Generation symmetry and E_6 unification

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The group E_6 for grand unification is combined with the generation symmetry group $SO(3)_g$. The coupling matrices in the Yukawa interaction are identified with the vacuum expectation values of scalar fields which are representations of the generation symmetry. These values determine the hierarchy of the fermions as well as their mixings and CP violation. This generation mixing appears in conjunction with the mixing of the standard model fermions with the heavy fermions present in the lowest representation of E_6. A close connection between charged and neutral fermions is observed relating for instance the Cabibbo-Kobayashi-Maskawa (CKM) mixings with the mass splittings of the light neutrinos. Numerical fits with only a few parameters reproduce quantitatively all known fermion properties. The model predicts an inverted neutrino hierarchy and gives rather strict values for the light and heavy neutrino masses as well as for the $0\nu2\beta$ decay parameter. It also predicts that the masses of the two lightest of six "right handed" neutrinos lie in the low TeV region.

I. INTRODUCTION

The origin of the properties of quarks and leptons, the masses and the mixings of the three generations, forms still an open problem in particle physics. Grand unified theories (GUTs) [1] provide a general understanding of the structure and the quantum numbers of the standard model states and suggest new ideas. The hope is to find a consistent scheme which provides for intimate relations between the many observables. Lagrangians considered for a single generation have the fundamental property of chiral symmetry, i.e. they are invariant under chiral transformations of the fermion fields taken together with appropriate transformations of the Higgs fields. For more generations chiral symmetry requires an extended symmetry which relates the different generations: the fermions should be members of a non-Abelian generation symmetry. There are two possibilities to enforce generation symmetry: one can either enlarge the number of Higgs fields by giving them generation indices or one can identify the coupling matrices in front of the Higgs fields as vacuum expectation values of new scalar fields carrying generation quantum numbers.

We choose here the second alternative in connection with the grand unified symmetry group E_6 [2–6]. For three generations the coupling matrices are 3×3 matrices. The corresponding 9 scalar fields are taken to be Hermitian and are expressed in terms of the Hermitian 3×3 matrix $\Phi(x)$

$$\Phi(x) = \chi(x) + \mathrm{i}\xi(x). \qquad (1.1)$$

Here χ denotes a symmetric and ξ an antisymmetric 3×3 matrix. Clearly, this choice implies that conventional

Yukawa interactions containing these fields are effective ones with dimension 5 and thus have to be understood on a deeper level. This can be done by the introduction of additional heavy spinor fields. We will see that, by integrating out these heavy fields, one finds interesting consequences for the relation between quark and neutrino mass matrices.

The introduction of the Hermitian matrix field $\Phi(x)$ coupled to the fermion fields suggests to use the group $SO(3)_g$ to describe the generation symmetry. In addition we will make use of a discrete parity like symmetry, generation parity P_g. From the point of view of chiral symmetry the use of $SU(3)_g$ instead of $SU(2)_g \simeq SO(3)_g$ would be more consequent. We will not treat this extension because the subgroup $SO(3)_g$ of $SU(3)_g$, in combination with P_g, is sufficient to reach our aims. Together with the unification group E_6 it can be dealt with in a very economical way. In the literature non-Abelian continuous [7] and discrete [8] flavor symmetries have been discussed and applied in various models. In our approach *all* fermion fields are taken to transform as 3-vectors under the generation group $SO(3)_g$. Consequently, the symmetric part of Φ has to transform as $\mathbf{1} + \mathbf{5}$ and the antisymmetric part as $\mathbf{3}$. Spontaneous symmetry breaking of this generation symmetry leads to vacuum expectation values of Φ. By an orthogonal transformation the symmetric matrix $\langle\chi\rangle$ can be taken diagonal. As we will see its elements describe the up quark hierarchy:

$$\frac{\langle\chi\rangle}{M} = G = \frac{1}{m_t}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} = \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$(1.2)$$

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Here M denotes the scale at which the appropriate effective Yukawa interaction of dimension 5 is formed. We take the mass ratios $m_c/m_t = m_u/m_c = \sigma^2$ to be valid at a high scale. At the weak scale M_Z these ratios are somewhat modified. Taking $\sigma = 0.050$ gives good agreement with the experimental mass determinations. In the following we will use this parameter for expressing small quantities.

There remains only the vacuum expectation value of the antisymmetric matrix $i\xi$ to produce all mixing and CP-violating properties of quarks and leptons. (The use of a purely antisymmetric and Hermitian mixing matrix was suggested in [9]). We take for ξ the particular form which was abstracted from the analysis of the fermion spectrum and the Cabibbo-Kobayashi-Maskawa (CKM) matrix in [5] using E_6 symmetry for grand unification. It has a particular symmetry: it changes sign by exchanging the second with the third generation:

$$\frac{1}{M'}\langle i\xi\rangle = A = i\lambda_A \begin{pmatrix} 0 & \sigma & -\sigma \\ -\sigma & 0 & 1/2 \\ \sigma & -1/2 & 0 \end{pmatrix}. \quad (1.3)$$

The relative factor between the (1, 2) and (2, 3) elements determines the particle mixings. Its value 2σ gives a good description of the mixings of quarks and neutrinos as will be shown in this article. The coupling constant λ_A can be incorporated into the mass scale M' (except for renormalization group (RG) considerations).

As already mentioned above, besides the $SO(3)_g$ symmetry we also introduce a parity symmetry P_g, "generation parity." The (extended) standard model fermions are taken to be even under this symmetry while the Higgs fields directly coupled to these fermion fields as well as the fields χ and ξ have negative generation parity. To obtain the spontaneous symmetry breaking of $SO(3)_g \times P_g$ we use in the Lagrangian for the field Φ $SO(3)_g \times P_g$ invariant potentials up to 4th order in the fields χ and ξ. Adding also the $SO(3)_g \times P_g$ invariant Coleman-Weinberg potential a complete breaking of the generation symmetry can be achieved. Moreover, by selecting properly the coefficients of these potentials the numerical values of (1.2) and (1.3) give the absolute minimum of the total potential.

It is clearly a challenge to obtain the masses, mixings, and CP properties of all fermions with only the two generation matrices G and A together with a few vacuum expectation values of the corresponding Higgs fields. Suited for this task is the grand unification symmetry E_6 [5]. Here we can postulate that particle mixings are caused by the mixing of the standard model particles with the heavy particles occurring in the lowest representation **27** of E_6. The up quarks cannot mix since they have no heavy partners in this representation. Thus, it follows immediately that—apart from a constant factor—their mass spectrum is simply given by $\langle\chi\rangle/M = G$ [as presented in (1.2)].

As we will see the breaking of generation symmetry and E_6 symmetry gives a hold on the complete fermion spectrum which includes new heavy fermions. In particular, we find a very strong hierarchy for the heavy right-handed neutrinos: their mass matrix is proportional to the square of the up quark mass matrix. By integrating out these heavy neutrinos, their masses appear in the denominator compensating the square of the Dirac mass matrix in the nominator. This mechanism gives the light neutrino spectrum a less pronounced hierarchy than the one of the up quarks. A possible consequence is an inverted neutrino spectrum. Such an interesting situation is not obtained in more conventional treatments of grand unified theories. It will be seen that very few parameters are sufficient to describe all the known properties of charged and neutral fermions in a quantitative way.

The use of E_6 as a grand unified theory has many virtues [2–6]. The fermions are in the lowest representation of this group and an elegant cyclic symmetry connects quark fields, lepton fields, and antiquark fields. In particular, as shown in [5], E_6 combines the mixings among fermion generations with the mixing of the standard model particles with heavy charged and neutral states.

Thus, our starting symmetry at the GUT scale is

$$E_6 \times SO(3)_g \times P_g. \quad (1.4)$$

In order to introduce notations and conventions we add at this place a short description of the **27** representation of E_6. In the E_6 grand unification model the fermions are contained in the **27** representation of the group, i.e. they are described by **27** two component (left-handed) Weyl fields for each generation:

$$\psi_r^\alpha, \qquad \alpha = 1, 2, 3, \qquad r = 1, \cdots, 27. \quad (1.5)$$

r denotes the E_6 flavor index and α labels the generations. These fields—even under P_g—describe the fermions of the standard model plus additional quark and antiquark fields with the same charge as the down quarks and new heavy charged and neutral leptons. All fermions are in singlet and triplet $SU(3)$ representations of the maximal subgroup of E_6

$$SU(3)_L \times SU(3)_R \times SU(3)_C \equiv G_{333}, \quad (1.6)$$

which plays an important role in our approach. In terms of G_{333} we have

$$\mathbf{27} = Q_L(x) + L(x) + Q_R(x), \quad (1.7)$$

where the quantum number assignments are

$$\text{Quarks: } Q_L(x) = (3, 1, \bar{3}),$$
$$\text{Leptons: } L(x) = (\bar{3}, 3, 1), \quad (1.8)$$
$$\text{Antiquarks: } Q_R(x) = (1, \bar{3}, 3).$$

For each generation one has

$$(Q_L)_i^a = \begin{pmatrix} u^a \\ d^a \\ D^a \end{pmatrix},$$

$$L_k^i = \begin{pmatrix} L_1^1 & E^- & e^- \\ E^+ & L_2^2 & \nu \\ e^+ & \hat{\nu} & L_3^3 \end{pmatrix}, \qquad (1.9)$$

$$(Q_R)_a^k = (\hat{u}_a, \hat{d}_a, \hat{D}_a),$$

where $i, k, a = 1, 2, 3$. In this description $SU(3)_L$ acts vertically (index i) and $SU(3)_R$ horizontally (index k) and a is a color index. It is seen, that in each generation there exist 12 new fields extending the standard model: colored quarks D, $\hat{D}$, charged leptons E^+, E^-, and 4 new neutral leptons $\hat{\nu}$, L_3^3, L_1^1, L_2^2, which all must correspond to heavy, but not necessarily very heavy, particles.

The group G_{333} can serve as an intermediate gauge symmetry below the E_6 breaking scale (the unification scale). It can be unbroken only at and above the scale where the two electroweak gauge couplings combine. In the nonsupersymmetric E_6 model, which we adopt here, this point occurs at $M_I \simeq 1.3 \times 10^{13}$ GeV according to the extrapolation of the standard model couplings. Interestingly, this is just the scale relevant for the neutrino masses using the seesaw mechanism. Above this scale the united electroweak couplings and the QCD coupling run at first separately until the electroweak coupling bends to meet the QCD coupling at the E_6 unification point. This happens at the scale $\approx 3 \times 10^{17}$ GeV (see Fig. 1 and more details in Appendix A 1).

