# Unified geometric description of black hole thermodynamics

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In the space of thermodynamic equilibrium states we introduce a Legendre invariant metric which contains all the information about the thermodynamics of black holes. The curvature of this thermodynamic metric becomes singular at those points where, according to the analysis of the heat capacities, phase transitions occur. This result is valid for the Kerr-Newman black hole and all its special cases and, therefore, provides a unified description of black hole phase transitions in terms of curvature singularities.

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## I. INTRODUCTION

According to the no-hair theorems of Einstein-Maxwell theory, electro-vacuum black holes are completely described by three parameters only: mass $M$, angular momentum $J$, and electric charge $Q$. The corresponding gravitational field is described by the Kerr-Newman metric which in Boyer-Lindquist coordinates can be expressed as [1]

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma} dt^2 - \frac{2a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma} dt d\varphi + \frac{(r^2 + a^2)^2 - a^2\sin^2\theta\Delta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \quad (1)$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = (r - r_+)(r - r_-), \qquad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad (2)$$

where $a = J/M$ is the specific angular momentum. Bekenstein [2] discovered in 1973 that the horizon area $A$ of a black hole behaves as the entropy $S$ of a classical thermodynamic system. This was the beginning of what is now called thermodynamics of black holes [3–5]. Although its statistical origin is still very unclear, black hole thermodynamics has been the subject of intensive research for the past three decades, due in part to its possible connection to a hypothetical theory of quantum gravity.

It has been established that the physical parameters of the Kerr-Newman black hole satisfy the first law of black hole thermodynamics [3]

$$dM = TdS + \phi dQ + \Omega_H dJ, \quad (3)$$

where $T$ is the Hawking temperature which is proportional to the surface gravity on the horizon, $S = A/4$ is the entropy, $\Omega_H$ is the angular velocity on the horizon, and $\phi$ is the electric potential. As in ordinary thermodynamics, all the thermodynamic information is contained in the fundamental equation which was first derived by Smarr [6]

$$M = \left[\frac{\pi J^2}{S} + \frac{S}{4\pi}\left(1 + \frac{\pi Q^2}{S}\right)^2\right]^{1/2}. \quad (4)$$

In the entropy representation, this fundamental equation can be rewritten as

$$S = \pi(2M^2 - Q^2 + 2\sqrt{M^4 - M^2Q^2 - J^2}). \quad (5)$$

Davies [5] argued that black holes undergo a second order phase transition at the points where the heat capacity diverges. This argument is supported by the result that some critical exponents related to the singular points obey scaling laws [7–12]. Following Davies, we assume in this work that the structure of the phase transitions of the Kerr-Newman black hole is determined by the corresponding heat capacity $C = T(\partial S/\partial T)$:

$$C_{Q,J} = -\frac{4TM^3S^3}{2M^6 - 3M^4Q^2 - 6M^2J^2 + Q^2J^2 + 2(M^4 - M^2Q^2 - J^2)^{3/2}}. \quad (6)$$

On the other hand, differential geometric concepts have been applied in ordinary thermodynamics since the seventies. First, Weinhold [13] introduced on the space of equilibrium states a metric whose components are given as the Hessian of the internal thermodynamic energy. Later, Ruppeiner [14,15] introduced a metric which is defined as minus the Hessian of the entropy, and is conformally equivalent to Weinhold's metric, with the inverse of the temperature as the conformal factor. One of the aims of the application of geometry in thermodynamics is to describe phase transitions in terms of curvature singularities so that the curvature can be interpreted as a measure of thermody-

