Dimensional reduction and the Yang-Mills vacuum state in 2 + 1 dimensions

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We propose an approximation to the ground state of Yang-Mills theory, quantized in temporal gauge and 2 + 1 dimensions, which satisfies the Yang-Mills Schrödinger equation in both the free-field limit, and in a strong-field zero mode limit. Our proposal contains a single parameter with dimensions of mass; confinement via dimensional reduction is obtained if this parameter is nonzero, and a nonzero value appears to be energetically preferred. A method for numerical simulation of this vacuum state is developed. It is shown that if the mass parameter is fixed from the known string tension in 2 + 1 dimensions, the resulting mass gap deduced from the vacuum state agrees, to within a few percent, with known results for the mass gap obtained by standard lattice Monte Carlo methods.

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I. INTRODUCTION

Confinement is a property of the vacuum state of quantized non-Abelian gauge theories, and it seems reasonable that something could be learned about the origin of confinement, and the origin of the mass gap, if we knew the form of the Yang-Mills vacuum wave functional in some physical gauge. There have, in fact, been a number of efforts along those lines, in temporal gauge [1–7], Coulomb gauge [8,9], axial gauge [10], and in a Bars corner-variable formulation [11,12].

In this article we will pursue this investigation in temporal gauge and in $D = 2 + 1$ dimensions, our strongest influences being Refs. [1,7]. Our claim is that the ground state wave functional $\Psi_0[A]$ can be approximated by the form

$$\Psi_0[A] = \exp\Big[-\frac{1}{2} \int d^2x d^2y B^a(x) \times \Big(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \Big)^{ab}_{xy} B^b(y) \Big] \qquad (1)$$

where $B^a = F^a_{12}$ is the color magnetic field strength, $D^2 = D_k D_k$ is the two-dimensional covariant Laplacian in the adjoint color representation, λ_0 is the lowest eigenvalue of $-D^2$, and m is a constant, with dimensions of mass, proportional to g^2. To support this claim, we will argue that the above expression

(1) is the ground state solution of the Yang-Mills Schrödinger equation in the $g \to 0$ limit;
(2) solves the zero-mode Yang-Mills Schrödinger equation in the zero-mode strong-field limit;
(3) confines if $m > 0$, and that $m > 0$ is energetically preferred;
(4) results in the numerically correct relationship between the mass gap and string tension.

A very similar proposal for the vacuum wave functional, with λ_0 absent, was put forward by Samuel in Ref. [7], generalizing the earlier "dimensional reduction" proposal of Ref. [1].

Our paper is organized as follows: In Sec. II, below, we find an approximate solution of the zero-mode Yang-Mills Schrödinger equation in 2 + 1 dimensions, and compare this to our proposed wavefunctional in an appropriate limit. The dimensional reduction and confinement properties are discussed in Sec. III. Section IV outlines a procedure for numerical simulation of our vacuum wavefunctional; in Sec. V this procedure is applied to calculate the mass gap, with parameter m chosen to give the correct string tension as a function of coupling. Confinement, in our approach, relies on $m^2 > 0$; in section VI we will discuss why this choice lowers the vacuum energy in the non-Abelian theory, while the minimum is at $m^2 = 0$ in the free Abelian theory. Section VII contains a few results and critical comments regarding certain other proposals for the Yang-Mills vacuum wave functional. Some brief remarks about Casimir scaling and N-ality are found in Sec. VIII, with conclusions in Sec. IX.

We would like to note here that the work in Sec. II, concerning the zero-mode strong-field limit, was motivated by a private communication from D. Diakonov to one of the authors [13].

II. THE FREE FIELD AND ZERO MODE LIMITS

In temporal gauge and $D = d + 1$ dimensions, the problem is to find the ground state of the Yang-Mills Schrödinger equation

$$H\Psi_0 = E_0 \Psi_0 \qquad (2)$$

where

$$H = \int d^dx \Big\{ -\frac{1}{2} \frac{\delta^2}{\delta A^a_k(x)^2} + \frac{1}{4} F^a_{ij}(x)^2 \Big\} \qquad (3)$$

and all states in temporal gauge, in SU(2) gauge theory, are subject to the physical state condition

$$(\delta^{ac}\partial_k + g\epsilon^{abc}A_k^b)\frac{\delta}{\delta A_k^c}\Psi = 0. \tag{4}$$

This condition requires invariance of $\Psi[A]$ under infinitesimal gauge transformations.

Our proposed vacuum wave functional, Eq. (1), obviously satisfies the physical state condition, since the kernel

$$K_{xy}^{ab} = \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\right)_{xy}^{ab} \tag{5}$$

transforms bilinearly, $K_{xy} \to U(x)K_{xy}U^{-1}(y)$, under a gauge transformation, with U a transformation matrix in the adjoint representation. In the $g \to 0$ limit, with both λ_0, $m \to 0$ in the same limit, the vacuum state becomes

$$(\Psi_0[A])_{g\to 0} = \exp\Big[-\frac{1}{2}\int d^2x d^2y(\partial_1 A_2^a(x) - \partial_2 A_1^a(x)) \times \left(\frac{\delta^{ab}}{\sqrt{-\nabla^2}}\right)_{xy}(\partial_1 A_2^b(y) - \partial_2 A_1^b(y))\Big] \tag{6}$$

which is the known ground state solution in $2+1$ dimensions, in the Abelian, free-field case.

The Yang-Mills Schrödinger equation is also tractable in a quite different limit, which is, in a sense, diametrically opposed to the free-field situation. Let us restrict our attention to gauge fields which are constant in the two space directions, and vary only in time (analogous to the minisuperspace approximation in quantum gravity). The Lagrangian is

$$L = \frac{1}{2}\int d^2x[\partial_t A_k \cdot \partial_t A_k - g^2(A_1 \times A_2)\cdot(A_1 \times A_2)] = \frac{1}{2}V[\partial_t A_k \cdot \partial_t A_k - g^2(A_1 \times A_2)\cdot(A_1 \times A_2)] \tag{7}$$

where V is the area of a time slice, leading to the Hamiltonian operator

$$H = -\frac{1}{2}\frac{1}{V}\frac{\partial^2}{\partial A_k^a \partial A_k^a} + \frac{1}{2}g^2 V(A_1 \times A_2)\cdot(A_1 \times A_2). \tag{8}$$

The factors of V in the Hamiltonian suggest the use of a $1/V$ expansion. Let us write

$$\Psi_0 = \exp[-VR_0 + R_1 + V^{-1}R_2 + \ldots)] \tag{9}$$

with R_0 chosen such that the leading order (in $1/V$) "kinetic" term contained in $H\Psi_0$

$$-\frac{1}{2}V\frac{\partial R_0}{\partial A_k^a}\frac{\partial R_0}{\partial A_k^a}\Psi_0 \tag{10}$$

cancels the potential term

$$\tfrac{1}{2}g^2 V(A_1 \times A_2)\cdot(A_1 \times A_2)\Psi_0 \tag{11}$$

at $O(V)$. Let

$$R_0 = \frac{1}{2}g\frac{(A_1 \times A_2)\cdot(A_1 \times A_2)}{\sqrt{|A_1|^2 + |A_2|^2}}. \tag{12}$$