Before E_6 symmetry breaking equivalent forms of (1.9) can be obtained by applying left and right $SU(3)$ $\mathcal{U}$-spin rotations. We fix the basis by using vacuum expectation values (VEV) of the lowest Higgs representation, the **27** of E_6. In this representation possible vacuum expectation values are restricted to the 5 neutral members sitting in positions corresponding to the ones of the neutral leptons
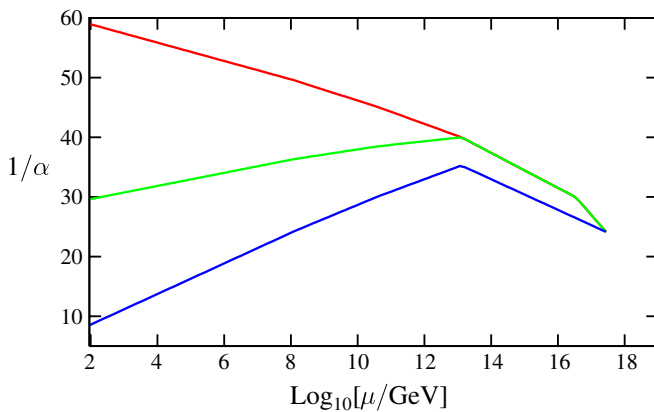

FIG. 1 (color online). "Concorde"—Unification of gauge couplings.
$M_I \simeq 1.27 \times 10^{13}$ GeV, $M_6 \simeq 3.47 \times 10^{16}$ GeV, $M_{\rm GUT} \simeq 2.8 \times 10^{17}$ GeV, and $\alpha_{\rm GUT}^{-1} \simeq 24.1$.

in (1.9). Below the scale M_I only two light $SU(2)_L$ doublets are assumed to be active. We incorporate them in two different scalar **27**-plet fields. Thus we introduce two scalar fields H_{27} and $\tilde{H}_{27}$ with the following transformation properties under $E_6 \times SO(3)_g \times P_g$:

$$H_{27} \sim (27, 1, -), \qquad \tilde{H}_{27} \sim (27, 1, +). \qquad (1.10)$$

Only H_{27} can couple to the ψ fields in the form $\chi \psi^T H \psi$. Thus, it is convenient to choose a basis in which the VEV $\langle H_{27} \rangle_k^i$ forms a diagonal matrix. In the scalar potential of the E_6 Lagrangian the bilinear terms with respect to both scalar **27**-plets can contain the singlet part $\chi_{(1)}$ of the fields χ:

$$\mu^2 H_{27}^\dagger H_{27} + \tilde{\mu}^2 \tilde{H}_{27}^\dagger \tilde{H}_{27} + \mu' \chi_{(1)} (H_{27}^\dagger \tilde{H}_{27} + H_{27} \tilde{H}_{27}^\dagger). \qquad (1.11)$$

After $\chi_{(1)}$ develops a VEV, and for μ' sufficiently small (i.e. $|\mu' \langle \chi_{(1)} \rangle| \ll |\mu^2 - \tilde{\mu}^2|$), the mass eigenstates H^u and H^d following from (1.11) are linearly related to H_{27} and $\tilde{H}_{27}$

$$H_{27} = H^u + z H^d \quad \text{and} \quad \tilde{H}_{27} = -z H^u + H^d$$

$$\text{with} \quad z = -\frac{\mu' \langle \chi_{(1)} \rangle}{(\mu^2 - \tilde{\mu}^2)}, \qquad |z| \ll 1. \qquad (1.12)$$

We take the light up type doublet to be in H^u: $((H^u)_1^1, (H^u)_2^1)$ and the light down type doublet $((H^d)_1^2, (H^d)_2^2)$ to occur in H^d. These Higgs doublets are important for generating the correct fermion mass pattern. $\langle (H_{27})_1^1 \rangle \simeq \langle (H^u)_1^1 \rangle = e_1^1$ determines the scale of the up quarks and $\langle (H_{27})_2^2 \rangle \simeq z \langle (H^d)_2^2 \rangle = z \epsilon_2^2$ the scale of the down quarks and charged leptons of the standard model. ϵ_2^2 is expected to be of the same order of magnitude as e_1^1. The factor z, which vanishes before P_g-symmetry breaking, is responsible for the small values of bottom and tau masses compared to the top mass. Components of H^u, H^d which are standard model singlets have large VEVs. For instance $\langle (H_{27})_3^3 \rangle \simeq \langle (H^u)_3^3 \rangle = e_3^3$ provide high (Dirac type) masses for all new quarks and leptons with the exception of the "right-handed" neutrinos $\hat{\nu} = L_2^3$ and L_3^3. Their masses arise from a different mechanism involving both H_{27} and $\tilde{H}_{27}$. e_3^3 is not a new scale parameter but is determined by the onset of G_{333}, the meeting point of the electroweak gauge couplings: $g_1(M_I) = g_2(M_I)$ with the result (see Appendix A 1):

$$M_I = 1.27 \times 10^{13} \text{ GeV},$$
$$e_3^3 = \frac{M_I}{g_2(M_I)} = 2.27 \times 10^{13} \text{ GeV}. \qquad (1.13)$$

We will see that this value of e_3^3 provides for the correct mass scale of the light neutrinos.

Because of the linear Yukawa coupling all fermions—except the right-handed neutrinos L_2^3 and L_3^3—are Dirac particles and have the same G hierarchy as the up quarks.

The hierarchy of the right-handed neutrinos, on the other hand, becomes super strong due to the combined action of H_{27} and $\tilde{H}_{27}$. Particle mixings will occur by taking into account the antisymmetric generation matrix A combined with an antisymmetric (in E_6 indices) Higgs field. These mixings modify the hierarchy of down quarks and charged and neutral leptons in agreement with the experimentally observed particle spectra.

II. GENERATION SYMMETRY

For the construction of the Yukawa interaction symmetric under $E_6 \times SO(3)_g \times P_g$ the existence of a set of new fields is necessary. First of all, as mentioned in the introduction, we introduce scalar fields represented by the 3×3 matrix $\Phi(x)$. The members of the symmetric part χ transforms as $(1, 1 + 5, -)$ [under $E_6 \times SO(3)_g \times P_g$] and the ones of the antisymmetric part ξ according to $(1, 3, -)$. To have a renormalizable interaction we need new heavy vectorlike fermionic fields F, $\bar{F}$ and F', $\bar{F}'$ which transform as

$$F \sim (27, 3, -), \qquad \bar{F} \sim (\overline{27}, 3, -),$$
$$F' \sim (27, 3, -), \qquad \bar{F}' \sim (\overline{27}, 3, -). \qquad (2.1)$$

The Yukawa interaction involving the Higgs fields arises from the vertices

$$(\psi^T (g_{(1)} \chi_{(1)} + g_{(5)} \chi_{(5)}) \bar{F}), \qquad M(F^T \bar{F}), \qquad (F^T H_{27} \psi). \qquad (2.2)$$

In (2.2) we used matrix notation with regard to generation indices, but suppressed E_6 indices and Clebsch Gordan coefficients. $\chi_{(1)}$ and $\chi_{(5)}$ denote the singlet and 5-plet parts of χ, respectively. By integrating out the massive fields F, $\bar{F}$ one gets the wanted effective interaction

$$\frac{1}{M} (\psi^T (g_{(1)} \chi_{(1)} + g_{(5)} \chi_{(5)}) H_{27} \psi). \qquad (2.3)$$

The corresponding diagram is shown in Fig. 2(a). Clearly, the antisymmetric matrix ξ does not contribute to the symmetric (in E_6 indices) field H_{27}. It couples, however, to the antisymmetric Higgs representation $H_A(351)$ of E_6 with negative generation parity. The vertices are

$$((\psi^T g_{(3)} i\xi \bar{F}'), M'(F'^T \bar{F}'), (F'^T H_{A351}) \psi)$$
$$\Rightarrow \frac{g_{(3)}}{M'} (\psi^T i\xi H_A \psi). \qquad (2.4)$$

The corresponding diagram containing the heavy fields F', $\bar{F}'$ is shown in Fig. 2(b).

The P_g symmetry forbids couplings such as $\psi H \psi$, $F\tilde{H}\psi$, FFH, etc. In the following we will not separate the singlet and 5-plet parts of χ but simply use the combination χ as occurring in Φ (i.e. $g_{(1)} = 1/3 \, g_{(5)}$) and absorb $g_{(5)}$ and the coupling $g_{(3)}$ in the values of M and M', respectively. Now we can introduce number valued generation matrices by taking the vacuum expectation values of χ and ξ as described in the introduction:

$$\frac{1}{M} \langle \chi \rangle = G, \qquad \frac{1}{M'} \langle i\xi \rangle = A. \qquad (2.5)$$

As mentioned before, the generation matrix G in Eq. (1.2) combined with vacuum expectation values of the Higgs field H_{27} gives Dirac masses to all fermions except the two heavy leptons L_2^3 and L_3^3. The latter require Higgs fields transforming as $(6, \bar{6})$ with respect to $SU(3)_L \times SU(3)_R$, components, which are not contained in H and H_A. Instead of introducing another high dimensional Higgs field it is plausible to use the Higgs field H_{27} together with $\tilde{H}_{27}$. This avoids the appearance of a new unknown generation matrix. With the help of the generation symmetry it is possible to derive the relevant generation matrix of the "compos-
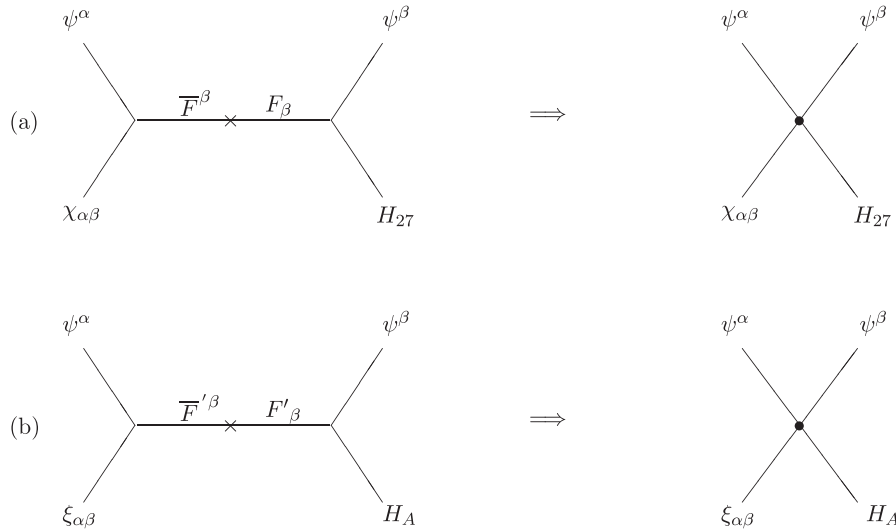

FIG. 2. Diagrams responsible for operators $\chi_{\alpha\beta} \psi^\alpha \psi^\beta H_{27}$ and $\xi_{\alpha\beta} \psi^\alpha \psi^\beta H_A$.

