PHYSICAL REVIEW D **77,** 074505 (2008)

Effective field theories for QCD with rooted staggered fermions

Claude Bernard,[1] Maarten Golterman,[2] and Yigal Shamir[3]

[1]*Department of Physics, Washington University, Saint Louis, Missouri 63130, USA*

[2]*Department of Physics and Astronomy, San Francisco State University, San Francisco, California 94132, USA*

[3]*Raymond and Beverly Sackler School of Physics and Astronomy, Tel-Aviv University, Ramat Aviv, 69978 Israel*

(Received 27 December 2007; published 14 April 2008)

Even highly improved variants of lattice QCD with staggered fermions show significant violations of taste symmetry at currently accessible lattice spacings. In addition, the ''rooting trick'' is used in order to simulate with the correct number of light sea quarks, and this makes the lattice theory nonlocal, even though there is good reason to believe that the continuum limit is in the correct universality class. In order to understand scaling violations, it is thus necessary to extend the construction of the Symanzik effective theory to include rooted staggered fermions. We show how this can be done, starting from a generalization of the renormalization-group approach to rooted staggered fermions recently developed by one of us. We then explain how the chiral effective theory follows from the Symanzik action, and show that it leads to ''rooted'' staggered chiral perturbation theory as the correct chiral theory for QCD with rooted staggered fermions. We thus establish a direct link between the renormalization-group based arguments for the correctness of the continuum limit and the success of rooted staggered chiral perturbation theory in fitting numerical results obtained with the rooting trick. In order to develop our argument, we need to assume the existence of a standard partially-quenched chiral effective theory for any local partially-quenched theory. Other technical, but standard, assumptions are also required.

DOI: 10.1103/PhysRevD.77.074505 PACS numbers: 12.38.Gc

I. INTRODUCTION

On a hypercubic lattice in four dimensions, the continuum limit of lattice QCD with staggered fermions [1] contains four ''tastes'' of mass-degenerate quarks per staggered fermion field [2–5].[1] Hence, if we introduce a separate staggered fermion field for each physical light-quark flavor (up, down, and strange), the continuum limit consists of QCD containing four up, four down, and four strange quarks.

A simple solution to this problem is to adjust for the excessive multiplicity by taking the fourth root of the fermion determinant for each staggered fermion field [6]. Heuristically, if the staggered determinant factorizes into four identical determinants in the continuum limit, one for each taste, taking the fourth root corrects for the taste multiplicity. The desired theory, QCD with one up, down, and strange quark each is then obtained in the continuum limit. Since the staggered determinant is positive for any real, nonzero bare quark mass m, and the continuum determinant is (formally) positive for positive quark mass, the positive fourth root should be chosen.[2] The continuum quark mass is proportional to $|m|$, which undergoes only a multiplicative renormalization, because staggered fermions have one exact chiral symmetry.

This procedure, the ''fourth-root trick,'' raises a number of questions [9–11]. The fourth root of a determinant cannot in general be written as a Grassmann integral with a local action. Therefore, the first question is whether the theory defined by the fourth-root trick is local and unitary.

In Ref. [12] we showed that, as might be expected, the fourth-root staggered theory is not local at nonzero lattice spacing a. Continuing correlation functions defined in the Euclidean theory to Minkowski space will lead to violations of unitarity at $a \neq 0$, on a distance scale set by the lightest particles in the theory, the Goldstone bosons. For examples of this, see Ref. [13], as well as Sec. 6 of Ref. [14], which we will revisit later in this paper.

The origin of these diseases can be traced back to the taste-symmetry-breaking part of the staggered Dirac operator. This taste-breaking part corresponds to a dimension-five irrelevant operator. Thus, in the local, unrooted staggered theory, all taste-symmetry-breaking effects are expected to vanish in the continuum limit, where exact $U(4)$ taste symmetry will be restored for each of the four up, four down, and four strange quarks present in that theory.

The leading power-law scaling of irrelevant operators is characteristic of any local and renormalizable theory, such as, in particular, the unrooted staggered theory. This brings us to the second question: Does the same scaling persist in the fourth-root theory? Two related considerations make it natural to address this question via a renormalization-group (RG) approach. To begin with, the RG framework allows us to define what we mean by the continuum limit. This is done by performing $n + 1$ blocking steps[3] on the original lattice theory, with its fine spacing $a_f = a$, each

[1]We assume the usual choice of only a single-site bare mass term.

[2]For the case of an odd number of quarks with negative quark mass, see Refs. [7,8].

[3]See Sec. III for an explanation of the convention $a_c/a_f = 2^{n+1}$ [15].

time increasing the lattice spacing by a factor of 2, to arrive at an RG-blocked theory formulated on a lattice with a coarse spacing $a_c = 2^{n+1} a_f$. Keeping a_c fixed and small in physical units, $a_c \ll \Lambda_{\rm QCD}^{-1}$, while sending $n \to \infty$ (and thus $a_f \to 0$), one obtains a coarse-lattice theory describing the continuum physics. An RG framework is also natural because the restoration of taste symmetry is only expected to occur on distance scales much larger than the original lattice cutoff a_f. RG blocking removes the short-distance fluctuations while modifying the action of the remaining degrees of freedom by local terms only. When we increase the number of blocking steps n, the blocked theory becomes more taste symmetric, and we eventually recover exact taste symmetry in the continuum limit $n \to \infty$.

Using this RG framework, it was argued in Ref. [15] that the continuum limit of QCD with rooted staggered fermions is a local theory that belongs to the correct universality class. There are strong arguments that the fourth-root theory, while nonlocal, is nevertheless renormalizable [10,16,17], and this is the fundamental reason behind the validity of its continuum limit. The detailed reasoning is based on a number of technical assumptions, all of which are very similar to the assumptions needed to establish the nature of the continuum limit for the *unrooted* staggered theory. Further analytic and numerical work aimed at confirming the technical assumptions of Ref. [15] would add direct and strong evidence for the validity of the fourth-root trick. For full details, we refer to Ref. [15]; for shorter, more intuitive accounts, we refer to Refs. [10,11]. We stress that one key element—the anticipated scaling of the taste-breaking effects—has been corroborated by extensive numerical studies [18–21].

Assuming that the rooted staggered theory has the correct continuum limit, this leaves us with a third question. While the anticipated scaling of taste-breaking effects is observed, these effects are clearly not negligible at present [13,18–21]. It is therefore imperative to take lattice artifacts into account in the effective continuum field theories (EFTs) such as the Symanzik effective theory (SET) or chiral perturbation theory (ChPT). The latter provides a central tool for analyzing the numerical data and performing the chiral and continuum extrapolations in the light-quark sector. In the case of rooted staggered fermions, we thus need to construct EFTs that take the discretization effects into account, including those that correspond to the nonlocal behavior of the theory at $a \neq 0$. The construction of such EFTs is the subject of this paper.

For the pseudoscalar Goldstone-boson physics, a candidate EFT already exists; it is provided by staggered ChPT [22] with the replica rule (rSChPT), or "rooted staggered ChPT" [23]. (Extensions to higher order [24], and to heavy-light meson [25] and baryon [26] rSChPT were recently given.) An argument for the validity of rSChPT was presented in Ref. [14], and reviewed in Refs. [10,11]. The key feature of Ref. [14] is that the argument takes place completely within the context of chiral effective theories, and the replica rule is justified only in that context. Here we will need to introduce a somewhat different version of the replica rule, which will be justified in addition at the level of the fundamental lattice theory, but which will ultimately give the same results in the chiral theory. A detailed comparison of the two approaches will be made in Sec. V C.

The overall goal in the current paper is to extend the standard procedure for the construction of ChPT for a local lattice theory to QCD with rooted staggered fermions. The standard procedure consists of two steps. The SET [27] is constructed first. This can be done order-by-order in perturbation theory, but it is generally assumed that the SET is valid nonperturbatively as well. We will assume throughout that this includes partially-quenched theories [16]. In particular, we will assume that locality suffices, and that unitarity (which may be lost in partially-quenched theories) is not necessary. Once the correct form of the SET has been established, its symmetries can be used to construct ChPT. Since the SET organizes the low-energy effective theory as a systematic expansion in the lattice spacing, one automatically obtains the chiral theory as an expansion in the lattice spacing as well.

Establishing that EFTs can be constructed following the usual rules for QCD with rooted staggered fermions thus constitutes a fundamental step in understanding the effects of rooting at nonvanishing lattice spacing. The main thrust of this paper is the construction of the SET for the rooted theory; obtaining the corresponding ChPT is then straightforward, and we show that it is indeed given by rSChPT. We emphasize that our construction applies to all commonly used versions of staggered fermions: standard (unimproved) staggered [1], Asqtad [28], HYP [29], Fat7bar [30], HISQ [31], etc. The only requirement is that the actions have the usual staggered symmetries. The size of the discretization effects is of course different with different versions of staggered fermions, but their form (and appearance at each order in a_f) is the same.

It is also important to note that the effective theories we ultimately construct are those for the relevant rooted staggered theory on the original (fine) lattice. The RG framework is used only as a tool in the derivation of these effective theories. Nevertheless, it is an indispensable tool: the conclusions of Ref. [15] have to be valid in order for our construction of the EFTs to make any sense. We will assume this to be the case.

The difficulty in constructing EFTs for the rooted theory is the following. Consider for simplicity a staggered theory with a common power, denoted n_r, of the fermion determinant for each staggered flavor in the theory. As long as n_r is a positive integer the lattice theory is local, and the construction of EFTs proceeds as usual. In order to arrive at the fourth-root theory,[4] however, we must set $n_r = 1/4$. Our task is to ensure that a *replica continuation* may be performed: a well-defined procedure must be devised to reach the value $n_r = 1/4$ at the level of an EFT, and the procedure must be consistent with the n_r dependence of the underlying lattice theory.

In a diagrammatic EFT calculation, the dependence on the number of (sea) quarks arises in two ways. First, there is explicit dependence arising through loop diagrams. In addition, the coupling constants of the EFT (the Symanzik coefficients in the case of the SET, and the low-energy constants in the case of ChPT) depend in an unknown way on the underlying lattice theory, including, in particular, on the number of replicas n_r. It is the latter dependence that makes our task nontrivial. In principle, one may envisage two basic obstructions to the replica continuation of the coupling constants in the EFT. Mathematically, a unique analytic continuation off the positive integers (which in the case at hand is where the theory is local) does not exist. Also, it could be that the replica continuation we have in mind will encounter a singularity precisely at the desired point $n_r = 1/4$.

The dependence of the underlying lattice theory on the number of replicas n_r is both perturbative and nonperturbative; this means that proving that no obstacle to the replica continuation is present would be tantamount to solving the theory nonperturbatively. The key observation that makes our task nevertheless tractable is that, after a large number n of RG-blocking steps, the taste-symmetry-breaking effects are very small: the unrooted staggered theory with integer n_r is very close to a $U(4)$ taste-invariant theory. The rooted theory, with $n_r = 1/4$, is then also very close to a *local* lattice theory, for which the standard construction of EFTs is valid. Indeed, the ‘‘reweighted’’ taste-invariant theories introduced in Ref. [15] are local whenever n_r is a multiple of $1/4$. The proximity of these local theories makes it possible to construct the SET and, later, ChPT, for the rooted theory.

We will reach the SET for the rooted theory starting from the SET for the corresponding reweighted, taste-invariant theory. The flavors of the taste-invariant theory will always be kept in one-to-one correspondence with those of the continuum-limit theory. In the taste-invariant theory the dependence of the Symanzik coefficients on the physical quarks is nonperturbative, and unknown, as usual. This does not pose any difficulty, because the number of physical flavors is never varied.

During the intermediate steps of the derivation, the parameter n_r will take on a related, but different technical meaning. The precise definitions will be given and explained in Sec. III below. As already mentioned above, we first approximate the staggered theory by a local, taste-invariant theory that belongs to the correct universality class. The (rooted) staggered theory will then be reached from the taste-invariant theory by ‘‘turning on’’ smoothly the taste-breaking effects. The dependence on n_r of the lattice theory will come only from the taste-breaking effects, which are nonlocal (for noninteger n_r) but small. The difference between corresponding taste-invariant and staggered theories is of order the fine-lattice spacing a_f of the original (unblocked) lattice. This will allow us to show that all the lattice correlation functions are polynomials in n_r to any fixed order in the expansion in a_f. The degree of the n_r polynomial is less than the order of the a_f expansion. The n_r dependence of the Symanzik coefficients can then be determined unambiguously. It follows that the replica continuation is nowhere singular in the complex n_r plane, to the given order in a_f. Finally, after performing the replica continuation, the parameter n_r resumes its original role as the power of the staggered determinant in the lattice theory. The further transition to ChPT is essentially a repeat of the same reasoning. As will become clear later on, we do have to assume that a chiral effective theory can be constructed for any local, but partially-quenched, theory. This was already emphasized in Refs. [10,14].

The outline of this paper is as follows. In Sec. II we consider the symmetries of staggered fermions in some detail. We derive the form in which shift symmetry [4] is realized in the SET, and thus in any other EFT derived from the SET. A quick overview of the most important observations of that section is given at its beginning, and any reader not interested in the details can skip the remainder of the section.

In Sec. III we come to the main part of this paper, the construction of the SET for QCD with rooted staggered fermions. We generalize the staggered theory to a class of partially-quenched theories in which it is possible to implement the program outlined above. In Sec. IV we discuss the SET to quadratic order in the lattice spacing in more detail, in order to illustrate the general construction. In Sec. V we make the transition to the chiral effective theory, and demonstrate that it is indeed given by rSChPT. As an example, we work out in rSChPT the leading-order contribution to the connected scalar two-point function, following the calculation in Ref. [14]. We then compare the present derivation of rSChPT to that of Ref. [14], using the respective discussions of the scalar two-point function to make the comparison concrete. The final section contains our conclusions. A brief account of this work was presented at Lattice 2007 [32].