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namic interaction. This turns out to be true in the case of the ideal gas, whose curvature vanishes, and the van der Waals gas for which the curvature of Weinhold's and Ruppeiner's metric becomes singular at those points where phase transitions occur. This is an encouraging result that illustrates the applicability of geometry in thermodynamics. It is then natural to try to describe the phase transitions of black holes in terms of curvature singularities in the space of equilibrium states. Unfortunately, the obtained results are contradictory. For instance, for the Reissner-Nordström black hole the Ruppeiner metric is flat [16], whereas the Weinhold metric with the mass as thermodynamic potential presents a curvature singularity only in the limit of an extremal black hole. None of these results reproduces the phase transitions as predicted by Davies using the heat capacity. Nevertheless, a simple change of the thermodynamic potential [17] affects Ruppeiner's geometry in such a way that the resulting curvature singularity now corresponds to a phase transition. A dimensional reduction of Ruppeiner's curvature seems to affect its properties too [18]. The situation is similar in the case of the Kerr black hole: Weinhold's metric is flat [16], and the original Ruppeiner metric does not present curvature singularities at the points of phase transitions of the Kerr black hole. Nevertheless, with a change of thermodynamic potential [17], Ruppeiner's metric reproduces the structure of the phase transitions of the Kerr black hole. These results seem to indicate that, in the case of black holes, geometry and thermodynamics are compatible only for a very specific thermodynamic potential. However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential. We believe that a geometric description of thermodynamics should preserve this property, i.e., it should be invariant with respect to Legendre transformations.

Recently [19], the formalism of geometrothermodynamics (GTD) was proposed as a geometric approach that incorporates Legendre invariance in a natural way, and allows us to derive Legendre invariant metrics in the space of equilibrium states. Since Weinhold and Ruppeiner metrics are not Legendre invariant, one of the first results in the context of GTD was the derivation of simple Legendre invariant generalizations of these metrics and their application to black hole thermodynamics. It turned out [20] that the thermodynamics of the Reissner-Nordström black hole is compatible with both Weinhold and Ruppeiner generalized metric structures. However, in the case of the Kerr black hole both generalized geometries are flat and, therefore, cannot reproduce its thermodynamic behavior. This was considered as a negative result for the use of geometry in black hole thermodynamics.

In the present work we use GTD to derive a Legendre invariant metric which completely and consistently reproduces the thermodynamic behavior of black holes, including the Kerr-Newman black hole. This result finishes the controversy regarding the application of geometric structures in black hole thermodynamics. The phase transition structure contained in the heat capacity of black holes becomes completely integrated in the scalar curvature of the Legendre invariant metric so that a curvature singularity corresponds to a phase transition.

This paper is organized as follows. In Sec. II we introduce the general formalism of GTD for black holes. A particular Legendre invariant metric is given in the thermodynamic phase space which is the starting point of our analysis. In Sec. III we apply 2-dimensional GTD in its entropy representation to the Reissner-Nordström and Kerr black holes. The analysis of the Kerr-Newman black hole requires 3-dimensional GTD and it is presented in Sec. IV. Finally, Sec. V is devoted to discussions of our results and suggestions for further research. Throughout this paper we use units in which $G = c = k_B = \hbar = 1$.

## II. GEOMETROTHERMODYNAMICS OF BLACK HOLES

The starting point of GTD is the thermodynamic phase space $\mathcal{T}$ which in the case of Einstein-Maxwell black holes can be defined as a 7-dimensional space with coordinates $Z^A = \{M, S, Q, J, T, \phi, \Omega_H\}$, $A = 0, \ldots, 6$. In the cotangent space $\mathcal{T}^*$, we introduce the fundamental one-form

