Casimir force for a scalar field in warped brane worlds

Román Linares,[1,*] Hugo A. Morales-Técotl,[1,†] and Omar Pedraza[1,2,‡]

[1]*Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa, San Rafael Atlixco 186, C.P. 09340, México D.F., México*
[2]*Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada, Unidad Legaria,*
Instituto Politécnico Nacional, Av. Legaria 694, C.P. 11500, México D.F., México.

In looking for imprints of extra dimensions in braneworld models one usually builds these so that they are compatible with known low energy physics and thus focuses on high energy effects. Nevertheless, just as submillimeter Newton's law tests probe the mode structure of gravity other low energy tests might apply to matter. As a model example, in this work we determine the 4D Casimir force corresponding to a scalar field subject to Dirichlet boundary conditions on two parallel planes lying within the single brane of a Randall-Sundrum scenario extended by one compact extra dimension. Using the Green's function method such a force picks the contribution of each field mode as if it acted individually but with a weight given by the square of the mode wave functions on the brane. In the low energy regime one regains the standard 4D Casimir force that is associated to a zero mode in the massless case or to a quasilocalized or resonant mode in the massive one while the effect of the extra dimensions gets encoded as an additional term.

I. INTRODUCTION

Considering extra spatial dimensions making our observable 4D universe a subspace of a higher dimensional space-time has a long tradition that started with the works of G. Nördstrom, T. Kaluza, and O. Klein. (see e.g. [1] and references therein). Amongst the main motivations for such approaches we find attempts to unify fundamental interactions, in particular including gravity through Kaluza-Klein theories [1], supergravities and string/M-theory [2]. On the other hand, it has also been proposed extra dimensions may help to come to terms with the cosmological constant and the hierarchy problems [3–12]. The current status of the braneworld idea, as it became popular to call the field, can be seen in some reviews [13–16] providing also some remarkable phenomenological aspects going from astrophysics to cosmology. Most of these assume known low energy physics remains unaltered thus focusing in the high energy regime. However, just as Newton's law tests at submillimeter scale have allowed us to probe braneworld scenarios [17,18] it is of interest to consider other precise low energy experiments. Since physics in the presence of extra dimensions is linked to the mode structure of matter or gravity fields, a natural candidate test to study is the Casimir force (other possibilities are high precision atomic experiments [19–21]).

In 1948 H. B. G. Casimir predicted that two uncharged perfectly conducting flat plates, placed in vacuum and separated by a distance l, should attract each other with a force per unit area A given by $F(l)/A = -\frac{\pi^2}{240} \times \frac{\hbar c}{l^4} = -1.3 \times 10^{-27} \frac{\text{Nm}^2}{l^4}$. This force is a purely quantum

effect caused by the alteration of the electromagnetic field modes due to the plates and it is described by QED [22]. We will refer to it as the standard or 4D Casimir force since it is obtained in Minkowski space-time. Although the effect is very weak it becomes measurable when $l \sim 1~\mu$m. Indeed it has been convincingly demonstrated by many experiments over the years [23–27] and its measurements have reached very high precision (see [28] for a review of the current experimental situation); also the Casimir force is convenient to consider experimentally in that it implies macroscopic bodies as opposed to atomic size systems.

Over the years the Casimir effect has been extended to different fields, geometries, materials, and models (see [29] and references therein for a review of the Casimir effect in different contexts). In general, it may be defined as the stress on the bounding surface when a quantum field in vacuum state is confined to a finite volume of space. In any case, the boundaries restrict the modes of the quantum field giving rise to a force which can be either attractive or repulsive, depending on the field and the type of boundaries.

In particular, the Casimir effect has received a great deal of attention within space-time models including extra spatial dimensions. For example, it has been discussed in the context of string theory [30–33]. Also in the Randall-Sundrum model, the Casimir effect has been considered to stabilize the separation between branes (radion) [34–38] as well as within inflationary braneworld universe models [39].

It is noteworthy that some of these models can also affect the standard 4D Casimir force. A simple way to see this is to consider a field defined on a higher dimensional scenario and then extracting a 4D effective dynamics for it. The Casimir force is computed in 4D with dispersion relations modified by the presence of the extra

*lirr@xanum.uam.mx
†hugo@xanum.uam.mx
‡opedrazao@ipn.mx

dimensions, say

$$\frac{\omega^2}{c^2} = k^2 + \Delta k_{\text{extra}}^2, \tag{1}$$

where ω is the frequency and $k := |\vec{k}|$ is the magnitude of the wave vector of the mode. Usually $\Delta k_{\text{extra}}^2$ includes parameters of the higher dimensional model and by demanding agreement of the corresponding Casimir force with experiments it is either possible to set bounds for the parameters or limit the phenomenological viability of the model. Within this approach, the 4D Casimir force between parallel plates has been computed for a scalar field in the presence of one compactified universal extra dimension [40–42], for the effective 4D QED [43] that comes from a Nielsen-Olesen vortex solution of the Abelian Higgs model with fermions coupled to gravity in 6D [44] and for a massless scalar field in the Randall-Sundrum models [45]. In all these models extra dimensions produce different kind of contributions to the dispersion relations. In the first case the scenario is 5D, has topology $M_4 \times S^1$ and the second term in Eq. (1) consists of tower of Kaluza-Klein massive modes of the scalar field, $\Delta k_{\text{extra}}^2 = n^2/R^2$, where $n \in Z$ and R is the radius of S^1 [40–42]. In the second case, there are two extra dimensions that contribute to the dispersion relations of the electromagnetic modes near the core of the vortex that represents our 4D world. A continuous one associated with a radial extra dimension and a discrete one corresponding to an angular coordinate labeled by a vortex number, $n_v \in N$. Explicitly, $\Delta k_{\text{extra}}^2 \approx k_r^2 + 1/n_v^2 \ell^2$. Here ℓ is a length scale defined by the ratio of the 6D Newton constant and the 6D gauge coupling [43,44]. The resulting 4D Casimir force is in conflict with experiments, thus reducing the phenomenological viability of the model. For a massless scalar field in the 5D RSI scenario which includes two 3-branes separated by a compact dimension, the contribution to the dispersion relation is a tower of Kaluza-Klein modes exponentially suppressed, $\Delta k_{\text{extra}}^2 \approx \kappa^2 (n + 1/4)^2 e^{-2\pi\kappa r}$. κ is the brane tension and r is the separation distance between the branes. A massless scalar field in the RSII model including a single brane, yields a tower of continuous Kaluza-Klein massive scalar modes, $\Delta k_{\text{extra}}^2 = m^2$, $m \geq 0$. Upon correcting by the polarization in higher dimensions to go from a massless scalar field to an electromagnetic one, the experiments imply an upper limit to κr for RSI. In the RSII case the effect seems to be too small to be probed by experiment [45].