ite'' $(6, \bar{6})$-plet in terms of the G matrix. This increases the predictive power of the model. To obtain the corresponding effective interaction from a renormalizable interaction, another massive Dirac field is required. It is a vector in generation space and an E_6 singlet: $N(1, 3, +)$, $\tilde{N}(1, 3, -)$ with N even under P_g and $\tilde{N}$ odd under P_g. In addition we need a total singlet field $(1, 1, -)$ which is odd under P_g. It can be identified with the singlet part $\chi_{(1)}$ of χ with vacuum expectation value $\langle \chi_{(1)} \rangle = M_N \simeq M$. The vertices are

$$(F^T H_{27}^\dagger N), \qquad \langle \chi_{(1)} \rangle (\tilde{N} N) = M_N (\tilde{N} N), \qquad (\tilde{N} \tilde{H}_{27}^\dagger F).$$
(2.6)

Integrating out the fields N, $\tilde{N}$ the vertex $(F^T H^\dagger) \times (\tilde{H}^\dagger F)/M_N$ emerges. Below the masses of the F states the effective Yukawa interaction

$$\frac{1}{M^2} \frac{1}{M_N} (\psi^T \chi H_{27}^\dagger)(\tilde{H}_{27}^\dagger \chi \psi)$$
(2.7)

is generated. The corresponding diagram is shown in Fig. 3. Because in (2.7) the E_6 indices are not shown, one has to keep in mind that the combinations $\psi H^\dagger$ and $\psi \tilde{H}^\dagger$ are E_6 singlets. It is easily seen that by this interaction only neutral leptons can get masses, notably the right-handed neutrinos L_2^3 and L_3^3. They are coupled to the large elements of H_{27} and $\tilde{H}_{27}$, which are standard model singlet fields and thus will get large vacuum expectation values. The most interesting feature is, however, the appearance of the square of χ/M. It implies that after generation symmetry breaking the mass hierarchy of these heavy neutrinos is dramatic, namely, equal to G^2, the square of the mass hierarchy of the up quarks which is already a very strong one.

H_{27} and $\tilde{H}_{27}$, which both appear now in the Yukawa interaction, can be expressed in terms of the mass eigenstates H^u and H^d. From (1.12) we have $H_{27} = H^u + z H^d$ and $\tilde{H}_{27} = -z H^u + H^d$, with the mixing parameter $|z| \ll 1$. We choose the VEVs of H^u and H^d as follows:

$$\langle H^u \rangle = \begin{pmatrix} e_1^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -z\epsilon_2^3 & e_3^3 \end{pmatrix}, \qquad \langle H^d \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_2^2 & 0 \\ 0 & \epsilon_2^3 & 0 \end{pmatrix}.$$
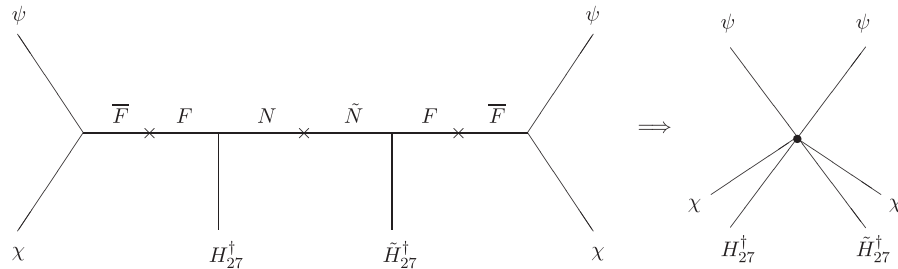(2.8)

Thus, the VEV of the Higgs field H_{27} has the diagonal form taken by convention

$$\langle H_{27} \rangle = \langle H^u + z H^d \rangle \simeq \begin{pmatrix} e_1^1 & 0 & 0 \\ 0 & z\epsilon_2^2 & 0 \\ 0 & 0 & e_3^3 \end{pmatrix},$$
(2.9)

while $\tilde{H}_{27}$'s VEV has the structure

$$\langle \tilde{H}_{27} \rangle = \langle H^d - z H^u \rangle \simeq \begin{pmatrix} -ze_1^1 & 0 & 0 \\ 0 & \epsilon_2^2 & 0 \\ 0 & \epsilon_2^3 & -ze_3^3 \end{pmatrix}.$$
(2.10)

Since $\langle H_{27} \rangle$ has no off diagonal element one expects a large $(3, 2)$ element for $\tilde{H}_{27}$. We take $\epsilon_2^3 \simeq M_N \simeq M$, where M is identified with $e_3^3 \simeq M_I$. In principle, however, ϵ_2^3 together with M and M_N could be of a lower scale (but still $\gg M_Z$). The dominant VEVs of H_{27} and $\tilde{H}_{27}$ fix now the mass terms of the right-handed neutrinos L_2^3 and L_3^3

$$F^{\{2,3\}} G^2 ((L_2^3)^T L_3^3) + F^{\{3,3\}} G^2 ((L_3^3)^T L_3^3),$$

with $F^{\{2,3\}} \simeq \dfrac{e_3^3 \epsilon_2^3}{M_N} \simeq e_3^3$, $\quad F^{\{3,3\}} \approx \dfrac{-z(e_3^3)^2}{M_N} \approx -z e_3^3$. (2.11)

$F^{\{3,3\}}$ is the only Majorana mass term occurring so far.

The effective Yukawa interaction below (M, M_N, M') now reads

$$\mathcal{L}_Y^{\text{eff}} = G_{\alpha\beta}(\psi^{\alpha T} H_{27} \psi^\beta) + A_{\alpha\beta}(\psi^{\alpha T} H_{A_{351}} \psi^\beta)$$
$$+ \frac{1}{M_N}(G^2)_{\alpha\beta}(\psi^{\alpha T} H_{27}^\dagger \tilde{H}_{27}^\dagger \psi^\beta).$$
(2.12)

This effective Yukawa interaction together with the VEV configurations (2.9) and (2.10) contains all the necessary information about the generation structure: Generation hierarchy and generation mixing are now completely fixed. At this stage we do not need to specify the scales of $M, M_N \simeq M, M' \gg M_Z$. However, these scales become important when we take renormalization effects into account.

Let us now discuss the breaking of the generation symmetry $SO(3)_g$. The Lagrangian for the field Φ is

$$\mathcal{L}_\Phi = \frac{1}{2} \text{Tr}\{(\partial_\mu \Phi - e[B_\mu, \Phi])(\partial_\mu \Phi - e[B_\mu, \Phi])\} - V(\chi, \xi),$$

FIG. 3. Diagram responsible for the $\chi^2 (H_{27}^\dagger \psi)(\tilde{H}_{27}^\dagger \psi)$ operator.

with $B_\mu = B_\mu^i t^i$, where B_μ^i denote the vector potential and t^i the 3 antisymmetric generators of the generation symmetry $SO(3)_g$. $V(\chi, \xi)$ stands for an $SO(3)_g$ invariant potential. Gauge invariance allows to choose χ, the symmetric part of Φ, to be a diagonal matrix with 3 real elements. This defines a direction in symmetry space for a possible spontaneous symmetry breaking. Normalizing Φ for this basis one can write

$$\Phi = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}. \tag{2.13}$$

In this form the vertices we have shown above simplify drastically since the orthogonal transformation diagonalizing χ can be absorbed by the fields ψ, F, and F'.

The scalar potential $V(\chi, \xi)$ in (2.12) can easily be made to have a minimum for specific values of the three invariants: the trace of χ^2, the trace of the square of the traceless part of χ, and the trace of ξ^2. At this minimum we have then two relations for the 3 fields forming χ and one relation for the other 3 fields forming ξ. There still remains the freedom of such $SO(3)_g$ transformations which do respect this minimum and keep χ diagonal. This remaining symmetry is necessarily a discrete subgroup of $SO(3)_g$. It is the discrete symmetry group A_4. As a generation symmetry A_4 has been suggested in many publications [8]. In our approach this symmetry occurs naturally in connection with the starting symmetry $SO(3)_g \times P_g$. It is the remaining symmetry after using appropriate potentials invariant under $SO(3)_g \times P_g$ and the choice of a symmetry breaking direction.

Radiative corrections add to the potential $V(\chi, \xi)$ new $SO(3)_g$ invariant parts which are of logarithmic type. The Coleman-Weinberg potential [10] is of this form. By including it the total potential can lead to a complete spontaneous symmetry breaking of $SO(3)_g$ which results in vacuum expectation values for χ and ξ of the form (1.2) and (1.3). We demonstrate this here by using the potential

$$\begin{aligned} V(\chi, \xi) = {}& -\frac{M^2}{2} \text{Tr}[\chi^2] - \frac{M'^2}{4} \text{Tr}[(i\xi)^2] + c_1 (\text{Tr}[\chi])^4 \\ & + c_T (\text{Tr}[\chi_T^2])^2 + c_3 \text{Tr}[\chi^3] \text{Tr}[\chi] \\ & + c_\xi (\text{Tr}[(i\xi)^2])^2 + c_{\chi\xi0} \text{Tr}[\chi.(i\xi)^2] \text{Tr}[\chi] \\ & + c_{\chi\xi1} \text{Tr}[\chi.i\xi.\chi.i\xi] + \frac{3e^2}{64\pi^2} \Big[M_{B1}^4 \ln\Big(d_1 \frac{M_{B1}^2}{M^2}\Big) \\ & + M_{B2}^4 \ln\Big(d_2 \frac{M_{B2}^2}{M'^2}\Big) + M_{B3}^4 \ln\Big(d_3 \frac{M_{B3}^2}{M'^2}\Big) \Big]. \end{aligned} \tag{2.14}$$

Here χ_T denotes the traceless part of χ and M_{Bi} stands for the vector boson masses as field dependent functions (we take in (2.14) $e = 1$ for simplicity). The coefficients in

(2.14) can be tuned such that the $SO(3)_g \times P_g$ symmetry breaks spontaneously and produces G and A with $\sigma = 0.050$. To achieve this one has to use the six relations following from the first derivatives of the potential at the proposed minimum: $\langle \chi \rangle = MG$ and $\langle \xi \rangle = M'A$. Because of the large hierarchy a high accuracy of this calculation is necessary. We use here $M'/M = 10^3$ i.e., the mass scale for forming the antisymmetric matrix A is large compared to the one for forming the symmetric matrix G (this is required for the gauge coupling unification, see the renormalization group treatment in the appendix). Setting then e.g. $d_1 = 1$, $d_2 = 237\,854/10^5$, and $d_3 = 237\,850/10^5$ all other coefficients are determined by putting the first derivatives of the potential to zero. These coefficients are sufficiently small ($c_1, c_T, c_3, c_\xi < 1$ for instance) to allow perturbative treatments. The log terms are small near the minimum. With this choice of the potential the six eigenvalues of the 6×6 matrix for the second derivatives of the potential are all positive at the wanted values of $\langle \Phi \rangle$. The minimum obtained is an absolute one (but degenerate with respect to different signs of the three elements in $\langle \xi \rangle$).

The above formulae for the vacuum expectation values and the corresponding potential would have a different form if we would have used other values for the couplings $g_{(1)}$ and $g_{(5)}$. However, also for this more general situation potentials can be constructed to produce the required spontaneous symmetry breaking.