II. SYMMETRIES OF THE SYMANZIK EFFECTIVE ACTION FOR STAGGERED FERMIONS

Here we discuss the symmetries of unrooted staggered fermions that are most relevant for this paper, and the way

[4]The discussion generalizes easily to the isospin limit $m_u = m_d \equiv m_\ell$, where one takes the square root of a single staggered flavor with (bare) mass m_ℓ.

they appear at the level of the SET. We begin with an overview of the main results of this section. In the following subsections we will then give a more detailed discussion.

(1) The staggered fermion action is invariant under shift symmetry, which, in the continuum limit, enlarges to the product of $SU(4)$ taste-symmetry and translation symmetry. At the level of the SET, the taste part of shift symmetry takes the form of the 32-element group Γ_4 generated by a set of four-dimensional Dirac gamma matrices ξ_μ, with

$$\{\xi_\mu, \xi_\nu\} = 2\delta_{\mu\nu}, \qquad \mu, \nu \in \{1, 2, 3, 4\}. \tag{2.1}$$

This result was derived to order a^2 in Ref. [22]. Here we give a general argument that makes it clear that the result is true to all orders in a. On the continuum quark fields q used in the SET, the generating elements of Γ_4 can be chosen to act according to

$$q \to \xi_\mu q, \qquad \bar{q} \to \bar{q}\xi_\mu. \tag{2.2}$$

Here the field $q_{\beta b}$ has a Dirac spin index β and an $SU(4)$ taste index b, with the matrices ξ_μ acting on the latter.

(2) On the lattice, a taste-basis field ψ carrying the same indices as the continuum quark field q is related to the one-component field χ by a unitary transformation [5,33]

$$\psi = Q\chi, \qquad \bar{\psi} = \bar{\chi}Q^\dagger. \tag{2.3}$$

The field ψ lives on a coarse lattice whose spacing is twice that of the original staggered action. The ultralocal, unitary matrix Q maps the one-component variables χ on the 16 sites of each even hypercube to the 16 components of ψ on the single corresponding coarse-lattice site. The transformation Q is required to be gauge covariant, and its choice is not unique. As a result hypercubic rotational symmetry is somewhat complicated in the taste basis.[5] Of course, since the one-component and taste bases are related by a unitarity transformation, the physical consequences of all staggered symmetries are preserved.

A somewhat different taste-basis operator, that we will refer to as the "RG-taste-basis" Dirac operator [15,34], is defined by a Gaussian smearing of the unitary transformation (2.3). The resulting inverse Dirac operator satisfies

$$D_{\rm taste}^{-1} = \frac{1}{\alpha} + QD_{\rm stag}^{-1}Q^\dagger, \tag{2.4}$$

where $D_{\rm stag}$ is the Dirac operator in the one-component formulation, and α is a parameter of order $1/a$. Even though the theories described by $D_{\rm stag}$ and $D_{\rm taste}$ are no longer related by a simple, unitary basis transformation, they are physically completely equivalent, because the propagators differ only by a contact term. The advantage of the Gaussian-smeared transformation is that discarding the taste-breaking part of $D_{\rm taste}$ does not introduce any fermion doublers [12,34]. Because the staggered theory and the taste-invariant theory have a similar fermion content, one can interpolate smoothly between them. This will prove useful for the derivation of the SET.

In the one-component formulation, shift symmetry is a unitary transformation on the fields χ and $\bar{\chi}$ [cf. Eq. (2.11) in the next subsection]. Since Q is unitary, the same is also true for the fields ψ and $\bar{\psi}$, and from this it follows that the theory in the RG-taste-basis is also invariant under shift symmetry.

(3) Because of staggered symmetries, discretization errors for theories with staggered fermions start at order a^2 [35]. This is not obvious if one considers staggered fermions in the taste basis of Refs. [5,33], or in the modified form used in the RG analysis of Refs. [15,34], where taste-breaking terms occur in the action starting at order a. In this case, shift symmetry connects the leading, taste-invariant term in the lattice action with the order a taste-breaking term, i.e., their relative strength is fixed. There exists a local field redefinition that brings the taste-basis lattice action into a form where the taste violations are explicitly of order a^2, and shift symmetry is realized as in Eq. (2.2), again up to order a^2 terms [36]. More generally, the momentum-space basis used in the derivation of Eq. (2.2) can be related to the taste basis by a nonlocal field redefinition. Because the construction of the SET proceeds order-by-order in a, the field redefinition in effect becomes local. Therefore, the SETs constructed in the taste basis and in the staggered (or momentum-space) basis are always related by a local field redefinition.

(4) Staggered fermions have an exact chiral symmetry when $m = 0$, often referred to as $U(1)_\epsilon$ symmetry, taking the form [2]

$$\chi(x) \to e^{i\theta\epsilon(x)}\chi(x), \qquad \bar{\chi}(x) \to e^{i\theta\epsilon(x)}\bar{\chi}(x), \qquad \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}. \tag{2.5}$$

For $m = 0$, this implies that

$$\{D_{\rm taste}, \gamma_5 \otimes \xi_5\} = \frac{2}{\alpha}D_{\rm taste}(\gamma_5 \otimes \xi_5)D_{\rm taste}, \tag{2.6}$$

where γ_5 acts on the spin index, and $\xi_5 = \xi_1\xi_2\xi_3\xi_4$ acts on the taste index [12,34]. In other words, $D_{\rm taste}$ is a Ginsparg–Wilson operator [37] with respect to $U(1)_\epsilon$ symmetry.

[5] For a detailed discussion of rotational symmetry in this framework, see Ref. [15].

Before we proceed, we return to the relation of our analysis and that of Ref. [22]. The SET at order a^2 was determined in Ref. [22] by enumerating the allowed dimension-6 lattice operators consistent with the lattice symmetries, including shift symmetry. It was then shown that shift symmetry is represented on the corresponding continuum operators as a Γ_4 symmetry. A more direct method of determining the SET, which we follow here, is to enumerate continuum operators. This leads to the result of point 1, that shift symmetry always implies a taste Γ_4 symmetry of the SET.

In the subsections following below, we will discuss some of these observations in more detail. These subsections are not needed for the construction of the SET for rooted staggered fermions, which can be found in Sec. III.

A. Diagrammatic argument

Our first argument for claim 1 above is essentially perturbative, and assumes that we are working in the momentum-space representation of the one-component basis. This result may be considered a corollary of Ref. [4]. To keep it self-contained, a summary of relevant facts from Ref. [4] has been included in the discussion below.

We will consider diagrams with n external fermion and r external gauge-field lines, corresponding to an operator which appears at a certain order in the SET. On the lattice, because of the phase factors which appear in the staggered action, momentum is conserved modulo π (in this section we work in lattice units), and any such diagram will have an overall delta function for momentum conservation of the form

$$\delta(p_1 + \ldots + p_n + k_1 + \ldots + k_r + \Pi), \qquad (2.7)$$

where Π is a vector with components 0 or π. The delta function is the periodic delta function with period 2π. The (lattice) quark and antiquark momenta are p_i, $i = 1, \ldots, n$ and the gluon momenta k_j, $j = 1, \ldots, r$.

Because we are interested in an operator in the SET, we may take all physical external momenta small. Fermion doubling then implies that on every quark line we need to split the momenta as

$$p_i = q_i + \pi_{A_i}, \qquad (2.8)$$

in which q_i lives in the reduced Brillouin zone ($-\pi/2 < q_{i\mu} \le \pi/2$), and $\pi_{A_i} = \pi A_i$, with

$$A_i \in \{(0,0,0,0), (1,0,0,0), \ldots, (1,1,1,1)\}. \qquad (2.9)$$

We now take all physical momenta q_i and k_j small—so small that their sum has no components as large as $\pm\pi$. The delta function in Eq. (2.7) thus factorizes into

$$\delta(q_1 + \ldots + q_n + k_1 + \ldots + k_r) \times \delta(\pi_{A_1} + \ldots + \pi_{A_n} + \Pi). \qquad (2.10)$$

Now consider what happens to this diagram under a shift

$$\begin{aligned} \chi(x) &\to \zeta_\mu(x)\chi(x+\hat\mu), \\ \bar\chi(x) &\to \bar\chi(x+\hat\mu)\zeta_\mu(x), \\ U_\nu(x) &\to U_\nu(x+\hat\mu), \\ \zeta_\mu(x) &= (-1)^{x_{\mu+1}+\ldots+x_4} = e^{i\pi_{\zeta_\mu}\cdot x}, \end{aligned} \qquad (2.11)$$

where the last equality defines π_{ζ_μ}. In momentum space (with $\chi(x) = \int_{-\pi}^{\pi} d^4p/(2\pi)^4 e^{ip\cdot x}\chi(p)$), this takes the form

$$\chi(p_i) = \chi(q_i + \pi_{A_i}) \to e^{i(q_i+\pi_{A_i})_\mu}\chi(q_i + \pi_{A_i} + \pi_{\zeta_\mu}). \qquad (2.12)$$

Applying a shift in the μ direction to all external legs of our diagram, and noting that the jth external gluon line is multiplied by a factor $e^{i(k_j)_\mu}$ under a shift, we obtain the total factor

$$e^{i(q_1+\ldots+q_n+k_1+\ldots+k_r)_\mu}, \qquad (2.13)$$

which, by virtue of the first delta function in Eq. (2.10), is equal to one. Therefore, we may omit these (small-momentum) phase factors in the shift (2.12). We conclude that the diagram is invariant under the modified symmetry

$$\chi(q_i + \pi_{A_i}) \to e^{i(\pi_{A_i})_\mu}\chi(q_i + \pi_{A_i} + \pi_{\zeta_\mu}), \qquad i = 1, \ldots, n, \qquad (2.14)$$

which does not act on the gluon fields. The transformation (2.14) generates a representation of the group Γ_4 acting on the quark fields. Indeed, applying the transformation first in the μ direction, and then in the ν direction, one obtains (dropping the index i)

$$\chi(q + \pi_A) \to e^{i(\pi_A + \pi_{\zeta_\mu})_\nu} e^{i(\pi_A)_\mu}\chi(q + \pi_A + \pi_{\zeta_\mu} + \pi_{\zeta_\nu}). \qquad (2.15)$$

For $\mu = \nu$, we have $(\pi_{\zeta_\mu})_\mu = 0$ [cf. Eq. (2.11)], and thus Eq. (2.15) reduces to the identity. For $\mu \ne \nu$,

$$\begin{aligned} \zeta_\mu(x+\nu) &= \zeta_\mu(x) \Rightarrow e^{i(\pi_{\zeta_\mu})_\nu} = +1, \qquad \mu > \nu, \\ \zeta_\mu(x+\nu) &= -\zeta_\mu(x) \Rightarrow e^{i(\pi_{\zeta_\mu})_\nu} = -1, \qquad \mu < \nu, \end{aligned} \qquad (2.16)$$

implying that shifts anticommute, just like the generators of Γ_4. We may make contact with Eq. (2.2) by introducing

$$\phi_A(q) \equiv \chi(q + \pi_A). \qquad (2.17)$$

The transformation Eq. (2.14) can now be written as

$$\phi_A(q) \to \sum_B (\Xi_\mu)_{AB}\phi_B(q), \qquad (2.18)$$

for some 16×16 matrices Ξ_μ satisfying the Dirac algebra

$$\{\Xi_\mu, \Xi_\nu\} = 2\delta_{\mu\nu}. \qquad (2.19)$$

Finally, we can perform a basis transformation such that

$\Xi_\mu = 1 \otimes \xi_\mu$, and transform back to position space to obtain Eq. (2.2).

Our argument shows that the diagram is invariant under the symmetry (2.2) if it is invariant under shift symmetry (2.12). The group Γ_4 may thus be used to restrict the form of the SET in accordance with the shift symmetry of the underlying lattice theory. This is a considerable simplification, because the group Γ_4 does not mix operators of different dimensions, i.e., of different orders in the Symanzik expansion.

The same reasoning goes through in a theory in which the staggered fermion fields carry a flavor index $\ell = 1, \ldots, n_f$: one simply labels the fields χ_ℓ and $\bar{\chi}_\ell$ in Eq. (2.11) with the extra index ℓ. Since the gauge fields also transform under shift symmetry, the same shift symmetry acts on all staggered fields simultaneously. It thus follows that the discrete symmetry Γ_4 acts in the same way on all staggered fields χ_ℓ, and does not enlarge to the group $(\Gamma_4)^{n_f}$ [23].

As an aside, we note that the invariance of the diagram under shift symmetry has implications for the second delta function in Eq. (2.10). Naively, it would seem to follow that Π just has to be equal to the sum over all π_{A_i}, but in general this is not sufficient. The reason is that the vertex can contain explicit periodic functions of the external momenta, which leads to additional sign factors under a shift. This is best illustrated with an example. Consider a lattice vertex of the form

$$\sum_{A,B} \delta(q_1 + q_2 + k)\delta(\pi_A + \pi_B + \Pi) \times \cos(q_1 + k + \pi_A)_\nu \bar{\chi}(q_2 + \pi_B)\chi(q_1 + \pi_A)A_\nu(k), \tag{2.20}$$

in which we split $p_1 = q_1 + \pi_A$, $p_2 = q_2 + \pi_B$, and take $q_{1,2}$ and k to be small. Performing a shift on the χ and $\bar{\chi}$ fields results in (dropping a factor $\delta(q_1 + q_2 + k)$)

$$\begin{aligned}\sum_{A,B} &\delta(\pi_A + \pi_B + \Pi)\cos(q_1 + k + \pi_A)_\nu e^{i(\pi_A+\pi_B)_\mu} \\ &\times \bar{\chi}(q_2 + \pi_B + \pi_{\zeta_\mu})\chi(q_1 + \pi_A + \pi_{\zeta_\mu})A_\nu(k) \\ = &\sum_{A,B} \delta(\pi_A + \pi_B + \Pi)\cos(q_1 + k + \pi_A)_\nu \\ &\times e^{i(\pi_{\zeta_\mu})_\nu} e^{i\Pi_\mu} \bar{\chi}(q_2 + \pi_B)\chi(q_1 + \pi_A)A_\nu(k), \end{aligned} \tag{2.21}$$

where we used that $(\pi_{\zeta_\mu})_\mu = 0$ and that $2\pi_{\zeta_\mu} = 0 \bmod 2\pi$. The vertex is thus invariant if $\Pi_\mu + (\pi_{\zeta_\mu})_\nu = 0 \bmod 2\pi$. An example of such a Π is π_{η_ν}, which is defined by the phase factors which appear in the staggered action

$$\eta_\nu(x) \equiv e^{i\pi_{\eta_\nu} x} \equiv (-1)^{x_1+\ldots+x_{\nu-1}}. \tag{2.22}$$

B. Group-theoretical argument

There is a very simple group-theoretical way to derive the same result. Let S_μ be the shift in the μ direction. All elements of the shift-symmetry group can be generated from the basic four shifts, and it is thus sufficient to consider only the S_μ. In any irreducible representation of the group, S_μ looks like

$$S_\mu \to e^{iq_\mu}\Xi_\mu, \tag{2.23}$$

with $-\pi/2 < q_\mu \le \pi/2$ the physical momentum in lattice units, and the matrices Ξ_μ generate a representation of Γ_4 [38]. All irreducible representations are either "bosonic," if each Ξ_μ is mapped onto ± 1 (sixteen choices), or "fermionic," if the Ξ_μ are chosen to satisfy the Dirac algebra (2.19). Any field appearing in an EFT for the staggered theory (such as the SET or ChPT) transforms in some representation of S_μ under a shift (i.e., with some choice of q_μ and Ξ_μ).