Implicit in the previous analysis is the assumption that the massless scalar field Casimir force can be translated into the electromagnetic one which is the one that is actually tested experimentally. For this to be the case both scalar and electromagnetic fields should have zero modes localized to our brane. This holds for some but not all of the above scenarios. Moreover, whereas in 4D Minkowski space-time different methods yield the same Casimir force, it is not obvious whether the same results hold for the effective models of braneworld scenarios considered so far. As a first step in this direction in this paper we compute the 4D Casimir force for a scalar field coming from a 6D scenario RSII-1, consisting of a single 3-brane and 1 additional compact extra dimension using the Green's function method, as opposed to the dimensional regularization of previous analysis. RSII-1 owns the nontrivial property of localizing gauge fields [46].

A salient feature of the modes corresponding to the noncompact dimension is linked to whether the scalar field has a 6D mass or not. The massive case does not contain a zero mode and there are not true localized modes with $m \neq 0$ but one that is quasilocalized. In contrast the massless case has a spectrum incorporating a zero mode with a continuum of massive modes [47,48]. This specific relation between the mass spectrum and the bulk mass of the field is a characteristic intrinsic to a noncompact dimension not shared by models containing compact extra dimensions only [46].

The structure of the paper is as follows. Section II sketches some basic features of warped and Kaluza-Klein models. Special attention is payed to RSII and RSII-1. In Sec. III we present our analysis of the Casimir force for the massive scalar field whereas Sec. IV contains the massless case. Finally we discuss our results in Sec. V. Unless otherwise stated we use units in which $\hbar = 1$, $c = 1$.

II. EXTRA SPATIAL DIMENSIONS

An important issue in considering higher dimensional scenarios is the mechanism by which extra dimensions are hidden in such a way that the space-time is effectively 4D. There are two different ways to implement this idea depending on whether the extra dimensions are either *compact* or *noncompact*. Both possibilities can be accommodated by means of the following $(4 + d)$ dimensional metric which is consistent with Poincaré invariance in 4D

$$ds_{4+d}^2 = \sigma(x^c) g_{\mu\nu}(x^\rho) dx^\mu dx^\nu - \gamma_{ab}(x^c) dx^a dx^b. \tag{2}$$

Here Greek indices denote the usual 4D coordinates whereas Latin indices denote extra dimensions, therefore in (2), $g_{\mu\nu}$ is the metric of our world while γ_{ab} is the metric associated with the d extra dimensions.

The first possibility arises in Kaluza-Klein type theories [1]. Within this approach the $(4 + d)$ space-time manifold is assumed to be separable in the form $M_{4+d} = M_4 \times M_d$, where M_4 is our 4D world and M_d is the manifold associated with the small extra dimensions which are *compact* and essentially homogeneous. The metric (2) describes this possibility by taking $\sigma(x^a) = 1$, implying that M_4 is described by a factorizable geometry, independent of x^a. Compactness of extra dimensions ensures that space-time is effectively 4D at distances exceeding the compactification scale R. This conclusion arises because from the 4D

point of view, every multidimensional field (matter, gravity, and gauge fields) corresponds to a Kaluza-Klein tower of particles with increasing masses. At low energies $E < R^{-1}$, only massless particles can be produced, whereas at $E \sim R^{-1}$, the tower of massive states is manifest and extra dimensions show up. Since experimentally the Kaluza-Klein massive states have not been observed, the energy scale R^{-1} must be at least in the TeV range, so in the Kaluza-Klein models the size of the extra dimensions must be microscopic $R \leq 10^{-17}$ cm. [8,9]. The 4D effective Casimir effect coming from a 5D massless scalar field in this geometry has been discussed in [40–42].

The second possibility considers *noncompact* extra dimensions but still unobservable at low energies. There exist basically two ways to obtain noncompact extra dimensions. One way is to consider the metric (2) where $\sigma(x^a)$ is a conformal factor depending on the extra coordinates only, implying that the metric is nonfactorizable, i.e., it does not correspond to a product of M_4 and a manifold of extra dimensions. It was proposed for the first time as a space-time Ansatz to solve Einstein equations with a positive cosmological constant in 6D [5]. The second way is to identify our 4D world with the internal space of a topological defect residing in a higher dimensional space-time [3,4,6], for instance, a domain-wall in 5D, a string in 6D, a monopole in 7D, etc. Generically all these types of backgrounds admit localization of both fermionic and scalar field massless zero modes which are associated with the 4D particles that we observe. It has also been established that gravity can be localized on several topological defect backgrounds. For instance, it was realized in [12] that gravity can be localized on a 3-brane (domain-wall), with positive tension and located at $y = 0$ and embedded in a 5D space-time whose metric is given by two patches of the symmetric space AdS_5 of radius κ^{-1} and has the structure of Eq. (2), namely,

$$ds_{4+1}^2 = e^{-2\kappa|y|} n_{\mu\nu} dx^\mu dx^\nu - dy^2. \quad (3)$$

Here the extra dimension y is noncompact and the parameter κ is determined by the 5D Planck mass and bulk cosmological constant. This metric obeys the full 5D Einstein equations with negative cosmological constant and the model is known as Randall-Sundrum II model (RSII). One important property of the geometry (3) is that every field in this background can be decomposed in 4D plane waves, due to its 4D Poincaré invariance