For the three vector boson masses at the minimum of the potential one gets

$$M_B \sim \begin{Bmatrix} 1.40eM \\ 0.505eM' \\ 0.505eM' \end{Bmatrix}. \tag{2.15}$$

III. CHARGED FERMION MASSES AND MIXINGS

As we have already mentioned, the diagonal generation matrix $\langle \chi \rangle$ (i.e. G) is taken such that the up quark masses have their observed hierarchy. The top quark mass at M_Z is determined by the vacuum expectation value e_1^1 and the coupling λ_t at this scale

$$m_t = e_1^1 \lambda_t = e_1^1 G_{3,3}^{u\hat{u}} = 170.9 \text{ GeV}. \tag{3.1}$$

The notation (the superscript $u\hat{u}$ of G) indicates that at low energy one has to distinguish between different channels. The renormalization group effects are treated in the appendix. We take for $\lambda_t(M_Z)$ a value such that the top coupling constant near M_I is close to 1,

$$\lambda_t(M_Z) = 1.30, \qquad e_1^1 = 131.5 \text{ GeV}. \tag{3.2}$$

Having for all diagonal G couplings the same spacing at the scale M_I, we are able to calculate the spacing for each channel at the scale M_Z. As obtained in the appendix we find

$$G^{u\hat{u}}(M_Z) = \mathrm{Diag}(r_t^{-1}\sigma^4, r_t^{-1}\sigma^2, 1) \cdot \lambda_t(M_Z) \quad \text{with}$$

$$r_t = 0.717. \tag{3.3}$$

Comparing with the measured values of up and charmed quark masses suggests

$$\sigma = 0.050. \tag{3.4}$$

Equation (3.3) gives for this value of σ $m_u = 1.5$ MeV and $m_c = 0.60$ GeV in good agreement with the experiment. We will use (3.4) in all further calculations.

The case for the down quark and charged lepton masses is not as straightforward. Here the light fermions will mix with the heavy states D and L. We take this as the source of the particle mixings. It involves the indices 2 and 3 ($\mathcal{U}$-spin mixing) of $SU(3)_L$ and $SU(3)_R$ of the Higgs field H_A. According to Eq. (2.12) this mixing of light and heavy fermions occurs together with the generation mixings by the coupling matrix $i\langle\xi\rangle$ (i.e. the matrix A).

A. The down quark masses and mixings

The matrix elements for down quarks are part of a 6×6 mass matrix. It has the form

$$M_{d,D} = \begin{matrix} & \hat{d} & \hat{D} \\ d \\ D \end{matrix} \begin{pmatrix} z\epsilon_2^2 G + f_2^2 A, & f_3^2 A \\ f_2^3 A, & e_3^3 G \end{pmatrix}. \tag{3.5}$$

In this equation the constants f_j^i stand for the vacuum expectation values of $H_A(\bar{3}, 3, 1)$:

$$f_j^i = \langle (H_A)_j^i \rangle, \qquad i, j = 2, 3. \tag{3.6}$$

ϵ_2^2 and f_3^2 have values of the order of the weak scale, while e_3^3 and f_2^3 will be very large because they arise from standard model singlets. Further below we will find that the vacuum expectation values of the standard model singlet components of H_A, like f_2^3, have values which are small compared to e_3^3 (smaller than $\sigma^3 e_3^3$). This allows to make use of the seesaw formula and also justifies the neglection of f_3^3 in (3.5). Integration of the heavy D-states leads to the 3×3 down quark mass matrix at the scale M_Z

$$\hat{m}_d(M_Z) = z\epsilon_2^2 G^{d\hat{d}} + A^{d\hat{d}} f_2^2 - \frac{f_2^3 f_3^2}{e_3^3} A^{d\hat{D}} (G^{D\hat{D}})^{-1} A^{D\hat{d}}. \tag{3.7}$$

We distinguished different G and A matrices when coupled to different channels ($d\hat{d}$, $d\hat{D}$), etc. Different renormalization factors arise when starting from the original G and A matrices at high scales. The calculations are deferred to the appendices. The matrices appearing in (3.7) are defined in (A12) and (A14).

For the $d\hat{d}$ matrix element in (3.7) with the generation index (3, 3) we write

$$z\epsilon_2^2 G_{33}^{d\hat{d}} = \epsilon_2^2 \lambda_b = m_b^0. \tag{3.8}$$

The down quark mass matrix leads to 7 observables: 3 mass eigenvalues, 3 mixing angles, and the CP-violating phase. According to (3.7) we have 3 parameters for a fit of the experimental results. We use the notation of the various coupling constants provided in (A12) and (A14). Taking then

$$m_b^0 = 2.95 \text{ GeV}, \qquad \lambda_A^{d\hat{d}} f_2^2 \simeq -0.23 \text{ GeV},$$

$$\frac{f_2^3 f_3^2}{e_3^3} \frac{\lambda_A^{d\hat{D}} \lambda_A^{D\hat{d}}}{\lambda^{D\hat{D}}} = 1.62 \times 10^{-4} \text{ GeV}, \tag{3.9}$$

one gets

$$m_d(M_Z) \simeq 2.6 \text{ MeV}, \qquad m_s(M_Z) \simeq 50 \text{ MeV},$$

$$m_b(M_Z) \simeq 2.89 \text{ GeV}, \qquad |V_{us}| \simeq 0.228, \tag{3.10}$$

$$|V_{cb}| \simeq 0.042, \qquad |V_{ub}| \simeq 0.0039.$$

The angles in the quark unitarity triangle have the values

$$\alpha_q \simeq 97°, \qquad \beta_q \simeq 23°, \qquad \gamma_q \simeq 60°. \tag{3.11}$$

Our results are very satisfying. Quark masses, the CKM mixing angles, and the three angles of the unitarity triangle have values within experimental errors [11,12].

B. The charged lepton masses and mixings

The charged lepton sector is constructed in a similar way. The light leptons mix with the heavy L's through the vacuum expectation values of the $H_A(\bar{3}, \bar{6}, 1)$ multiplet (the vacuum expectation values of $H_A(6, 3, 1)$ are considered to be negligible because of its **6** representation for the $SU(3)_L$ symmetry):

$$\langle H_A^{i\{j,k\}} \rangle = f^{i\{j,k\}}. \tag{3.12}$$

With the abbreviations

$$f^{2\{1,3\}} = g_2^2, \qquad f^{2\{1,2\}} = g_3^2, \qquad f^{3\{1,3\}} = g_2^3 \tag{3.13}$$

and applying again the effective Lagrangian (2.12) the 6×6 mass matrix for charged leptons has the form

$$M_{e,E} = \begin{matrix} & e^+ & E^+ \\ e^- \\ E^- \end{matrix} \begin{pmatrix} -z\epsilon_2^2 G - g_2^2 A, & g_2^3 A \\ -g_3^2 A, & -e_3^3 G \end{pmatrix}. \tag{3.14}$$

Integrating out the heavy L states leads at the scale $\mu \simeq M_Z$ to the 3×3 matrix

$$\hat{m}_e \simeq -z\epsilon_2^2 G^{e^- e^+} - g_2^2 A^{e^- e^+}$$

$$- \frac{g_3^2 g_2^3}{e_3^3} A^{e^- E^+} (G^{L\hat{L}})^{-1} A^{E^- e^+}. \tag{3.15}$$

Here appear again renormalization group coefficients defined in (A12) and (A14). We choose

$$zG_{33}^{e^-e^+}\epsilon_2^2 \simeq \epsilon_2^2 \lambda_\tau = m_\tau^0 = 1.593 \text{ GeV},$$

$$\lambda_A^{e^-e^+} g_2^2 \simeq -0.1898 \text{ GeV}, \tag{3.16}$$

$$\frac{g_3^2 g_2^3}{e_3^3}\frac{\lambda_A^{e^-E^+}\lambda_A^{E^-e^+}}{\lambda^{L\bar{L}}} \simeq 3.16 \cdot 10^{-4} \text{ GeV}.$$

These three numbers determine the charged lepton masses and, as in the quark case, determine also the 3 mixing angles and the *CP*-violating phase. The charged lepton mixings are not directly observable, but will play an important role in the discussion of the neutrino properties. The masses come out precisely

$$m_e(M_Z) = 0.487 \text{ MeV}, \qquad m_\mu(M_Z) = 102.8 \text{ MeV},$$
$$m_\tau(M_Z) = 1.747 \text{ GeV}. \tag{3.17}$$

For the mixing angles and the *CP*-violating phase γ^e we obtain

$$\theta_{12}^e \simeq 4.8°, \qquad \theta_{23}^e \simeq 5.1°,$$
$$\theta_{13}^e \simeq 0.35°, \qquad \gamma^e \simeq -23°. \tag{3.18}$$

We have defined these four parameters in complete analogy to the quark CKM mixing angles and the unitarity angle gamma.

C. Estimates of ϵ_2^2, f_3^2, f_2^3, g_3^2, and g_2^3

So far only the combinations $\lambda_\tau \epsilon_2^2$, $f_3^2 f_2^3$, and $g_3^2 g_2^3$ are known numerically. However, there is one relation due to the known mass of the vector boson W of the standard model. It connects all vacuum expectation values which belong to $SU(2)_L$ doublets:

$$(e_1^1)^2 + (\epsilon_2^2)^2 + (f_2^2)^2 + (g_2^2)^2 + (f_3^2)^2 + (g_3^2)^2$$
$$= (174.1 \text{ GeV})^2. \tag{3.19}$$

Because of the nearly identical interaction of down quarks and charged leptons we can estimate the ratio f_3^2/g_3^2 from the fit values given in (3.9) and (3.16) by taking the large standard model singlet VEV's f_2^3 and g_2^3 to be equal. One gets $\frac{f_3^2}{g_3^2} \simeq 0.5$ which is close to m_τ^0/m_b^0 as one could have expected. Using Eq. (3.19) the only parameter needed then is the ratio g_3^2/ϵ_2^2. This is a ratio of two weak scale quantities with the same weak isospin quantum numbers. We therefore take it to be equal to one. This choice gives a value for g_2^3/e_3^3 small enough to justify the application of the seesaw mechanism we used above. It also provides for small values for the couplings λ_τ and λ_b necessary for renormalization group stability of the neutrino sector to be discussed in the appendices. All couplings and VEV's introduced are now fixed:

$$f_3^2 \simeq 39 \text{ GeV},$$

$$g_3^2 = \epsilon_2^2 \simeq 76 \text{ GeV},$$

$$zG_{3,3}^{e^-e^+} \simeq \lambda_\tau \simeq \frac{1.6 \text{ GeV}}{\epsilon_2^2} \simeq 2 \times 10^{-2}, \tag{3.20}$$

$$\lambda_b \simeq 4 \times 10^{-2},$$

$$\frac{f_2^3}{e_3^3} = \frac{g_2^3}{e_3^3} \simeq 4.16 \times 10^{-6}.$$

It is useful to set

$$\frac{f_2^3}{e_3^3} = \frac{g_2^3}{e_3^3} = \sigma^3 x_g$$

giving

$$x_g \simeq 0.033. \tag{3.21}$$

The factor σ^3 removes the singularities in the mass matrices with respect to the formal limit $\sigma \to 0$.