$$\Theta_M = dM - TdS - \phi dQ - \Omega_H dJ, \qquad (7)$$

which satisfies the condition $\Theta_M \wedge (d\Theta_M)^3 \neq 0$. Furthermore, in $\mathcal{T}$ we introduce a nondegenerate metric $G$. The triplet $(\mathcal{T}, \Theta_M, G)$ is said to form a Riemannian contact manifold. Let $\mathcal{E}$ be a 3-dimensional subspace of $\mathcal{T}$ with coordinates $E^a = \{S, Q, J\}$, $a = 1, 2, 3$, defined by means of a smooth mapping $\varphi_M: \mathcal{E} \to \mathcal{T}$. The subspace $\mathcal{E}$ is called the space of equilibrium states if $\varphi_M^*(\Theta_M) = 0$, where $\varphi_M^*$ is the pullback induced by $\varphi_M$. Furthermore, a metric structure $g$ is naturally induced on $\mathcal{E}$ by applying the pullback on the metric $G$ of $\mathcal{T}$, i.e., $g = \varphi_M^*(G)$. It is clear that the condition $\varphi_M^*(\Theta_M) = 0$ leads immediately to the first law of thermodynamics of black holes as given in Eq. (3). It also implies the existence of the fundamental equation $M = M(S, Q, J)$ and the conditions of thermodynamic equilibrium

$$T = \frac{\partial M}{\partial S}, \qquad \phi = \frac{\partial M}{\partial Q}, \qquad \Omega_H = \frac{\partial M}{\partial J}. \qquad (8)$$

Legendre invariance is an important ingredient of GTD. It allows us to change the thermodynamic potential without affecting the results. If we denote the intensive thermodynamic variables as $I^a = \{T, \phi, \Omega_H\}$, then a Legendre transformation is defined by [21]

$$\{M, E^a, I^a\} \to \{\tilde{M}, \tilde{E}^a, \tilde{I}^a\}, \qquad (9)$$

$$M = \tilde{M} - \delta_{ab}\tilde{E}^a\tilde{I}^b, \qquad E^a = -\tilde{I}^a, \qquad I^a = \tilde{E}^a. \qquad (10)$$

It is easy to see that the fundamental one-form $\Theta_M$ is invariant with respect to Legendre transformations. Furthermore, if we demand that the metric $G$ be Legendre invariant, it can be shown [19] that the induced metric $g = \varphi_M^*(G)$ is also Legendre invariant.

Another advantage of the use of GTD is that it allows us to easily implement different thermodynamic representations. The above description is called the $M$-representation because the fundamental equation is given as $M = M(S, Q, J)$. However, one can rewrite this equation as $S = S(M, Q, J)$, $Q = Q(S, M, J)$ or $J = J(S, M, Q)$, and redefine the coordinates in $\mathcal{T}$ and the smooth mapping $\varphi$ in such a way that the condition $\varphi^*(\Theta) = 0$ generates on $\mathcal{E}$ the corresponding fundamental equation in the $S$-, $Q$-, or the $J$-representation, respectively. As an example of this procedure we will present the $S$-representation which turned out to be the most appropriate for the description of black hole thermodynamics. It must be emphasized, however, that the results obtained with different representations of the same fundamental equation are completely equivalent.

For the $S$-representation we consider the fundamental one-form

$$\Theta_S = dS - \frac{1}{T}dM + \frac{\phi}{T}dQ + \frac{\Omega_H}{T}dJ, \qquad (11)$$

so that the coordinates of $\mathcal{T}$ are $Z^A = \{S, E^a, I^a\} = \{S, M, Q, J, 1/T, -\phi/T, -\Omega_H/T\}$. The space of equilibrium states $\mathcal{E}$ can then be introduced with the smooth mapping

$$\varphi_S: \{M, Q, J\} \mapsto \{M, S(M, Q, J), Q, J, I^a(M, Q, J)\}, \qquad (12)$$

which, from the condition $\varphi_S^*(\Theta_S) = 0$, generates the first law of thermodynamics of black holes (3) and the equilibrium conditions

$$\frac{1}{T} = \frac{\partial S}{\partial M}, \qquad \frac{\phi}{T} = -\frac{\partial S}{\partial Q} \qquad \frac{\Omega_H}{T} = -\frac{\partial S}{\partial J}. \qquad (13)$$

In this representation the fundamental equation is given as in Eq. (5).