$$\phi \propto \exp(ip_\mu x^\mu)\phi_p(z), \quad (4)$$

where the 4-momentum p_μ coincides with the physical momentum on the brane ($p^2 = m^2$). The key point for the localization of gravity is that a normalizable graviton zero mode ($m^2 = 0$) residing on the domain-wall reproduces 4D gravity, while the continuum massive spectrum of 5D gravitons living on the bulk, gives only a small correction to the Newton's law at large distances. It is worthwhile to

mention that there are other models in the noncompact extra dimensions approach that also localize gravity (see e.g. [14,16] and references therein for a review of them). If one or more extra dimensions are infinite, one naturally expects that particles may eventually leave the brane and escape into the extra dimensions. In the RSII model, this process is possible for gravitons [13]. If other fields have bulk modes, the corresponding particles may also leave the brane. As an example, fermions bound to the brane, even in the absence of gravity, are capable of leaving the brane provided they are given enough energy. From the 4D point of view this process would show up as a process in which the charge is not conserved, for example $e^- \rightarrow$ nothing. It is remarkable that an AdS metric allows for such a process at low enough energies [49]. On the other hand the effective 4D Casimir effect produced by a 5D massless scalar field in the geometry (3) has been recently considered in [45] using the dimensional regularization technique.

Now in order to make the whole construction realistic it is also important to have localization of gauge fields. There are several scenarios that achieve this goal [13], and in this paper we are interested in the one that describes a 6D geometry with a compact warped additional dimension. The metric, that we refer to as RSII-1, away from the brane is

$$ds_{4+2}^2 = e^{-2\kappa|y|}(\eta_{\mu\nu} dx^\mu dx^\nu - R^2 d\theta^2) - dy^2, \quad (5)$$

where θ is a compact extra coordinate taking values in the interval $[0, 2\pi)$, and y is the coordinate along a single noncompact extra dimension. This metric can be obtained in two different ways, either as an asymptotic solution to the 6D Einstein equations with negative bulk cosmological constant and a 3-brane (local string defect) with an appropriately tuned energy-momentum tensor [50,51], or as in the RSII model [12], considering a codimension one brane (a 4-brane) with both positive tension and one compact dimension embedded in a 6D space-time. In this case the metric (5) obeys the full Einstein equations with essentially the same fine-tuning condition between the tension of the brane and the negative bulk cosmological constant as in the RSII model. Gravity is localized on the brane because there exists a graviton zero mode which is independent of θ outside the brane and decrease at large y as $e^{-2\kappa|y|}$, in complete analogy with the RSII model. As in the Kaluza-Klein picture, the compact dimension θ, is invisible at low energies $E < R^{-1}$.

In this geometry it is possibility to localize not only spin 0 and spin 2 fields, but also spin 1 fields [49,52]. This result is in contrast with the RSII model for which is not possible to localize gauge fields [47]. The key point to have the localization is that at low energies, $E \ll R^{-1}$, the relevant gauge field configurations are independent of θ and there exists a zero mode gauge field also independent of y, which corresponds to a massless vector boson localized on the stringlike defect.

Localization of the gauge field besides gravity is the characteristic that makes attractive the geometry (5). To have a complete picture of the model, it should be pointed out that if we want to localize also particles of spin $1/2$ or $3/2$, it is necessary to introduce additional interactions. Generalizations of the metric (5) to metrics with more than two extra dimensions that also localize gauge fields can be found in [49]. In this case a $(3 + n)$-brane with n compact coordinates is embedded in a $(3 + n + 2)$-dimensional space-time,

$$ds^2_{5+n} = e^{-2\kappa|y|}\left[\eta_{\mu\nu}dx^\mu dx^\nu - \sum_{i=1}^{n} R_i^2 d\theta_i^2 \right] - dy^2. \quad (6)$$

We denote this metric as RSII-p. In this paper we shall restrict ourselves to the study of RSII-1 and we will only comment in the discussion section some aspects of RSII-p relevant to the effective 4D Casimir force.

III. MASSIVE SCALAR FIELD

In this section we obtain the Casimir force for a massive scalar field in the background metric RSII-1, Eq. (5). We start by computing both the eigenfunctions and eigenvalues of the differential operators for each independent coordinate that is associated to the scalar field equation. With them we go on then to compute the corresponding Green's function which is used to determine the effective Casimir force.

Let us consider a massive scalar field Φ described by the action in 6D

$$S = \int R d\theta dy d^4x \sqrt{-g}\left(\frac{1}{2} g^{MN}\partial_M \Phi \partial_N \Phi - \frac{1}{2} m_6^2 \Phi^2\right),$$
$$(7)$$

where the metric g_{MN} is given by (5). Here m_6 is the 6D mass of Φ.

The field equation for Φ in the background (5) hence becomes

$$e^{2\kappa|y|}\Box_4\Phi - \frac{e^{2\kappa|y|}}{R^2}\partial_\theta^2\Phi - \frac{1}{\sqrt{-g}}\partial_y[\sqrt{-g}\partial_y\Phi] + m_6^2\Phi = 0.$$
$$(8)$$

$\Box_4$ here stands for the flat 4D Dalambertian corresponding to $\eta_{\mu\nu}$. Assuming solutions of the form $\Phi(x, \theta, y) = \varphi(x)\Theta(\theta)\psi(y)$ and performing separation of variables it is straightforward to obtain the three differential equations

$$(\partial_\theta^2 + m_\theta^2 R^2)\Theta(\theta) = 0, \quad (9)$$

$$(\partial_y^2 - 5\kappa\,\mathrm{sgn}(y)\partial_y - m_6^2 + m^2 e^{2\kappa|y|})\psi(y) = 0, \quad (10)$$

$$(\Box_4 + m_\theta^2 + m^2)\varphi(x) = 0, \quad (11)$$

where m_θ and m are separation constants with units of mass. From the 4D point of view Eq. (11) corresponds to an

effective massive scalar field $\varphi(x)$ whose mass, $m_4^2 := m_\theta^2 + m^2$, picks up two independent contributions corresponding to the compact and noncompact extra dimensions, respectively. Our first task is to determine this mass spectrum and the associated eigenfunctions.