IV. NEUTRINO MASSES AND MIXINGS

In each generation one has to deal with 5 neutral leptons (see (1.9)). Thus, the matrix for neutral leptons is a 15×15 matrix.

$$M_L = \begin{array}{c} \\ L_3^2 \\ L_2^3 \\ L_3^3 \\ L_1^1 \\ L_2^2 \end{array}\overset{\displaystyle L_3^2 \quad\quad L_2^3 \quad\quad L_3^3 \quad\quad L_1^1 \quad\quad L_2^2}{\begin{pmatrix} 0 & -e_1^1 G & 0 & -g_2^3 A & 0 \\ -e_1^1 G & 0 & M_1 & 0 & 0 \\ 0 & M_1^T & M_2 & 0 & e_1^1 G \\ -g_2^3 A^T & 0 & 0 & 0 & M_0 \\ 0 & 0 & e_1^1 G & M_0^T & 0 \end{pmatrix}}.$$

$$\tag{4.1}$$

Each entry stands for a 3×3 matrix. In the 12×12 submatrix for the heavy leptons we neglected small terms like g_2^2 and g_3^2. They play no role in the evaluation of the light neutrino properties. The following abbreviations are used:

$$M_0 = e_3^3 G, \qquad M_1 = F^{\{2,3\}} G^2 + F_A A,$$
$$M_2 = F^{\{3,3\}} G^2. \tag{4.2}$$

M_1 and M_2 have the superstrong hierarchy G^2 according to $\mathcal{L}_Y^{\text{eff}}$ and Eq. (2.11). The only new element we had to introduce arises from the vacuum expectation value of $H_A(6, 3, 1)$ which we argued to be negligible in the previous section. But here it appears together with very small elements occurring in G^2. Thus, the constant F_A defined according to

$$F_A = \langle (H_A)_{\{3,3\},1} \rangle = f_{\{3,3\}1}$$

is expected to be tiny compared to $F^{\{2,3\}}$ even though it is a singlet with respect to standard model transformations. Nevertheless it has to be kept as a parameter.

It is useful to rewrite the matrix (4.1) in the form

$$M_L = \begin{pmatrix} 0 & \Omega \\ \Omega^T & \hat{M} \end{pmatrix}, \qquad (4.3)$$

where 0 stands for the 3×3 zero block matrix, while Ω and $\hat{M}$ are 3×12 and 12×12 matrices, respectively. They read

$$\Omega = (-e_1^1 G, 0, -g_2^3 A, 0), \qquad (4.4)$$

$$\hat{M} = \begin{pmatrix} 0 & M_1 & 0 & 0 \\ M_1^T & M_2 & 0 & e_1^1 G \\ 0 & 0 & 0 & M_0 \\ 0 & e_1^1 G & M_0^T & 0 \end{pmatrix}. \qquad (4.5)$$

The matrix $\hat{M}$ contains the masses of the heavy neutral states which should be integrated out. This can be done analytically since g_2^3/e_3^3 is sufficiently small according to (3.20). In doing so the light neutrino 3×3 mass matrix is given by

$$m_\nu = -\mathcal{U}^T \Omega \hat{M}^{-1} \Omega^T \mathcal{U}. \qquad (4.6)$$

In this expression the matrix $\mathcal{U}$ provides for a first order correction to the generalized seesaw result $\Omega \hat{M}^{-1} \Omega^T$. One finds with sufficient accuracy

$$\mathcal{U} \simeq 1 - \tfrac{1}{2} \Omega^* \hat{M}^{-2} \Omega^T. \qquad (4.7)$$

The inverse of the matrix $\hat{M}$ is

$$\hat{M}^{-1} = \begin{pmatrix} -\frac{1}{M_1^T} M_2 \frac{1}{M_1} & \frac{1}{M_1^T} & -\frac{e_1^1}{M_1^T} G \frac{1}{M_0} & 0 \\ \frac{1}{M_1} & 0 & 0 & 0 \\ -\frac{e_1^1}{M_0^T} G \frac{1}{M_1} & 0 & 0 & \frac{1}{M_0^T} \\ 0 & 0 & \frac{1}{M_0} & 0 \end{pmatrix}. \qquad (4.8)$$
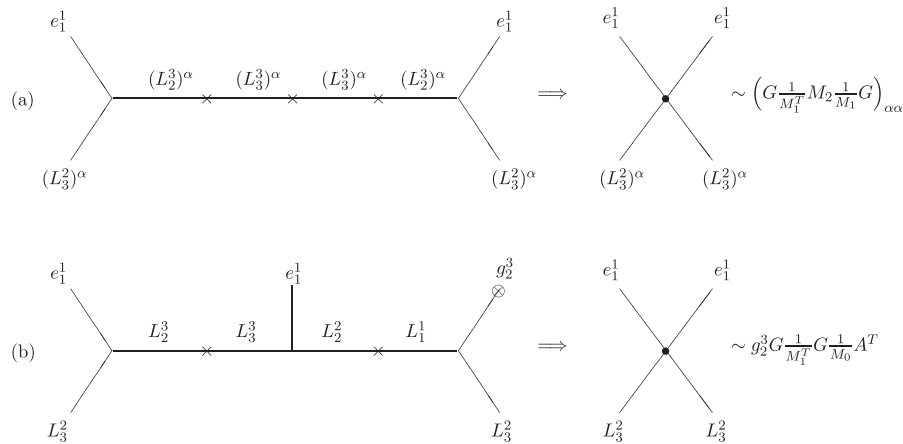
From (4.4) and (4.8) one obtains

$$\Omega \hat{M}^{-1} \Omega^T = -(e_1^1)^2 \Big(G \frac{1}{M_1^T} M_2 \frac{1}{M_1} G + g_2^3 G \frac{1}{M_1^T} G \frac{1}{M_0} A^T$$
$$+ g_2^3 A \frac{1}{M_0^T} G \frac{1}{M_1} G \Big). \qquad (4.9)$$

Figure 4 shows how the terms in (4.9) are generated. The generation of the first term in (4.9) is exhibited by the diagram of Fig. 4(a), while the diagram of Fig. 4(b) and its transpose shows the formation of the second and third terms of (4.9), respectively. In linear approximation with respect to F_A the matrix M_1^{-1} becomes

$$M_1^{-1} \simeq \frac{1}{F^{\{2,3\}}} \Big(\frac{1}{G^2} - \frac{F_A}{F^{\{2,3\}}} \frac{1}{G^2} A \frac{1}{G^2} \Big). \qquad (4.10)$$

The expression of $\mathcal{U}$ appearing in Eq. (4.6) can be approximated by

$$\mathcal{U} \simeq 1 - \frac{1}{2} g_2^3 A \frac{1}{M_0^T} \frac{1}{M_0} g_2^3 A \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{x_g^2}{2} & \frac{x_g^2}{2} \\ 0 & \frac{x_g^2}{2} & 1 - \frac{x_g^2}{2} \end{pmatrix}. \qquad (4.11)$$

For consistency the quantity x_g, at this place responsible for the correction to the seesaw formula, must be small compared to 1 which is indeed the case as we have seen before. Finally, taking all together, one obtains for the light neutrino mass matrix

$$m_\nu = \frac{(e_1^1)^2}{F^{\{2,3\}}} \mathcal{U}^T \Big(\frac{F^{\{3,3\}}}{F^{\{2,3\}}} \mathbf{1} + \frac{g_2^3}{e_3^3} \Big(A \frac{1}{G} + \frac{1}{G} A^T \Big)$$
$$- \frac{F_A}{F^{\{2,3\}}} \frac{g_2^3}{e_3^3} \Big(A \frac{1}{G^2} A \frac{1}{G} + \frac{1}{G} A^T \frac{1}{G^2} A^T \Big) \Big) \mathcal{U}. \qquad (4.12)$$

In order to compensate the powers of σ in the denominators of (4.12) we extract powers of σ from F_A as we did for g_2^3,

FIG. 4. Diagrams responsible for neutrino $d = 5$ operators.

$$\frac{g_2^3}{e_3^3} = \sigma^3 x_g, \qquad \frac{F_A}{F^{\{2,3\}}} = \sigma^5 x_A. \qquad (4.13)$$

Like x_g also x_A needs to be small compared to one.

For a convenient description and discussion we use now the abbreviations

$$m = x_g \frac{(e_1^1)^2}{F^{\{2,3\}}}, \qquad \rho = \frac{F^{\{3,3\}}}{x_g F^{\{2,3\}}}. \qquad (4.14)$$

Let us at first discuss the case with only linear terms in x_g and σ, putting also $x_A = 0$. The neutrino mass matrix takes then the very simple form

$$m_\nu \simeq \rho m \mathbf{1} + m \begin{pmatrix} 0 & -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & 0 & -\mathrm{i}\frac{\sigma}{2} \\ \mathrm{i} & -\mathrm{i}\frac{\sigma}{2} & 0 \end{pmatrix}. \qquad (4.15)$$

The eigenvalues of $m_\nu.m_\nu^\dagger$ are to first order in σ

$$(m_2)^2 \simeq \left(\rho^2 + 2 + \frac{\sigma}{\sqrt{2}}\right) m^2,$$

$$(m_1)^2 \simeq \left(\rho^2 + 2 - \frac{\sigma}{\sqrt{2}}\right) m^2, \qquad (m_3)^2 \simeq \rho^2 m^2. \qquad (4.16)$$

It is now easy to see the following properties of the light neutrinos:

(i) The neutrino mass spectrum has the inverted form.

(ii) In the no mixing limit $x_g \to 0$ the 3 neutrino masses are degenerate.

(iii) The ratio between the solar and atmospheric mass squared differences is $\frac{\sigma}{\sqrt{2}} \simeq 0.035$ independent of the

parameters in good agreement with experiments.

(iv) The experimentally observed atmospheric mass squared difference can be used to fix the mass parameter m for the light neutrinos and to estimate the mass parameter $F^{\{2,3\}}$ for the heavy neutrinos:

$$m \simeq \frac{1}{\sqrt{2}}\sqrt{\Delta m_{\mathrm{atm}}^2} \simeq 0.035 \text{ eV},$$

$$F^{\{2,3\}} \simeq \sqrt{2}x_g \frac{(e_1^1)^2}{\sqrt{\Delta m_{\mathrm{atm}}^2}} \approx 1.8 \times 10^{13} \text{ GeV}. \qquad (4.17)$$

Here we used for x_g the estimate (3.21). Thus, as expected from (2.11), the largest heavy neutrino mass $F^{\{2,3\}}$ has indeed a value close to the intermediate mass scale M_I.

(v) In the approximation used so far, and without taking renormalization effects into account, the neutrino mixing matrix is of the *bimaximal* form.