Consider now the following metric on $\mathcal{T}$:

$$G = \left(dS - \frac{1}{T}dM + \frac{\phi}{T}dQ + \frac{\Omega_H}{T}dJ\right)^2 + \left(\frac{M}{T} - \frac{Q\phi}{T} - \frac{J\Omega_H}{T}\right) \times \left[dMd\left(\frac{1}{T}\right) + dQd\left(\frac{\phi}{T}\right) + dJd\left(\frac{\Omega_H}{T}\right)\right]. \qquad (14)$$

It is easy to show that this metric is invariant with respect to Legendre transformations (10). The first term of this metric can be written in the form $\Theta_S \otimes \Theta_S$ so that its projection on $\mathcal{E}$ vanishes, due to the condition $\varphi_S^*(\Theta_S) = 0$. Nevertheless, this term is necessary in order for the metric $G$ to be nondegenerate. For the metric induced on $\mathcal{E}$ by means of $g = \varphi_S^*(G)$, only the second term of $G$ is relevant. A straightforward computation leads to

$$g = (MS_M + QS_Q + JS_J)(S_{MM}dM^2 - S_{QQ}dQ^2 - S_{JJ}dJ^2 - 2S_{QJ}dQdJ), \qquad (15)$$

where for simplicity we introduced the notation that a subindex represents partial derivative with respect to the corresponding coordinate. This metric is Legendre invariant and nondegenerate and therefore can be used to introduce a Legendre invariant, Riemannian metric structure in the space of equilibrium states $\mathcal{E}$. This turns $\mathcal{E}$ into a well-defined Riemannian submanifold of the thermodynamic phase space $\mathcal{T}$. In the next sections we will show that metric (15) correctly reproduces the thermodynamic behavior of Einstein-Maxwell black holes.

## III. BLACK HOLES WITH TWO DEGREES OF FREEDOM

From the above description of GTD, it follows that the dimension of the phase space is $2n + 1$, where $n$ is the number of thermodynamic degrees of freedom which coincides with the dimension of the subspace $\mathcal{E}$. The case $n = 1$ corresponds to the Schwarzschild black hole with the mass $M$ as the only nonvanishing thermodynamic degree of freedom. In this case the Riemannian structure of $\mathcal{E}$ is trivial. For $n = 2$ the geometric structure of $\mathcal{E}$ is nontrivial and corresponds to the Reissner-Nordström black hole ($J = 0$) or to the Kerr black hole ($Q = 0$). Notice that in this case the metric $g$ on $\mathcal{E}$ becomes diagonal and that drastically simplifies the calculations. The general Kerr-Newman black hole corresponds to a 3-dimensional manifold $\mathcal{E}$ with a nondiagonal metric $g$. It requires a separate analysis that will be performed in Sec. IV.

### A. The Reissner-Nordström black hole

The Reissner-Nordström metric can be obtained from Eq. (1) by imposing the condition $J = 0$. It describes a static, spherically symmetric black hole with two horizons situated at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \qquad (16)$$

We assume that $Q \leq M$ in order to avoid naked singularities. The thermodynamic information of this black hole is contained in the fundamental equation which, in the entropy representation we are using in this work, becomes

$$S = \pi(M + \sqrt{M^2 - Q^2})^2. \qquad (17)$$

According to Davies [5], the phase transition structure of the Reissner-Nordström black hole can be derived from the heat capacity

$$C_Q = \frac{4TM^3S^3}{-2M^6 + 3M^4Q^2 - 2(M^4 - M^2Q^2)^{3/2}} = -\frac{2\pi^2 r_+^2 (r_+ - r_-)}{r_+ - 3r_-}. \quad (18)$$

For our geometric approach to black hole thermodynamics all that is needed is the fundamental equation as given in (17) from which we can calculate the thermodynamic metric

$$g_{ab}^{\rm RN} = (MS_M + QS_Q)\begin{pmatrix} S_{MM} & 0 \\ 0 & -S_{QQ} \end{pmatrix} = \frac{8\pi^2 r_+^3}{(r_+ - r_-)^3}\begin{pmatrix} 2r_+(r_+ - 3r_-) & 0 \\ 0 & r_+^2 + 3r_-^2 \end{pmatrix}. \quad (19)$$