A. Mode decomposition

In this subsection we focus on the θ, y dependence, Eqs. (9) and (10), whereas next one contains the x dependence, Eq. (11), which accounts for the Dirichlet boundary conditions on flat planes. Now, in Eq. (9), Θ is subject to periodic boundary conditions so we obtain the well-known eigenfunctions and eigenvalues for a particle in a circle

$$\Theta_n = \frac{1}{\sqrt{2\pi R}} e^{in\theta} \quad \text{where } n = m_\theta R \in \mathbb{Z}. \quad (12)$$

Therefore the contribution of the extra compact coordinate to m_4, is given in terms of the discrete spectrum, $m_\theta^2(n) = n^2/R^2$. Hence lower dimensional physics is associated to the massless mode $n = 0$. For $n \neq 0$ we have an infinite tower of discrete massive modes as manifestation of the extra warped compact dimension which, however, are considered to be suppressed in the low energy regime as long as $R^{-1} \ll \kappa$. It must be stressed that Eq. (9) does not depend on m_6 so its solutions will hold in the massless scalar field case.

As for the y dependence, Eq. (10), the question arises of whether there is a localized scalar field mode on the brane. As it will be shown below the key observation to answer this question is that for a massive Φ a zero (massless) mode is precluded by RSII-1 and there are not true localized modes with $m \neq 0$ either [46]. The best we can hope for then is a quasilocalized mode, which happens to actually be the case as it is argued below. In contrast, as it is shown in Sec. IV, a massless Φ has an spectrum incorporating a zero mode and a Kaluza-Klein continuum [47,48].

Lacking a zero mode the next best thing we can have is a quasilocalized or metastable mode. The corresponding state is associated with a complex eigenvalue. There are several ways to prove the existence of a metastable state (see [46] for a detailed discussion). One of them is to solve the Eq. (10) imposing the radiation boundary conditions at $y \rightarrow \pm\infty$. It turns out that the solutions are linear combinations of the Hankel functions $H_\gamma^{(1)}(me^{\pm\kappa y}/\kappa)$. Requiring continuity of these solutions and its derivatives on the brane one arrives at an eigenvalue equation for m. The Hankel functions can be expanded in the regime $m \ll \kappa$ and assuming additionally that $m_6 \ll \kappa$, it is possible to show that

$$m = m_q - i\Delta, \quad (13)$$

is a solution to the eigenvalue equation with

$$m_q^2 = \frac{3}{5}m_6^2, \quad \text{and} \quad \frac{\Delta}{m_q} = \frac{1}{6}\left(\frac{m_q}{\kappa}\right)^3. \quad (14)$$

Such a state can decay into the continuum modes and the physical interpretation from the point of view of a 4D observer is that the state corresponds to massive particle propagating in 3 spatial dimensions for some time, and then disappears into the y direction [49]. Notice that the width Δ is suppressed with respect to the mass m_q by a factor $(m_q/\kappa)^3$ at small m_q/κ.

To proceed with our analysis it is convenient to have explicitly the eigenfunctions for the modes. To do so let us observe Eq. (10) is invariant under reflection in y and therefore it is enough to solve the equation in the region $y > 0$. Performing the change of variable $\tilde{y} = e^{\kappa y}/\kappa$ and redefining the function ψ as $\tilde{\psi}(\tilde{y}) = \tilde{y}^{5/2}\psi(\tilde{y})$, we obtain

$$\partial_{\tilde{y}}^2 \tilde{\psi} + \frac{1}{\tilde{y}}\partial_{\tilde{y}}\tilde{\psi} + \left(m^2 - \frac{\gamma^2}{\tilde{y}^2}\right)\tilde{\psi} = 0. \qquad (15)$$

Here the parameter γ is given by $\gamma^2 = (\frac{5}{2})^2 + (\frac{m_6}{\kappa})^2$. When $m = 0$, Eq. (15) does not admit a solution consistent with the corresponding boundary conditions. Thus there is no zero mode. However, for $m > 0$, the normalized modes are

$$\psi_m(y) = e^{(5/2)\kappa y}\sqrt{\frac{m}{2\kappa}}\left[a_m J_\gamma\left(\frac{me^{\kappa y}}{\kappa}\right) + b_m N_\gamma\left(\frac{me^{\kappa y}}{\kappa}\right)\right]. \qquad (16)$$

The coefficients a_m, b_m can be obtained from both the normalization condition and the Neumann boundary condition [the latter following from the reflection invariance of Eq. (10)]

$$\int_{-\infty}^\infty dy\, e^{-3\kappa|y|}\psi_m\psi_{m'} = \delta(m - m') \Rightarrow a_m^2 + b_m^2 = 1, \qquad (17)$$

$$\partial_y \psi_m(y)|_{y=0} = 0, \qquad (18)$$

where the weight factor in the measure (17) comes from the Sturm-Liouville form of (10). The resulting expressions for a_m and b_m are

$$a_m = -\frac{A_m}{\sqrt{1 + A_m^2}}, \qquad b_m = \frac{1}{\sqrt{1 + A_m^2}}, \qquad (19)$$

where

$$A_m = \frac{N_{\gamma-1}(\frac{m}{\kappa}) - (\gamma - \frac{5}{2})\frac{\kappa}{m}N_\gamma(\frac{m}{\kappa})}{J_{\gamma-1}(\frac{m}{\kappa}) - (\gamma - \frac{5}{2})\frac{\kappa}{m}J_\gamma(\frac{m}{\kappa})}. \qquad (20)$$

These states are not localized modes and therefore cannot represent scalar particles in 4D. However we are interested in the low energy regime and only modes with $m \ll \kappa$ are relevant, making it possible to expand Bessel functions at small arguments. Assuming additionally a light Φ, namely $m_6 \ll \kappa$, the coefficients A_m become

$$A_m \approx -\frac{2\Gamma(\gamma + 1)\Gamma(\gamma - 1)}{\pi(\gamma + \frac{5}{2})}\left(\frac{m}{2\kappa}\right)^{2-2\gamma}$$
$$\times \left(1 - 2(\gamma - 1)\left(\gamma - \frac{5}{2}\right)\left(\frac{\kappa}{m}\right)^2\right), \qquad (21)$$

and the squared wave functions of the modes at the brane behave like

$$\psi_m^2(y \to 0) \approx \frac{9}{\pi}\left(\frac{m}{\kappa}\right)^{-4}\frac{1}{1 + A_m^2}. \qquad (22)$$