Now we use that any continuum EFT is also invariant under continuum translations, which, on a continuum field Φ with momentum q, act as

$$\Phi(q) \to e^{iq\cdot r}\Phi(q), \tag{2.24}$$

for a translation over a displacement r. We may thus choose r such that $q \cdot r = -q_\mu$, follow S_μ by this translation, and again obtain a symmetry of the EFT. This symmetry is precisely the one generated by the Ξ_μ, i.e., a representation of Γ_4.

C. Taste basis

The arguments in the previous subsections made use of the momentum basis of the one-component formalism. The Feynman rules for the staggered theory in the one-component basis [4] were (assumed to have been) used in the derivation of the SET. Also, the group-theoretical argument works naturally on the momentum basis, since that is where irreducible representations of the staggered symmetry group live [38]. Alternatively, one could have started from the taste basis. The SET derived from the taste basis will not look the same as that derived from the one-component formalism; but the two SETs should be physically equivalent. Since the one-component and taste bases are related by a (nonlocal) unitary transformation in momentum space [5], one expects that the SETs derived from them, too, will be related by a field redefinition. Moreover, to any finite order in a, the SET-level field redefinition should be local, because the same is true for the unitary transformation between the two bases, when expanded to the corresponding finite order in a.

We illustrate this in the free massless theory, working to order p^2 in the Symanzik expansion. On the taste basis, shift symmetry takes on the form [5,36]

$$\begin{aligned}\psi(y) \to \tfrac{1}{2}((&\xi_\mu + \gamma_5\gamma_\mu\xi_5)\psi(y) \\ &+ (\xi_\mu - \gamma_5\gamma_\mu\xi_5)\psi(y + \hat{\mu})). \end{aligned} \tag{2.25}$$

The field ψ, introduced in Eq. (2.3), is in this case given explicitly by

$$\psi_{\beta b}(y) = \frac{1}{2^{3/2}} \sum_A (\gamma_A)_{\beta b} \chi(2y + A), \tag{2.26}$$

where $\gamma_A = \gamma_1^{A_1}\gamma_2^{A_2}\gamma_3^{A_3}\gamma_4^{A_4}$, and A runs over the set (2.9). The normalization factor in Eq. (2.26) differs from that in Ref. [5] because we take ψ to be in lattice units of the coarser lattice; whereas Ref. [5] works in physical units. In momentum space, Eq. (2.25) looks like

$$\begin{aligned}\psi(p) &\to e^{ip_\mu/2}(\xi_\mu \cos(p_\mu/2) - i\gamma_5\gamma_\mu\xi_5 \sin(p_\mu/2))\psi(p) \\ &= e^{ip_\mu/2}\Big(\xi_\mu - \frac{i}{2}\gamma_5\gamma_\mu\xi_5 p_\mu + \mathcal{O}(p^2)\Big)\psi(p).\end{aligned} \tag{2.27}$$

The factor $e^{ip_\mu/2}$ corresponds to the factor e^{iq_μ} in Eq. (2.12), because the lattice spacings differ by a factor of two. Dropping the factor $e^{ip_\mu/2}$ on the same grounds as in Sec. II A, it is easily verified that the transformation (2.27) becomes a generating element of Γ_4, and that it is a symmetry of the order-a SET in the taste representation,

$$S_{\rm free} = \sum_\mu \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \bar\psi(p)\Big(i\gamma_\mu p_\mu + \frac{1}{2}\gamma_5\xi_5\xi_\mu p_\mu^2 + \mathcal{O}(p^3)\Big)\psi(p). \tag{2.28}$$

This may also be written as

$$S_{\rm free} = \sum_\mu \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4}\Big(\mathrm{tr}[\bar\psi(p) i\gamma_\mu p_\mu \psi(p)] + \frac{1}{2}\mathrm{tr}[\bar\psi(p)\gamma_5 p_\mu^2 \psi(p)(\gamma_5\gamma_\mu)^\dagger] + \mathcal{O}(p^3)\Big), \tag{2.29}$$

where we consider the field $\psi_{\beta b}$ as a 4×4 matrix.

In momentum space, the transformation relating the one-component and taste representations is [5,36,39]

$$\begin{aligned}\psi(p) &= \frac{1}{2^{11/2}}\sum_{A,B}(-1)^{A\cdot B}\gamma_A \phi_B(q) e^{iq\cdot A}, \\ \bar\psi(p) &= \frac{1}{2^{11/2}}\sum_{A,B}(-1)^{A\cdot B}\gamma_A^\dagger \bar\phi_B(q) e^{-iq\cdot A},\end{aligned} \tag{2.30}$$

where again A and B take values in the set (2.9), and where $q = p/2$. The field $\phi(q)$ was defined in Eq. (2.17). The transformation (2.30) is indeed nonlocal, but its expansion to any finite order in a is local. For instance, upon expanding $e^{\pm iq\cdot A} = 1 \pm iq\cdot A + \mathcal{O}(q^2)$, and starting from Eq. (2.29), this field redefinition brings the action (2.28) into the form

$$S_{\rm free} = \sum_\mu \sum_{AB} \int_{-\pi/2}^{\pi/2} \frac{d^4q}{(2\pi)^4} \bar\phi_A(q)(i(\Gamma_\mu)_{AB} q_\mu + \mathcal{O}(q^3))\phi_B(q), \tag{2.31}$$

where the Γ_μ matrices form a 16-dimensional representation of the Dirac algebra and commute with the taste matrices Ξ_ν defined in Eq. (2.18). Note that Eq. (2.31) is expressed in units of the fine-lattice spacing.

Let us also briefly consider the RG-taste representation defined by Eq. (2.4) in the massless free theory. To order p^2 the action is given by [12]

$$\sum_\mu \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \bar\psi(p)\Big(i\gamma_\mu p_\mu + \frac{1}{\alpha}p_\mu^2 + \frac{1}{2}\gamma_5\xi_5\xi_\mu p_\mu^2 + \mathcal{O}(p^3)\Big)\psi(p). \tag{2.32}$$

This action is invariant under $U(1)_\epsilon$ symmetry in "Ginsparg–Wilson–Lüscher' (GWL) [37,40] form. In the free theory, this symmetry looks like (again to order a)

$$\begin{aligned}\delta\psi(p) &= \gamma_5\xi_5\Big(1 - \frac{2}{\alpha}\sum_\mu i\gamma_\mu p_\mu + \mathcal{O}(p^2)\Big)\psi(p), \\ \delta\bar\psi(p) &= \bar\psi(p)\gamma_5\xi_5.\end{aligned} \tag{2.33}$$

In this case, we may first carry out a field redefinition

$$\begin{aligned}\psi(p) &\to \Big(1 + \frac{1}{\alpha}\sum_\mu i\gamma_\mu p_\mu\Big)\psi(p), \\ \bar\psi(p) &\to \bar\psi(p),\end{aligned} \tag{2.34}$$

followed by (2.30), to bring the action into a form without terms of order a. Note that Eq. (2.34) is nothing but the free-field, order-a form of the field redefinition

$$\psi \to (1 - D/\alpha)^{-1}\psi \tag{2.35}$$

with here $D = D_{\rm taste}$, which transforms the GWL form of $U(1)_\epsilon$ symmetry into a standard $\gamma_5\xi_5$ symmetry, see, e.g., Ref. [41].

As a final note, we observe that to this order in a, field redefinitions can be carried out such that the resulting action is invariant under the full $U(4)$ taste symmetry. This turns out to be true to all orders in a in the free theory [4], but not in the interacting theory.

III. DERIVATION OF THE SYMANZIK EFFECTIVE ACTION

We begin by considering a theory with n_r replicas of one staggered fermion with bare mass m, in the RG taste basis. For now, n_r will be a positive integer. We perform $n + 1$ RG-blocking steps, labeled $k = 0, 1, \ldots, n$, following the blocking procedure of Ref. [15]. The special $k = 0$ step is used to carry out the transition from the one-component to the taste basis, cf. Eq. (2.4). In this step the number of fermion degrees of freedom is not thinned out; in each subsequent step they are thinned out by a factor $2^4 = 16$. The partition function for this theory can be written as

$$Z(n_r) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^{n} \mathcal{D}\mathcal{V}^{(k)} \boldsymbol{B}_n(n_r; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \mathrm{Det}^{n_r}(D_{\mathrm{taste},n}). \tag{3.1}$$

The notation here is as follows: The gauge field on the original lattice, with spacing a_f, is denoted by $\mathcal{U}$. The spacing of the kth blocked lattice is $a_k = 2^{k+1}a_f$, and the gauge field on that lattice is $\mathcal{V}^{(k)}$. The spacing of the final, coarse lattice is $a_c = 2^{n+1}a_f$. The Boltzmann weight for the collection of gauge fields, original and blocked, is $\boldsymbol{B}_n(n_r; \mathcal{U}, \{\mathcal{V}^{(k)}\})$. It is composed of three parts: the original gauge action, the gauge-field blocking kernels,[6] and a short-distance contribution to the effective gauge-field action, $n_r\delta S_{\mathrm{eff}}$, coming from integrating out the fermions on all lattices except the last one, where

$$e^{-\delta S_{\mathrm{eff}}} = \prod_{k=0}^{n} \mathrm{Det}(G_k^{-1}). \tag{3.2}$$

The operators $D_{\mathrm{taste},k}$ and G_k^{-1} are recursively given by

$$\begin{aligned} D_{\mathrm{taste},k}^{-1} &= \alpha_k^{-1} + Q^{(k)} D_{\mathrm{taste},k-1}^{-1} Q^{(k)\dagger}, \qquad k = 1, \ldots, n, \\ G_k^{-1} &= D_{\mathrm{taste},k-1} + \alpha_k Q^{(k)\dagger} Q^{(k)}, \qquad k = 1, \ldots, n. \end{aligned} \tag{3.3}$$

The blocking parameter α_k is of order $1/a_k$. The blocking kernel at the kth step, $Q^{(k)} = Q^{(k)}(\mathcal{V}^{(k-1)})$, gauge covariantly averages the fermion fields over 2^4 hypercubes on the $(k-1)$st lattice. For the $k = 0$ step, $D_{\mathrm{taste},0} = D_{\mathrm{taste}}$ is defined in Eq. (2.4), and $G_0^{-1} = D_{\mathrm{stag}} + \alpha_0 Q^{(0)\dagger} Q^{(0)}$, where $\alpha_0 = \alpha$ and $Q^{(0)} = Q$ are those introduced in Eq. (2.4). Recall that the special $k = 0$ blocking kernel is unitary; all other blocking kernels are not.

For small momenta, $Q^{(k)\dagger} Q^{(k)} \approx \mathbf{1}$, and with $\alpha_k \sim 1/a_k$ it thus follows that the eigenvalues of G_k^{-1} are at least of order $1/a_k$, making δS_{eff} a short-distance contribution to the effective gauge action.[7] While this can be proved in the free case [34], in the interacting case this is an assumption that is already necessary for the conventional RG picture to work in local, renormalizable theories. The nature of this assumption is discussed in detail in Ref. [15]; here we will assume it to be correct. It follows that δS_{eff} remains local when we take n_r to be any real number.[8]

The fermionic contribution to long-distance physics then resides entirely in the n_rth power of the determinant of $D_{\mathrm{taste},n}$ in Eq. (3.1). The problems with locality of the rooted theory originate with taking $n_r \to 1/4$ in this power. Our task will be to perform a faithful replica continuation at the level of the SET. As explained in the introduction, this is not straightforward. Calculations in the effective theories, the SET or ChPT, lead to explicit dependence on n_r (for instance, through loops). But there is also implicit dependence through the couplings that appear in the effective theory, which is in general nonperturbative, and not known.

Our strategy will be to first approximate the fourth-root theory by a local ("reweighted") theory. The fermions of this theory do not carry a taste degree of freedom; they are taste singlets. The multiplicity of taste-singlet fermions, n_s, will always be chosen to match the fermion spectrum of the target continuum theory. Therefore we will never have to perform any "replica continuation" in n_s; rather, n_s will always be kept a positive integer. In our construction, the unknown dependence of the couplings in the effective theory on the fermions will be due to the taste-singlet fermions only.

The fourth-root theory will be reached from the taste-singlet theory by "turning on" the taste-breaking effects that introduce the nonlocal behavior. This is where a replica continuation away from the integers will be needed. Because of the smallness of the taste-breaking effects, the replica continuation will be under control. Indeed, we will show that to any order in a_f, the dependence of the taste-breaking effects on n_r is polynomial, with a degree less than the order in the a_f expansion.