Then, defining

$$T_0 = V\Big[-\frac{\partial R_0}{\partial A_k^a}\frac{\partial R_0}{\partial A_k^a} + g^2(A_1 \times A_2)\cdot(A_1 \times A_2)\Big] \tag{13}$$

it is not hard to verify that

$$T_0 = 0 + \frac{7}{4}g^2 V\frac{[(A_1 \times A_2)\cdot(A_1 \times A_2)]^2}{(|A_1|^2 + |A_2|^2)^2} = \frac{7VR_0^2}{|A_1|^2 + |A_2|^2}. \tag{14}$$

Now for A-fields for which Ψ_0 is non-negligible, it is easy to see that $T_0\Psi_0$ is of order no greater than $1/V$, except in the immediate neighborhood of the origin ($A_k = 0$) of field space. That is because $\Psi_0 \approx \exp[-VR_0]$, which is non-negligible only if VR_0 is $O(1)$. For comparison with Eq. (1) we are interested in a strong-field limit, far from the origin of field space. In that case, since $R_0 \sim 1/V$, then the rhs of (14) is at most of order $1/V$, which can be neglected. It follows that R_0 in Eq. (12) accomplishes the required cancellation at leading order, and provides the leading contribution to the logarithm of the vacuum wave function.

Now consider the proposal (1) for the vacuum wave-functional of the full theory, in a corner of field space where the nonzero momentum modes of the A-field are negligible compared to the zero modes, and in fact the zero modes are so large in magnitude that we can approximate $D_k^{ac} \approx g\epsilon^{abc}A_k^b$. In this region

$$(-D^2)_{xy}^{ab} = g^2\delta^2(x-y)M^{ab} \tag{15}$$

where

$$M^{ab} = (A_1^2 + A_2^2)\delta^{ab} - A_1^a A_1^b - A_2^a A_2^b. \tag{16}$$

In SU(2) gauge theory, the two zero-mode fields A_1, A_2 define a plane in three-dimensional color space. Take this to be, e.g., the color $x-y$ plane, i.e.

$$A_1 = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}. \tag{17}$$

Then

$$M = \begin{pmatrix} a_2^2 + b_2^2 & -a_1a_2 - b_1b_2 & 0 \\ -a_1a_2 - b_1b_2 & a_1^2 + b_1^2 & 0 \\ 0 & 0 & A_1^2 + A_2^2 \end{pmatrix}. \tag{18}$$

Now M has three eigenstates

$$\phi_1 = \begin{bmatrix} \phi_1^1 \\ \phi_1^2 \\ 0 \end{bmatrix}, \qquad \phi_2 = \begin{bmatrix} \phi_2^1 \\ \phi_2^2 \\ 0 \end{bmatrix}, \qquad \phi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

with corresponding eigenvalues

$$\mu_1 = \tfrac{1}{2}(S - \sqrt{S^2 - 4C}), \quad \mu_2 = \tfrac{1}{2}(S + \sqrt{S^2 - 4C}), \quad \mu_3 = S \tag{20}$$

where

$$S = A_1^2 + A_2^2, \qquad C = (A_1 \times A_2) \cdot (A_1 \times A_2). \tag{21}$$

Then

$$\left(\frac{1}{\sqrt{M - (\mu_1 - m^2)I}}\right)^{ab} = \sum_{n=1}^{3} \frac{\phi_n^a \phi_n^{*b}}{\sqrt{\mu_n - \mu_1 + m^2}}. \tag{22}$$

We have

$$\begin{aligned}\Psi_0 &\approx \exp\left[-\frac{1}{2} \int d^2x d^2y B^a(x) \right. \\ &\quad \left. \times \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\right)_{xy}^{ab} B^b(y) \right] \\ &= \exp\left[-\frac{1}{2} g^2 V (A_1 \times A_2)^a \right. \\ &\quad \left. \times \left(\frac{1}{\sqrt{g^2(M - \mu_1 I) + m^2 I}}\right)^{ab} (A_1 \times A_2)^b \right].\end{aligned} \tag{23}$$

Taking account of Eqs. (17), (19), and (22), we get

$$\begin{aligned}\Psi_0 &= \exp\left[-\frac{1}{2} g V (A_1 \times A_2)^3 \left(\frac{1}{\sqrt{M - \mu_1 I + m^2 I}}\right)^{33} \right. \\ &\quad \left. \times (A_1 \times A_2)^3 \right] \\ &= \exp\left[-\frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{\mu_3 - \mu_1 + m^2}} \right].\end{aligned} \tag{24}$$

Now by assumption, in the strong-field limit,

$$g^2 \mu_3 = g^2 (A_1^2 + A_2^2) \gg m^2 \tag{25}$$

and

$$\mu_1 = \frac{1}{2} S \left(1 - \sqrt{1 - 4\frac{C}{S^2}}\right) \approx \frac{C}{S} \approx \frac{2}{g} R_0. \tag{26}$$

We recall that the ground-state solution of the zero-mode Schrödinger equation $\Psi_0 = \exp[-V R_0]$ with R_0 given in Eq. (12) is valid for $R_0 \sim 1/V$, where the wave function is non-negligible. In this same region of configuration space, μ_1 is negligible compared to μ_3, and Eq. (24) becomes

$$\begin{aligned}\Psi_0 &= \exp\left[-\frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{\mu_3}} \right] \\ &= \exp\left[-\frac{1}{2} g V \frac{(A_1 \times A_2) \cdot (A_1 \times A_2)}{\sqrt{A_1^2 + A_2^2}} \right]\end{aligned} \tag{27}$$

which is identical to the solution found for the ground state of the zero-mode Schrödinger equation, in the region of validity of that solution, where $V R_0 \sim O(1)$. Therefore, in a small region of configuration space where a nonperturbative treatment is possible, we find that our ansatz for the vacuum state agrees with the ground state of the zero-mode Yang-Mills Schrödinger equation.[1]

The argument above can also be extended to $3 + 1$ dimensions, as outlined in Appendix A.

III. DIMENSIONAL REDUCTION AND CONFINEMENT

Assuming that our proposal (1) for the Yang-Mills vacuum wave functional in $2 + 1$ dimensions is at least approximately correct, then where does the confinement property appear?