There are two relevant mass regimes: one that turns out to correspond to a quasilocalized mode, a regime where ψ_m^2 is peaked, or equivalently where the A_m are small,

$$A_m \approx \frac{1}{\Delta}(m_q - m). \qquad (23)$$

The other regime is defined by the light modes, $m \ll m_6$, for which

$$1 + A_m^2 \approx \left(\frac{9}{5}\frac{\kappa^3 m_6^2}{m^5}\right)^2. \qquad (24)$$

Explicitly we have

$$\psi_m^2(0) \approx \begin{cases} \frac{3}{2}\kappa\delta(m_q - m) & \text{for } m \sim m_q, \\ \frac{m^6}{\pi\kappa^2 m_q^4} & \text{for } m \ll m_q. \end{cases} \qquad (25)$$

An illustrative representation of this idea is presented in Fig. 1.

Once we have obtained the mass spectrum of the scalar field modes by looking at the θ, y dependence, our next task is to solve Eq. (11) for the x dependence and incorporate them all to get the Green's function.
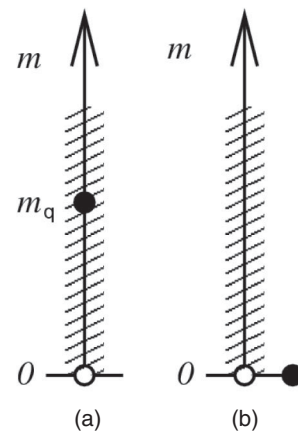

FIG. 1. The figure shows the mass spectrum corresponding to Eq. (10). (a) For $m_6 \neq 0$ the spectrum is continuous but it does not include the zero mode. m_q represents the mass of a quasilocalized mode. (b) For $m_6 = 0$ the spectrum is continuous and it does include the value $m = 0$.

B. Green's function

We want to apply Green's function formalism to compute the Casimir force and hence, given the field equation for Φ, Eq. (8), the corresponding Green's function, G_{6D}, should fulfill

$$\left(e^{2\kappa|y|}\left[\Box_4 - \frac{1}{R^2}\partial_\theta^2\right] - \frac{1}{\sqrt{-g}}\partial_y[\sqrt{-g}\partial_y] + m_6^2\right)G_{6D}$$

$$= \frac{\delta(x-x')\delta(R\theta - R\theta')\delta(y-y')}{\sqrt{-g}}. \quad (26)$$

This can be expressed in terms of the eigenfunctions of the differential operators for the different coordinates. In the previous subsection we have presented the modes Θ_n, ψ_m accounting for the θ, y dependence, respectively, so we still have to solve (11) depending on the 4D x coordinates. Now, since our aim is to compute the Casimir effect for the scalar field in the standard setting of parallel plates, which in our case lie along the brane, it is convenient to split up the 3D position vector $\vec{x}$ in the way $\vec{x} = (\vec{x}_\perp, z)$, where z denotes the coordinate orthogonal to the plates and $\vec{x}_\perp$ denotes a 2D vector orthogonal to the z coordinate. Equation (11) has to be solved in two different regions, between planes ($0 < z < l$) and to the right of the plane $z = l$ (or equivalently to the left of the plane $z = 0$). Upon using such a parametrization Eq. (11) becomes

$$(-\partial_z^2 - \lambda^2)\varphi(z) = 0, \quad (27)$$

where the following definitions have been made

$$\varphi(x) = \varphi(t, \vec{x}_\perp, z) =: \varphi(z)e^{-i\omega t + i\vec{k}_\perp \cdot \vec{x}_\perp}, \quad (28)$$

$$\lambda^2 := \omega^2 - \vec{k}_\perp^2 - \frac{n^2}{R^2} - m^2. \quad (29)$$

The Green's function including only the information of the z coordinate, say $G(z, z')$, is defined through Eq. (27), that is to say

$$(-\partial_z^2 - \lambda^2)G(z, z') = \delta(z - z'). \quad (30)$$

$G(z, z')$ is usually termed reduced Green's function.

For the region between plates ($0 \leq z, z' \leq l$) Eq. (30) is solved subject to the boundary conditions

$$G(0, z') = G(l, z') = 0, \quad (31)$$

obtaining

$$G_{in}(z, z') = -\frac{1}{\lambda \sin\lambda l}\sin\lambda z_< \sin\lambda(z_> - l), \quad (32)$$

where $z_>(z_<)$ represents the greater (lesser) of z and z'.

For the region to the right of the plates ($l \leq z, z'$) the solution is

$$G_{out}(z, z') = \frac{1}{\lambda}(\sin\lambda(z_< - l)e^{i\lambda(z_> - l)}), \quad (33)$$

which vanishes at $z = l$ and has outgoing boundary conditions as $z \to \infty$, $g(z, z') \sim e^{ikz}$. Notice that this solution is different from the free Green's function, where plates are not present.

Going back to the full Green's function and using the eigenfunction expansion we have

$$G_{6D} = \sum_n \int \frac{dm}{\kappa}\int \frac{d\omega}{2\pi}\frac{d^2k_\perp}{(2\pi)^2}\frac{1}{2\pi R}e^{-i\omega(t-t') + i\vec{k}_\perp \cdot (\vec{x}_\perp - \vec{x}'_\perp)}$$

$$\times e^{in(\theta - \theta')}\psi_m(y)\psi_m(y')G(z, z'), \quad (34)$$

where $k_\perp$ is defined also transversal to the z direction and dm/κ is the proper measure on the continuum states [12]. Also it is understood that $G(z, z')$ is either $G_{in}(z, z')$ or $G_{out}(z, z')$ depending on whether the region of interest is between or outside the plates.

