Clustering, angular size, and dark energy

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The influence of dark matter inhomogeneities on the angular size-redshift test is investigated for a large class of flat cosmological models driven by dark energy plus a cold dark matter component (XCDM). The results are presented in two steps. First, the mass inhomogeneities are modeled by a generalized Zeldovich-Kantowski-Dyer-Roeder distance which is characterized by a smoothness parameter $\alpha(z)$ and a power index γ, and, second, we provide a statistical analysis to angular size data for a large sample of milliarcsecond compact radio sources. As a general result, we have found that the α parameter is totally unconstrained by this sample of angular-diameter data.

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I. INTRODUCTION

An impressive convergence of recent astronomical observations are suggesting that our world behaves like a spatially flat scenario dominated by cold dark matter (CDM) plus an exotic component endowed with large negative pressure, usually named dark energy [1–3]. In the framework of general relativity, besides the cosmological constant, there are several candidates for dark energy, among them: a vacuum decaying energy density, or a time varying $\Lambda(t)$ [4], the so-called ''X-matter'' [5], a relic scalar field [6], and a Chaplygin gas [7]. Some recent review articles discussing the history, interpretations, as well as the major difficulties of such candidates have also been published in the past few years [8].

In the case of X-matter, for instance, the dark energy component is simply described by an equation of state $p_x = \omega\rho_x$. The case $\omega = -1$ reduces to the cosmological constant, and together the CDM defines the scenario usually referred to as ''cosmic concordance model'' (ΛCDM). The imposition $\omega \geq -1$ is physically motivated by the classical fluid description [9]. However, as discussed by several authors, such an imposition introduces a strong bias in the parameter determination from observational data. In order to take into account this difficulty, superquintessence or phantom dark energy cosmologies have been recently considered where such a condition is relaxed [10]. In contrast to the usual quintessence model, a decoupled phantom component presents an anomalous evolutionary behavior. For instance, the existence of future curvature singularities, a growth of the energy density with the expansion, or even the possibility of a rip-off of the structure of matter at all scales are theoretically expected ([11] for a thermodynamic discussion). Although possessing such strange features, the phantom behavior is theoretically allowed by some kinetically scalar field driven cosmology [12], as well as by brane world models [13], and, perhaps, more important to the present work, a PhantomCDM cosmology provides a better fit to type Ia Supernovae observations than does the ΛCDM model [14]. Many other observational and theoretical properties phantom driven cosmologies (more generally, of XCDM scenarios) have been successfully confronted to standard results (see, for instance [15–19]).

In this context, one of the most important tasks for cosmologists nowadays is to confront different cosmological scenarios driven by cold dark matter (CDM) plus a given dark energy candidate with the available observational data. As widely known, a key quantity for some cosmological tests is the angular distance-redshift relation, $D_A(z)$, which for a homogeneous and isotropic background can readily be derived by using the Einstein field equations for the Friedmann-Robertson-Walker (FRW) geometry. From $D_A(z)$, one obtains the expression for the angular diameter $\theta(z)$ which can be compared with the available data for different samples of astronomical objects [20].

Nevertheless, the real Universe is not perfectly homogeneous, with light beams experiencing mass inhomogeneities along their way. Actually, from small to intermediate scales (≤ 100 Mpc), there is a lot of structure in the form of voids, clumps, and clusters which is probed by the propagating light [21]. Since the perturbed metric is unknown, an interesting possibility to account for such an effect is to introduce the smoothness parameter α which is a phenomenological representation of the magnification effects experienced by the light beam. From general grounds, one expects a redshift dependence of α since the degree of smoothness for the pressureless matter is supposed to be a time varying quantity [17,18]. When $\alpha = 1$ (filled beam), the homogeneous FRW case is fully recovered; $\alpha < 1$ stands for a defocusing effect while $\alpha = 0$ represents a totally clumped universe (empty beam). The distance relation that takes these mass inhomogeneities into account was discussed by Zeldovich [22] followed by Kantowski [23], although a clear-cut application for cosmology was given only in 1972 by Dyer and Roeder [24]. Later on, by considering a perturbed Friedmannian model Tomita [25] performed N-body simulations with the CDM spectrum in order to determine the distribution for α

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(see also Ref. [26] for a more general analysis involving distances in perturbed models). Many references may also be found in the textbook by Schneider, Ehlers, and Falco [27], as well as Kantowski [28–30].