A long time ago it was suggested that the effective Yang-Mills vacuum wave functional at large scales, in $D = d + 1$ dimensions, has the form [1]

$$\Psi_0^{\text{eff}} \approx \exp\left[-\mu \int d^d x F_{ij}^a(x) F_{ij}^a(x) \right] \tag{28}$$

(see also [3,5]). This vacuum state has the property of "dimensional reduction": Computation of a large spacelike loop in $d + 1$ dimensions reduces to the calculation of a large Wilson loop in d Euclidean dimensions. Suppose $\Psi_0^{(3)}$ is the ground state of the $3 + 1$ dimensional theory, and $\Psi_0^{(2)}$ is the ground state of the $2 + 1$ dimensional theory. If these ground states both have the dimensional reduction form, and $W(C)$ is a large planar Wilson loop, then the area law falloff in $D = 3 + 1$ dimensions follows from confinement in two Euclidean dimensions in two steps:

$$\begin{aligned}W(C) &= \langle \mathrm{Tr}[U(C)] \rangle^{D=4} \\ &= \langle \Psi_0^{(3)} | \mathrm{Tr}[U(C)] | \Psi_0^{(3)} \rangle \sim \langle \mathrm{Tr}[U(C)] \rangle^{D=3} \\ &= \langle \Psi_0^{(2)} | \mathrm{Tr}[U(C)] | \Psi_0^{(2)} \rangle \sim \langle \mathrm{Tr}[U(C)] \rangle^{D=2}\end{aligned} \tag{29}$$

In $D = 2$ dimensions the Wilson loop can of course be calculated analytically, and we know there is an area-law falloff, with Casimir scaling of the string tensions. The dimensional reduction form of the ground state wave functional can be demonstrated explicitly in strong-coupling lattice gauge theory [2]; Monte Carlo support for the hypothesis has also been obtained at intermediate couplings [14,15].

It is natural to try and improve on the dimensional reduction idea by considering wave functionals which interpolate, in some natural way, between free-field dynamics at short distance scales, and the dimensional reduction form at large scales. In Ref. [7], Samuel suggested that the vacuum state in $D = 2 + 1$ dimensions might have the

[1]We learned from D. Diakonov that he had obtained this result in unpublished work, which considered a wavefunctional of similar form to (1) but without the λ_0, m terms in the kernel [13]. Those terms are not important in the region of configuration space discussed in this section.

form

$$\Psi_0[A] = \exp\Big[-\frac{1}{2}\int d^2x d^2y B^a(x) \times \Big(\frac{1}{\sqrt{-D^2 + m_0^2}}\Big)^{ab}_{xy} B^b(y)\Big]. \quad (30)$$

Our proposal differs from Samuel's in that m_0^2 is replaced by $-\lambda_0 + m^2$, with the lowest eigenvalue λ_0 being field-dependent and gauge-invariant. The rationale is that we should allow for a subtraction in the operator $-D^2$ appearing in the vacuum kernel; a subtraction will be absolutely required if the spectrum of $-D^2$, starting with λ_0, diverges in the continuum limit. On the other hand, if $m_0^2 < 0$ is a negative constant, then the wave functional in Eq. (30) is not necessarily real throughout configuration space, and can oscillate. Now the true vacuum state must be real up to a constant factor, and it is forbidden to pass through zero by the "no node" theorem for quantum-mechanical ground states. Requiring a subtraction which respects the reality of the wave functional, and avoids oscillations anywhere in field configuration space, dictates the replacement

$$m_0^2 \rightarrow -\lambda_0 + m^2 \quad (31)$$

with $m^2 \geq 0$.

The dimensional reduction form is obtained by dividing the field strength into "fast" and "slow" components, defined in terms of a mode cutoff. Let $\{\phi_n^a\}$ and $\{\lambda_n\}$ denote the eigenmodes and eigenvalues, respectively, of the covariant Laplacian operator in adjoint color representation, i.e.

$$-(D^2)^{ab}\phi_n^b = \lambda_n \phi_n^a. \quad (32)$$

The field strength can be expanded as a mode sum

$$B^a(x) = \sum_{n=0}^{\infty} b_n \phi_n^a(x) \quad (33)$$

and we define the slow component to be

$$B^{a,\mathrm{slow}}(x) = \sum_{n=0}^{n_{\max}} b_n \phi_n^a(x) \quad (34)$$

where $n_{\max}$ is a mode cutoff chosen such that $\Delta\lambda \equiv \lambda_{n_{\max}} - \lambda_0 \ll m^2$ remains fixed as $V \rightarrow \infty$. In that case, the portion of the (squared) vacuum wave functional Gaussian in B^{slow} is approximately

$$\exp\Big[-\frac{1}{m}\int d^2x B^{a,\mathrm{slow}} B^{a,\mathrm{slow}}\Big] \quad (35)$$

which is just the probability measure for Yang-Mills theory in two Euclidean dimensions, with a particular type of ultraviolet cutoff. The string tension for fundamental representation Wilson loops in $D = 2$ Yang-Mills theory, with coupling $g^2 m$, is easily computed:

$$\sigma = \tfrac{3}{16} g^2 m \quad (36)$$

or in lattice units, with lattice coupling β,

$$\sigma = \frac{3}{4}\frac{m}{\beta}. \quad (37)$$

In the next sections we will address two questions. First, suppose we fix m to give the known string tension at a given lattice coupling. What is then the value of the mass gap predicted by the vacuum wave functional, and to what extent does this agree with the corresponding value determined by standard lattice Monte Carlo methods? Second, since confinement depends on having $m \neq 0$, is there any reason why the mass parameter m should be nonzero?

IV. NUMERICAL SIMULATION OF THE VACUUM WAVEFUNCTIONAL

The mass gap implied by the vacuum state (1) can, in principle, be extracted from the equal-times connected correlator

$$\mathcal{D}(x-y) = \langle (B^a B^a)_x (B^b B^b)_y \rangle - \langle (B^a B^a)_x \rangle^2 \quad (38)$$

where the expectation value is taken with respect to the probability distribution $P[A]$ defined by the vacuum wave functional, i.e.

$$\langle Q \rangle = \int DA_1 DA_2 Q[A] P[A] \quad (39)$$

with

$$P[A] = |\Psi_0[A]|^2 = \exp\Big[-\frac{1}{g^2}\int d^2x d^2y B^a(x) K^{ab}_{xy}[A] B^b(y)\Big] \quad (40)$$

and

$$K^{ab}_{xy}[A] = \Big(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\Big)^{ab}_{xy}. \quad (41)$$

Here we have absorbed a factor of g into the definition of A_i, which accounts for the factor of $1/g^2$ in the exponent in eq. (40).

It not easy to see how $\mathcal{D}(x-y)$ could be computed analytically beyond the level of weak-coupling perturbation theory, but computation by numerical simulation of $P[A]$ also seems hopeless, at least at first sight. Not only is the kernel K^{ab}_{xy} nonlocal, it is not even known explicitly for arbitrary $A^a_k(x)$. However, suppose that after eliminating the wild variations of K along gauge orbits via a gauge choice, $K[A]$ has very little variance among thermalized configurations. In that case, things are more promising.

