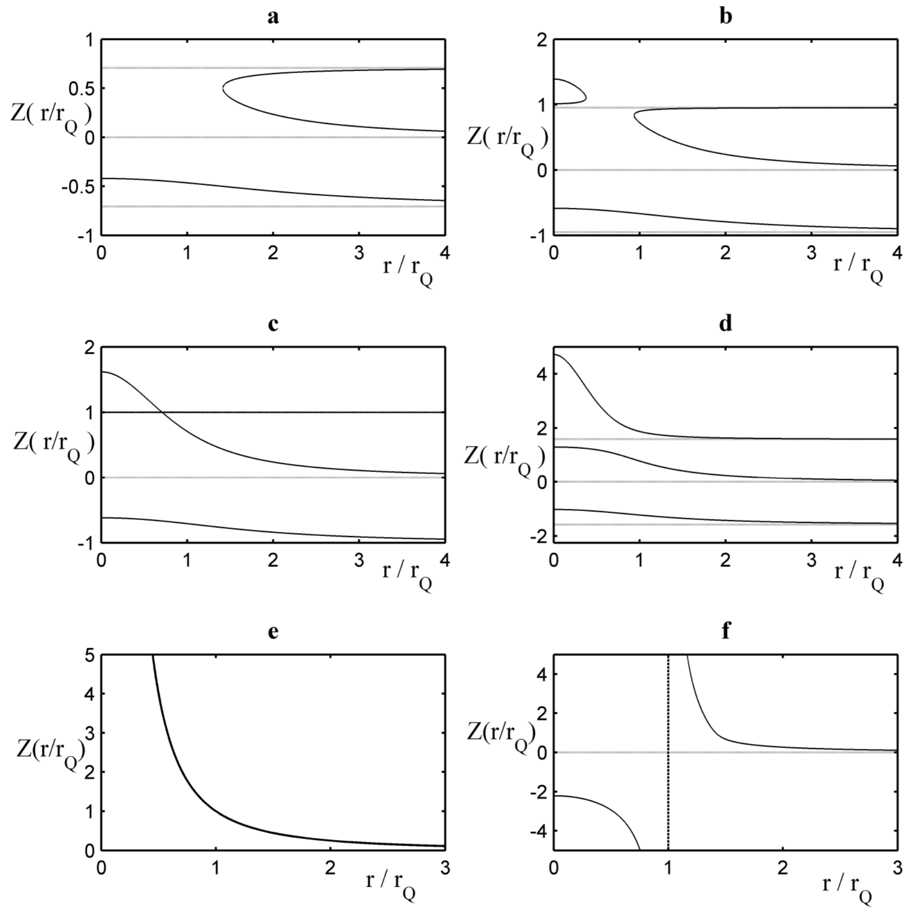

FIG. 5. It is shown, in the nonminimal integrable model, with $q_1 = -q_2 \equiv -q$ and $q_3 = 0$, the rescaled electric field $Z(x, a)$, $(Z(x, a) \equiv E(r)/E_Q)$, as a function of the scaled radius x $(x \equiv r/r_Q)$ of gravitational electrically charged objects characterized by $a = \frac{2q}{r_Q^2}$. (a) Displays the solution $Z(x, a)$ when the nonminimal quantity a satisfies the inequality $\frac{32}{27} < a < \infty$. There are three real branches of the solution $Z(x, a)$. Nevertheless, only one of them is defined on the whole interval $0 \leq x < \infty$. Only one branch tends to the horizontal asymptote $Z = 0$ at $x \to \infty$. (b) Displays the solution $Z(x, a)$ in the case $1 < a \leq \frac{32}{27}$. There are three real starting points $Z_1(0, a)$, $Z_2(0, a)$, $Z_3(0, a)$ for the three corresponding branches of the solution $Z(x, a)$. Nevertheless, two branches are not continuous, only the third being defined on the whole interval $0 \leq x < \infty$. (c) Displays the solution $Z(x, a)$ for the important case $a \equiv a_0 = 1$, the Fibonacci soliton. (d) Displays the solution $Z(x, a)$ in the case $0 < a < 1$. (e) Displays the solution $Z(x, a)$ in the case $a = 0$ which is a Coulombian electric field. The curve is not continuous. (f) Gives an example of the solution for negative a with a vertical asymptote. When a tends to zero remaining negative, the vertical asymptote shifts towards the line $x = 0$, and the solution $Z(x, a \to 0_-)$ converts, finally, into the Coulombian solution. At large values of x the plots of the function $Z(x, a < 0)$ tend to the Coulombian curve $Z(x, 0)$ for all values of the quantity a.

do not characterize as extremal. Rather, they are of the Schwarzschild type, with one horizon, a spacelike singularity, but here, different from Schwarzschild, the singularity is mild, it is a conical singularity.