Many studies involving the Zeldovich-Kantowski-Dyer-Roeder (ZKDR) distances in dark energy models have been published in the literature. Analytical expressions for a general background in the empty beam approximation ($\alpha = 0$) were derived by Sereno *et al.* [31]. By assuming that both dominant components may be clustered they also discussed how the critical redshift, i.e., the value of z for which $D_A(z)$ is a maximum [or $\Theta(z)$ minimum], and compared to the homogeneous background results as given by Lima and Alcaniz [32], and further discussed by Lewis and Ibata [33], and Araújo and Stoeger [34]. More recently, Demianski *et al.* [35] derived a useful analytical approximate solution for a clumped concordance model (ΛCDM) valid on the interval $0 \leq z \leq 10$. Additional studies on this subject involving time delay (Lewis and Ibata [33]; Giovi and Amendola [36]), gravitational lensing (Kochanek; Kochanek and Schechter [37]) or even accelerated models driven by particle creation have also been considered [38,39].

Although carefully investigated in many of their theoretical and observational aspects, an overview in the literature shows that a quantitative analysis on the influence of dark energy in connection with inhomogeneities present in the observed universe still remains to be studied. Analytical expression for a general applied for the $\theta(z)$ statistics with basis on a ΛCDM cosmology with constant α [40]. It was concluded that the best fit model occurs at $\Omega_M = 0.2$ and $\alpha = 0.8$ whether the characteristic angular size l of the compact radio sources is marginalized. More recently, the smoothness α parameter was constrained through a statistical analysis involving Supernovae Ia data [41]. A χ^2-analysis based on the 182 SNe Ia data of Riess *et al.* [2] constrained the pair of parameters (Ω_M, α) to be $\Omega_M = 0.33^{+0.09}_{-0.07}$ and $\alpha \geq 0.42$ (2σ). Such an analysis has also been carried out in the framework of a ΛCDM cosmology.

In this paper, we focus our attention on X-matter cosmologies with special emphasis to phantom models ($\omega < -1$) by taking into account the presence of a clustered cold dark matter. The mass inhomogeneities will be described by the ZKDR distance characterized by a smoothness parameter $\alpha(z)$ which depends on a positive power index γ. The main objective is to provide a statistical analysis to angular size data from a large sample of milliarcsecond compact radio sources [42] distributed over a wide range of redshifts ($0.011 \leq z \leq 4.72$) whose distance is defined by the ZKDR equation. As an extra bonus, it will be shown that a pure CDM model ($\Omega_M = 1$) is not compatible with these data even for the empty beam approximation ($\alpha = 0$).

The manuscript is organized as follows. In Sec. II we outline the derivation of the ZKDR equation for a XCDM cosmology. We also provide some arguments (see the Appendix) for a locally nonhomogeneous Universe where the homogeneous contribution of the dark matter obeys the relation $\rho_h = \alpha\rho_o(\rho_M/\rho_o)^\gamma$, where γ is a positive number, ρ_M is the average matter density, and ρ_o its present value. In Sec. III we analyze the constraints on the free parameters α and Ω_M from angular size data. We end the paper by summarizing the main results in Sec. IV.

II. THE EXTENDED ZKDR EQUATION

Let us now consider a flat FRW geometry ($c = 1$)

$$ds^2 = dt^2 - R^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \quad (1)$$

where $R(t)$ is the scale factor. Such a space-time is supported by the pressureless CDM fluid plus a X-matter component of densities ρ_M and ρ_x, respectively. Hence, the total energy momentum tensor, $T^{\mu\nu} = T^{\mu\nu}{}_{(M)} + T^{\mu\nu}{}_{(x)}$, can be written as

$$T^{\mu\nu} = [\rho_M + (1+\omega)\rho_x]U^\mu U^\nu - \omega\rho_x g^{\mu\nu}, \quad (2)$$

where $U^\mu = \delta^\mu_o$ is the hydrodynamics 4-velocity of the comoving volume elements. In this framework, the independent components of the Einstein field equations