Amusingly, Eq. (34) can be rewritten as

$$G_{6D} = \sum_n \int \frac{dm}{\kappa}\Theta_n(\theta)\Theta_n(\theta')\psi_m(y)\psi_m(y')G_{4D}(x, x'; m_4),$$

$$(35)$$

where G_{4D} is the standard Green's function in 4D for a massive scalar field of mass $m_4^2 = \frac{n^2}{R^2} + m^2$ (see e.g. [29]),

$$G_{4D}(x, x'; m_4) := \int \frac{d\omega}{2\pi}$$

$$\times \frac{d^2k_\perp}{(2\pi)^2}e^{-i\omega(t-t') + i\vec{k}_\perp \cdot (\vec{x}_\perp - \vec{x}'_\perp)}G(z, z').$$

$$(36)$$

The interpretation of the 6D Green's function (34) is a nice one: it is just a combination of 4D Green's functions individual massive modes would produce irrespective of whether they are discrete or continuous weighted by the extra dimensional wave functions' modes.

We can push the analytic calculation in the low energy regime by considering light modes approximation $m \ll R^{-1}$, κ and therefore only the zero mode for the compact dimension, $n = 0$, will be considered. Also, since we shall calculate the Casimir force between plates lying along the brane we restrict our analysis to $y, y' \to 0$. The effective 4D Green's function in this approximation splits into two pieces: one for quasilocalized, or resonant, mode and the other for the light modes contribution proper, as follows

$$G_{eff}(x, x') = \frac{1}{2\pi R}\int_{m \sim m_q}\frac{dm}{\kappa}\psi_{m_q}^2(0)G_{4D}(x, x'; m)$$

$$+ \frac{1}{2\pi R}\int_{m \ll m_6 \ll \kappa}\frac{dm}{\kappa}\psi_m^2(0)G_{4D}(x, x'; m).$$

$$(37)$$

Since the first term includes the factor $\psi_{m_q}^2(0)$ which is a resonant state peaked at m_q, in light of (25), it yields a contribution having the form of the standard 4D Green's function of a massive scalar field whose mass is m_q. We will show below it gives rise to the standard 4D Casimir

force. The second term is the contribution due to the rest of light massive modes.

C. Casimir force

In order to calculate the Casimir force between plates from the Green's function one uses the stress tensor [29]. Let us focus on the plate located at $z = l$. This can be done evaluating the discontinuity in the flux of the stress tensor across the plate, i.e., we have to take into account the discontinuity of the normal-normal component of the stress tensor

$$F = \int_0^A dx_\perp \int_0^{\kappa^{-1}} dy \int_0^{2\pi} Rd\theta [\langle T_{zz} \rangle_{\text{in}}|_{z=l} - \langle T_{zz} \rangle_{\text{out}}|_{z=l}].$$

(38)

Here A is the area of the plate.

For our massive scalar field the stress tensor is given by

$$T_{MN} = \partial_M \Phi \partial_N \Phi - \frac{1}{2} g_{MN} \partial^P \Phi \partial_P \Phi + \frac{1}{2} m_6^2 g_{MN} \Phi^2, \quad (39)$$

and its vacuum expectation value may be obtained applying a differential operator to the expression of the Green's function in terms of the vacuum expectation value of the time ordering product of two scalar fields

$$G(x, \theta, y; x', \theta', y') = i \langle T[\Phi(x, \theta, y)\Phi(x', \theta', y')] \rangle. \quad (40)$$

Computing the normal-normal component of the stress tensor to the left of the plate, the obtained result in $z = l$ is

$$\langle T_{zz} \rangle_{\text{in}}|_{z=l} \sim \frac{1}{2i} \partial_z \partial_{z'} G_{in}(z, z')|_{z \to z' = l} = \frac{i}{2} \lambda \cot \lambda l. \quad (41)$$

To the right of the plate we obtain

$$\langle T_{zz} \rangle_{\text{out}}|_{z=l} \sim \frac{1}{2i} \partial_z \partial_{z'} G_{\text{out}}(z, z')|_{z \to z' = l} = \frac{\lambda}{2}. \quad (42)$$

Combining (35), (41), and (42) with (38) gives the 4D force per unit area of the plates (recall $y, y' \to 0$ on the brane)

$$f_T = \sum_n \int_0^\infty \frac{dm}{\kappa} \psi_m^2(0) f_{4D}(m_4), \quad (43)$$

with $f_{4D}(m_4)$ is the standard 4D Casimir force for a 4D massive scalar field whose mass is $m_4^2 = \frac{n^2}{R^2} + m^2$, i.e. [29]

$$f_{4D}(m_4) = \int \frac{d\omega d^2 k_\perp}{(2\pi)^3} \left(\frac{i}{2} \lambda \cot \lambda l - \frac{\lambda}{2} \right)$$

$$= -\frac{1}{32\pi^2 l^4} \int_{2lm_4}^\infty dx \frac{x^2 \sqrt{x^2 - 4l^2 m_4^2}}{e^x - 1}. \quad (44)$$

At this point Eq. (43) for f_T does not seem to reproduce the standard 4D Casimir force, in particular, due to the fact that ψ_m are not true localized states. Nevertheless by alluding to the existence of a quasilocalized resonant mode and a light mode sector given by (25) there are two contributions giving the following effective Casimir force

$$f_{\text{eff}} = \frac{3}{2} f_{4D}(m_q) + f_{\text{light}}, \quad (45)$$

$$f_{\text{light}} := \frac{1}{\pi \kappa^3 m_q^4} \int_{m \ll m_q} m^6 f_{4D}(m) dm.$$