Let us define a probability distribution for gauge fields A which is controlled by a second, independent configuration A'

$$P[A;K[A']] = \det{}^{1/2}\left[\frac{1}{g^2}K[A']\right]\exp\left[-\frac{1}{g^2}\times\int d^2x d^2y B^a(x)K^{ab}_{xy}[A']B^b(y)\right] \quad (42)$$

where the field strength B is computed from the A-configuration, and both A and A' are fixed to some appropriate gauge. Now, assuming that the variance of $K[A]$ in the probability distribution $P[A]$ is small after the gauge choice, we can approximate

$$P[A] \approx P[A,\langle K\rangle] = P\left[A,\int DA' K[A']P[A']\right] \approx \int DA' P[A,K[A']]P[A'] \quad (43)$$

where the step from the first to the second line follows from assuming that the variance of K in the distribution $P[A]$ is small. If this assumption about $K[A]$ is correct, and Eq. (43) holds, then the probability distribution could in principle be generated by solving (43) iteratively:

$$P^{(1)}[A] = P[A;K[0]] \qquad P^{(n+1)}[A] = \int DA' P[A;K[A']]P^{(n)}[A']. \quad (44)$$

A numerical version of this approach would be to use equilibrium configurations of $P^{(n)}[A]$, generated at the nth step, to generate equilibrium configurations of $P^{(n+1)}$ at the $(n+1)$th step.

We may use the remaining gauge freedom to fix to an axial gauge in the two-dimensional time slice. This allows us to change variables in the functional integral over two-dimension configurations from A^a_1, A^a_2 to B^a, without introducing a field-dependent Jacobian. Let eigenvalues λ_n, and eigenmodes ϕ^a_n solve the eigenvalue equation

$$-D^2\phi_n = \lambda_n\phi_n \quad (45)$$

for the covariant Laplacian $-D^2$ determined from the fixed A' configuration, and let $\{b_n\}$ be the mode amplitudes of the B-field, as seen in the mode expansion (33). Then the probability distribution for the $\{b_n\}$, which follows from $P[A;K[A']]$ at fixed A', is Gaussian

$$\text{prob}\,[b_n] \propto \exp\left[-\frac{\beta}{4}\frac{b_n^2}{\sqrt{\lambda_n-\lambda_0+m^2}}\right]. \quad (46)$$

In practice we use a lattice regularization on an $L\times L$ lattice with periodic boundary conditions, and the gauge field $A^a_k(x,y)$ is initialized to zero at the first iteration. We then generate gauge fields recursively; the procedure at the nth iteration is as follows:

(1) From one of the lattice configurations generated at the $(n-1)$th iteration, compute the link variables in the adjoint representation, and then determine numerically the eigenvalues and eigenmodes of the two-dimensional lattice covariant Laplacian operator $-D^2$.

(2) Generate a set of $3L^2$ normally-distributed random numbers with unit variance, denoted $\{r_n\}$. From these, we obtain a new set of mode amplitudes

$$b_n = \sqrt{\frac{2}{\beta}}(\lambda_n-\lambda_0+m^2)^{1/4}r_n \quad (47)$$

and a corresponding B-field

$$B^a(x) = \sum_{n=0}^{3L^2} b_n\phi^a_n(x). \quad (48)$$

From the field strength $B^a(x)$, and the axial gauge condition, determine the corresponding gauge field $A^a_k(x)$. This step can be repeated to generate as many thermalized configurations of $P[A,K[A']]$ as desired.

(3) The gauge fields are exponentiated to give link variables

$$U_k(x,y) = \exp[iA^a_k(x,y)\sigma_a/2] \quad (49)$$

and any observables of interest are computed. This concludes the nth iteration.

Lattice configurations generated by this procedure will be referred to as "*recursion lattices*."

Details about our particular choice of axial gauge on a finite lattice, and the procedure for obtaining the A-field from the B-field in that gauge, may be found in Appendix B.

V. THE MASS GAP

The simulation procedure outlined in the last section leans heavily on the assumption that there is little variance in the kernel K^{ab}_{xy} in a fixed gauge, or, equivalently, that there is negligible variance, among thermalized configurations, in gauge-invariant combinations of the kernel such as $\text{Tr}[K^{-1}_{xy}K^{-1}_{yx}]$, or in the gauge-invariant spectrum of K. The absence of significant fluctuations in these quantities, when evaluated numerically, is a self-consistency requirement of the method we have proposed. The quantity $\text{Tr}[K^{-1}_{xy}K^{-1}_{yx}]$ is of particular interest, because its rate of falloff at large $|x-y|$ is determined by the mass gap.

We begin with the spectrum $\{\lambda_n-\lambda_0+m^2\}$ of the operator $-D^2-\lambda_0+m^2$, with m chosen, at a given β, to reproduce the string tension $\sigma(\beta)$ known from Monte Carlo simulations of the standard Wilson action in three Euclidean dimensions [16]. From Eq. (37), this means choosing

$$m = \tfrac{4}{3}\beta\sigma(\beta). \quad (50)$$

The result for the spectrum at $\beta = 18$ on a 50×50 lattice is shown in Fig. 1. The figure displays our results for ten separate recursion lattices, as well as the zero-field result $-\nabla^2+m^2$ for a very large volume lattice, with the eigenmode numbers rescaled by the factor $50^2/V$, so as to fit in

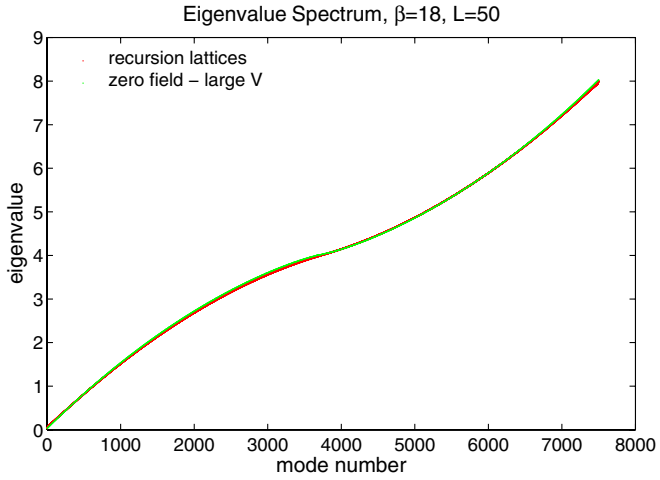

FIG. 1 (color online). Ten sets of eigenvalue spectra of the operator $-D^2 - \lambda_0 + m^2$, at $\beta = 18$, from ten independent 50×50 recursion lattices. Also plotted, but indistinguishable from the other spectra, is the rescaled spectrum of the large-volume zero-field operator $-\nabla^2 + m^2$.

the same range on the x-axis as the other ten data sets. It can be seen that, at the resolution of this figure, the spectra essentially all fall on top of one other. The ten separate data sets cannot be distinguished, and the spectrum of $-D^2 - \lambda_0$ looks identical to the (suitably rescaled) spectrum of $-\nabla^2$ at large volume. At higher resolution (Fig. 2) some fluctuation in the eigenvalue spectrum is observable, and the eigenvalues of the lowest-lying modes appear to deviate slightly from the zero-field large-volume spectrum.