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = 8\pi G T^{\mu\nu}, \quad (3)$$

take the following forms:

$$\left(\frac{\dot{R}}{R}\right)^2 = H_o^2\left[\Omega_M\left(\frac{R_o}{R}\right)^3 + \Omega_x\left(\frac{R_o}{R}\right)^{3(1+\omega)}\right], \quad (4)$$

$$\frac{\ddot{R}}{R} = -\frac{1}{2}H_o^2\left[\Omega_M\left(\frac{R_o}{R}\right)^3 + (3\omega+1)\Omega_x\left(\frac{R_o}{R}\right)^{3(1+\omega)}\right], \quad (5)$$

where an overdot denotes derivative with respect to time and $H_o = 100h$ Kms^{-1} Mpc^{-1} is the Hubble parameter. By the flat condition, $\Omega_x = 1 - \Omega_M$ is the present day dark energy density parameter. As one may check from (2)–(5), the case $\omega = -1$ describes effectively the favored "cosmic concordance model" (ΛCDM).

On the other hand, in the framework of a conformally flat FRW metric, the optical scalar equation in the geometric optics approximation reads (optical shear neglected) [43]

$$\sqrt{A}'' + \tfrac{1}{2}R_{\mu\nu}k^\mu k^\nu\sqrt{A} = 0, \quad (6)$$

where A is the beam cross sectional area, plicas means derivative with respect to the affine parameter describing the null geodesics, and k^μ is a 4-vector tangent to the photon trajectory whose divergence determines the optical scalar expansion [17,31,36]. The circular frequency of the light ray as seen by the observer with 4-velocity U^α is $\omega = U^\alpha k_\alpha$, while the angular-diameter distance, D_A, is proportional to $\sqrt{A}$ [27].

As widely known, there is no acceptable averaging procedure for smoothing out local inhomogeneities. After Dyer and Roeder [24], it is usual to introduce a phenomenological parameter, $\alpha(z) = 1 - \frac{\rho_{cl}}{\langle\rho_M\rangle}$, called the "smoothness" parameter. For each value of z, such a parameter quantifies the portion of matter in clumps (ρ_{cl}) relative to the amount of background matter which is uniformly distributed (ρ_M). As a matter of fact, such authors examined only the case for constant α; however, the basic consequence of the structure formation process is that it must be a function of the redshift. Combining Eqs. (2), (3), and (6), after a straightforward but lengthy algebra, one finds that the angular-diameter distance, $D_A(z)$, obeys the following differential equation:

$$(1+z)^2 \mathcal{F} \frac{d^2 D_A}{dz^2} + (1+z)\mathcal{G}\frac{dD_A}{dz} + \mathcal{H} D_A = 0, \quad (7)$$

which satisfies the boundary conditions:

$$\begin{cases} D_A(0) = 0, \\ \left.\frac{dD_A}{dz}\right|_0 = 1. \end{cases} \quad (8)$$

The functions $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ in Eq. (7) read

$$\mathcal{F} = \Omega_M(1+z)^3 + (1-\Omega_M)(1+z)^{3(\omega+1)}$$
$$\mathcal{G} = \frac{7}{2}\Omega_M(1+z)^3 + \frac{3\omega+7}{2}(1-\Omega_M)(1+z)^{3(\omega+1)}$$
$$\mathcal{H} = \frac{3\alpha(z)}{2}\Omega_M(1+z)^3 + \frac{3(\omega+1)}{2}(1-\Omega_M)(1+z)^{3(\omega+1)}. \quad (9)$$

The smoothness parameter $\alpha(z)$, appearing in the expression of $\mathcal{H}$, assumes the form below (see the Appendix for a detailed discussion)

$$\alpha(z) = \frac{\beta_o(1+z)^{3\gamma}}{1+\beta_o(1+z)^{3\gamma}}, \quad (10)$$

where β_o and γ are constants. Note that the fraction $\alpha_o = \beta_o/(1+\beta_o)$ is the present day value of $\alpha(z)$. In Fig. 1 we show the general behavior of $\alpha(z)$ for some selected values of β_o and γ.