We start by splitting $D_{\mathrm{taste},n}$ into a taste-singlet part and a taste-breaking part with vanishing trace in taste space,

$$\begin{aligned} D_{\mathrm{taste},n} &= D_{\mathrm{inv},n} + \Delta_n, \\ D_{\mathrm{inv},n} &= \tilde{D}_{\mathrm{inv},n} \otimes \mathbf{1}, \\ \tilde{D}_{\mathrm{inv},n} &= \tfrac{1}{4}\mathrm{tr}_{ts}(D_{\mathrm{taste},n}), \end{aligned} \tag{3.4}$$

where tr_{ts} denotes the trace in taste space, and $\mathbf{1}$ is the taste identity matrix. Following Ref. [15] we assume that, in the coarse-lattice theory, Δ_n scales like

$$\| a_c \Delta_n \| \lesssim \frac{a_f}{a_c}. \tag{3.5}$$

This estimate is valid modulo logarithmic corrections to the leading power-law scaling. For extensive discussions of this scaling assumption, we refer to Ref. [15] (see also Refs. [10,11]). Here we only observe that, in any theory with integer n_r, this assumption is needed to establish that unrooted staggered fermions have the usually assumed continuum limit. However, by exploiting the proximity of the local reweighted theory after a large number n of blocking steps, it was argued that the scaling (3.5) is also valid in theories with fractional n_r. In this paper, we will assume this to be the case.

Using this split, we generalize the determinant in Eq. (3.1) to

$$\mathrm{Det}^{n_r}(D_{\mathrm{taste},n}) \to \mathrm{Det}^{n_s}(\tilde{D}_{\mathrm{inv},n}) \frac{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n} + t\Delta_n)}{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n})}, \tag{3.6a}$$

[6] We do not integrate over any of the gauge fields; this can be postponed to the end. The explicit expression for $\boldsymbol{B}_n(n_r; \mathcal{U}, \{\mathcal{V}^{(k)}\})$ is given in Ref. [15].

[7] Much smaller eigenvalues are allowed, as long as the corresponding eigenmodes are localized on a distance of at most order a_k. Such modes would not affect the long-distance physics.

[8] We keep n_r in the range where the gauge coupling is asymptotically free.

while also replacing

$$\boldsymbol{B}_n(n_r; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \to \boldsymbol{B}_n(n_s/4; \mathcal{U}, \{\mathcal{V}^{(k)}\}). \qquad (3.6b)$$

The generalized theory reduces to Eq. (3.1) if we set $n_s = 4n_r$ and $t = 1$. This generalization has two important properties. First, if $n_s = 4n_r$ and n_r assumes physically interesting values, i.e., multiples of $1/4$, then n_s is an integer. Second, when n is large enough, $D^{-1}_{\mathrm{inv},n}\Delta_n$ is small enough (in an ensemble-average sense) that we may expand

$$\frac{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n} + t\Delta_n)}{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n})} = \exp[n_r \mathrm{Tr} \log(1 + tD^{-1}_{\mathrm{inv},n}\Delta_n)] = \exp\left[-n_r \mathrm{Tr}\left(\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} \times t^\ell (D^{-1}_{\mathrm{inv},n}\Delta_n)^\ell\right)\right]. \qquad (3.7)$$

The parameter t interpolates between the taste-invariant operator $D_{\mathrm{inv},n}$ at $t = 0$ and the (blocked) staggered operator at $t = 1$. In addition, t is a book-keeping device. The power of t is, evidently, the same as the power of $D^{-1}_{\mathrm{inv},n}\Delta_n$. As we explain in detail in Sec. III B below, for the construction of the effective theories we may use the bound

$$\| D^{-1}_{\mathrm{inv},n}\Delta_n \| \lesssim \frac{a_f}{a_c} = \frac{1}{2^{n+1}} \equiv \epsilon_n. \qquad (3.8)$$

(The $\sim$ sign has a meaning similar to that in Eq. (3.5).) We conclude that the t expansion is an expansion in powers of a_f for the taste-breaking effects.

For $t = 0$, the determinant ratio (3.7) collapses to one. The taste-invariant theory at $t = 0$ is thus local for any integer n_s, and independent of n_r. The staggered theory is reached by expanding as in Eq. (3.7), eventually setting $t = 1$. The rooted staggered theory is obtained by setting n_r to a quarter-integer value. When we construct the SET to any finite order in a_f, the maximal power of t will be limited by that order.[9] By Eq. (3.7), the maximal power of n_r is bounded by the power of t. (Because of taste tracelessness of Δ_n, the maximal power of n_r is in fact strictly less than the power of t.) The maximal power of n_r is thus (strictly) less than the order in a_f. Therefore, at fixed n_s and to any finite order in a_f, the dependence of any correlation function on n_r, and thus of the SET that reproduces it, will be polynomial. This implies that, at the level of the SET, the replica continuation in n_r to quarter-integer values will be well-defined, resulting in the "staggered SET with the replica rule." What this means is the following: We start with integer n_r. The effective action is then given in terms of a set of Symanzik coefficients which are unknown functions of n_s, but depend polynomially on n_r (we may already set $t = 1$). With this action, one calculates correlation functions which again depend polynomially on n_r (to any finite order in a_f), with n_r dependence coming from the Symanzik coefficients and from loops. Finally, one sets $n_r = n_s/4$, and the resulting correlation function is precisely that of the rooted staggered theory. The following subsections contain a more detailed argument on how this works.

We comment in passing that, for $t = 1$, we may also interpolate between the taste-singlet local theory at $n_r = 0$, and the (rooted) staggered theory at $n_r = n_s/4$ by varying n_r instead of t. While the two ways of moving from the taste-singlet to the staggered theory are mathematically equivalent, we find the argument more transparent if the transition is done by varying t.

A. Generalized theory

In order to define the SET we first need a complete definition of the generalized staggered theory, coupled to sources in order to generate all correlation functions. Returning to integer n_r, the theory defined by Eq. (3.6) contains n_s taste-singlet fermions with Dirac operator $\tilde{D}_{\mathrm{inv},n}$, n_r generalized staggered fermions with Dirac operator $D_{\mathrm{inv},n} + t\Delta_n$, and $4n_r$ ghosts with Dirac operator $\tilde{D}_{\mathrm{inv},n}$. Introducing sources $H = (\hat{\eta}, \eta, \tilde{\eta})$ and $\bar{H} = (\bar{\hat{\eta}}, \bar{\eta}, \bar{\tilde{\eta}})$ for the taste-singlet, generalized staggered, and ghost fields, respectively, we define the partition function of the generalized theory as

$$\begin{aligned} Z_n(t, n_r, n_s; H, \bar{H}) &= \int \mathcal{D}\mathcal{U} \prod_{k=0}^{n} \mathcal{D}\mathcal{V}^{(k)} \boldsymbol{B}_n\left(\frac{n_s}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \times \mathrm{Det}^{n_s}(\tilde{D}_{\mathrm{inv},n}) \frac{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n} + t\Delta_n)}{\mathrm{Det}^{n_r}(D_{\mathrm{inv},n})} \exp[\bar{\hat{\eta}}(\tilde{D}^{-1}_{\mathrm{inv},n} \times \boldsymbol{I}_{n_s})\hat{\eta}] \\ &\quad \times \exp[\bar{\eta}((D_{\mathrm{inv},n} + t\Delta_n)^{-1} \otimes \boldsymbol{I}_{n_r})\eta + \bar{\tilde{\eta}}(D^{-1}_{\mathrm{inv},n} \otimes \boldsymbol{I}_{n_r})\tilde{\eta}] \\ &= \int \mathcal{D}\mathcal{U} \prod_{k=0}^{n} \mathcal{D}\mathcal{V}^{(k)} \boldsymbol{B}_n\left(\frac{n_s}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \times \mathrm{Det}^{n_s}(\tilde{D}_{\mathrm{inv},n}) \exp\left[-n_r \mathrm{Tr}\left(\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} t^\ell (D^{-1}_{\mathrm{inv},n}\Delta_n)^\ell\right)\right] \\ &\quad \times \exp[\bar{\hat{\eta}}(\tilde{D}^{-1}_{\mathrm{inv},n} \times \boldsymbol{I}_{n_s})\hat{\eta} + \bar{\eta}(\tilde{D}^{-1}_{\mathrm{inv},n} \otimes \boldsymbol{I}_{4n_r})\eta + \bar{\tilde{\eta}}(\tilde{D}^{-1}_{\mathrm{inv},n} \otimes \boldsymbol{I}_{4n_r})\tilde{\eta}] \\ &\quad \times \exp\left[\bar{\eta}\left(\sum_{\ell=1}^{\infty} (-1)^\ell t^\ell (D^{-1}_{\mathrm{inv},n}\Delta_n)^\ell D^{-1}_{\mathrm{inv},n} \otimes \boldsymbol{I}_{n_r}\right)\eta\right]. \end{aligned} \qquad (3.9)$$

[9] Note that a_f dependence which does not involve taste-symmetry breaking may result from other sources besides the determinant ratio (3.7).

Here $\boldsymbol{I}$ stands for the identity matrix, with dimensions as indicated by the subscript. This is a theory with two lattice parameters, a_c and a_f. Alternatively, we may trade a_f for the small parameter ϵ_n of Eq. (3.8). In the second expression we give the explicit expansions in the book-keeping parameter t. As explained above, for fixed n_s correlation functions expanded to some finite power in a_f are polynomial in n_r. For $t = 1$, $n_r = n_s/4$, and $\hat{\eta} = \bar{\hat{\eta}} = \tilde{\eta} = \bar{\tilde{\eta}} = 0$, Eq. (3.9) is precisely the theory of n_s degenerate, fourth-rooted staggered fermions.

The generalized theory has a vectorlike $U(n_s|4n_r) \times U(n_r)$ graded symmetry; $U(n_s|4n_r)$ acts on the taste-singlet and ghost fields, and $U(n_r)$ on the generalized staggered field. For $t = 0$ the symmetry enlarges to $U(n_s + 4n_r|4n_r)$. The discrete symmetries include hypercubic rotations and axis reversal [4]. In the staggered sector, for $t = 1$ this is augmented by shift symmetry, and (softly broken) $U(1)_\epsilon$ symmetry in GWL form for each flavor. The vector and axial staggered symmetries expand to a $U(n_r)_\ell \times U(n_r)_r$ chiral symmetry group [23]. There is no chiral symmetry in the taste-singlet and ghost sectors, because the GWL version of $U(1)_\epsilon$ symmetry mixes the taste-invariant and non-invariant parts of the blocked staggered Dirac operator [12].[10]

We are now ready to discuss the SET for the generalized theory. As long as n_r is a positive integer, the lattice theory is partially quenched but local, and we will assume that an SET for this theory exists in Euclidean space.[11] The effective theory can be written in terms of continuum fields $\Psi = (\hat{q}, q, \tilde{q})$ and $\bar{\Psi} = (\bar{\hat{q}}, \bar{q}, \bar{\tilde{q}})$ for the taste-singlet, generalized staggered, and ghost fields, respectively, as well as a continuum gluon field A_μ. As explained above, its parameters (the couplings multiplying each operator in the Symanzik expansion) are polynomials in n_r if we work to a finite order in a_f; while their dependence on n_s is unknown. Only the n_s dependence survives in the continuum limit, where the determinant ratio (3.7) collapses to one.[12]

For general t, n_s, and n_r, the fundamental cutoff is the lattice spacing of the generalized theory, a_c. The SET is the effective theory for quarks and gluons with momenta much smaller than $1/a_c$. However, the lattice theory contains an additional small parameter, $\epsilon_n = a_f/a_c$, cf. Eq. (3.8). It will be useful for our purposes to think of the Symanzik expansion as an expansion in $a_f = \epsilon_n a_c$, with Symanzik coefficients that depend on a_c.[13] The effective theory can be divided into three different sectors, corresponding to three different types of operators that can occur. The (generalized) staggered sector consists of operators made out of staggered fields q and $\bar{q}$ only. Likewise, the taste-singlet–ghost sector consists of operators made out of the "auxiliary" fields $\hat{\Psi} = (\hat{q}, \tilde{q})$ and $\bar{\hat{\Psi}} = (\bar{\hat{q}}, \bar{\tilde{q}})$ only. Finally there is the mixed sector, where each operator is made out of both staggered and auxiliary fields. (Of course, all operators may contain gluon fields.)

In order to establish the validity of rSChPT in Sec. V, we will not need to know the explicit form of the SET in full generality. In fact, we need only consider the staggered sector of the SET. Disregarding the auxiliary and mixed sectors, the resulting SET, defined in terms of the quark fields q and $\bar{q}$ and the gluon fields, is invariant under all symmetries of the generalized staggered operator $D_{\mathrm{inv},n} + t\Delta_n$. For $t = 0$ this includes taste-replica symmetry $U(4n_r)$, while for $t = 1$ this includes the smaller group Γ_4, as well as softly broken $U(1)_\epsilon$ symmetry.

For the remainder of this subsection we set $t = 1$, and thus $D_{\mathrm{inv},n} + \Delta_n = D_{\mathrm{taste},n}$ reduces to the RG-blocked operator of Eq. (3.1). Symmetries that act on the space-time coordinates often take a complicated form under RG blocking. In particular, shift symmetry is realized in a complicated way. First, the RG blocking leading to Eqs. (3.1) and (3.9) was started in the RG taste basis defined in Eq. (2.4), and shift symmetry is thus realized as a gauge-covariant form of Eq. (2.25). Second, the transition to the RG taste basis was followed by n additional RG-blocking steps.

The physical consequences of any exact lattice symmetry of the underlying staggered theory, nevertheless, cannot be lost by RG blocking. The reason is the existence of a pull-back mapping of every coarse-lattice operator to a fine-lattice operator [15]. For $n_r = n_s/4$, where the taste-singlet and ghost determinants drop out, this mapping gives rise to exact equality of corresponding observables. In other words, the coarse-lattice observables are a subset of the original fine-lattice staggered observables.