Next we turn to the computation of the mass gap. According to Eq. (43),

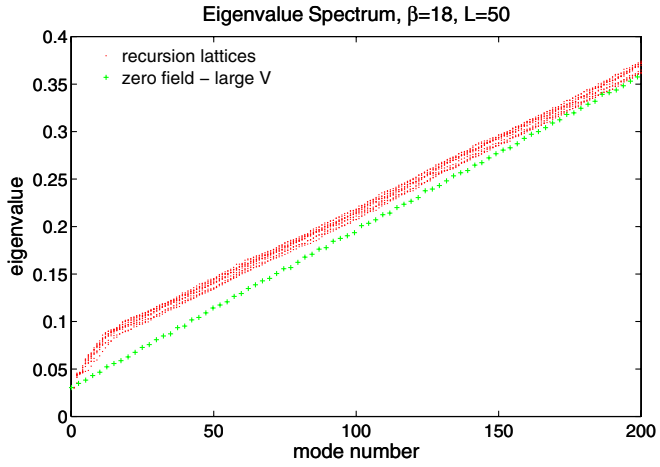

FIG. 2 (color online). Same as Fig. 1, for the lowest 200 eigenmodes. The closely spaced dots are from ten sets of eigenvalue spectra. The "+" symbols are taken from the rescaled spectrum of the large-volume zero-field operator.

$$\begin{aligned}\langle Q\rangle &= \int DA_1 DA_2 Q[A]P[A] \\ &\approx \int DA_1 DA_2 DA_1' DA_2' Q[A]P[A, K[A']]P[A'] \\ &= \int DBDA_1' DA_2' Q[A(B)]P[A(B), K[A']]P[A'] \quad (51)\end{aligned}$$

where we have changed variables, in an axial gauge, from gauge field A to field strength B as discussed in the last section. Evaluating in this way the rhs of (38) with $P[A, K[A']]$ as defined by Eq. (42), the integration over B is gaussian, and we find

$$\mathcal{D}(R) = \frac{8}{\beta^2} G(R) \quad (52)$$

where $R = |x - y|$ and (no sum over x, y)

$$\begin{aligned}G(R) &= \langle (K^{-1})^{ab}_{xy} (K^{-1})^{ba}_{yx} \rangle, \\ K^{-1} &= \sqrt{-D^2 - \lambda_0 + m^2}.\end{aligned} \quad (53)$$

Of course, the expectation value of $(K^{-1})^{ab}_{xy}(K^{-1})^{ba}_{yx}$ can also be evaluated by standard lattice Monte Carlo methods based on the $D = 3$ dimensional Wilson action. A number of thermalized lattices are generated by the usual heat bath procedure, and K^{-1} is evaluated on a two-dimensional constant-time slice of each three-dimensional lattice. The two-dimensional lattices generated in this way will be referred to as "MC lattices". They can be thought of as having been drawn from a probability weighting $P[U] = \Psi^2_{E,0}[U]$, where $\Psi_{E,0}[U]$ is the ground state of the transfer matrix of the $D = 3$ dimensional Euclidean lattice gauge theory.

Figure 3 shows the data for $G(R)$ at $\beta = 18$, averaged from a set of ten 50×50 recursion lattices, and, for comparison, corresponding data averaged from a set of ten 50×50 MC lattices at $\beta = 18$. Note the very small [$\sim O(10^{-12})$] magnitude of the observable at $R = 20$, yet even at this magnitude there seems to be very little noisiness in the data. Once again, this absence of noise is only possible if the variance in the $K^{-1}K^{-1}$ observable is negligible, which supports our original hypothesis. Moreover, the data obtained on recursion and MC lattices obviously agree very well with each other.

The mass gap is obtained by fitting the data for $G(R)$ to an appropriate functional form, and extracting the exponential falloff. Define

$$\begin{aligned}G_0(R) &= \delta^{ab}\delta^{ba}[(\sqrt{-\nabla^2 + \mu^2})_{xy}]^2 \\ &= \frac{3}{4\pi^2}(1 + \mu R)^2 \frac{e^{-2\mu R}}{R^6}. \quad (54)\end{aligned}$$

We have seen (Fig. 1) that the spectrum of $-D^2 - \lambda_0$ is almost identical to that of the zero-field Laplacian $-\nabla^2$. With this motivation, we introduce the fitting function

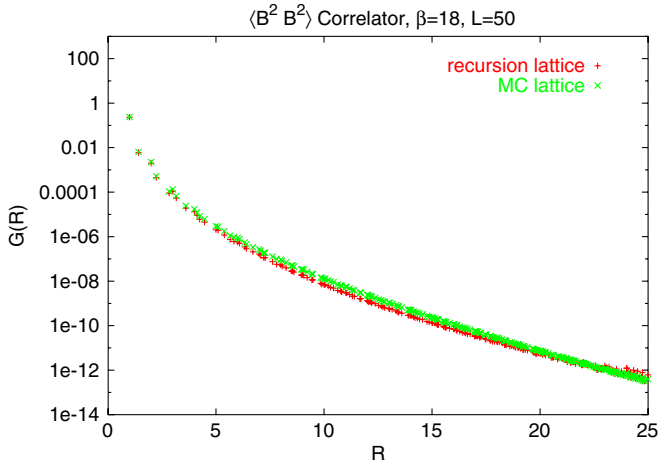

FIG. 3 (color online). The correlator $G(R)$ computed (i) on two-dimensional lattice configurations generated from the vacuum wave functional by the method described in the text; and (ii) on constant-time slices of three-dimensional lattice configurations generated by the usual lattice Monte Carlo method. Lattices generated by the first method are denoted "recursion," and by the second as "MC." In each case, the lattice extension is 50 sites at $\beta = 18$.

$$f_0(R) = \log\left[a\left(1 + \frac{1}{2}MR\right)^2 \frac{e^{-MR}}{R^6}\right] \tag{55}$$

and carry out a two parameter (a and M) best fit of $\log[G(R)]$ by $f_0(R)$.[2] The resulting value for M is an estimate of the mass gap. The best fit of the data for $G(R)$ at $\beta = 18$ on a 50^2 lattice by the fitting function $\exp[f_0(R)]$ is shown in Fig. 4.

In an old paper which anticipates the work in this section, Samuel [7] argued that $M \approx 2m_0$, where m_0 is the mass parameter in the vacuum state (30) which he proposed. This result is obtained if the covariant operator $-D^2$ in (30) is replaced by $-\nabla^2$. We believe that a more natural approximation is the replacement of $-D^2 - \lambda_0$ by $-\nabla^2$, since the lowest eigenvalue in the spectrum of each operator begins at zero. Thus the "naive" estimate for the mass gap, in our proposal, is $M = 2m$.