This idea is also clear from Fig. 2 showing conditions under which the quasilocalized or resonant mode dominates. The physical interpretation now is obvious. The first term corresponds to 3/2 of the standard 4D Casimir force of a massive scalar field whose mass is m_q, i.e., we regain the functional form of the standard Casimir effect up to a numerical factor. The second term accounts for the effect of the light modes in the low energy regime: $m \ll R^{-1}$, κ, which imply $n = 0$. Clearly this result is connected to the form the Green's function gets in the light mode approximation as presented in Eq. (37). Moreover we can evaluate numerically an approximation of this integral noticing that for large m: $m \gg l^{-1}$, the standard Casimir force $f_{4D}(m)$ vanishes exponentially [29]. This fact allows the extension of the integration interval in Eq. (45) from $\int_{m \ll m_q} \to \int_0^\infty$, obtaining

$$f_{\text{light}} \approx \frac{2645}{\pi^3 (\kappa l)^3 (m_q l)^4} f_{4D}(0) \quad (46)$$

where we have done the change $m = \alpha/l$ and evaluated numerically the integral $\int_0^\infty \alpha^6 f_{4D}(\alpha) d\alpha = -\frac{5.51}{l^4}$. $f_{4D}(0) = -\frac{\pi^2}{480 l^4}$ represents the standard Casimir force of a massless scalar field. This approximation is the same one that is realized for the scalar potential in [46]. Recalling the
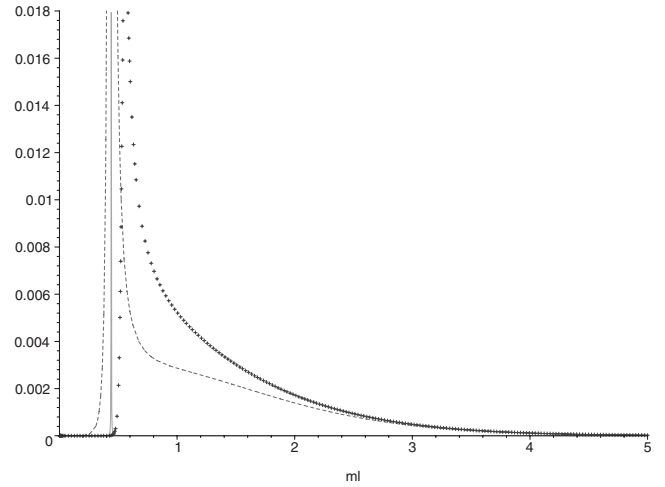

FIG. 2. The figure shows in the vertical axis the contribution to the Casimir force of the continuous modes ($n = 0$): $l^4 \psi_m^2(0) f_{4D}(m_4)$ which is the integrand of Eq. (43) times l^4, as a function of mass in units of l^{-1}. The curve in crosses represents the case $\kappa l = 1/50$, the dashed one is associated with $\kappa l = 1$ whereas the continuous one corresponds to $\kappa l = 50$. It can be seen that the larger the brane tension κl the more peaked the curve. A value of $m_6 l = \sqrt{5/8}$ has been used for the scalar field mass.

light scalar field approximation $m_q l \ll 1 \Rightarrow f_{4D}(m_q) \approx f_{4D}(0)$ and considering $f_{\text{light}} \ll \frac{3}{2} f_{4D}(0)$ in Eq. (45) one gets the lower bound $(\kappa l)^3 (m_q l)^4 \gg 56.9$.

IV. MASSLESS SCALAR FIELD

In this section we describe how the mass spectrum for the scalar field changes when its bulk mass m_4 is zero. The basic difference is that whereas for the massive case there does not exist a zero mode but a quasilocalized or resonant one, for the massless case there does exist a zero mode state. We shall also determine the Casimir force for the standard setting of two parallel plates.

A. Green's function

Let us start by considering the field equation (8) by setting $m_6 = 0$. One can keep track of this condition resulting again in (9) and (11) but changing Eq. (10) for $\psi(y)$ which naturally leads to

$$(\partial_y^2 - 5\kappa sgn(y)\partial_y + m^2 e^{2\kappa|y|})\psi = 0, \qquad (47)$$

and is equivalent, after the change of variable $\tilde{y} = e^{\kappa y}/\kappa$ and $\tilde{\psi}(\tilde{y}) = \tilde{y}^{5/2}\psi(y)$ to

$$\partial_{\tilde{y}}^2 \tilde{\psi} + \frac{1}{\tilde{y}} \partial_{\tilde{y}} \tilde{\psi} + \left(m^2 - \frac{\gamma_0^2}{\tilde{y}^2}\right)\tilde{\psi} = 0, \qquad (48)$$

where $\gamma_0 = \frac{5}{2}$. Mathematically the fact that γ is a rational number is the feature that allows us to have a zero mode solution. Notice that this equation can be obtained from (15) by setting $m_6 = 0$ so that $\gamma|_{m_6=0} = \gamma_0$. When $m = 0$ Eq. (48) includes as a solution

$$\psi_0(y) = \sqrt{\frac{3}{2}}\kappa, \quad \text{which satisfies} \quad \int_{-\infty}^{\infty} dy e^{-3\kappa|y|}\psi_0^2 = 1. \qquad (49)$$

When $m > 0$, the normalized eigenstates are

$$\psi_m(y) = e^{(5/2)\kappa y}\sqrt{\frac{m}{2\kappa}}\left[a_m J_{\gamma_0}\left(\frac{me^{\kappa y}}{\kappa}\right) + b_m N_{\gamma_0}\left(\frac{me^{\kappa y}}{\kappa}\right)\right], \qquad (50)$$

where the constants are given by

$$a_m = -\frac{A_m}{\sqrt{1 + A_m^2}},$$

$$b_m = \frac{1}{\sqrt{1 + A_m^2}}, \quad \text{and} \quad A_m = \frac{N_{\gamma_0-1}\left(\frac{m}{\kappa}\right)}{J_{\gamma_0-1}\left(\frac{m}{\kappa}\right)}, \qquad (51)$$

as can be shown by recalling that all these solutions satisfy the same normalization and boundary conditions as in the massive case $m_6 \neq 0$, i.e. (17) and (18). The existence of the zero mode ψ_0 associated to the noncompact coordinate y is the main difference between the massless and the massive scalar field cases. This result implies that, in the

massless Φ case, the contribution of the noncompact extra dimension to the 4D mass includes as a possible value $m_4^2 = (n/R)^2$ in contrast with the massive case. In particular we have a state corresponding to $m_4 = m = n = 0$, which is a true localized massless state. See Fig. 1 for an illustration of this fact.