In the next subsection we take renormalization into account by running the mass matrix down from M_I to M_Z. Moreover, the effect of x_A and the correction terms x_g^2 (emerging from $\mathcal{U}$) will be included.

Neutrino masses and mixings with RG effects

Now we make full use of Eq. (4.12) and take the renormalization effects discussed in Appendix B into account. Neglecting higher powers of the small quantity σ, the light neutrino 3×3 mass matrix at the scale M_Z has the form

$$m_\nu \simeq m \mathcal{U}^T \begin{pmatrix} \rho(1+r_1) & -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & \rho(1+r_2) - 2x_A & -\mathrm{i}\frac{\sigma}{2}r_{23} + x_A \\ \mathrm{i} & -\mathrm{i}\frac{\sigma}{2}r_{23} + x_A & \rho \end{pmatrix} \mathcal{U}, \qquad (4.18)$$

where r_1, r_2, r_{23} denote renormalization factors derived in Appendix B [see Eqs. (B3) and (B4)]. Their numerical values are

$$r_1 = 0.1325, \qquad r_2 = 0.0582, \qquad r_{23} = 0.9426. \qquad (4.19)$$

Without renormalization the value of ρ has no influence on the neutrino mixing pattern. Now, however, ρ plays a role. It should not be large in order not to change the mixing pattern drastically. In fact, $F^{\{3,3\}}/F^{\{2,3\}}$ must be as small as x_g in accord with $\rho \approx -z/x_g$ (2.11) and our value for x_g, i.e. $|\rho|$ must be around 1. As a consequence, the inverted spectrum found above still prevails.

With the following choice of parameters we obtain a very good description of the known neutrino properties. In this fit we adjust x_A and ρ and decrease m slightly. For x_g we take the value used already,

$$m = 0.0324 \text{ eV}, \qquad \rho = 1.132,$$
$$x_g = 0.033, \qquad x_A = -0.0601. \qquad (4.20)$$

For the neutrino masses one gets

$$m_1 = 0.0616 \text{ eV}, \qquad m_2 = 0.0623 \text{ eV},$$
$$m_3 = 0.037 \text{ 4eV}. \qquad (4.21)$$

Accordingly, the mass square differences are

$$\Delta m_{\mathrm{sol}}^2 = m_2^2 - m_1^2 = 8 \times 10^{-5} \text{ eV}^2,$$
$$|\Delta m_{\mathrm{atm}}^2| = m_2^2 - m_3^2 = 2.5 \times 10^{-3} \text{ eV}^2. \qquad (4.22)$$

For the mixing angles emerging from the neutrino mass matrix one finds $\theta_{12}^\nu \simeq 35.3°$, $\theta_{23}^\nu \simeq 39.4°$, and $\theta_{13}^\nu \simeq 3.2°$. By including the effect of the diagonalization of the charged lepton mass matrix as obtained in Sec. III, the neutrino mixing angles and the *CP* violating phase δ_l

(appearing in the neutrino oscillation amplitude) become

$$\theta_{12} \simeq 34°, \qquad \theta_{23} \simeq 43°, \qquad \theta_{13} \simeq 6.3°, \qquad \delta_l \simeq 67°. \tag{4.23}$$

The angles and the phase are given according to the standard parametrization. These mixing angles together with the mass squared differences (4.22) are in perfect agreement with global fits to neutrino oscillation data [13]. These results are very satisfactory. For the neutrino less double β-decay parameter one finds

$$|\langle m_{\beta\beta}\rangle| \simeq 0.046 \text{ eV}. \tag{4.24}$$

We note that there is not much freedom within this E_6 model to get larger or smaller values for the neutrino masses or the $0\nu2\beta$-decay parameter, or to change the neutrino hierarchy. We already observed that $F^{\{2,3\}}$, the mass of the heavy third generation right-handed neutrino turned out to be amazingly close to M_I, the point where the electroweak gauge couplings of the standard model meet.

As a consequence, we know now, at least approximately, the masses and mixings of all heavy fermions. A direct diagonalization of the 15×15 neutral lepton mass matrix reveals, in particular, that the two lightest of the right-handed neutrinos (i.e. the first generation) turn out to have masses of only ≈ 700 GeV. This is caused by the super strong hierarchy valid for these particles. On the other hand, the masses of the first generation of heavy quarks and heavy $SU(2)_L$ leptons are much larger ($\approx 10^8$ GeV). We also find that the neutrino mixing matrix for the light neutrinos is not strictly unitary. The light neutrinos mix to about 2% with the neutral leptons L_1^1 and L_2^2. The two lightest of the heavy neutrinos (a combination of the first generations of $\hat{\nu} = L_2^3$ and L_3^3) mix with an amplitude of about 8×10^{-7} with the first generation of the light neutrinos. These heavy neutrinos could be detected by their decay to the Higgs field $(H^u)_1^1$ [from the $SU(2)_L$ doublet $((H^u)_1^1, (H^u)_2^1)$] and a light neutrino provided the mass of the Higgs field $(H^u)_1^1$ lies below the heavy neutrino masses. If not, they can decay into W and Z gauge bosons and a light lepton via the mixings given above.

Our phenomenological treatment gives a clear picture of the spectrum and mixings of the standard model fermions and of their heavy E_6 partners. A quantitative fit reproducing all known properties of the standard model charged and neutral fermions could be performed. Several not yet measured quantities are predicted. However, the model is still incomplete, because an understanding of the scalar sector, in particular, of the vacuum expectation values of the E_6 Higgs fields, is not yet achieved.

V. SUMMARY

In this work we have addressed the problem of fermion masses and mixings. We took the gauge group E_6 for grand unification augmented with the generation symmetry $SO(3)_g \times P_g$. The fermion fields of the standard model are taken to be in the $(27, 3, +)$ representation of $E_6 \times SO(3)_g \times P_g$. The scalar fields transforming under the generation symmetry obtain vacuum expectation values by a complete spontaneous symmetry breaking. The corresponding values provide for the hierarchy of the fermions, for their mixings and CP violation. The generation mixings occur in conjunction with the $SU(3)$ $\mathcal{U}$-spin mixing of the standard model fermions with their heavy partners. To have a renormalizable model we had to introduce heavy Dirac fields. Integrating them out led to a very strong hierarchy of the right handed heavy neutrinos with important consequences for the light neutrino properties. The onset of the intermediate symmetry $SU(3)_L \times SU(3)_R \times SU(3)_c$ occurs at the well-known meeting point of the electroweak gauge couplings of the standard model. This scale determines the masses of the heavy neutrinos and turns out to be in full accord with the measured mass splittings of the light neutrinos. Also the formation of the Yukawa coupling matrices determining the fermion hierarchies can happen at this scale. Our model needs only few fit parameters to reproduce all known masses and mixing properties of the fermions. Because of the unique use of the up quark hierarchy G and the antisymmetric mixing matrix A for quarks, charged leptons, and neutrinos, the mixing in the quark sector determines the masses, the splittings, and the main part of the mixings in the neutrino sector. We obtain for the light neutrinos the inverted hierarchy and—for the unrenormalized case—bimaximal mixing. Taking renormalization group effects and the mixings coming from the charged lepton sector into account and by adjusting the small parameter x_A, the slight change to the experimentally observed mixing angles can be achieved. The entire spectrum of light and heavy fermions can be estimated. The two light neutrinos with "solar mass splitting" have masses $\simeq 0.06$ eV, while the lightest one weighs $\simeq 0.04$ eV. The angle θ_{13} is predicted to be $\approx 6°$. The CP-violating phase δ_l appearing in neutrino oscillations is large, around $70°$. For the parameter relevant for neutrino less double β-decays we find $\simeq 0.046$ eV. The two lightest members of the heavy neutrinos have a surprisingly low mass of about 700 GeV. The neutrino mixing matrix is not fully unitary, mixings to some high mass states have a magnitude of about 2%.

ACKNOWLEDGMENTS

APPENDIX A: GAUGE COUPLING UNIFICATION AND RENORMALIZATION OF FLAVOR MATRICES

1. Gauge coupling unification

In the previous section we have fixed the field contents of the fermion sector which are nonsinglets under E_6 and contribute to the gauge coupling running. Also, scalar representations play an important role. In this section we will give the full list of fields appearing at appropriate energy scales. This allows to study gauge coupling unification and to examine whether or not perturbativity is kept up to the $M_{\text{Pl}} \simeq 2.4 \times 10^{18}$ GeV (the reduced Planck mass).

We assumed that the scales e_3^3 and ϵ_2^3 are both equal to $M_I/g_2(M_I)$ (at M_I the couplings g_1 and g_2 meet). Below this scale the field content consists of the three fermion generations of the standard model together with two Higgs doublets. One light (up type) Higgs doublet $h_u = ((H^u)_1^1, (H^u)_1^2)$ comes from $H^u(\bar{3}, 3, 1)$ and has the VEV $\langle (H^u)_1^1 \rangle = e_1^1$. The second one, the down type Higgs doublet $h_d = ((H^d)_2^1, (H^d)_2^2)$, emerges from $H^d(\bar{3}, 3, 1)$ with the VEV $\langle (H^d)_2^2 \rangle = \epsilon_2^2$. Therefore, below the scale M_I, the b-factors relevant for the gauge coupling constants g_1, g_2, g_3 affected by these states are:

$$(b_1, b_2, b_3) = \left(\tfrac{21}{5}, -3, -7 \right). \tag{A1}$$

In addition, there are extra D and L fermion states from the 27^α fermion representation. Their masses have the scales $M_I(\sigma^4, \sigma^2, 1)$. The corresponding b-factors valid for each generation of D's and L's are

$$(b_1, b_2, b_3)^D = (\tfrac{4}{15}, 0, \tfrac{2}{3}), \qquad (b_1, b_2, b_3)^L = (\tfrac{2}{5}, \tfrac{2}{3}, 0), \tag{A2}$$

contributing at different scales. Above M_I, all components of 27^α can be considered light. Together with these, there are "light" $H^u(\bar{3}, 3, 1)$ and $H^d(\bar{3}, 3, 1)$ scalars. The b-factors corresponding to all these states are

$$(b_L, b_R, b_C)^{M_I} = (-4, -4, -5). \tag{A3}$$

Moreover, at the scale M [the masses of the states $F_\alpha(27)$, $\bar{F}^\alpha(\bar{27})$] which we identify with M_I, the RG b-factors receive the additions $b_i^F = (12, 12, 12)$. Between the scales M_I and M_{GUT} we have $L \leftrightarrow R$ ($\mathcal{D}$-symmetry) which insures the equality $g_L(\mu) = g_R(\mu)$. It starts at M_I, the meeting point of g_1 and g_2. At some scale M_6 ($M_I < M_6 < M_{\text{GUT}}$) the scalar states $(6, 3, 1)$, $(\bar{3}, \bar{6}, 1)$ [arising from $H_A(351)$] have to come in. We take them with the common mass M_6. The leptonic states from F'_α, $\bar{F}^{\alpha'}$ $(L_{F'} + \bar{L}_{\bar{F}'})_\alpha$ are also assumed to have masses below the GUT scale. For convenience (see the discussion below) we will take their masses equal to M_6. All states with mass M_6 are important for the successful gauge coupling unification. The corresponding b-factors are

$$(b_L, b_R, b_C)^6 = (\tfrac{19}{2}, \tfrac{19}{2}, 0). \tag{A4}$$

By taking $M_6 = 3.47 \times 10^{16}$ GeV unification is completed at $M_{\text{GUT}} = 2.8 \times 10^{17}$ GeV. The mass scale M_6 has been chosen in such a way as to insure the perturbativity of the unified gauge coupling α_{GUT} up to the Planck scale M_{Pl}. The plot which shows the unification of the gauge couplings is given in Fig. 1. It has a "Concorde" shape pretty similar to the one first obtained in Ref. [5]. For the case considered here we have

$$M_I \simeq 1.27 \times 10^{13} \text{ GeV}, \qquad M_6 \simeq 3.47 \times 10^{16} \text{ GeV},$$

$$M_{\text{GUT}} \simeq 2.8 \times 10^{17} \text{ GeV}, \qquad \alpha_{\text{GUT}}^{-1}(M_{\text{GUT}}) \simeq 24.1. \tag{A5}$$