The results of extracting M via the best fit of f_0 to the data, for simulations of the vacuum wave functional at a variety of lattice couplings, are shown in Fig. 5. There we compare our values for the mass gap with those reported by Meyer and Teper in Ref. [16] (the values for $\sigma(\beta)$, used in Eq. (50), were also taken from this reference.) In Table I we list these mass gap results, as well as the mass gaps extracted from MC lattices, and the naive estimate $M(0^+) = 2m$. It can be seen that the agreement between

[2]The fits were carried out by the GNUPLOT package, which implements the Marquardt-Levenberg fitting algorithm. We have fit the data for $\log(G(R))$ on an $L \times L$ lattice in the interval $R \in [1, L/2]$. Error bars are estimated from the variance in mass gaps computed separately, at each β, on ten independent lattices.

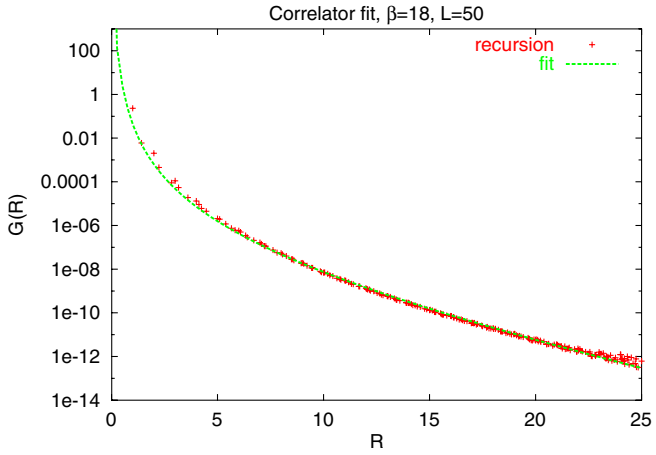

FIG. 4 (color online). Best fit (dashed line) of the recursion lattice data for $G(R)$ by the analytic form given in Eq. (54).

the reported values for the mass gap, and the masses we have obtained from simulation of our proposed wave functional (with parameter m fixed to give the observed asymptotic string tension), agree within a few (< 6) percent. This is a substantial improvement over the naive estimate of $M = 2m$, which disagrees with the Monte Carlo results by up to 20%.

VI. VACUUM ENERGY AND CONFINEMENT

Our proposed vacuum wave functional results in a nonvanishing asymptotic string tension, via the dimensional reduction argument, for any mass parameter $m > 0$. In this context, the question of why pure SU(2) gauge theory confines in $2 + 1$ dimensions boils down to why m is nonzero in that case, yet $m = 0$ in the Abelian theory. The answer must lie in energetics: For some reason the expectation value of $\langle H \rangle$ is lowered, in the non-Abelian theory, by having $m > 0$.

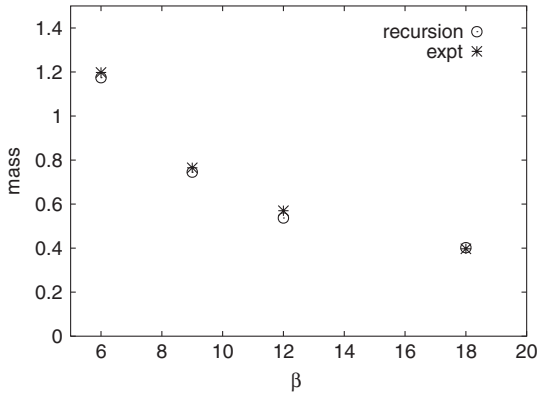

FIG. 5. Mass gaps extracted from recursion lattices at various lattice couplings, compared to the 0^+ glueball masses in $2 + 1$ dimensions obtained in Ref. [16] (denoted expt) via standard lattice Monte Carlo methods. Error bars are smaller than the symbol sizes.

TABLE I. The mass gaps in $D = 2 + 1$ dimensional Yang-Mills theory, at a variety of β values and lattice sizes L^2. Column 3 shows the values derived from the estimate $M = 2m$, and the values extracted from $G(R)$ computed on MC lattices are shown in column 4. Column 5 displays the results extracted from $G(R)$ computed from recursion lattices; these are the predictions obtained from numerical simulation of the vacuum wave functional. All of these values can be compared to the mass gaps reported in Ref. [16], shown in column 6, which were obtained by conventional lattice Monte Carlo methods.

		mass gap			
β	L^2	naive ($M = 2m$)	MC lattices	recursion lattices	"expt" Ref. [16]
6	24^2	1.031	1.269(5)	1.174(8)	1.198(25)
9	24^2	0.627	0.775(3)	0.745(5)	0.765(8)
12	32^2	0.445	0.562(5)	0.537(5)	0.570(11)
18	50^2	0.349	0.436(3)	0.402(4)	0.397(8)

The calculation of $\langle H \rangle$ is complicated by functional derivatives of the kernel $K[A]$. In this initial study we will simply ignore these derivatives, on the grounds that variance of the gauge-invariant product $K^{-1}K^{-1}$ among thermalized configurations has been found, in numerical simulations, to be negligible. In fact, this product seems to be remarkably well approximated, in any thermalized configuration, by the free-field expression $G_0(R)$ of Eq. (54). This insensitivity to the A-field suggests that the variation of $K[A]$ in the neighborhood of thermalized configurations is extremely small, and therefore the neglect of functional derivatives of K might be justified. However, we are not as yet able to quantify the actual error which is made by dropping those derivatives.

Writing $\Psi_0 = \exp(-R[A])$ where

$$R = -\frac{1}{2g^2}\int d^2x d^2y B^a(x) K^{ab}_{xy} B^b(y) \tag{56}$$

we find

$$H\Psi_0 = \left(T_0 - T_1 + \frac{1}{2g^2}\int d^2x B^2\right)\Psi_0 \tag{57}$$

where

$$\begin{aligned} T_0 &= \frac{g^2}{2}\int d^2x \frac{\delta^2 R}{\delta A^c_k(x)^2} \\ T_1 &= \frac{g^2}{2}\int d^2x \frac{\delta R}{\delta A^c_k(x)}\frac{\delta R}{\delta A^c_k(x)}. \end{aligned} \tag{58}$$

Carrying out the indicated functional derivatives of R, but dropping terms involving functional derivatives of the kernel K leads to

$$\begin{aligned} T_0 &= \frac{1}{2}\int d^2x d^2y \delta(x-y)(-D^2)^{ab} K^{ba}_{xy} \\ &= \frac{1}{2}\,\mathrm{Tr}\left[(-D^2)\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\right] \end{aligned} \tag{59}$$

and

$$\begin{aligned} T_1 &= \frac{1}{2g^2}\int d^2x d^2y B^a(x)\left[\frac{-D^2}{-D^2 - \lambda_0 + m^2}\right]^{ab}_{xy} B^b(y) \\ &= \frac{1}{2g^2}\int d^2x d^2y B^a(x)\left[1 + \frac{\lambda_0 - m^2}{-D^2 - \lambda_0 + m^2}\right]^{ab}_{xy} B^b(y). \end{aligned} \tag{60}$$