Taking into account all the ingredients together we are now in position to write down the 6D Green's function, which in this case satisfies the Eq. (26) with $m_6 = 0$ and its expression in eigenfunctions, analogue of (35), becomes

$$G_{6D}^0 = \sum_n \Theta_n(\theta)\Theta_n(\theta')\left[\frac{1}{\kappa}\psi_0(y)\psi_0(y')\right.$$

$$\left. + \int \frac{dm}{\kappa}\psi_m(y)\psi_m(y')\right]G_{4D}(x, x'; m_4). \qquad (52)$$

Clearly one can read this full Green's function as made up of individual massive modes' 4D Green's functions weighted by the modes' wave functions. In particular in this massless Φ case there is a localized zero mode contribution as opposed to the massive case.

In the light modes approximation $m \ll R^{-1}$, κ hence $n = 0$, the effective 4D Green's function is

$$G_{\text{eff}}^0(x, x') \approx \frac{1}{2\pi R}\frac{1}{\kappa}\psi_0^2(0)G_{4D}(x, x'; m_4 = 0)$$

$$+ \frac{1}{2\pi R}\int_{m\ll\kappa}\frac{dm}{\kappa}\psi_m^2(0)G_{4D}(x, x'; m_4), \qquad (53)$$

where we have taken the limit $y, y' \to 0$, which gives the physics on the brane.

The first term corresponds to the zero mode of the massless scalar field Φ and is proportional to the 4D Green's function of a massless scalar field. Consequently this term is associated with standard 4D physics. The second term is the contribution of the massive modes of the continuous tower of states.

B. Casimir force

The total Casimir force between plates in the stringlike defect can be computed from (52) following the same procedure as in the previous section only extended by the presence of the zero mode. In this manner one obtains

$$f_T^0 = \sum_n\left(\frac{\psi_0^2(0)}{\kappa} + \int_{m\ll\kappa}\frac{dm}{\kappa}\psi_m^2(0)\right)f_{4D}(m_4). \qquad (54)$$

Notice that although the second term for the light modes here is formally the same as the one in the massive scalar field case discussed in the previous section, they truly differ due to the fact that in that case $\psi_m^2 \overset{m_6\neq 0}{\sim} m^6$, whereas here $\psi_m^2 \overset{m_6\neq 0}{\sim} m^2$. Explicitly, in the light modes approximation $m \ll R^{-1}$, κ

$$A_m \approx -3\left(\frac{m}{\kappa}\right)^{-3} \quad \text{and} \quad \psi_m^2(0) \approx \frac{m^2}{\pi\kappa^2}, \qquad (55)$$

resulting in an effective Casimir force

$$f_{\text{eff}}^0 \approx \tfrac{3}{2} f_{\text{4D}}(0) + f_{\text{light}}^0. \tag{56}$$

$$f_{\text{light}}^0 := \frac{1}{\pi \kappa^3} \int_{m \ll \kappa} m^2 f_{\text{4D}}(m) dm.$$

The light modes contribution can be evaluated further as in the massive case obtaining finally

$$f_{\text{light}}^0 \approx \frac{27.1}{\pi^3 (\kappa l)^3} f_{\text{4D}}(0). \tag{57}$$

Considering again $f_{\text{light}}^0 \ll \tfrac{3}{2} f_{\text{4D}}(0)$ in (56) one gets the lower bound $\kappa l \gg 8.3 \times 10^{-1}$. By taking $l \sim 10^{-6}$ m of typical Casimir experiments one gets an upper bound for the anti-de Sitter radius of $\kappa^{-1} \ll 1.2 \times 10^{-6}$ m.

V. DISCUSSION

To avoid conflict with well tested low energy physics braneworld models are usually built up accordingly, namely, they include a big enough mass gap of a given field separating the zero mode, which yields standard 4D physics, from a continuum sector of massive modes in the case of noncompact extra dimensions, which produce corrections to 4D physics. In this way physical effects are investigated mostly in the high energy regime including also astrophysics or cosmology. Interestingly, just for the same reason submillimeter experiments of Newton's gravity law [17,18] probe the mode structure of gravity in the low energy regime for braneworlds, the case for matter fields can be raised in relation with precision tests like the Casimir force or other atomic experiments [19–21,28].

As a specific model proposal of low energy test in this paper we consider the Casimir force produced by a scalar field with and without mass in a Randall-Sundrum type of braneworld consisting of a single brane extended by one compact extra dimension or RSII-1. Such a higher dimensional scenario is interesting because it allows localization of gauge fields [46]. We adopt the standard setting of the Casimir effect which involves two parallel flat plates on which the scalar field is subject to Dirichlet boundary conditions. The plates lie along the single brane of the RSII-1 model.

To calculate the Casimir force we use the Green's function method. The first result that makes a crucial difference between the massive and massless scalar field is related to the mass spectrum they present. Whilst both include a continuous sector, the former lacks a zero (massless) mode possessing instead a quasilocalized or resonant mode, see Eq. (25). The latter case contains a zero mode. The Green's function and the Casimir force it produces depend in detail on this fact. Nonetheless, and this is our second result, they both turn out to be expressed in the low energy regime as the combination of the individual modes acting as in 4D but weighted by the values of the modes' wave functions on the brane as shown in Eqs. (37), (43), (53), and (54). Investigating the dependence of the correction terms to the standard Casimir force led us to Eqs. (45) and (56) for the massive and massless cases, respectively. By looking at the conditions under which such corrections do not dominate standard Casimir force produced two kinds of bounds. In the massive scalar field case we got the lower limit of the product $(\kappa l)^3 (m_q l)^4 \gg 56.9$ whereas in the massless case the upper bound for the anti-de Sitter radius $\kappa^{-1} \ll 1.2 \times 10^{-6}$ m. Incidentally this limit turns out to be weaker than the others obtained based upon the Lamb shift for hydrogen in a similar braneworld [21].

We should keep in mind our model proposal is too simple at present to be compared with actual experimental data. Indeed we are giving only the first steps in this direction to fill in the gap for low energy tests of braneworld models in current literature.

As a further development it would be interesting to compare the analysis following dimensional regularization like in Refs. [40,43,45] with those adopting Green's function techniques like in the present paper. Moreover it seems possible to generalize our results to an arbitrary number of extra compact dimensions or RSII-p, and it should be possible to study the mass spectrum as well as Greens' function and the corresponding Casimir force [53].

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