The pull-back mapping extends to $n_r \neq n_s/4$. Consider the expectation value of a product of coarse-lattice staggered fermion (and gauge) fields. By undoing the RG-blocking Gaussian transformations of the fermions, this can be rewritten as an expectation value of a corresponding product of fine-lattice staggered fields (that depends in addition on the original and blocked gauge fields). Because the Boltzmann weight of the generalized theory contains the taste-singlet and ghost determinants, expectation values will not be the same as in the original staggered theory. But since the fine-lattice symmetries are unchanged, pulled-back coarse-lattice observables will still transform under all the staggered symmetries. Together with other observables constructed from the fine-lattice staggered fields, they must fall into representations of all these symmetries. This implies that the physical conse-

[10] We remind the reader that $D_{\mathrm{inv},n}$ and Δ_n in Eq. (3.9) are defined in the RG taste basis, cf. Eq. (2.4), and not in the standard taste basis of Refs. [5,33].

[11] It is sufficient to consider the SET in Euclidean space, since we will postpone the continuation to Minkowski space until after the continuum limit has been taken [11].

[12] We observe that at nonzero a_c but $a_f \to 0$, i.e., in the limit $n \to \infty$, the lattice action is a perfect action.

[13] In the following subsection, we will argue that no negative powers of a_f can appear.

quences of the full set of staggered symmetries remain intact.

The $t = 1$ staggered-sector SET must therefore be invariant under all the symmetries listed in Sec. II. If we derive the SET using the taste basis some of these symmetries will take a complicated form. In particular, shift symmetry will mix different orders in $a = a_f$. But other continuum fields can always be chosen by suitable field redefinitions such that shift symmetry resumes the simple form of Eq. (2.2) at the level of the SET. Moreover, a SET-level field redefinition will also eliminate any a_c dependence of the SET that originates from the matching to the coarse-lattice interpolating fields.[14] The only remaining dependence of the staggered-sector SET on a_c originates at this stage from the presence of the taste-singlet and ghost determinants in the underlying theory (3.9).

Recall now that the group generated by the four elementary shifts S_μ contains translations by $2a_f$. At the level of the SET shift symmetry enlarges to the direct product of the group Γ_4 and the continuous translation group. In the continuum limit $a_f \to 0$ the discrete group Γ_4 enlarges to the full taste/replica symmetry group $SU(4n_r)$ (with Γ_4 embedded such that it acts identically on all n_r replicas).

The conclusion of the above arguments is that, for $t = 1$ and for any positive integer values of n_s and n_r, the generalized staggered sector of the SET assumes exactly the same structure, as an expansion in the fine-lattice spacing a_f, as the standard staggered SET for n_r staggered fields. To order a_f^2, this SET is derived in Ref. [22] (for $n_r = 1$) and Ref. [23] (for arbitrary n_r), and is written down explicitly in Ref. [24]. However, the Symanzik coefficients of the staggered-sector SET of the generalized theory are not the same functions of the parameters of the underlying theory as in the ordinary staggered SET. In the generalized theory, the Symanzik coefficients depend on n_s and a_c, parameters not present in the ordinary staggered theory. Dependence on n_s arises because of contributions from taste-singlet loops. In addition, the n_r dependence (at fixed n_s) of the Symanzik coefficients is different from that of the ordinary staggered SET, because of contributions from ghost loops. Indeed, the reason why the auxiliary sector was introduced in the first place, is that—unlike the original staggered theory—the SET of the generalized theory depends polynomially on n_r to any order in a_f, as long as n_s is held fixed.

We are now ready to make contact with the rooted theory. In order to reach the SET of the rooted theory we hold n_s fixed and choose $t = 1$. For any t, we may perform the replica continuation $n_r \to n_s/4$ in any correlation function at any given order in the loop expansion.[15] Indeed, because the Symanzik coefficients are polynomials in n_r to any desired order in a_f, this continuation from integer values of n_r is well-defined. Now, recall that the taste-singlet and ghost sectors of the generalized theory (3.9) cancel (for vanishing sources) when we set $n_r = n_s/4$. As explained above, this finally eliminates all the remaining dependence of the staggered-sector SET on the coarse spacing a_c, leaving only the dependence on the fine spacing a_f. We have thus succeeded in constructing the replica-continued SET for the original blocked theory, Eq. (3.1), for any quarter-integer value of n_r, and to the desired order in a_f.

Putting everything together, we have shown that the familiar staggered SET for integer n_r, derived to order a_f^2 in Refs. [22,23], and written down explicitly and extended to order a_f^4 in Ref. [24], can be used to compute any correlation function of interest to the desired order in a_f. The result should then be replica-continued to quarter-integer values of n_r. This continuation provides the correct prescription for calculating any correlation function in the rooted theory from the staggered SET. Of course, in practice we will not know the precise coefficients of powers of n_r in the Symanzik coefficients; indeed in practical situations the Symanzik coefficients must be treated as unknown numbers, to be fitted from numerical data. However, it suffices for our argument to know that the dependence is polynomial. When we continue in n_r, we then need only continue the explicit n_r dependence coming from loops, giving a result as usual in terms of unknown Symanzik coefficients.

B. Power counting

A cornerstone in the argument of the previous section is the expansion in Eq. (3.9), which is convergent if the norm of $D_{\mathrm{inv},n}^{-1}\Delta_n$ is small enough. In this subsection, we consider this condition in more detail. There are two issues to be considered: the effect of insertions of Δ_n, as well as the size of the full object in which we expand, $D_{\mathrm{inv},n}^{-1}\Delta_n$.

In general, the SET for a lattice theory with lattice spacing a is constructed by matching correlation functions in an expansion in ap, with $p \ll 1/a$ a generic momentum, to the underlying lattice theory. To make the matching possible in perturbation theory, one should also take $p \gg \Lambda_{\mathrm{QCD}}$. The Symanzik coefficients are extracted by computing suitable one-particle irreducible correlation functions in the lattice theory, taking all the (nonexceptional) external momenta to be of order p [27]. For the part coming from the fermions, this amounts to expanding D_{latt}^{-1} around D_{cont}^{-1}, namely, to an expansion in $D_{\mathrm{cont}}^{-1}(D_{\mathrm{latt}} - D_{\mathrm{cont}})$,

[14] Via the pull-back, the coarse-lattice operators may be regarded as a particular set of interpolating fields on the fine lattice as well. The freedom in making field redefinitions at the level of the SET thus parallels the freedom, discussed in Appendix B of Ref. [10], to choose different sets of interpolating fields on the fine lattice.

[15] For further discussion of the replica continuation, see Sec. III C.

where $D_{\rm cont}$ is the Dirac operator for the continuum-limit theory, and $D_{\rm latt}$ is the Dirac operator of the lattice theory. Because $D_{\rm latt} - D_{\rm cont}$ is an irrelevant operator, we expect $\| D_{\rm latt} - D_{\rm cont} \| \lesssim ap^2$. Also, on dimensional grounds, $\| D_{\rm cont}^{-1} \| \sim 1/p$. Putting it together we conclude that $\| D_{\rm cont}^{-1}(D_{\rm latt} - D_{\rm cont}) \| \sim ap$ is the relevant estimate for the construction of the SET. Observe that this argument is insensitive to the long-distance physics, because the effective infrared cutoff on the loop momenta is p, and by assumption $p \gg \Lambda_{\rm QCD}$. In particular, the estimates are independent of the quark masses.

In the above argument we have implicitly assumed that the momentum flowing through a particular (sub-)diagram is of order p. This need not be true for subdiagrams with a non-negative degree of divergence, where all ultraviolet momenta may contribute significantly to the loop integrals. In general, counterterms will need to be added in order to absorb contributions from such diagrams; in a renormalizable theory there are only a finite number of counterterms that need to be adjusted. Symmetries may exclude (some of) these counterterms.

Let us now study how these general considerations enter the construction of the SET for the generalized theory (3.9). Our starting point will be the $t = 0$ taste-singlet theory. This theory is local, because n_s is integer. In order to reach the generalized staggered theory from the taste-singlet theory, we have to expand the propagator $(D_{{\rm inv},n} + t\Delta_n)^{-1}$ around $D_{{\rm inv},n}^{-1}$, and eventually set $t = 1$. The object in which we are expanding is thus $D_{{\rm inv},n}^{-1}\Delta_n$. Since Δ_n is an irrelevant operator [cf. Eq. (3.5)], repeating the above general arguments leads to the estimate $\| D_{{\rm inv},n}^{-1}\Delta_n \| \sim a_f p$, if the momentum flowing through the diagram is of order p.

As noted above, we must separately consider subdiagrams with a non-negative degree of divergence. The contributions of such subdiagrams depend crucially on the number of blocking steps n, as we now explain.

Consider first what happens for $k = n = 0$, namely, when we have performed only the first special RG step that takes the fermions from the one-component to the taste basis. We then have $a_c = 2a_f$. When we extract the Symanzik coefficients from a lattice calculation, the loop momenta live on the coarse lattice. But since the coarse and fine-lattice spacings differ only by a factor of 2, the loop momentum can go as high as $p \sim 1/a_f$. In the divergent subdiagrams we thus have $\| D_{{\rm inv},n}^{-1}\Delta_n \| \sim 1$. Indeed, for $a_c = 2a_f$, the generalized staggered theory will develop $\mathcal{O}(1/a_c) = \mathcal{O}(1/a_f)$ mass terms, since shift symmetry and $U(1)_\epsilon$ symmetry (for any $t \neq 1$) are broken at the (common) lattice scale.[16]

The situation is qualitatively different after a large number n of RG steps has been performed. Because the lattice calculation is performed on the coarse lattice,[17] the maximal momentum that can flow through any subdiagram is now of order $1/a_c$, and one arrives at the estimate (3.5) for the magnitude of insertions of Δ_n. The estimate $\| D_{{\rm inv},n}^{-1}\Delta_n \| \sim a_f p$ still holds, but, what has changed is that now the maximal value that p can reach is $1/a_c \ll 1/a_f$. The conclusion is that, for extracting the Symanzik coefficients, the appropriate estimate is just that of Eq. (3.8),

$$\| D_{{\rm inv},n}^{-1}\Delta_n \| \lesssim a_f/a_c. \tag{3.10}$$

This estimate is valid in the taste-singlet, $t = 0$ theory, on the same grounds as for any other local theory, and we will thus assume that it is valid nonperturbatively as well. This is all we need, because the staggered theory is constructed as an expansion in $D_{{\rm inv},n}^{-1}\Delta_n$ around the taste-singlet theory.

We end this subsections with three comments. First, it should be noted that, in Ref. [15], the bound

$$\| D_{{\rm inv},n}^{-1}\Delta_n \| \lesssim a_f/(ma_c^2) \tag{3.11}$$

was used, with m the renormalized quark mass after n RG steps. Clearly, the bound (3.11) is far weaker than (3.10), and it implies that the chiral ($m \to 0$) limit can be taken only after the continuum ($a_f \to 0$) limit. In Ref. [15], this was necessary in order to place a uniform bound on the difference between any taste-singlet correlation function and the corresponding rooted correlation function on any (including the most infrared) scale, thereby establishing the existence of the (correct) continuum limit for the rooted theory. In contrast, assuming that the scaling (3.5) holds, the bound (3.11) is much too generous for the derivation of the SET for the generalized theory (3.9), as we have seen above. In particular, it follows that this SET is well-defined in the chiral limit, as is the chiral effective theory that can be derived from the SET. The requirement that the chiral limit for staggered fermions be taken after the continuum limit [8,43–46] is then reproduced by calculations within staggered ChPT [45]. Note that, while Ref. [45] finds many standard quantities for which the limits commute in SChPT, other quantities for which the limits do not commute are also discussed.

Our second comment is that the original staggered theory has no power divergences, because of shift and $U(1)_\epsilon$ symmetry. This is therefore also true for the n-times blocked staggered theory (3.1), and for the corresponding SET. Moreover, for large n, the SET for the generalized theory (3.9) at arbitrary values for $t \in [0, 1)$ is related to the SET at $t = 1$ by a convergent expansion in t, equivalently in $\epsilon_n = a_f/a_c$. The implication is that, for all t, the SET for the generalized theory (3.9) has no power diver-

[16]The breaking of shift symmetry is qualitatively the same as in the theory studied in Ref. [42].

[17]See Ref. [15] for a detailed discussion on how the coarse-lattice diagrammatic calculation is related to a calculation in the underlying fine-lattice staggered theory.

gences in $1/a_f$, but only in $1/a_c$. Examples of this are given in Sec. IV below.

Finally, we remark that the framework introduced here resolves a concern, discussed in Ref. [10], about the renormalizability of the rooted staggered theory. The concern is the following: the complete notion of renormalizability requires not only that (infinite) counterterms can be chosen to make amplitudes finite, but also that the finite parts of counterterms can be chosen to bring the theory into a given scheme. While we know that the staggered theory is renormalizable for integer n_r, for non-integer n_r this notion of renormalizability requires that the finite parts of counterterms, as well as the infinite parts, are polynomial in n_r to any finite order in perturbation theory. In Ref. [10], the condition on the finite parts was introduced as an additional assumption, *albeit* a plausible one. Here, such a separate assumption is unnecessary. Under the assumptions of the RG approach [15], the taste-singlet (reweighted) theory, defined by setting $t = 0$ in Eq. (3.9), is a local theory of n_s fermions, that moreover becomes a perfect-action lattice theory in the limit $a_f \to 0$, for any fixed a_c. Thus one expects its renormalizability to follow straightforwardly by standard arguments. The rooted staggered theory is then reached by expanding in t, and setting $t = 1$ and $n_r = n_s/4$. Because of the bound (3.10), the expansion in t just brings in positive powers of a_f, and all finite (and infinite) parts of the counterterms are unaffected for any n_r. Thus the rooted staggered theory is renormalizable if the taste-singlet theory is. In addition, the two theories have the same counterterms.