Altogether

$$\begin{aligned} \langle H \rangle = \frac{1}{2}\Bigg\langle &\mathrm{Tr}\left[\frac{-D^2}{\sqrt{-D^2 - \lambda_0 + m^2}}\right] - (\lambda_0 - m^2)\frac{1}{g^2} \\ &\times \int d^2x d^2y B^a(x)\left(\frac{1}{-D^2 - \lambda_0 + m^2}\right)^{ab}_{xy} B^b(y)\Bigg\rangle. \end{aligned} \tag{61}$$

Expanding $B(x)$ in eigenstates of $-D^2$

$$B^a(x) = \sum_n b_n \phi^a_n(x) \tag{62}$$

the second term on the rhs of Eq. (61) becomes

$$\text{2nd term} = (\lambda_0 - m^2)\frac{1}{g^2}\sum_n b_n^2 \frac{1}{\lambda_n - \lambda_0 + m^2}. \tag{63}$$

In the previous section, it was found that the eigenvalue spectrum $\{\lambda_n\}$ is almost unchanged from one equilibrium lattice to the next. Then, in the VEV shown in (61), we may replace b_n^2 by its VEV with the $\{\lambda_n\}$ fixed, which is $\frac{1}{2}g^2\sqrt{\lambda_n - \lambda_0 + m^2}$. Then

$$\begin{aligned} \text{2nd term} &= \frac{1}{2}(\lambda_0 - m^2)\sum_n \frac{1}{\sqrt{\lambda_n - \lambda_0 + m^2}} \\ &= \frac{1}{2}(\lambda_0 - m^2)\,\mathrm{Tr}\left[\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\right] \end{aligned} \tag{64}$$

which leads to

$$\langle H\rangle = \frac{1}{2}\Big\langle \mathrm{Tr}\frac{-D^2}{\sqrt{-D^2-\lambda_0+m^2}} - \frac{1}{2}\,\mathrm{Tr}\frac{\lambda_0-m^2}{\sqrt{-D^2-\lambda_0+m^2}}\Big\rangle = \frac{1}{2}\Big\langle \mathrm{Tr}\sqrt{-D^2-\lambda_0+m^2} + \frac{1}{2}\,\mathrm{Tr}\frac{\lambda_0-m^2}{\sqrt{-D^2-\lambda_0+m^2}}\Big\rangle. \quad (65)$$

Defining

$$\tilde{k}_n^2 \equiv \lambda_n - \lambda_0 \quad (66)$$

we finally obtain

$$\langle H\rangle = \frac{1}{2}\Big\langle \sum_n \Big(\sqrt{\tilde{k}_n^2+m^2} + \frac{1}{2}\,\frac{\lambda_0-m^2}{\sqrt{\tilde{k}_n^2+m^2}}\Big)\Big\rangle. \quad (67)$$

Suppose that the expectation value of the eigenvalues λ_n were independent of m^2, with zero variance, as in the free theory. Setting $\partial\langle H\rangle/\partial m^2 = 0$, the minimum vacuum energy is obtained trivially, at $m^2 = \langle\lambda_0\rangle$. In the Abelian free-field limit we have $\lambda_0 = 0$, so $m = 0$ at the minimum and the theory is not confining. In the non-Abelian theory, in contrast, $\lambda_0 > 0$, so $m^2 = \lambda_0 > 0$ at the minimum, and confinement is obtained. Of course, this simple result neglects both the m^2-dependence of the eigenvalue spectrum, as well as contributions arising from functional derivatives of the kernel. The situation can be improved on somewhat, at least regarding the m^2 dependence, by a numerical treatment.

A Monte Carlo evaluation of the energy density $\langle H\rangle/L^2$ as a function of m, for $\beta = 6$ and $L = 16$ and $\langle H\rangle$ as given in Eq. (67), is shown in Fig. 6. The minimum is away from zero, at roughly $m = 0.3$. This gives a string tension which is a little low; the known string tension of the Euclidean theory at $\beta = 6$ would require $m = 0.515$. This quantitative disagreement should not be taken too seriously, because the estimate for vacuum energy on which it is based, Eq. (67), is of unknown accuracy. Once again, in deriving (67), we have neglected some terms deriving from functional derivatives of the kernel. Even assuming, as we have, that those contributions are quite small (and this has not been shown), they could still have a large effect on the position of the minimum of a rather flat potential. The main point of this section is not to obtain m with any degree of accuracy (although that would have been desirable), but rather just to see that a nonzero value of m, which implies both confinement and a mass gap, is the natural outcome of a variational calculation.

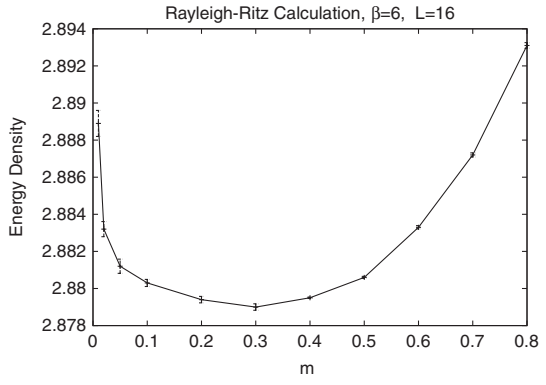

FIG. 6. Vacuum energy $\langle H\rangle$ of Eq. (67), per lattice site, computed at a variety of mass parameters m on a 16×16 lattice at lattice coupling $\beta = 6$.

VII. OTHER PROPOSALS

There have been other approaches to the Yang-Mills vacuum state in 2 + 1 dimensions. In particular, the vacuum wave functional proposed by Karabali, Kim, and Nair (KKN) in Ref. [11] has some strong similarities to ours, and the method we have developed for numerical simulation can be applied to the KKN vacuum state, as well as to our own proposal. This application is important, because we would like to test the claim that a string tension can be derived from the KKN state which agrees, to within a few percent, with the continuum limit of string tensions extracted from lattice Monte Carlo [17].

The KKN approach is formulated in terms of gauge-invariant field variables first introduced by Bars [18], and the idea is to solve for the ground state of the Hamiltonian, in these variables, in powers of the inverse coupling $1/g^2$. To lowest order, when reexpressed in terms of the usual A-field variables, their state has the dimensional reduction form

$$\Psi_0^{(0)} = \exp\Big[-\frac{1}{4mg^2}\int d^2x B^a(x)B^a(x)\Big] \quad (68)$$

where

$$m = \frac{g^2 C_A}{2\pi} \quad (69)$$

and C_A is the quadratic Casimir for the SU(N) group in the adjoint representation. Because this state has the dimensional reduction form, the corresponding string tension is easily deduced. In lattice units, for the SU(2) group, the predicted string tension is

$$\sigma_{\mathrm{KKN}}^{(0)} = \frac{6}{\pi\beta^2} \quad (70)$$

which is in rather close agreement with the lattice Monte Carlo results.