(ii) $a = 0$.—It is the Reissner-Nordström case. For this case the electric field is irregular at the center.

(iii) $a < 0$.—For these cases the electric field is discontinuous or irregular at the center, like the Reissner-Nordström case, and we do not discuss these models here.

4. Exact solution: $a \equiv a_0 = 1$, the Fibonacci soliton

Let $a \equiv a_0 = 1$. Then the cubic equation (74) takes the form

$$(Z - 1)[(1 + x^2)Z^2 + (x^2 - 1)Z - 1] = 0. \tag{83}$$

One sees that Eq. (83) splits into one linear equation and one quadratic equation. One branch of solutions of (83), the linear one, describes a constant electric field $Z_{\rm const}(x, 1) = 1$, or, equivalently, $E(r) = E_Q$. This branch is of no great interest. Another branch $Z_{\rm nonCoulomb}(x, 1)$ is given by the function $Z_{\rm nonCoulomb}(x, 1) = \frac{1}{2(1+x^2)}[1 - x^2 - \sqrt{x^4 + 2x^2 + 5}]$, which is bounded. The graph of this function starts from $Z_{\rm nonCoulomb}(0, 1) = -\frac{\sqrt{5}-1}{2}$ and tends asymptotically to the line $Z = -1$. The behavior of such electric field is not of Coulombian type, and will be not discussed further. Yet, there is a third branch. The branch

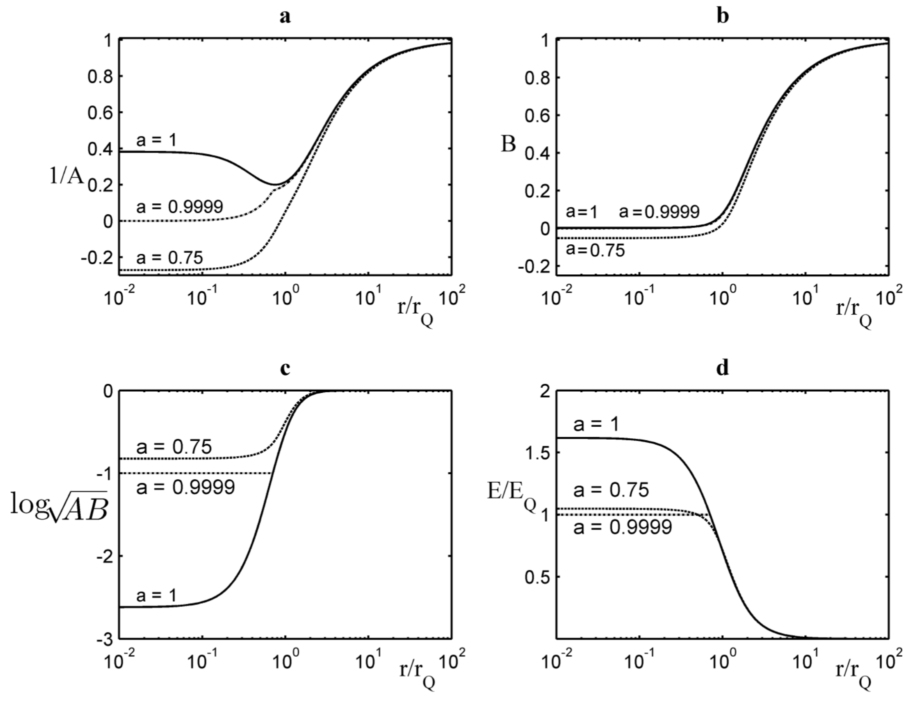

FIG. 6. Nonminimal solution of the integrable model, with $q_1 = -q$, $q_2 = q$, and $q_3 = 0$, of gravitational electrically charged objects characterized by $a = \frac{2q}{r_Q^2}$. Plots (a), (b), (c), and (d) depict the functions $\frac{1}{A}$, B, $\log\sqrt{AB}$ and E/E_Q, respectively, as functions of $x = \frac{r}{r_Q}$, for three typical values of the quantity a, when $0 < a \leq 1$. These functions are regular and take finite values at $r = 0$. The solution with $a \equiv a_0 = 1$ is a solution without horizons, since $\frac{1}{A}$ and B are positive everywhere. Indeed the $a \equiv a_0 = 1$ solution is a soliton of the model, the Fibonacci soliton. It has a mild conical singularity at the center, and is a solution without horizons.