Notice that this metric is singular in the extremal limit $r_+ = r_-$. It could indicate a breakdown of our geometric approach. However, the analysis of the corresponding scalar curvature

$$R^{\rm RN} = \frac{(r_+^2 - 3r_- r_+ + 6r_-^2)(r_+ + 3r_-)(r_+ - r_-)^2}{\pi^2 r_+^3 (r_+^2 + 3r_-^2)^2 (r_+ - 3r_-)^2} \quad (20)$$

shows that in the extremal limit the space of equilibrium states becomes flat. This means that there must exist a different coordinate system in which the metric (19) does not diverge in the extremal limit. Moreover, we see from the expression for the scalar curvature that the only singular point corresponds to the value $r_+ = 3r_-$ which is exactly the point where a phase transition occurs in the heat capacity (18).

### B. The Kerr black hole

The Kerr metric corresponds to the limit $Q = 0$ of the Kerr-Newman metric (1). It describes the gravitational field of a stationary, axially symmetric, rotating black hole with two horizons situated at the radial distances

$$r_\pm = M \pm \sqrt{M^2 - J^2/M^2}. \quad (21)$$

The corresponding thermodynamic fundamental equation in the entropy representation becomes

$$S = 2\pi(M^2 + \sqrt{M^4 - J^2}). \quad (22)$$

Furthermore, second order phase transitions occur at the points where the heat capacity

$$C_J = \frac{4TM^3S^3}{6M^2J^2 - 2M^6 - 2(M^4 - J^2)^{3/2}} = \frac{2\pi^2 r_+ (r_+ + r_-)^2 (r_+ - r_-)}{r_+^2 - 6r_+ r_- - 3r_-^2} \quad (23)$$

diverges. We assume values of the mass in the range $M^2 \geq J$, the equality being the extremal limit of the Kerr black hole in which the two horizons coincide.

The Legendre invariant metric reduces in this case to

$$g_{ab}^{K} = (MS_M + JS_J)\begin{pmatrix} S_{MM} & 0 \\ 0 & -S_{JJ} \end{pmatrix} = \frac{16\pi^2 r_+^2 (r_+ + r_-)}{(r_+ - r_-)^4} \times \begin{pmatrix} r_+(r_+^2 - 6r_+ r_- - 3r_-^2) & 0 \\ 0 & r_+ + r_- \end{pmatrix}. \quad (24)$$

We obtain again a metric that becomes singular at the extremal limit $r_+ = r_-$. The scalar curvature for the thermodynamic metric of the Kerr black hole can be expressed as

$$R^K = \frac{(3r_+^3 + 3r_+^2 r_- + 17r_+ r_-^2 + 9r_-^3)(r_+ - r_-)^3}{2\pi^2 r_+^2 (r_+ + r_-)^4 (r_+^2 - 6r_+ r_- - 3r_-^2)^2}. \quad (25)$$

This shows that the metric singularity at $r_+ = r_-$ is only a coordinate singularity. On the other hand, the curvature singularities are situated at the roots of the polynomial equation $r_+^2 - 6r_+ r_- - 3r_-^2 = 0$. According to the expression for the heat capacity (23), these are exactly the roots that determine the critical points where phase transitions take place.

## IV. THE GENERAL KERR-NEWMAN BLACK HOLE

The Kerr-Newman metric (1) describes the gravitational field of the most general rotating, charged black hole. It possesses an outer horizon at $r_+$ and an inner horizon at $r_-$, with $r_\pm$ given as in Eq. (2). According to our results of Sec. II, the space of thermodynamic equilibrium states is 3-dimensional and the corresponding Legendre invariant metric can be written as

$$g_{ab}^{\rm KN} = (MS_M + QS_Q + JS_J)\begin{pmatrix} S_{MM} & 0 & 0 \\ 0 & -S_{QQ} & -S_{QJ} \\ 0 & -S_{QJ} & -S_{JJ} \end{pmatrix}. \quad (26)$$