At and above M_{GUT} we include, together with the states mentioned above, the remaining states of $F'_\alpha(27)$, $\bar{F}^{\alpha'}(\bar{27})$ in order to complete the E_6 states. Together with all E_6 gauge bosons, the complete E_6 scalar multiplets $H(27)$, $\tilde{H}(27)$, and $H_A(351)$, also the scalar field $H(650)$ is taken into account. The role of the 650-multiplet is to break E_6 down to G_{333} at the scale M_{GUT}. The b-factor above M_{GUT}, corresponding to all these states, is $b_{E_6} = 63$. With this input we can evolve the E_6 unified gauge coupling above M_{GUT} and compute its value at the Planck mass:

$$\alpha_{\text{GUT}}^{-1}(M_{\text{Pl}}) = \alpha_{\text{GUT}}^{-1}(M_{\text{GUT}}) - \frac{63}{2\pi} \ln \frac{M_{\text{Pl}}}{M_{\text{GUT}}}. \tag{A6}$$

With the values of M_{GUT} and $\alpha_{\text{GUT}}(M_{\text{GUT}})$ given in (A5) we find from (A6) $\alpha_{\text{GUT}}(M_{\text{Pl}})/4\pi \simeq 0.03$ (which may be viewed as an effective loop expansion parameter). It is seen that within our approach the gauge couplings remain perturbative up to the Planck scale.

2. Flavor coupling renormalization

Let us start with the renormalization of the top, bottom, and tau Yukawa couplings. They are generated at the scale $\langle \chi_{33} \rangle \simeq M_I$ after integrating out the states F, $\bar{F}$. In order to have RG stability in the neutrino sector we need to have a rather small value for the τ coupling constant λ_τ. This is satisfied because of the small value of z. In a notation appropriate for the G_{333} symmetry, the Yukawa interaction involving the G matrices reads

$$G^{QQ} Q_L Q_R H^u(\bar{3}, 3, 1) + z G^{QQ} Q_L Q_R H^d(\bar{3}, 3, 1)$$
$$+ \tfrac{1}{2} z G^{LL} L L H^d(\bar{3}, 3, 1). \tag{A7}$$

Since the up type quark masses are generated by $e_1^1 \subset H^u$ and the down type and charged lepton masses through $\epsilon_2^2 \subset H^d$, we have

$$\lambda_t = (G^{QQ})_{33}, \qquad \lambda_b = z(G^{QQ})_{33}, \qquad \lambda_\tau = z(G^{LL})_{33}. \tag{A8}$$

We estimated $zG^{e\hat{e}} \simeq 0.02$ in (3.20) using $\epsilon_2^2 \simeq 76$ GeV. Thus, RG effects for λ_τ and λ_b are small and the only

strong scale dependence below M_I occurs for λ_t. For a detailed description we will now use the appropriate notation for all Yukawa couplings. Namely, $e_1^1 G \rightarrow e_1^1 G^{u\hat{u}}$, $z\epsilon_2^2 G \rightarrow z\epsilon_2^2 (G^{d\hat{d}}, G^{e^-e^+})$, $f_2^2 A \rightarrow f_2^2 A^{d\hat{d}}$ etc, and $(G^{u\hat{u}})_{33} = \lambda_t$, $z(G^{d\hat{d}})_{33} = \lambda_b$, $z(G^{e^-e^+})_{33} = \lambda_\tau$, $(A^{d\hat{d}})_{23} = i\lambda_A^{d\hat{d}}/2$, etc.

In the charged lepton sector we had already fixed some Yukawa couplings and products of Yukawa couplings with the corresponding VEVs. The values for λ_t, λ_b, and λ_τ are

$$\lambda_t(M_Z) = 1.30, \qquad \lambda_b(M_Z) \simeq 4 \times 10^{-2},$$
$$\lambda_\tau(M_Z) \simeq 2 \times 10^{-2}. \tag{A9}$$

The Yukawa coupling in front of the matrices A is defined according to

$$A = i\begin{pmatrix} 0 & \sigma & -\sigma \\ -\sigma & 0 & 1/2 \\ \sigma & -1/2 & 0 \end{pmatrix}\lambda_A. \tag{A10}$$

In our estimates we take for all λ_A's the same value, however.

In our model in which the mixing of E_6 flavors is a $\mathcal{U}$-spin mixing effect, one can expect that the VEV's of ϵ_2^2, f_3^2, and g_3^2 which appear at the weak scale have roughly the same magnitude. This implies using (3.19) and the discussion at the end of Sec. III $\epsilon_2^2 \simeq g_3^2 \simeq 76$ GeV. Together with the fit values quoted in (3.9) and (3.16) all VEV's encountered are known at least approximately. Using $\lambda^{D\hat{D}} \approx \lambda^{L\bar{L}} \approx 1$ and $\lambda_A = 1$ we exhibit them here all together:

$$e_1^1 = 131.5 \text{ GeV}, \qquad f_2^2 \simeq -0.23 \text{ GeV},$$
$$g_2^2 \simeq -0.19 \text{ GeV}, \qquad f_3^2 \simeq 39 \text{ GeV}, \tag{A11}$$
$$g_3^2 = \epsilon_2^2 \simeq 76 \text{ GeV}, \qquad \frac{f_2^3}{e_3^3} = \frac{g_2^3}{e_3^3} \simeq 4.16 \times 10^{-6}.$$

It is important that the couplings λ_b and λ_τ are small in this scenario. It implies that their RG dependence can be ignored. We could select $\lambda_A \simeq 1$ because in the RG treatment at scales below M_I this coupling does not explicitly enter. The reason is that the scalar states (6, 3, 1) and $(\bar{6}, \bar{3}, 1)$ decouple already at the higher scale M_6 and thus cannot appear in appropriate loops. The only renormalization effects relevant below the scale M_I are due to λ_t and the gauge couplings $g_{1,2,3}$. The renormalization of the flavor matrices is performed in the following paragraph taking this fact into account.

The flavor matrices G and A as introduced in Sec. I did not include RG effects. We start with the study of the renormalization of the G matrices which are generated at the scale M_I. At this point the spacing of different channels (i.e. $u\hat{u}$, $d\hat{d}$, e^-e^+, etc.) is universal. We suggested for the splitting equal spacing between first and second and between second and third generations (1.2). At the scale M_Z, however, for different channels different RG factors

emerge affecting the spacings. The G-couplings can be divided into two groups. One group corresponds to couplings involving H_{27}^u. The second group corresponds to couplings involving H_{27}^d. At M_Z they are split as follows: $G_{H^u} \rightarrow (G^{u\hat{u}}, G^{D\hat{D}}, G^{L\bar{L}})$, $zG_{H^d} \rightarrow z(G^{d\hat{d}}, G^{e^-e^+}, \cdots)$ (we will keep only those quantities which are relevant for us). Our aim is to calculate the relative factors between appropriate entries in these matrices. The relative RG factors between the 3rd generation components of $G^{u\hat{u}}$, $G^{d\hat{d}}$, and $G^{e^-e^+}$ are described by the coupling constants λ_t, λ_b, λ_τ. However, different generations have different Yukawa interactions. Because we can take $z \approx 0.02$ all Yukawa couplings involving this prefactor do not participate in the change of the spacings. With the boundary condition that all G-couplings at M_I have the form (1.2), one gets at the scale M_Z

$$\frac{G^{u\hat{u}}}{\lambda_t}(M_Z) = \text{Diag}(r_t^{-1}\sigma^4, r_t^{-1}\sigma^2, 1),$$

$$\frac{G^{d\hat{d}}}{\lambda_b}(M_Z) = \text{Diag}(r_t^{-1/3}\sigma^4, r_t^{-1/3}\sigma^2, 1),$$

$$\frac{G^{e^-e^+}}{\lambda_\tau}(M_Z) = \text{Diag}(\sigma^4, \sigma^2, 1), \tag{A12}$$

$$\frac{G^{D\hat{D}}}{\lambda_D}(M_Z) = \text{Diag}(\kappa_1^D\sigma^4, \kappa_2^D\sigma^2, 1),$$

$$\frac{G^{L\bar{L}}}{\lambda_L}(M_Z) = \text{Diag}(\kappa_1^L\sigma^4, \kappa_2^L\sigma^2, 1).$$

The appropriate RG factors are

$$r_t = \exp\left[-\frac{1}{16\pi^2}\int_{t_z}^{t_3}\frac{3}{2}\lambda_t^2 dt\right],$$

$$\kappa_1^D = \exp\left[\frac{1}{16\pi^2}\int_{t_1}^{t_3}\left(\frac{2}{5}g_1^2 + 8g_3^2\right)dt\right],$$

$$\kappa_2^D = \exp\left[\frac{1}{16\pi^2}\int_{t_2}^{t_3}\left(\frac{2}{5}g_1^2 + 8g_3^2\right)dt\right],$$

$$\kappa_1^L = \exp\left[\frac{1}{16\pi^2}\int_{t_1}^{t_3}\left(\frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)dt\right],$$

$$\kappa_2^L = \exp\left[\frac{1}{16\pi^2}\int_{t_2}^{t_3}\left(\frac{9}{10}g_1^2 + \frac{9}{2}g_2^2\right)dt\right],$$

with $t = \ln\mu$, $t_z = \ln M_Z$, $t_i = \ln\mu_i$,

$\mu_1 = M_I\sigma^4$, $\mu_2 = M_I\sigma^2$, $\mu_3 = M_I$.

In $G^{D\hat{D}}$ and in $G^{L\bar{L}}$ the renormalization factors are due to gauge interactions. The reason is that the heavy states M_{D_i} and M_{L_i} decouple at different scales. Below the decoupling scale the mass of the corresponding state "freezes" out and does not run, and the leg in the diagram involving the decoupled state will not be "dressed" by the loop.