C. Partial quenching

Unlike other lattice discretizations of QCD, the continuum limit of the rooted staggered theory is, inherently, a partially-quenched theory [10,14,16,46]. This remains true when we consider the staggered sector of our generalized lattice theory (3.9) all by itself. Let us work out the example of a target theory with n_s degenerate quarks. Our starting point is the generalized lattice theory with the same n_s, and with $t = 1$. In order to obtain the set of all correlation functions of the physical n_s-flavor theory in the continuum limit, we need to let the combination of replica and taste indices of the external lines assume precisely n_s distinct values. This can, for example, be accomplished by fixing the taste index of the external legs to a single value (for example, 1), and letting the replica indices take on n_s values (for example $1, 2, \ldots, n_s$). Alternatively, we could use all four taste indices and only $[n_s/4]$ replica indices, where the square brackets denote rounding up to the next integer. (In this case, unless $n_s/4$ is already an integer, not all taste indices will be used in conjunction with each replica index.) Many other similar choices, as well as other types of embeddings for certain classes of physical correlation functions [8,14], are also possible. Prior to the replica continuation, the lattice theory is local. The source term in Eq. (3.9) must accommodate all the degrees of freedom, as specified above, that will be used in physical correlation functions. Therefore, we must consider only theories where n_r, the (still integer) number of staggered replicas, is not smaller than $[n_s/4]$.

When we perform the replica continuation we set the power of the staggered and ghost determinants in Eq. (3.9) to $n_r = n_s/4$. Since we have already set $t = 1$, if we turn off all sources, the partition function of the generalized theory reduces to the rooted partition function, in its RG-blocked dress (3.1). During the replica continuation of any correlation function, by definition we hold fixed all indices of the external legs, including in particular the replica (and taste) indices. This means that the number of replicas in the source term of Eq. (3.9) must stay equal to or larger than $[n_s/4]$. The mismatch created between the power of the staggered (or ghost) determinant and the multiplicity of the corresponding external sources means that the staggered sector has in itself been partially-quenched unless n_s is a multiple of 4.

After the replica continuation, the correlation functions of the EFT reproduce those of the rooted lattice theory to the same order in a_f. We stress again that the replica continuation at the level of the EFT is well-defined because, as we have shown, to any order in a_f the n_r-dependence in the underlying lattice theory (3.9) assumes the form of a finite-degree polynomial.

In our above example, be it before or after the replica continuation, the $replica \times taste$ multiplicity of the staggered fields used to generated physical correlation functions is equal to or larger than $4[n_s/4]$, which is to be compared with the n_s physical flavors of the target theory. As a result, the total number of available valence degrees of freedom will in general exceed the physical number, and, when we finally take the continuum limit, the physical correlation functions will form a proper subset of the set of all (partially-quenched) correlation functions.[18] This conclusion is in fact valid for any target theory. The only exception is a target theory in which the multiplicity of every mass-degenerate quark species is divisible by four, in which case the theory may be obtained in the continuum limit of an unrooted staggered theory.

Another conclusion is that the partially-quenched representation obtained in the continuum limit is not unique. The only restriction is that the set of all partially-quenched correlation functions must be large enough to accommodate all the physical correlation functions of the target continuum theory. With the minimal choice of replicas on the external lines, $[n_s/4]$, the vector $replica \times taste$ symmetries are represented as a $U(4[n_s/4]|4[n_s/4] - n_s)$ graded group on the continuum-limit correlation functions. Had we initially allowed for $n' > [n_s/4]$ values of the

[18]Correlation functions lying outside of the physical subset may exhibit various types of pathological behavior [16,46].

replica index on the external legs, all the physical correlation functions of the target theory would still be reproduced once we performed the replica continuation (followed by the continuum limit). But there would be more ways of embedding a given physical correlation function in the space of all correlation functions. Correspondingly, the *replica* $\times$ *taste* symmetries would be represented as an $U(4n'|4n'-n_s)$ graded group. The arbitrariness in picking a range $n' \geq [n_s/4]$ for the external-legs replica index thus entails the existence of infinitely many partially-quenched representations in the continuum limit, all of which share the same physical subspace.

In the rooted theory, closed (‘‘sea-quark’’) fermion loops as well as (‘‘valence-quark’’) fermion lines attached to external legs both originate from the same staggered fields. Therefore the sea and valence masses are equal, and there is no clear-cut distinction between the sea and valence sectors. This is a necessary condition for the emergence of a unitary, physical subspace in the continuum limit.

In practice, it is often useful to explore unitarity-violating correlation functions in which the valence-quark mass is allowed to vary away from the sea-quark mass. This situation is what is usually referred to as partial quenching. As we have just explained, the continuum limit of the rooted theory is automatically a partially-quenched theory, *albeit* with equal sea and valence masses. If it is desired to study different sea and valence masses, it is straightforward to add a (generalized-)staggered valence sector to the generating functional (3.9), by simply inserting a factor

$$\exp[\bar{\eta}_v(D^v_{\mathrm{inv},n} + t_v\Delta_n)^{-1} \otimes \boldsymbol{I}_{n_v})\eta_v] \tag{3.12}$$

into the integrand. The superscript v on $D^v_{\mathrm{inv},n}$ indicates that a different quark mass may have been chosen in the valence sector. For $t_v = 1$ the valence sector has all staggered symmetries. Again, for a_f small enough, an expansion can be set up in t_v, just as before.

In Eq. (3.12), η_v and $\bar{\eta}_v$ are sources for any desired number n_v of valence (generalized) staggered fields. To avoid confusion we stress that, even if the valence-sector source term (3.12) has been added to the generating functional (3.9), we cannot dispose of the original source terms. The reason is that, if we want to consider the SET for both sea and valence quarks, we need sources for both in order to match the complete set of partially-quenched correlation functions between the lattice and the effective theory. With the valence sector (3.12) in place, the *replica* $\times$ *taste* symmetries form an $U(4n'+4n_v|4n'+4n_v-n_s)$ graded group in the continuum limit. (Of course, these symmetries will be softly broken by unequal sea and valence masses.) As before, n' is the number of distinct values of the replica index that we have allowed for the staggered fields with sea-quark mass on the external legs.

In summary, we have seen that partial quenching occurs at *three* distinct levels. The generalized theory (3.9) is partially quenched to begin with, because, to keep the taste-breaking effects under control, we had to introduce a taste-singlet sector and a taste-invariant ghost sector. During the replica continuation, the staggered sector undergoes a second-stage partial quenching, created by the mismatch between the power of the determinant and the multiplicity of the sources. Last, if we are interested in different valence and sea masses, we need to introduce a ‘‘conventional’’ valence sector, cf. Eq. (3.12).

IV. EXAMPLES

It is instructive to consider some aspects of the SET to second order in a_f in more detail.[19] The SET can be written as an expansion in a_f, t, and n_r, and thus takes the general form

$$S(\Psi, \bar{\Psi}, A; a_f, t, n_r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j-1}(a_f)^i t^j (n_r)^k S_{i,j,k}(\Psi, \bar{\Psi}, A). \tag{4.1}$$

Here we already took into account that each power of t has to come with at least one power of a_f, and that each power of n_r has to be lower than the power of t (it cannot be equal because $\mathrm{tr}_{ts}(\Delta_n) = 0$). Equation (4.1) is manifestly polynomial in n_r to any fixed, finite order in a_f. Here we allow all types of quarks (taste-singlet, generalized staggered, and ghost) to appear on the external legs. The staggered sector is obtained by setting $\hat{q} = \bar{\hat{q}} = \tilde{q} = \bar{\tilde{q}} = 0$. The coefficients in $S_{i,j,k}$ depend on both a_c and n_s in all sectors. Because of this, one cannot in general conclude that terms linear in a_f have to be multiplied by dimension-five operators, etc. As already explained in Sec. III A, for $t = 1$ we may assume that a correlation function calculated in the SET does not depend on a_c if we set $n_r = n_s/4$ after the calculation. In this section, we will consider n_r integer.

Because of the way the RG-blocked theory is constructed, for general t the preferred basis for (the generalized staggered sector of) the SET is the RG-taste basis. Using this basis while restricting ourselves to the (generalized) staggered sector, and to $i \leq 2$, the expansion (4.1) takes the explicit form

$$\begin{aligned} S^{\mathrm{quad}}(q, \bar{q}, A; a_f, t, n_r) = {} & S_{0,0,0}(q, \bar{q}, A) + a_f[S_{1,0,0}(q, \bar{q}, A) \\ & + tS_{1,1,0}(q, \bar{q}, A)] \\ & + a_f^2[S_{2,0,0}(q, \bar{q}, A) \\ & + tS_{2,1,0}(q, \bar{q}, A) \\ & + t^2 S_{2,2,0}(q, \bar{q}, A) \\ & + n_r t^2 S_{2,2,1}(q, \bar{q}, A)]. \end{aligned} \tag{4.2}$$

[19] In this section we return to the theory defined by Eq. (3.9). The inclusion of valence quarks with a mass unequal to that of the sea quarks, as described in Sec. III C, is straightforward.

The n_r-dependent term (the last term) is at this order the only one coming from the expansion of the determinant ratio in Eq. (3.9). The other t-dependent terms come from the expansion of the staggered source term in that equation. We note that $S_{i,0,0}$ is taste invariant, because of taste invariance of the $t = 0$ theory. Furthermore, $S_{2,2,1}$ is taste invariant too, because the factor of $n_r t^2$ originates from the determinant ratio in Eq. (3.9), which does not affect the symmetry structure of the SET. The taste structure of the SET is determined by the external legs, which correspond to the source terms in Eq. (3.9). Since the two allowed insertions of Δ_n have been "used up" by the determinant ratio, only the taste-invariant part of the source term contributes to $S_{2,2,1}$.

If we set $t = 1$ then, as discussed in Sec. III A, there exists a field redefinition that brings $S^{\rm quad}$ to the familiar form of Ref. [22] for $n_r = 1$, or to the form of Refs. [23,24] for $n_r > 1$. In particular, the redefinition removes the terms linear in a_f. The Symanzik coefficients are equal to those of Refs. [22–24] if one also chooses $n_s = 4n_r$, a multiple of four. For general n_s and n_r, the staggered SET is that of Refs. [23,24], but the coefficients are different functions of n_r.[20] This form of $S^{\rm quad}$ is the one needed for the construction of rSChPT [23], which we will discuss in Sec. V.

The taste-invariant operator $D_{{\rm inv},n}$ has no chiral symmetry, even when the chiral limit is taken in the underlying staggered theory, and we would thus naively expect a linearly divergent mass term of the form $\bar{q}q/a_c$. However, for large n, the taste-invariant theory is close to the theory with $t = 1$ in the sense explained in Sec. III B. In order to deviate from the $t = 1$ staggered theory, at least one power of a_f, coming from an insertion of Δ_n, is needed. Equivalently, the $1/a_c$ linear divergence has to be multiplied by at least one factor of $\epsilon_n = a_f/a_c$. In fact, even a mass term with magnitude $\sim\epsilon_n/a_c = a_f/a_c^2$ cannot occur. To see this, note that we may write

$$D^{-1}_{{\rm taste},n} = D^{-1}_{{\rm inv},n} - D^{-1}_{{\rm inv},n}\Delta_n D^{-1}_{{\rm inv},n} + \ldots. \qquad (4.3)$$

This shows that the order a_f difference between the $t = 0$ and $t = 1$ theories has to break taste, and therefore a taste-singlet difference has to be of order a_f^2. Singlet mass terms can thus only occur in $S_{2,0,0}$ and $S_{2,2,0}$, with opposite coefficients such that they cancel at $t = 1$.

Next, let us consider nonsinglet mass terms, i.e., terms of the form $\bar{q}Kq/a_c$ with some (momentum-independent) kernel K for which ${\rm tr}_{ts}(K) = 0$. At order a_f a nonsinglet mass term can only be part of $S_{1,1,0}$, because $S_{1,0,0}$ is taste invariant. However, staggered symmetries at $t = 1$ forbid such terms in $S_{1,1,0}$, thus excluding this possibility. At order a_f^2, a nonsinglet mass term can only appear in $tS_{2,1,0} + t^2S_{2,2,0}$ because $S_{2,0,0}$ and $S_{2,2,1}$ are taste invariant. Let us assume that a bilinear $\bar{q}Kq$ appears with coefficient c_1 in $S_{2,1,0}$, and with coefficient c_2 in $S_{2,2,0}$. Staggered symmetries then imply that $tc_1 + t^2c_2 = 0$ at $t = 1$, and thus $c_1 + c_2 = 0$. Any nonsinglet mass term at order a_f^2 is therefore proportional to $t(t-1)$. Simply put, there has to be a factor t in order to break taste symmetry, and a factor $t - 1$ to break staggered symmetries, which include Γ_4 and $U(1)_\epsilon$.

In order to exclude various contributions to the nonsinglet mass terms in the above argument, we used the fact that mass terms cannot be introduced or removed by field redefinitions. As we now explain, the same is not true for operators of dimension five or higher: they cannot be excluded by arguments based on field redefinitions. With the taste basis of Eq. (4.2), we know from Eq. (2.28) that taste nonsinglet Wilson-like dimension-five operators will already appear in $S_{1,1,0}$. Of course, being nonsinglet, such terms will have to vanish at $t = 0$. In addition, because of staggered symmetries, a local field redefinition can be found removing such terms at $t = 1$. However, this same field redefinition applied to the SET at $t \neq 1$ will, in general, *introduce* taste-breaking terms at $t = 0$. So, all we can conclude is that *before* the field redefinition such terms are proportional to t, while *after* the field redefinition they are proportional to $t - 1$. We cannot conclude that they are proportional to $t(t-1)$. In the case of the mass terms discussed above, stronger conclusions are possible, because dimension-three terms cannot be removed by a field redefinition.

V. STAGGERED CHIRAL PERTURBATION THEORY

In this section, we will discuss the transition from the SET to staggered ChPT, or SChPT. For integer n_r and $n_s = 4n_r$ the derivation was first given in Ref. [22] (for $n_r = 1$) and Ref. [23] (for $n_r > 1$), and we refer to those papers for details on the explicit construction of the SChPT chiral Lagrangian. Here we will focus on the continuation to $n_r = n_s/4$, with n_s as always a positive integer.

A. Transition to staggered chiral perturbation theory

In the previous sections we explained how the appropriate SET for a rooted staggered theory can be constructed. Holding n_s fixed, the Symanzik coefficients are polynomials in n_r, and thus have no singularities at quarter-integer values of n_r. The rooted staggered SET is obtained as a replica rule: calculate correlation functions to a given order in a_f, then set $n_r = n_s/4$. For the next step—the transition to ChPT—we must again retain both n_s and n_r as independent variables. In ChPT, as for the SET, the replica continuation in n_r will be well-defined at fixed n_s, and SChPT with the replica rule, namely, rSChPT, will be recovered after the continuation to $n_r = n_s/4$.