Inserting here the expression for the entropy (5) we obtain a rather cumbersome metric which cannot be written in a compact form. Moreover, the scalar curvature can be shown to have the form

$$R^{\rm KN} = \frac{N}{D}, \quad (27)$$
$$D = 4(MS_M + QS_Q + JS_J)^3 (S_{QJ}^2 - S_{QQ}S_{JJ})^3 S_{MM}^2$$

so that replacing the entropy formula we obtain

$$D \propto [2M^4 - 2M^2Q^2 - J^2 + (2M^2 - Q^2) \times (M^4 - M^2Q^2 - J^2)^{1/2}]^3[M^4 + (M^2 - Q^2) \times (M^4 - M^2Q^2 - J^2)^{1/2}]^3[2M^6 - 3M^4Q^2 - 6M^2J^2 + Q^2J^2 + 2(M^4 - M^2Q^2 - J^2)^{3/2}]^2. \quad (28)$$

The first two terms in squared brackets can be shown to be always positive in the range $M^4 \geq M^2Q^2 + J^2$, which is a condition that guarantees the nonexistence of naked singularities. The third term in squared brackets is exactly the denominator of the heat capacity (6). This proves that the curvature singularities of the thermodynamic metric $g^{KN}$ are situated at those points where phase transitions can occur. Moreover, it can be shown that the curvature vanishes in the case of an extremal black hole, $M^4 = M^2Q^2 + J^2$. This resembles the behavior of the curvature of the thermodynamic metrics of the Reissner-Nordström and Kerr black holes presented in the last section.

## V. DISCUSSION AND CONCLUSIONS

Using the formalism of GTD, in this work we derived a metric for the space of equilibrium states of black holes which reproduces the thermodynamic behavior of Reissner-Nordström, Kerr, and Kerr-Newman black holes. The thermodynamic metric is derived from a Legendre invariant metric which is introduced in the thermodynamic phase space. In contrast to other metrics used previously in the literature, the curvature singularities of our metric reproduce in a unified manner the phase transitions of black holes, if we assume that phase transitions correspond to divergences of the heat capacity. This result shows that the curvature of our thermodynamic metric can be used as a measure of thermodynamic interaction for black holes.

For all black holes of Einstein-Maxwell theory, the space of equilibrium states, equipped with our thermodynamic metric, becomes singular at those points where phase transitions occur, and it is flat in the limit of extreme black holes, i.e. when the two horizons coincide. This indicates that our thermodynamic metric is well defined in the region $M^4 - M^2Q^2 - J^2 \geq 0$, except at the phase transition points where it becomes singular. Outside this region, our thermodynamic metric is not well defined because the fundamental equation becomes complex and cannot be used to generate the geometric Riemannian structure of the space of equilibrium states. This is an indication that the thermodynamic description of black holes cannot be extended into the region of naked singularities. This is also an indication that classical thermodynamics cannot be used for black holes of the size of the Planck length, which is the extremal limit of applicability one would expect for classical thermodynamics.

We assumed in this work Davies' formulation of phase transitions for black holes. However, the interpretation of divergences in specific heats as phase transitions is not definitely settled and is still a subject of debate [22–25]. In fact, what is really needed is a microscopic description which would couple to the macroscopic thermodynamics of black holes. However, such a macroscopic description must be related to a theory of quantum gravity which is still far from being formulated in a consistent manner. In the meantime, we can only use the intuitive interpretation of phase transitions as it is known in classical thermodynamics.

The thermodynamic metric we propose in this work is intuitively simple, it can be written in a compact form, and it satisfies the mathematical compatibility conditions of GTD. However, we do not have whatsoever any interpretation of its components in terms of any physical theory. We believe that Ruppeiner's metric is the only know thermodynamic metric with a specific physical interpretation in the context of thermodynamic fluctuation theory. It would be interesting to investigate the stability of the metric derived in this work, especially the different scenarios available in black hole thermodynamics [26].

The computer algebra system REDUCE 3.8 was used for most of the calculations reported in this work.

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