At this point, it is interesting to compare Eq. (7) together with the subsidiary definitions (8)–(10) with other treatments appearing in the literature. For $\gamma = 0$ (constant α) and $\omega = -1$ (ΛCDM) it reduces to Eq. (2) as given by Alcaniz *et al.* [40]. In fact, for $\omega = -1$ the function $\mathcal{H}$ is given by $\mathcal{H} = \frac{3\alpha}{2}\Omega_M(1+z)^3$. Further, recalling the existence of a simple relation between the luminosity distance and the angular-diameter distance [from Etherington principle [44], $D_L = (1+z)^2 D_A$], it is easy to see that Eq. (3) of Santos *et al.* [41] is recovered. A more general expression for the ΛCDM model (by including the curvature term) has been derived by Demianski *et al.* [35]. As one may check, for α constant, by identifying $\omega \equiv m/3 - 1$, our Eq. (7) is exactly Eq. (10) as presented by Giovi and Amendola [36] in their time delay studies [see also Eq. (2) of Sereno *et al.* [45]]. Different from other approaches appearing in the literature (see, for instance, Refs. [25,26]), we stress that in this paper the α parameter is always smaller than unity. In addition, the α parameter may also depend on the direction along the line of sight (for a discussion of such effects see Linder [18], Sereno *et al.* [45], Wang [46]).

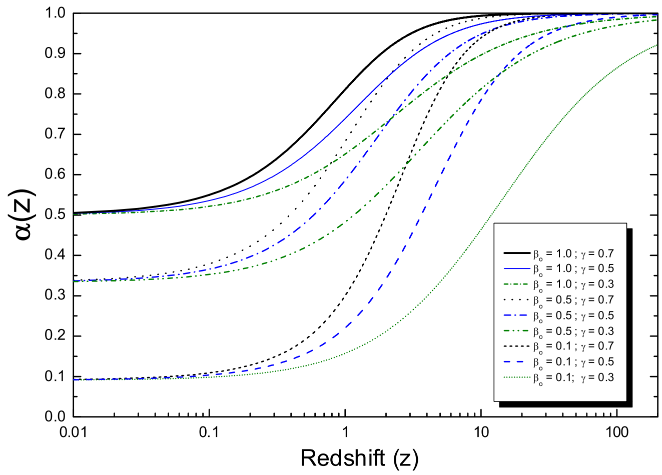

FIG. 1 (color online). The smoothness parameter as a function of the redshift for some selected values of β_o and γ. All curves approach the filled beam result ($\alpha = 1$) at high redshifts regardless of the values of β_o and γ. Note that β_o determines $\alpha_o = \alpha(z=0)$. For a given β_o the curves start at the same point but the rate approaching unit (filled beam) depends on the γ parameter.

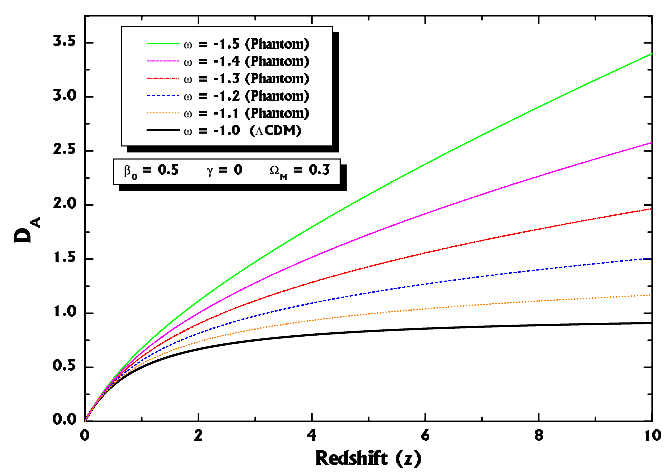

FIG. 2 (color online). Angular-diameter distance for a flat FRW phantom cosmology. The curves display the effect of the equation of state parameter for $\beta_o = 0.5$ and $\gamma = 0$. The thick curve corresponds to the ΛCDM model. Note that, for a given redshift, the distances always increase for ω beyond the phantom divide line ($\omega < -1$).