$Z_{\rm Coulomb}(x, 1) \equiv Z(x, 1)$ given by the formula

$$Z(x, 1) = \frac{1}{2(1 + x^2)}[1 - x^2 + \sqrt{x^4 + 2x^2 + 5}], \qquad (84)$$

describes a Coulombian type electric field. At $x \to \infty$, one has $Z(x, 1) \to \frac{1}{x^2}$, or equivalently, $E(r) \to \frac{Q}{r^2}$. The graph of this function starts from $Z(0, 1) = \frac{\sqrt{5}+1}{2}$. Interesting to note that the starting points $Z(0, 1)$ are associated to the well-known Fibonacci series and the "golden section" $\phi \equiv \frac{\sqrt{5}+1}{2} = \frac{2}{\sqrt{5}-1} = 1.618\ldots$. For the Coulombian type solution, the function $N(x, 1)$, which we write simply as $N(x)$ when suitable, is regular at the center only if the constant K satisfies (78). The corresponding quadrature for $N(x)$ is

$$N(x) = \frac{1}{2x\sigma(x)} \int_0^x d\xi \sigma(\xi)[\xi^2 + 3 - \sqrt{\xi^4 + 2\xi^2 + 5}], \qquad (85)$$

where $\sigma(x)$ is given by (86). Clearly, $N(\infty, 1) = N(\infty) = 1$ and $N(0, 1) = N(0) = \frac{3-\sqrt{5}}{2}$, so that $1 - N(0) = \frac{1}{\phi} \equiv \phi - 1$, and the relations (20) are satisfied. The plot of the function $N(x)$ for $a \equiv a_0 = 1$ is shown in Fig. 6. Clearly, the function $N(x)$ is positive in the interval $0 \leq x < \infty$. For the Coulombian type solution (84) the function $\sigma(x, 1)$, which we write simply as $\sigma(x)$ when suitable, is given by

$$\sigma(x) = \exp\left\{-\frac{3 + (1 - x^2)\sqrt{x^4 + 2x^2 + 5} + x^4}{2(1 + x^2)^2}\right\}, \qquad (86)$$

with $\sigma(0, 1) = \sigma(0)$ being equal to $\exp\{-(1 + \phi)\}$. Then one finds $1/A$ from $1/A = N$ and $B = \sigma^2/A$. The function $B(x)$ is also positive, and $B(0) = (1 - \frac{1}{\phi})e^{-2(1+\phi)} \simeq 0.002\,032$. Thus, this solution is a solution without horizons and is regular. Although the curvature scalars diverge, the singularity at the center is a mild one, it is a conical singularity. The asymptotic mass M of the object defined as

$$r_M \equiv 2GM = \lim_{r\to\infty}\{r[1 - B(r)]\}, \qquad (87)$$

is represented in this case by the integral

$$M = \frac{|Q|}{4\sqrt{G}} \int_0^\infty d\xi\left[\frac{1}{\sigma(\xi)} - \xi\frac{\sigma'(\xi)}{\sigma^2(\xi)} - \frac{1}{2}\sigma(\xi)(\xi^2 + 3 - \sqrt{\xi^4 + 2\xi^2 + 5})\right]. \qquad (88)$$

Numerical calculations give the value

$$M \simeq 0.442\frac{|Q|}{\sqrt{G}}, \qquad (89)$$

which yields in addition,

TABLE I. In this table it is displayed the main results on the studied models.

$K = r_M/r_Q$ \ $a = 2q/r_Q^2$	$q_1 = -q$, $q_2 = 2q$, $q_3 = -q$			$q_1 = -q$, $q_2 = q$, $q_3 = 0$
	$K = 1$, $a_0 = 7.49$	$K = 2$, $a_0 = 0.468$	$K = 2\sqrt{2}$, $a_0 = 0.117$	$K = 0.884$, $a_0 = 1$
$a < 0$	Finite electric field $E(r)$ Infinite $1/A(0)$, $B(0)$			Finite $E(0)$, Infinite electric barrier
	0 Horizon	1 Horizon	2 Horizons	
$a = 0$	Minimal Reissner-Nordstrom model			
$0 < a < a_0$	Discontinuous electric field $E(r)$			Finite $E(r)$, $1/A(r)$, $B(r)$ 1 Horizon
$a = a_0$	Finite $E(r)$, $1/A(r)$, $B(r)$			
	0 Horizons	1 Horizon	1 Horizon	0 Horizons
$a > a_0$	Finite $E(r)$, infinite $1/A(0)$ and $B(0)$			Discontinuous $E(r)$
	1 Horizon	1 Horizon	1 Horizon	

$$K = \frac{2\sqrt{G}M}{|Q|} \simeq 0.884. \tag{90}$$

This is thus a very interesting solution. It is a soliton in the sense that it is made of the very own fields of the theory, the gravitational and electric fields; it is a solution without horizons; and it is quasiregular, with a conical singularity at the center.