When we calculate correlation functions using the SET for the generalized theory (3.9), dependence on n_r occurs

[20]In particular, the Symanzik coefficients of all taste-breaking four-fermion operators in the $\mathcal{O}(a_f^2)$ SET are independent of n_r and depend only on n_s.

in two ways: through the polynomial dependence of the Symanzik coefficients, and through fermion loops. Once we have calculated a certain correlation function to some order in a_f and to a given order in the loop expansion, the dependence on n_r is thus explicitly known. Technically, this dependence will not be a polynomial, because only the inverse quark propagators, and not the quark propagators themselves, depend polynomially on n_r. However, each quark propagator can be reexpanded around that of the $t = 0$ theory in terms of a_f, and thus n_r, just as in the underlying lattice theory [Eq. (3.9)].

This sets the stage for the derivation of the appropriate chiral theory for QCD with rooted staggered fermions. The continuum chiral theory is an effective theory for low-energy scales where only Goldstone bosons can appear on the external lines. It can be organized as an expansion in p/Λ_χ, where $\Lambda_\chi \sim 1$ GeV is the chiral scale separating other hadrons from the Goldstone bosons [47]. The chiral effective theory can be generalized to include discretization errors, in an expansion in $a = a_f$. The chiral effective theory is to be constructed by matching its correlation functions to those of the underlying theory in a double expansion in p/Λ_χ and $a_f p$. In practice, the low-energy constants (LECs) of the chiral theory cannot be calculated by analytic methods, and are determined by fitting experimental or numerical data.

For positive integer n_s and n_r, the underlying lattice theory is local, as is the SET, and the transition to the chiral theory is more or less standard [22,23,48,49].[21] In addition, the estimate (3.10) is still expected to hold, even though it cannot be checked in perturbation theory, because in this case the correct degrees of freedom for $p \lesssim \Lambda_\chi$ are no longer quarks and gluons. Using the expansion (3.9) just as in Sec. III, this implies that the LECs of the chiral theory again have to be polynomials in n_r. Finally, setting $t = 1$ and performing the continuation to $n_r = n_s/4$ we recover the replica-continued SChPT, or rSChPT, of Refs. [14,23].

We assume here that the contributions of ghosts and taste-singlet quarks in the sea will cancel to all orders in the partially-quenched ChPT once we put $n_r = n_s/4$. All differences between the current rSChPT and the standard rSChPT [14,23] (which does not have the taste-singlet and ghost sectors) will then disappear in the limit $n_r = n_s/4$, as long as we choose not to put ghosts and taste-singlet quarks on the external lines. Since the ghost and taste-singlet Dirac operators and masses are identical, this cancellation is trivial at the QCD level, but not completely trivial beyond one loop at the chiral level.[22] We believe, though, that the cancellation is almost certainly true order-by-order in SChPT, and that it will probably be possible to construct a "quark flow" proof of this. This completes our argument that rSChPT is the correct chiral theory for QCD with rooted staggered fermions.

B. Example

It is instructive to see how our approach works in a concrete example. We will reconsider the leading-order contribution in rSChPT to the connected scalar two-point function, previously described in detail in Sec. 6 of Ref. [14]. Adding a scalar source $s(x)$ to the generating functional, this two-point function is defined as the connected part of the second derivative with respect to this source (setting $s = 0$ after taking the derivatives). Adapting it to our generalized theory, Eq. (27) of Ref. [14] takes the form[23]

$$Z(s) = \frac{\int \mathcal{D}\mathcal{U} \prod_{k=0}^{n} \mathcal{D}\mathcal{V}^{(k)} \boldsymbol{B}_n(\frac{n_s}{4}) \mathrm{Det}^{n_r}(D_{\mathrm{taste},n} + s \otimes \mathbf{1}) \mathrm{Det}^{(n_s - 4n_r)}(\tilde{D}_{\mathrm{inv},n} + s)}{\int \mathcal{D}\mathcal{U} \prod_{k=0}^{n} \mathcal{D}\mathcal{V}^{(k)} \boldsymbol{B}_n(\frac{n_s}{4}) \mathrm{Det}^{n_r}(D_{\mathrm{taste},n}) \mathrm{Det}^{(n_s - 4n_r)}(\tilde{D}_{\mathrm{inv},n})}, \tag{5.1}$$

where we only indicated the n_s dependence of $\boldsymbol{B}_n$ explicitly, cf. Eq. (3.9). Here we have chosen $t = 1$, but have not yet set $n_r = n_s/4$. It is important to keep n_r integral at this stage in order to develop the chiral theory; keeping $n_r \neq n_s/4$ also allows us to highlight the different ways in which n_s and n_r appear.

In Eq. (5.1), we are starting from the fact that correlation functions generated in the rooted staggered theory by the taste-singlet meson source $s(x) \otimes \mathbf{1}$ are identical, in the continuum limit, to the desired correlations generated by $s(x)$ in the target QCD theory. (See Eq. (12) of Ref. [8].) Note, however, that we have coupled $s(x)$ not only to the staggered quarks but also to the ghost and taste-singlet quarks. This keeps the expansion in n_r under control because the staggered and ghost contributions differ only by the small taste-violating term Δ_n. Requiring that the taste-singlet and ghost quarks cancel at $n_r = n_s/4$ then implies that $s(x)$ also couples to the taste-singlet quarks.

Even without a replica continuation, the lattice theory defined by Eq. (5.1) is, as we discussed already above, a partially-quenched theory with n_r staggered fermions, n_s taste-singlet fermions, and $4n_r$ taste-singlet ghosts. It differs from Eq. (3.9) in the way it is coupled to sources. Of course, the correlation functions that are generated by

[21] Again, the only element of this transition that is not absolutely standard is the assumption that all steps can be carried out for partially-quenched theories, since the generalized theory (3.9) is partially quenched.

[22] We thank S. Sharpe [50] for emphasizing this point to us.

[23] The connection with the method and notation of Ref. [14] is explained in Sec. V C.

taking derivatives with respect to $s(x)$ can also be generated by taking joint derivatives with respect to $H(x)$ and $\bar{H}(x)$ (with one each for each space-time point). Regardless of which type of source is used, the dynamics are that of the sea-quark loops, and are controlled by the determinants in Eq. (3.9). Since in this subsection we are only interested in the scalar two-point function, the formulation with the source $s(x)$ is simpler. Note that here we need the complete effective theory, including taste-singlet and mixed sectors, because the source $s(x)$ couples to all quarks.

At leading order in ChPT, the scalar two-point function consists of a sum over one-loop diagrams, with pseudoscalar mesons on the loop (cf. Fig. 2 of Ref. [14]). Since $s(x)$ couples to all bilinears, staggered, taste-singlet, and ghost, all types of pseudoscalar mesons contribute to these diagrams, including fermionic mesons made out of quarks and ghosts, and mesons made only out of ghosts. Because the taste-singlet quarks and ghost have the same Dirac operator $\tilde{D}_{\mathrm{inv},n}$, the result for the scalar two-point function that we will give below is that of a theory with $n_s - 4n_r$ taste-singlet quarks, irrespective of the value (and, in particular, sign) of $n_s - 4n_r$. In the interest of brevity, therefore, the discussion below will simply assume that we are dealing with a theory with a positive number $n_s - 4n_r$ of taste-singlet quarks (as well as n_r staggered quarks).

In Ref. [14] it was shown that, as expected, in the one-flavor theory (for which $n_s = 4n_r = 1$) only the non-Goldstone, heavy pseudoscalar taste-singlet state (the "η'") contributes to this two-point function in the continuum limit, despite the presence of 15 additional light pions in the underlying staggered theory. That this has to happen follows from the general discussion given in Ref. [8]. Here we will not repeat the details of the calculation given in Ref. [14], but only keep track of how the results change in the generalized setup of the present paper, and see how n_r and n_s appear in the final result. With Ref. [14], we keep the singlet pseudoscalar state in the calculation for pedagogical reasons.

There are now three kinds of pions, those made out of staggered quarks, those made out of taste-singlet quarks, and "mixed pions," made out of staggered and taste-singlet quarks. The leading-order masses of the pseudoscalars in the staggered sector are given by

$$M_\Xi^2 = 2\mu m + a_f^2 \Delta_\Xi, \tag{5.2}$$

where $\Xi \in \{I, \xi_\mu, i\xi_\mu\xi_\nu(\mu > \nu), i\xi_\mu\xi_5, \xi_5\}$ labels the taste of each of the 16 staggered pseudoscalars (for each replica), and the Δ_Ξ are four LECs[24] representing the taste splittings; m is the quark mass. Then there are pions made out of only taste-singlet quarks, with mass[25]

$$M_{ts}^2 = 2\mu m + a_f^2 \Delta_{ts}. \tag{5.3}$$

Finally, there are mixed pseudoscalars made out of one taste-singlet and one staggered quark. The mass of the latter can be parametrized, to leading order, as [51]

$$M_{\mathrm{mix}}^2 = 2\mu m + a_f^2 \Delta_{\mathrm{mix}}, \tag{5.4}$$

with, in general, $\Delta_{\mathrm{mix}} \neq \Delta_{ts}$. The fact that the mass of the mixed mesons does not depend on their staggered taste follows, as in Ref. [51], from shift symmetry, which forbids taste-violating staggered bilinears, and therefore forbids taste-violating four-quark operators with one staggered and one taste-singlet bilinear. Note that all the above masses (in particular, M_I) are the pseudoscalar masses before including the effect of the anomaly.

The LECs μ, Δ_Ξ, Δ_{ts}, and Δ_{mix} have unknown dependence on n_s, but do not depend on n_r. For μ this is obvious, because it is a continuum LEC, and the continuum theory does not depend on n_r at all, but only on n_s. (Recall that, in the continuum limit, the determinants ratio (3.7) goes to one.) Because Δ_Ξ represents an order a_f^2 effect, it can, according to our general arguments, be at most linear in n_r. In practice, it is independent of n_r, because symmetry-breaking terms of order a_f^2 in the SET do not originate from the determinant ratio but only from the source term in Eq. (3.9) [cf. the discussion below Eq. (4.2)]; similar arguments apply for Δ_{mix} and Δ_{ts}. At higher order there will be n_r-dependent corrections to Eqs. (5.2) through (5.4) coming from insertions of the operator $S_{2,2,1}$ in Eq. (4.2). The taste-singlet and mixed mesons also contribute to our scalar two-point function as long as $n_r \neq n_s/4$.

Of course, the singlet pseudoscalar (the "η'") will not be a Goldstone boson. It will pick up a mass that does not vanish in the chiral and continuum limits. In the continuum limit, the η' mass is given by

$$M_{\eta'}^2 = 2\mu m + n_s \frac{m_0^2}{3}, \tag{5.5}$$

where m_0^2 is the double-hairpin parameter (cf. Ref. [23]).

Again, since the continuum limit does not depend on n_r, the parameter m_0^2 does not depend on n_r.[26] Away from the continuum limit, mixing takes place in the neutral meson sector because of different scaling violations in M_I^2 and M_{ts}^2. This mixing leads to the appearance of pseudoscalar

[24] $\Delta_{\xi_5} = 0$ because this taste corresponds to the exact Goldstone bosons.

[25] The operator $\tilde{D}_{\mathrm{inv},n}$ has no chiral symmetry, and the taste-singlet quark mass is additively renormalized by an amount of order a_f^2 (see Sec. IV). The quantity Δ_{ts} represents the effect of this renormalization on the meson mass. In the case of the mixed pseudoscalar mass, Eq. (5.4), such renormalization is absorbed in Δ_{mix}, which must be present in any case.

[26] There are in general corrections of order a_f^2, as well as momentum-dependent contributions, to this parameter, but they do not invalidate our conclusions. Following Ref. [14], other hairpin contributions of order a_f^2 will be ignored as well.

mesons with masses $M_\pm$ given by

$$M_\pm^2 = \frac{1}{2}\left(n_s \frac{m_0^2}{3} + M_I^2 + M_{ts}^2 \pm \sqrt{\left(n_s \frac{m_0^2}{3}\right)^2 - 2(n_s - 8n_r)\frac{m_0^2}{3} a_f^2 \Delta + a_f^4 \Delta^2}\right),$$
$$a_f^2 \Delta \equiv M_I^2 - M_{ts}^2 = a_f^2(\Delta_I - \Delta_{ts}). \tag{5.6}$$

In the continuum limit, $\Delta = 0$ and $M_I^2 = M_{ts}^2 \equiv 2\mu m$, so the expression for M_+^2 simplifies to Eq. (5.5).

In order to give the expression for the scalar two-point function, we define single-particle propagators

$$D_A(p) = \frac{1}{p^2 + M_A^2}, \qquad A = \Xi, \text{ts}, \text{mix}, \tag{5.7}$$

and hairpin "double poles"

$$X_{I,I}(p) = \frac{1}{(p^2 + M_-^2)(p^2 + M_+^2)} \frac{p^2 + M_{ts}^2}{p^2 + M_I^2},$$
$$X_{ts,ts}(p) = \frac{1}{(p^2 + M_-^2)(p^2 + M_+^2)} \frac{p^2 + M_I^2}{p^2 + M_{ts}^2}, \tag{5.8}$$
$$X_{I,ts}(p) = X_{ts,I}(p) = \frac{1}{(p^2 + M_-^2)(p^2 + M_+^2)}.$$

For $\Delta = 0$, all hairpin double poles become equal, and $D_I(p) = D_{ts}(p)$.