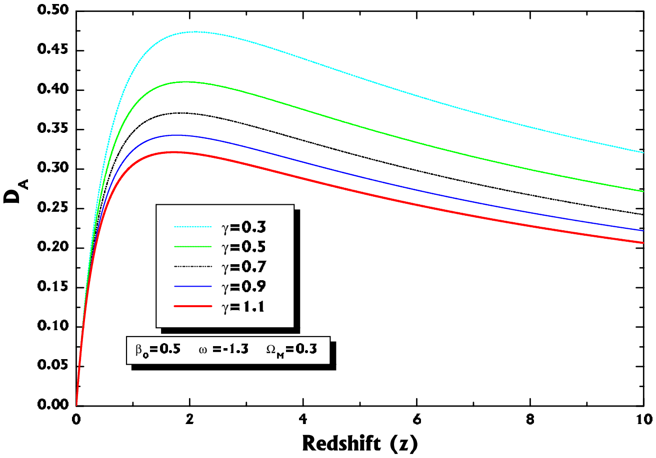

FIG. 3 (color online). Effects of the γ parameter on the angular-diameter distance. For all curves we fixed $\omega = -1.3$, $\beta_o = 0.5$, and $\Omega_M = 0.3$. Note that the distances increase for smaller values of γ.

Let us now discuss the integration of the ZKDR equation with emphasis in the so-called phantom dark energy model ($\omega < -1$). In what follows, assuming that ω is a constant, we have applied for all graphics a simple Runge-Kutta scheme (see, for instance, [47]).

In Fig. 2 one can see how the equation of state parameter, ω, affects the angular-diameter distance. For fixed values of $\Omega_M = 0.3$, $\beta_o = 0.5$, and $\gamma = 0$, all the distances increase with the redshift when ω diminishes and enters in the phantom regime ($\omega < -1$). For comparison, we have also plotted the case for ΛCDM cosmology ($\omega = -1$).

In Fig. 3 we show the effect of the γ parameter on the angular-diameter distance for a specific phantom cosmology with $\omega = -1.3$, as requested by some recent analyses of Supernovae data [2]. For this plot we have considered $\beta_o = 0.5$. As shown in the Appendix, $\beta_o = (\rho_h/\rho_{\rm cl})_{z=0}$ is the present ratio between the homogeneous (ρ_h) and the clumped ($\rho_{\rm cl}$) fractions. It was fixed in such a way that α_o assumes the value 0.33. Until redshifts of the order of 2, the distance grows for smaller values of γ, and after that, it decreases following nearly the same behavior.

In Fig. 4 we display the influence of the β_o parameter on the angular-diameter distance for two distinct sets of γ values. The cosmological framework is defined $\Omega_M = 0.3$ and the same equation of state parameter $\omega = -1.3$ (phantom cosmology). For each branch (a subset of 3 curves with fixed γ), the distance increases for smaller values of β_o, as should be expected.

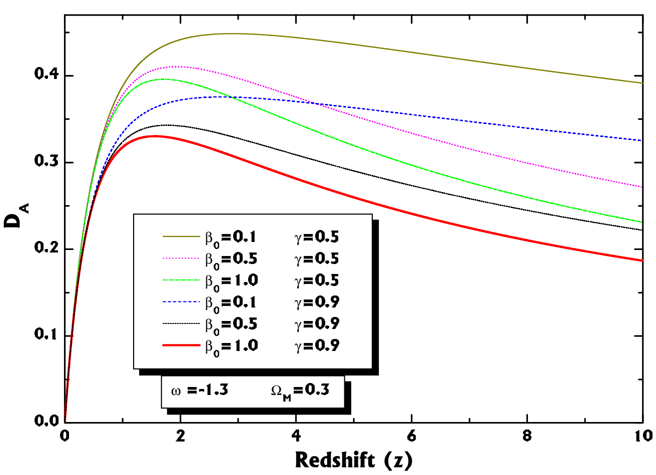

FIG. 4 (color online). Influence of the β_o parameter on the angular-diameter distance for $\Omega_M = 0.3$ and $\omega = -1.3$. The curves are separated in two sets corresponding to the values of $\gamma = 0.5$, 0.9 as indicated in the box. As expected, both sets present the same behavior at low redshifts.

III. ZKDR DISTANCE AND ANGULAR SIZE STATISTICS

As we have seen, in order to apply the angular-diameter distance to a more realistic description of the universe, it is necessary to take into account local inhomogeneities in the distribution of matter. Similarly, such a statement remains true for any cosmological test involving angular-diameter distances, as for instance, measurements of angular size, $\theta(z)$, of distant objects. Thus, instead of the standard FRW homogeneous diameter distance, one must consider the solutions of the ZKDR equation.