Note also that the value $a \equiv a_0 = 1$ can be regarded as a critical one. There are two reasons for this. First, it is clear from Fig. 5, that for the unique case $a \equiv a_0 = 1$ there exists a bifurcation point, in which the two branches of the curve $Z(x, a)$ or $E(r, a)$ intersect. When $a > 1$, the Coulombian branch of the electric field curve is discontinuous. When $0 < a < 1$, three continuous regular branches exist. The second reason is that at $a \equiv a_0 = 1$ the corresponding curve on Fig. 6 plays a role of a separatrix; when $0 < a < 1$, desirable curves, continuous and regular at the center, exist, otherwise they do not appear.

E. Summary of the results

In Table-I the results of the various models studied are summarized.

IV. CONCLUSIONS

We have shown that the original nonminimal Einstein-Maxwell theory with three parameters q_1, q_2, and q_3, is reducible in natural different ways to a theory with one parameter q only, in which the three parameters obey two relations between themselves. We have then studied two special models for static spherically symmetric solutions obeying the following requirements: the electric field $E(r)$ is regular everywhere in the interval $0 \leq r < \infty$, being Coulombian far from the center. From the solutions of this class we extract the ones, for which the metric coefficients $\frac{1}{A(r)}$ and $B(r)$ are regular at the center $r = 0$ and tend to one asymptotically as $r \rightarrow \infty$.

The first nonminimal model, the Gauss-Bonnet model (with $q_1 \equiv -q$, $q_2 = 2q$, $q_3 = -q$, q free), displays charged black hole solutions with one horizon only, when the dimensionless nonminimal quantity a, with $a = 2q/r_Q^2$ naturally appearing in the model, exceeds a critical value a_0, $a > a_0$. Although the black hole is electrically charged the solutions found have one horizon only, and are similar in this connection to the Schwarzschild solution. When $a < a_0$, the solutions are discontinuous in the interval $0 \leq r < \infty$, or irregular at the center $r = 0$. Another main result in this model is that there exists a unique solution, the solution for the critical value $a = a_0$, which does not possess horizons and is characterized by regular fields $E(r)$, $1/A(r)$, and $B(r)$, with $B(0) = 0$ and $A(0) \neq 1$, although the curvature invariants blow at the origin.

The second model, the integrable model (with $q_1 \equiv -q$, $q_2 = q$, $q_3 = 0$, q free), is also characterized by one critical value $a \equiv a_0 = 1$ of the nonminimal quantity a. When $a < 0$ or $a > 1$, the solutions are irregular. When $0 < a < 1$ one obtains black holes with electric field regular everywhere and with only one horizon, like the Schwarzschild solution. Finally, when $a \equiv a_0 = 1$, i.e., at the critical value of the quantity a, there exists a solitonic solution with a conical singularity at the center, but otherwise well behaved. This solution can be called the Fibonacci soliton, since the well-known ϕ number ($\phi = \frac{\sqrt{5}+1}{2} \simeq 1.618$) associated with the golden section appears naturally in the expressions for the central values of the electric field $G|Q|E(0) = \phi$, and the metric coefficients are also related to ϕ, namely $1 - \frac{1}{A(0)} = \frac{1}{\phi}$ and $B(0) = (1 - \frac{1}{\phi})\exp[-2(1 + \phi)]$.

Summing up, we can say that the nonminimal curvature induced interaction between the gravitational and electromagnetic fields provides an electric field of static spherically symmetric charged objects, which is regular everywhere for different relations between the coupling constants q_1, q_2, and q_3. As for additional regularity of the metric coefficients $B(r)$ and $A(r)$, the nonminimal interaction can provide models which have very specific, critical, values for the coupling constants, in which the geometry has at most a conical, and thus mild, singularity. This is in line with the problem posed by Bardeen [17], where one should look for theories with regular black hole solutions. We have partially solved it within these models.

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