The result for the Fourier transform $\tilde{G}(p)$ of the scalar two-point function is

$$\tilde{G}(q) = \mu^2 \int \frac{d^4p}{(2\pi)^4} \Big\{ 2n_r^2 \sum_\Xi D_\Xi(p) D_\Xi(p+q) + 16 n_r(n_s - 4n_r) D_{\text{mix}}(p) D_{\text{mix}}(p+q) + 2(n_s - 4n_r)^2 D_{ts}(p) D_{ts}(p+q)$$
$$- 8n_r \frac{m_0^2}{3}(D_I(p) X_{I,I}(p+q) + D_I(p+q) X_{I,I}(p)) - 2(n_s - 4n_r)\frac{m_0^2}{3}(D_{ts}(p) X_{ts,ts}(p+q) + D_{ts}(p+q) X_{ts,ts}(p))$$
$$+ \left(\frac{m_0^2}{3}\right)^2 [32 n_r^2 X_{I,I}(p) X_{I,I}(p+q) + 2(n_s - 4n_r)^2 X_{ts,ts}(p) X_{ts,ts}(p+q) + 16 n_r(n_s - 4n_r) X_{I,ts}(p) X_{I,ts}(p+q)]\Big\}. \tag{5.9}$$

The explicit factors $m_0^2/3$ can be eliminated from this expression by using the relation

$$\frac{m_0^2}{3} = \frac{1}{n_s}(M_+^2 + M_-^2 - M_I^2 - M_{ts}^2). \tag{5.10}$$

As discussed above, if we expand out the masses $M_\pm^2$ in powers of a_f^2, the n_r dependence of Eq. (5.9) is polynomial. The n_s dependence is not polynomial because the LECs μ, Δ_Ξ, Δ_{ts}, and Δ_{mix} depend on n_s implicitly in an unknown way.

Let us compare the result (5.9) to a similar calculation, done in the taste-singlet theory obtained by replacing $D_{\text{taste},n}$ with $D_{\text{inv},n}$ in Eq. (5.1). To order a_f^2, this corresponds to setting $M_I^2 = M_{\text{mix}}^2 = M_{ts}^2$. The expression for M_+^2 [cf. Eq. (5.6)] again simplifies to Eq. (5.5), except that $2\mu m$ is replaced with M_{ts}^2, because M_{ts}^2 may still include discretization errors. Instead of Eq. (5.9) we now arrive at

$$\tilde{G}(q) \to 2\mu^2 \int \frac{d^4p}{(2\pi)^4} \left\{ (n_s^2 - 1) \frac{1}{p^2 + M_{ts}^2} \frac{1}{(p+q)^2 + M_{ts}^2} + \frac{1}{p^2 + M_{\eta',ts}^2} \frac{1}{(p+q)^2 + M_{\eta',ts}^2} \right\},$$
$$M_{\eta',ts}^2 = M_{ts}^2 + n_s \frac{m_0^2}{3}. \tag{5.11}$$

As expected, this result is n_r-independent. The first term on the right-hand side is recognized as the anticipated contribution of the $n_s^2 - 1$ degenerate Goldstone pions of a theory with n_s (mass-degenerate) flavors.

Replacing $D_{\text{taste},n}$ with $D_{\text{inv},n}$ means that the product of determinants in the denominator of Eq. (5.1) collapses to $\text{Det}^{n_s}(\tilde{D}_{\text{inv},n})$, with a similar simplification in the numerator. Our calculation thus explicitly demonstrates how we may consider the rooted staggered theory as a local taste-singlet theory with small, nonlocal corrections of order a_f^2, which, to any fixed order in a_f, are polynomial in n_r.[27] Our example also illustrates how the nonlocality of the rooted staggered theory manifests itself in the low-energy EFT: while Eq. (5.11) satisfies unitarity, Eq. (5.9), at $a_f \neq 0$, does not. This is most easily seen by noting the presence of the minus signs multiplying various terms in Eq. (5.9), in what should be (in a unitary theory) a positive definite correlation function.

C. Comparison with Ref. [14]

The present work may be compared with the complementary argument for the validity of rSChPT given in Ref. [14]. That argument starts from ChPT for a rooted theory with four degenerate flavors of staggered fermions, which thus describes four mass-degenerate quark species. The underlying lattice theory is local, trivially, because it

[27] By making use of the general sources in Eq. (3.9) this conclusion applies to any physical correlation function of interest. A by-product is that the generalized theory (3.9) provides a alternative framework to that discussed in Appendix B of Ref. [10] for solving the "valence rooting" problem.

contains the fourth power of the fourth-rooted staggered determinant. Staying entirely within the ChPT framework, Ref. [14] then treats the nondegenerate case by perturbing in the quark masses. An assumption of the analyticity of the expansion around positive quark mass is required at this point. In addition, the replica rule (called the "replica trick" in Ref. [14]) needs to be introduced because the theory becomes nonlocal as one moves away from the degenerate limit. Finally, one of the four masses can be made so large that the corresponding quark decouples from the chiral effective theory (at which point it can be thought of as the charm quark). Using an assumption about the details of decoupling, one arrives at rSChPT for three light quarks. The decoupling assumption leaves a small potential loophole in the argument of Ref. [14]. While the three-flavor chiral theory goes over, in the continuum limit, to the standard three-flavor chiral theory of QCD, it is not guaranteed that the LECs have the same numerical values as in QCD. (In the initial four-flavor case, the correctness of the LECs is guaranteed, however.)

Here, we have started instead from the fundamental lattice theory (in RG-blocked form) and have shown how rSChPT may be derived from it, *via* the SET. The replica rule is given definite meaning in the fundamental theory, so its appearance in the EFTs is completely natural. In contrast, the replica rule in Ref. [14] has, by construction, meaning only at the chiral level. It is for that reason that a distinction was made in Ref. [14] between the power of the staggered determinant at the QCD level, which was called R, and the number of replicas introduced at the chiral level, n_r. Here, because the replica rule is justified at the QCD level, we need to make no such distinction. We do however need to introduce the number of flavors of the taste-singlet quarks, n_s, which affects LECs in a nonperturbative (and hence unknown) way, in order that the n_r dependence be completely controlled (indeed, polynomial). Thus Ref. [14] and the current work represent two different generalizations of the staggered theory. In the limit $R = n_r = n_s/4$, the two generalizations agree. Since this is the limit we need to take at the end of any rSChPT calculation, it is clear that the two versions of rSChPT give the same results.[28]

Another advantage of the present approach is that it allows us to dispense with the assumptions about decoupling and about the analyticity of the mass expansion. This means that the current argument closes the loophole mentioned above. The continuum low-energy constants are automatically those of QCD with the correct number of flavors.

On the other hand, the current argument, based as it is on Ref. [15], inherits the assumptions of that work. The key assumptions have already been mentioned in the Introduction and explained in Sec. III. They are that:

(i) The effective action $\delta S_{\rm eff}$, generated by integrating out fermions on finer lattices, is local.

(ii) The perturbative scaling laws apply, implying that the dimension-five taste-breaking operator Δ_n goes to zero like a_f (times logarithms) in the continuum limit. This in turn is based on the highly plausible assumption that the theory is renormalizable to all orders in perturbation theory for any n_r.

The assumption of taste-symmetry restoration is needed in Ref. [14] too, but only for integer n_r, where the scaling argument is completely standard. The argument of Ref. [14] works entirely within the chiral theory, and the resulting rSChPT then implies the symmetry restoration (in the chiral sector) for the rooted case. We also note that, in the RG framework, there is an alternative route to establish the validity of the continuum limit while relying only on the scaling of Δ_n in the taste-singlet (reweighted) theory [11]. Since the latter theory is local by the first assumption, the validity of the scaling assumption needed for the RG treatment is very plausible. We remind the reader that there is considerable numerical evidence for the continuum restoration of taste symmetry in the rooted case [13,18–21].

Both the present arguments and those of Ref. [14] rely heavily on the validity of the standard partially-quenched chiral theory [16] for describing partially-quenched fundamental theories that are local. We also need to assume here that the SET exists for partially-quenched theories, as long as the lattice theory is local.

The calculation of the scalar two-point function, presented in Sec. V B, may now be compared to the corresponding calculation in Sec. 6 of Ref. [14]. Note that Ref. [14] considers only the one-flavor case as an example, so to make the connection, we must put $n_s = 1$. The result here, Eq. (5.9), then corresponds directly to Eq. (41) of Ref. [14]. We can in fact make the connection at the quark flow level: The first two lines of Eq. (5.9) correspond to Figs. 3(a) and 3(d) of Ref. [14], the next two lines correspond to Figs. 3(b) and (c), and the last two lines correspond to Fig. 3(e). It is straightforward to check that, if we set $n_r = n_s/4 = 1/4$ in Eq. (5.9), and $R = n_r = 1/4$ in Eq. (41) of Ref. [14], the results are identical.

VI. CONCLUSION

In this paper we presented a theoretical argument that rSChPT [23] is the correct chiral theory for QCD with rooted staggered fermions. Much evidence in favor of this claim already existed, both on the theoretical side [14], as well as on the numerical side [13,18–20]. Here we showed that it is possible to extend the usual construction of the Symanzik effective theory and chiral perturbation theory, to the rooted staggered case. Our arguments apply equally

[28]We again are assuming that the contributions of ghosts and taste-singlet quarks in the sea cancel to all orders in partially-quenched ChPT once there are the same number of ghosts and taste-singlet quarks, i.e., once $n_r = n_s/4$.

well to any staggered quark action that has the usual staggered symmetries, for example, standard (unimproved) staggered [1], Asqtad [28], HYP [29], Fat7bar [30], or HISQ [31] quarks. The version of staggered quarks used will not effect the form of the discretization effects summarized by the effective theory, but does effect the size of these effects, which is reflected in the size of the LECs.

The effective theories are first constructed for a taste-singlet local theory with n_s physical fermion flavors [the $t = 0$ theory of Eq. (3.9)]. The rooted, nonlocal staggered theory is then reconstructed as an expansion in the lattice spacing of the underlying staggered theory (i.e., a_f), by moving smoothly from $t = 0$ to $t = 1$. In this framework, the dependence on n_r is polynomial to any finite order in a_f and to any finite order in the loop expansion.[29] The effective theories, however, are in the first instance only known at integer values of n_r, where they are fairly standard. The polynomial dependence on n_r allows us to make the replica continuation of any correlation function, computed order-by-order in the effective theory for integer n_r, to $n_r = n_s/4$. Once the value $n_r = n_s/4$ is reached, the correct correlation functions of the underlying rooted lattice theory are recovered.

The ability to extend standard techniques for the derivation of the SET and ChPT to rooted staggered fermions does not preclude various sicknesses in the rooted theory at nonzero a_f. Indeed, in Ref. [12] we argued that the rooted theory is nonlocal at nonzero a_f, due to the taste-breaking induced splittings in hadron taste multiplets. It is essential that the replica-continued SET and SChPT reproduce the nonlocal behavior. This happens because loop corrections calculated in these theories have to be continued to a non-integer number of staggered replicas as well, and the replica-continued amplitudes cannot be reproduced from any local Lagrangian. An explicit example of this was worked out in Sec. 6 of Ref. [14]; we revisited this example in Sec. V B in our generalized framework.

It is important to list the assumptions that underlie our arguments. The most important assumption is that QCD with rooted staggered fermions has the desired continuum limit. This conclusion, in turn, is based on a number of technical and testable assumptions, as explained in detail in Ref. [15] (see also Refs. [10,11]). If this conclusion were to turn out to be incorrect, that would also invalidate the analysis presented here. Turning this around, we consider the success of fitting high-precision numerical results with rSChPT as direct evidence that the conclusion of Ref. [15] is, in fact, valid.

In order to keep the replica continuation under control, in Eq. (3.9) we temporarily treated the number of dynamical quarks in the theory (n_s) and the power of the staggered determinant (n_r) as independent. Because $4n_r$ ghosts are needed, we also have to assume that the construction of the SET and ChPT goes through in the standard way for partially-quenched (but local) theories. This second assumption is very common in applications of EFTs to lattice QCD. However, one should keep in mind that, while partially-quenched ChPT [16] is by now standard, its foundations are not as firm as for ordinary, unquenched ChPT. See Ref. [52] for a discussion of this point.

A third assumption is the technical observation that $D^{-1}_{\mathrm{inv},n}\Delta_n$ has to scale as $a_f p$, with p the momentum scale at which a correlation function in the effective theory is matched to the underlying theory. An exception is short-distance contributions coming from subdiagrams with a non-negative degree of divergence in which $D^{-1}_{\mathrm{inv},n}\Delta_n$ can become as large as a_f/a_c at most. The end result, the estimate (3.10), is crucial for establishing that the n_r-dependence of the generalized theory (3.9) is polynomial, to any finite order in a_f.[30] Again, we consider this assumption as noncontroversial, because it underlies the standard derivation of EFTs for local lattice theories, and because it is used only in the $t = 0$ theory, which is local by our first assumption. The weaker, quark-mass dependent bound on $D^{-1}_{\mathrm{inv},n}\Delta_n$ used in Ref. [15] is not needed for the derivation of the effective theories, and both the SET and the chiral theory are valid in the chiral limit. We emphasize here that the physically sensible approach for any staggered theory (rooted or not) is to avoid the region $m \ll a_f^2\Lambda^3_{\mathrm{QCD}}$, where lattice artifacts may dominate [8,44,45].

In the actual construction of a SET or a chiral theory, use is made of the symmetries of the underlying theory. Particularly important symmetries for staggered fermions are $U(1)_\epsilon$ chiral symmetry and shift symmetry, and we discussed in detail how these are realized at the level of the SET. Generalizing a result previously derived to order a_f^2 in Ref. [22], we showed that for the SET, shift symmetry enlarges to the direct product of the continuum translation group and the finite discrete group Γ_4. Since this observation holds for the SET, it also holds for any EFT derived from the SET. Finally, we note that our arguments also apply to the cases of rSChPT with baryons or heavy-light mesons.

ACKNOWLEDGMENTS

We would like to thank Andreas Kronfeld and Steve Sharpe for helpful discussions and for their suggestions for improving a draft version of this manuscript. C. B. and M. G. were supported in part by the U.S. Department of Energy. Y. S. was supported by the Israel Science Foundation under Grant No. 173/05.

[29] For the SET, the relevant loop expansion is the one in fermion loops; for ChPT it is the chiral loop expansion.

[30] We note that the same assumption, coupled with the framework introduced in this paper, can be used to make more plausible the argument for perturbative renormalizability of the rooted theory. See the discussion at the end of Sec. III B.

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