Here we are concerned with angular diameters of light sources described as rigid rods and not isophotal diameters. In the FRW metric, the angular size of a light source of proper length l (assumed free of evolutionary effects) and located at redshift z can be written as

$$\theta(z) = \frac{\ell}{D_A(z)}, \tag{11}$$

where $\ell = 100lh$ is the angular size scale expressed in milliarcsecond (mas) while l is measured in parsecs for compact radio sources (see below).

Let us now discuss the constraints from angular size measurements of high z objects on the cosmological parameters. The present analysis is based on the angular size data for milliarcsecond compact radio sources compiled by Gurvits *et al.* [42] (see also [20] for applications to the unclustered FRW case). This sample is composed by 145 sources at low and high redshifts ($0.011 \le z \le 4.72$) distributed into 12 bins with 12–13 sources per bin (for more details see Gurvits *et al.* [42]). In Fig. 5 we show the binned data of the median angular size plotted as a function of redshift z to the case with $\gamma = 0$ and some selected values of Ω_M and $\alpha_o = \beta_o/(1 - \beta_o) =$ constant. As can be seen there, for a given value of Ω_M the corresponding curve is slightly modified for different values of the smoothness parameter α.

Now, in order to constrain the cosmic parameters, we first fix the central value of the Hubble parameter obtained

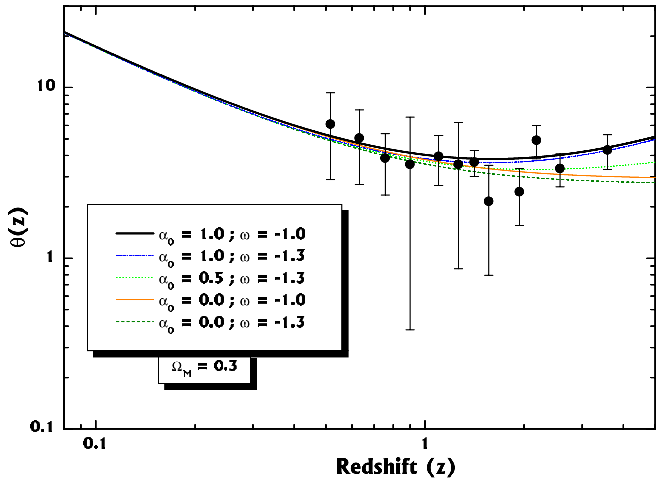

FIG. 5 (color online). Angular size versus redshift according to the ZKDR distance. Curves for $\Omega_M = 0.3$, $\gamma = 0$, and different values of ω are shown. The data points correspond to 145 compact radio sources binned into 12 bins (Gurvits *et al.* [42]). For comparison, the filled beam ΛCDM has been included.

by the Hubble Space Telescope (HST) key project $H_o = 72 \pm 8\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ (Freedman *et al.* [48]). Nowadays, this HST result has been confirmed by many different classes of estimators like the Sunyaev-Zeldovich effect and the ages of old high redshifts galaxies [49]. This value is also in accordance with the 3 years release of the WMAP team [3]; however, it is greater than the recent determination by Sandage and collaborators [50]. Following standard lines, the confidence regions are constructed through a χ^2 minimization

$$\chi^2(l, \omega, \alpha) = \sum_{i=1}^{12} \frac{[\theta(z_i, l, \omega, \alpha) - \theta_{oi}]^2}{\sigma_i^2}, \qquad (12)$$

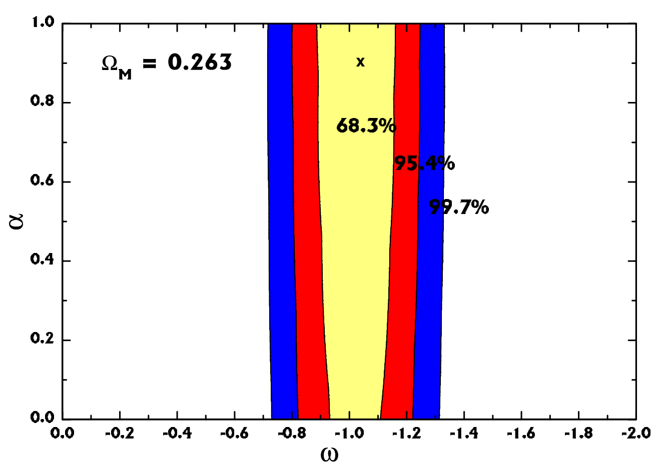

FIG. 6 (color online). Confidence regions in the $\omega - \alpha$ plane according to the sample of angular size data by Gurvits *et al.* [42] and fixed $\Omega_M = 0.263$ as shown in the panel. The confidence levels of the contours are indicated. The point "x" marks the best fit values, $\omega = -1.03$ and $\alpha = 0.90$.