Now we turn to the RG of the A couplings: At low energies instead of one A matrix we are dealing in general

with six A matrices: $A \rightarrow (A^{d\hat{d}}, A^{d\hat{D}}, A^{D\hat{d}}, A^{e^-e^+}, A^{e^-E^+}, A^{E^-e^+})$. The superscripts indicate the channels they couple to. Compared to the G couplings, the A matrices are formed at relatively high scales. These matrices are the only source for generation mixing, but through loop diagrams involving the scalar states $H_A^6(6, 3, 1)$ and $H_A^{\bar{6}}(\bar{6}, \bar{3}, 1)$, these matrices could nevertheless induce off diagonal entries into the G matrices. However, if the latter states are already decoupled when the A matrices are formed, the corresponding loops will not contribute to renormalization. This is one of the reasons why we take the masses of the $L_{F'} + \bar{L}_{\bar{F}'}$ states equal to M_6. It insures that in the lepton sector nondiagonal A couplings do not influence the G matrices through loop effects. The A matrices corresponding to the quark channels are formed at the GUT scale keeping the model intact since quarks do not couple with H_A^6, $H_A^{\bar{6}}$.

For the renormalization of the A matrices the relevant energy range is $M_Z - M_I$ where the G matrices are already formed (above the scale M_I the matrices A do not change their structural form). Only the coupling λ_t is relevant. Since only the quarks $q_3 = (u, d)_3$ are connected with λ_t, solely the $(3, 2)$ elements of the matrices $A^{d\hat{d}}$ and $A^{d\hat{D}}$ will pick up relative RG factors. The spacing of the other entries and of the remaining A matrices will remain unchanged. Therefore, we have at the scale M_Z

$$\frac{A^{d\hat{d}}}{\lambda_A^{d\hat{d}}} = \frac{A^{d\hat{D}}}{\lambda_A^{d\hat{D}}} = i \begin{pmatrix} 0 & \sigma & -\sigma \\ -\sigma & 0 & 1/2 \\ \sigma & -\kappa_A/2 & 0 \end{pmatrix},$$

$$\text{with} \quad \kappa_A = r_t^{-1/3}$$

and

$$\frac{A^{D\hat{d}}}{\lambda_A^{D\hat{d}}} = \frac{A^{e^-e^+}}{\lambda_A^{e^-e^+}} = \frac{A^{e^-E^+}}{\lambda_A^{e^-E^+}} = \frac{A^{E^-e^+}}{\lambda_A^{E^-e^+}}$$

$$= i \begin{pmatrix} 0 & \sigma & -\sigma \\ -\sigma & 0 & 1/2 \\ \sigma & -1/2 & 0 \end{pmatrix}. \quad \text{(A14)}$$

The numerical values of the RG factors appearing above are

$$r_t = 0.717, \qquad \kappa_A = 1.117, \qquad \kappa_1^D = 1.299,$$
$$\kappa_2^D = 1.127, \qquad \kappa_1^L = 1.137, \qquad \kappa_2^L = 1.066. \quad \text{(A15)}$$

These coefficients are used in the analysis of the charged fermion mass matrices in Sec. III.

Finally, using the above results, we are able to fix the values of the SM singlet VEVs

$$e_3^3 = \frac{M_I}{g_2(M_I)} \simeq 2.27 \times 10^{13} \text{ GeV,}$$
$$f_2^3 = g_2^3 = 9.5 \times 10^7 \text{ GeV.} \quad \text{(A16)}$$

On the other hand, from the neutrino sector (see Sec. IV) we have

$$F^{\{2,3\}} = x_g \frac{(e_1^1)^2}{m} \simeq 1.8 \times 10^{13} \text{ GeV,}$$
$$F^{\{3,3\}} = x_g \rho F^{\{2,3\}} \simeq 6.6 \times 10^{11} \text{ GeV,} \quad \text{(A17)}$$
$$F_A = x_A \frac{\sigma^5}{\lambda_A} F^{\{2,3\}} \approx -3.3 \times 10^5 \text{ GeV.}$$

We recall that by integrating out the N, $\tilde{N}$ states, we found for the mass scales in (A17)

$$F^{\{2,3\}} = \frac{\lambda_N \tilde{\lambda}_N}{M_N} e_3^3 \epsilon_2^3, \qquad F^{\{3,3\}} = -z \frac{\lambda_N \tilde{\lambda}_N}{M_N} (e_3^3)^2 \quad \text{(A18)}$$

(the couplings λ_N, $\tilde{\lambda}_N$ come from the vertices $\lambda_N F H^\dagger N$, $\tilde{\lambda}_N F \tilde{H}^\dagger \tilde{N}$). The comparison of (A16) with (A17) allows the estimate

$$\frac{M_N}{\lambda_N \tilde{\lambda}_N} \simeq 2.9 \times 10^{13} \text{ GeV,}$$
$$-z e_3^3 = \frac{F^{\{3,3\}}}{F^{\{2,3\}}} \epsilon_2^3 \simeq 8.5 \times 10^{11} \text{ GeV.} \quad \text{(A19)}$$

One recognizes again the importance of the scale $M_I = g_2(M_I) e_3^3$ for the heavy fermions and, in particular, for the right-handed neutrinos.

The VEV configuration discussed here is consistent with the charged fermion sector as well as the neutral lepton sector. The following picture of the symmetry breaking of our grand unified model emerges: In a first step E_6 is broken by an $H(650)$ scalar field. It reduces E_6 to the group $G_{333} = SU(3)_L \times SU(3)_R \times SU(3)_C$. At $M_I = g_2(M_I) e_3^3$ the G_{333} symmetry breaks down to the SM. The role of $H(650)$ fields is important for the phenomenology. These fields include the G_{333} singlet $S_+(1, 1, 1)$ state. The VEV $\langle S_+ \rangle = M_{\text{GUT}}/g_{\text{GUT}}$ gives the breaking $E_6 \rightarrow G_{333}$ in such a way that the $L - R$ symmetry $\mathcal{D}$ is unbroken at high energies. This way a self-consistent gauge coupling unification is guaranteed. The additional SM singlet VEVs are much smaller than e_3^3. They are important for the fermion sector. The leading role for the electroweak symmetry breaking is played by the VEVs e_1^1, ϵ_2^2, g_3^2, and f_3^2. The remaining electroweak VEVs (f_2^2, g_2^2) are small but still relevant for the light fermion masses and their mixings.

APPENDIX B: NEUTRINO MASS MATRIX RENORMALIZATION

We need to define m_ν at the scale M_Z. To calculate the corresponding renormalization effects it is convenient to consider the combination $(\mathcal{U}^T)^{-1} m_\nu \mathcal{U}^{-1}$ taking (4.12), (4.13), and (4.14) into account. Without RG effects this matrix has the form

$$(\mathcal{U}^T)^{-1}m_\nu\,\mathcal{U}^{-1} = \begin{pmatrix} \rho & -i & i \\ -i & \rho - 2x_A & -i\frac{\sigma}{2} + x_A \\ i & -i\frac{\sigma}{2} + x_A & \rho \end{pmatrix} \cdot m.$$

(B1)

In order to compute the RG factors we recall the origin of each entry in (B1). Let us start with the ρm entries in the diagonal part. The elements (1, 1), (2, 2), and (3, 3) emerge from dimension five ($d = 5$) operators at different scales (at $F^{\{2,3\}}\sigma^8$, $F^{\{2,3\}}\sigma^4$, and $F^{\{2,3\}}$, respectively) by integrating out the appropriate states of L_2^3, L_3^3 (this can be seen from Fig. 4(a)]. Above the corresponding scale only the Dirac Yukawa couplings run. After the formation of the effective dimension five operators, i.e. below the corresponding scale, new RG factors are relevant. They describe the running of couplings of $d = 5$ operators which are different from the running of the square of the Dirac Yukawa couplings. Therefore, relative (mismatching) RG factors between different entries have to be applied. In order to calculate these RG factors, we need to go through the entire energy interval considering the appropriate thresholds. After performing these procedures stepwise from M_I down to M_Z the diagonal ρm entries at M_Z can be parametrized as follows:

$$\rho m(1 + r_1), \qquad \rho m(1 + r_2), \quad \text{and} \quad \rho m, \quad \text{(B2)}$$

where the RG factors $r_{1,2}$ are given by

$$r_1 = \exp\left[\frac{1}{16\pi^2}\int_{t_{11}}^{t_{33}}\left(\frac{3}{2}g_2^2 + \frac{9}{10}g_1^2\right)dt\right] - 1,$$

$$r_2 = \exp\left[\frac{1}{16\pi^2}\int_{t_{22}}^{t_{33}}\left(\frac{3}{2}g_2^2 + \frac{9}{10}g_1^2\right)dt\right] - 1$$

(B3)

with $t_{ii} = \ln\mu_{ii}$, $\mu_{11} = F^{\{2,3\}}\sigma^8$,

$\mu_{22} = F^{\{2,3\}}\sigma^4$, $\mu_{33} = F^{\{2,3\}}$

(when RG effects are ignored these factors are zero and the diagonal ρm part of the matrix (B1) is a unit matrix in flavor space). In the parametrization of (B2) we did not write any RG factor for the third term ρm because an overall factor can be absorbed by the parameter ρ (or the scale m). Only the relative RG factors given in (B3) are needed.

Considering now the remaining entries of m_ν, it is easy to verify that the (1, 2), (1, 3) elements together with the $2mx_A$ and mx_A entries of (2, 2) and (2, 3) are formed at the same scales after the subsequent integration of the heavy neutral states (this can be seen from the second and third terms of (4.9) and the corresponding diagram in Fig. 4(b)]. Therefore, no relative RG factors between these entries will appear. The only additional factor occurs for the $-im\frac{\sigma}{2}$ term in the (2, 3) and (3, 2) elements. We can parametrize this term at the scale M_Z by $-im\frac{\sigma}{2}r_{23}$. The RG factor r_{23} can be calculated from the equation

$$r_{23} = \exp\left[-\frac{1}{16\pi^2}\int_{t_{11}}^{t_{22}}\left(\frac{3}{2}g_2^2 + \frac{9}{10}g_1^2\right)dt\right]. \quad \text{(B4)}$$

Taking all RG factors into account, the matrix (B1) at M_Z takes the form

$$(\mathcal{U}^T)^{-1}m_\nu\,\mathcal{U}^{-1}\big|_{\mu=M_Z} = \begin{pmatrix} \rho(1 + r_1) & -i & i \\ -i & \rho(1 + r_2) - 2x_A & -i\frac{\sigma}{2}r_{23} + x_A \\ i & -i\frac{\sigma}{2}r_{23} + x_A & \rho \end{pmatrix} \cdot m. \quad \text{(B5)}$$

Finally, one obtains the neutrino mass matrix at the scale $\mu = M_Z$ as shown by Eq. (4.18).

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