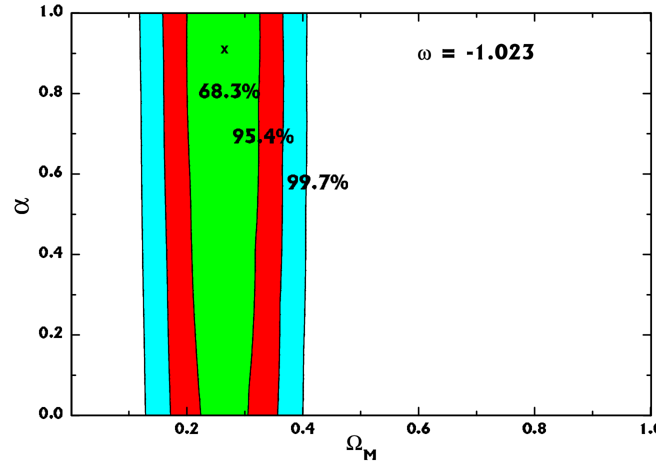

FIG. 7 (color online). Confidence regions in the $\Omega_M - \alpha$ plane according to the sample of angular size data by Gurvits *et al.* [42]. For a phantom cosmology with $\omega = -1.023$, the confidence levels of the contours are indicated. As in Fig. 6, the x also points to the best fit values shown in the panel.

where $\theta(z_i, l, \omega, \alpha)$ is defined from Eq. (7) and θ_{oi} are the observed values of the angular size with errors σ_i of the ith bin in the sample. The confidence regions are defined by the conventional two-parameters χ^2 levels. In this analysis, the intrinsic length l is considered a kind of "nuisance" parameter, and, as such, we have also marginalized over it.

In Fig. 6 we show confidence regions in the $\omega - \alpha$ plane fixing $\Omega_M = 0.263$, and assuming a Gaussian prior on the ω parameter, i.e., $\omega = -1 \pm 0.3$ (in order to accelerate the universe). The "×" indicates the best fit model that occurs at $\omega = -1.03$ and $\alpha \simeq 0.9$.

In Fig. 7 the confidence regions are shown in the $\Omega_M - \alpha$ plane. We have now assumed a Gaussian prior on Ω_M, i.e., $\Omega_M = 0.3 \pm 0.1$ from the large scale structure. From Figs. 6 and 7, it is also perceptible that, while the parameters ω and Ω_M are strongly restricted, the entire interval of α is still allowed. This shows the impossibility of tightly constraining the smoothness parameter α with the current angular size data. This result is in good agreement with the one found by Lima and Alcaniz [4], where the same data set were used to investigate constraints on quintessence scenarios in homogeneous background, and is also in line with the one obtained by Barber *et al.* [51] who argued in favor of $\alpha_o = \alpha(z = 0)$ near unity (see also Alcaniz, Lima, and Silva [40] for constraints on a clustered ΛCDM model).

IV. SUMMARY AND CONCLUDING REMARKS

All cosmological distances must be notably modified whether the space-time is filled by a smooth dark energy component with negative pressure plus a clustered dark matter. Here we have addressed the question of how the angular-diameter distance of extragalactic objects are modified by assuming a slightly inhomogeneous universe.

The present study complements our previous results [20] by considering that the inhomogeneities can be described by the Zeldovich-Kantowski-Dyer-Roeder distance (in this connection, see also Giovi and Amendola [36]; Lewis and Ibata [33]; Sereno *et al.* [45]; Demianski *et al.* [35]). The dark energy component was described by the equation of state $p_x = \omega \rho_x$. A special emphasis was given to the case of phantom cosmology ($\omega < -1$) when the dominant energy condition is violated. The effects of the local clustered distribution of dark matter have been described by the smoothness phenomenological parameter $\alpha(z)$, and a simple argument for its functional redshift dependence was given in the Appendix (see also Fig. 1).

The influence of the dark energy component was quantified by considering the angular diameters for sample of milliarcsecond radio sources (Fig. 5) as described by Gurvits *et al.* [42]. By marginalizing over the characteristic angular size l, fixing $\Omega_M = 0.263$, and assuming a Gaussian prior on the equation of state parameter, i.e., $\omega = -1 \pm 0.3$, the best fit model occurs at $\omega = -1.03$ and $\alpha = 0.9$. This phantom model coincides with the central value recently determined by the Supernova Legacy Survey (Astier *et al.* [3]). On the other hand, fixing $\omega = -1.023$ and assuming a Gaussian prior for Ω_M, that is, $\Omega_M = 0.3 \pm 0.1$, we obtained the best fit values ($\Omega_M = 0.29$, $\alpha = 0.9$).

Finally, in order to improve the present results, a statistical study is necessary for determining the intrinsic length of the compact radio sources. Further, unlike what happens with SNe data [41], the angular-diameter sample of compact radio sources of Gurvits *et al.* [42] does not provide useful constraints on the α parameter (see Figs. 6 and 7). Naturally, these results reinforce the interest for measurements of angular size from compact radio sources at intermediary and high redshifts in order to constrain the α parameter with basis on the ZKDR distance.

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APPENDIX: ON THE REDSHIFT DEPENDENCE OF $\alpha(z)$

In this Appendix, we discuss the functional redshift dependence of the smoothness parameter, $\alpha(z)$, adopted in this work. By definition

$$\alpha(z) = 1 - \frac{\rho_{\rm cl}(z)}{\rho_M(z)}, \tag{A1}$$

where $\rho_{\rm cl}$ denotes the clumped fraction of the total matter density, ρ_M, present in the considered FRW-type Universe. This means that the ratio between the homogeneous (ρ_h) and the clumped fraction can be written as $\rho_h/\rho_{\rm cl} = \alpha(z)/[1 - \alpha(z)]$. How does this ratio depend on the redshift? In this concern, we first remember that $\alpha(z)$ lies on the interval [0,1]. Second, in virtue of the structure formation process, one expects that the degree of homogeneity must increase for higher redshifts, or equivalently, the clumped fraction should be asymptotically vanishing at early times, say, for $z \geq 100$. This means that $\alpha(z) \rightarrow 1$ at high z. On the other hand, α must be zero for a completely clustered matter which is disproved at low redshifts by the data of galaxy clusters [3]. It thus follows that at present ($z = 0$), the related fraction assume an intermediate value, say β_o. In addition, it is also natural to suppose that the redshift dependence of the total matter density, ρ_M, must play an important role in the evolution of their fractions. In this way, for the sake of generality, we will assume a power law

$$\frac{\rho_h}{\rho_{\rm cl}} \equiv \frac{\alpha(z)}{1 - \alpha(z)} = \beta_o \left(\frac{\rho_M}{\rho_o}\right)^{\gamma}, \tag{A2}$$

where $\beta_o = (\rho_h/\rho_{\rm cl})_{z=0}$ and γ are dimensionless numbers. Finally, inserting $\rho_M(z)$, and solving for $\alpha(z)$ we obtain

$$\alpha(z) = \frac{\beta_o (1+z)^{3\gamma}}{1 + \beta_o (1+z)^{3\gamma}}, \tag{A3}$$

which is the expression adopted in this work [see Eq. (10)].

As one may check, for positive values of γ, the smoothness function (A3) has all the physically desirable properties above discussed. In particular, the limit for high values of z does not depend on the values of β_o and γ (both of the order of unity). Note also that if the clumped and homogeneous portions are contributing equally at present ($\beta_o = 1$), we see that $\alpha(z = 0) = 1/2$ regardless of the value of γ. Figure 1 displays the general behavior of $\alpha(z)$ with the redshift for different choices of β_o and γ. The above functional dependence should be compared with the other ones discussed in the literature (see, for instance, [17,18,39] and references therein). One of the most interesting features of (A3) is that its validity is not restricted to a given redshift interval.

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