However, the state $\Psi_0^{(0)}$ is only the first term in a strong-coupling series. As it stands, it implies an infinite glueball mass in 2 + 1 dimensions, and it cannot be even approximately correct at short distance scales. The question is whether inclusion of the higher-order terms in the series,

which are necessary in order to have a nonzero correlation length, will affect the long-distance structure, and move the prediction for the string tension away from the desired value. KKN resum all of the terms in the strong-coupling series which are bilinear in their field variables, and when this expression is converted back to ordinary A-field variables, their resummed vacuum state has the form

$$\Psi_0 \approx \exp\Big[-\frac{1}{2g^2}\int d^2x d^2y B^a(x) \times \Big(\frac{1}{\sqrt{-\nabla^2+m^2}+m}\Big)_{xy} B^a(y)\Big]. \quad (71)$$

This state is gauge noninvariant as it stands, and for that reason must be incomplete. However, KKN argue that the further terms in the strong-coupling series, involving higher powers of the field variables and their derivatives, supply the extra terms required to convert the ∇^2 operator in Eq. (71) to a covariant Laplacian. So, according to Ref. [11], the vacuum state when reexpressed in ordinary variables has the form

$$\Psi_0 \approx \exp\Big[-\frac{1}{2g^2}\int d^2x d^2y B^a(x) \times \Big(\frac{1}{\sqrt{-D^2+m^2}+m}\Big)^{ab}_{xy} B^b(y)\Big]. \quad (72)$$

In this form, the KKN vacuum state is amenable to the numerical methods described above.

At this point we see that there may be trouble ahead for the previous string tension prediction. The problem is that the coefficient $1/(4mg^2)$ in the dimensional reduction form (68) is only obtained if the lowest eigenvalue λ_0 of the covariant Laplacian would be zero. We know that this is not the case. The effective long-distance wave functional, Gaussian in the B-field, is obtained as before via a mode cutoff in the B-field, and the actual estimate for the KKN string tension, according to dimensional reduction, is

$$\sigma_{\rm KKN} = \frac{3}{4\beta}\langle m + \sqrt{\lambda_0 + m^2}\rangle. \quad (73)$$

A nonzero λ_0 will certainly move the predicted string tensions $\sigma_{\rm KKN}$ away from values given in Eq. (70); the question is by how much. This can only be determined, at any given β, by numerical simulation.

In Table II we display our results for the string tension $\sigma_{\rm KKN}$, obtained by evaluating Eq. (73) in the vacuum state (72) by the methods developed in this paper. It is clear that there is a very substantial discrepancy between the predicted string tension $\sigma_{\rm KKN}$ and the string tension $\sigma_{\rm MC}$, obtained by standard Monte Carlo methods in Ref. [16]. The disagreement becomes disastrous if λ_0 actually diverges, in physical units, in the continuum limit. In that case the percentage discrepancy at $\beta \to \infty$ will be infinite. The only way out, that we can see, is if Eq. (72) is for some reason *not* the true resummation of the KKN strong-coupling expansion.

TABLE II. A comparison of the string tension $\sigma_{\rm KKN}$ calculated numerically from the Karabali-Kim-Nair vacuum wave functional (72), by methods developed above, with the values of the string tension $\sigma_{\rm MC}$ in $D = 3$ dimensions, computed by standard lattice Monte Carlo methods in Ref. [16].

β	L^2	$\sigma_{\rm KKN}$	$\sigma_{\rm MC}$	discrepancy
9	24^2	0.0340(4)	0.0261(2)	30%
12	32^2	0.0201(6)	0.0139(1)	45%

An approach which is closely related to that of KKN, relying on the same change of field variables, has been followed by Leigh, Minic, and Yelnikov (LMY) in Ref. [12]. This again results in an expression for the vacuum state which is the exponential of a bilinear term, with field variables connected by a field-dependent kernel. Whereas KKN perform a partial resummation of the strong-coupling series to arrive at their result, LMY rely on a conjectured operator identity (eq. (56) of Ref. [12]) to derive a differential equation for the kernel. The hope is that this gives an exact expression for the bilinear term in the wave function (of course there must be other terms also, because the resulting expression for the vacuum is not an exact eigenstate of the Hamiltonian). Since the derivation relies on a certain conjecture, the justification for the LMY wave functional so far lies in its predictions.

On the one hand, a glueball mass spectrum resulting from the LMY vacuum state has been derived, and this spectrum appears to be in very good agreement with existing Monte Carlo data. On the other hand, as in the KKN case, the string tension (same as (70)) and the spectrum are arrived at by neglecting the field-dependence of the kernel, which involves a holomorphic-covariant Laplacian. We have seen above that neglect of the field dependence of the kernel can be dangerous, and we think it likely that inclusion of this field-dependence will affect the LMY string tension and spectrum significantly. It may be possible to use the methods developed here to go beyond the zero-field approximation for the kernel, as we have done for the KKN state, to get a better idea of the true predictions of the LMY state. This is left for future investigation.

VIII. THE PROBLEM OF N-ALITY

The Casimir scaling of string tensions is inevitable for the lattice Yang-Mills action in two spacetime dimensions, and therefore this scaling, out to infinite charged source separations, seems to be a consequence of dimensional reduction to two dimensions. This feature cannot be true for the asymptotic string tension in $2+1$ and $3+1$ dimensions, except in the $N_c = \infty$ limit. Asymptotic string tensions in $D = 2+1$ and $3+1$ dimensions must depend only on the N-ality of the charged source, due to color

screening by gluons. The absence of color screening in $D = 2$ dimensions can be attributed to the fact that a gluon has $D - 2$ physical degrees of freedom in D dimensions. In two dimensions there are no physical degrees of freedom corresponding to propagating gluons. If there are no gluons there can be no string-breaking via dynamical gluons, and hence no N-ality dependence.

However, the vacuum state of a $d + 1$-dimensional gauge theory in temporal gauge does not, in general, describe a d-dimensional Euclidean Yang-Mills theory, despite the fact that each is expressed in terms of a gauge-invariant combination of d-dimensional vector potentials. For example, the vacuum state of the $2 + 1$-dimensional Abelian theory, shown in Eq. (6), describes the ground state of a theory of free, noninteracting photon states with a global SU(2) invariance. Our proposed vacuum state in Eq. (1) interpolates between a theory of noninteracting gluons at short distances, and the dimensional reduction form (28) at large scales. If this is the correct vacuum, then at intermediate distance scales it describes the ground state of strongly interacting gluons with physical degrees of freedom; these gluons are free to bind with an external source. In that case, the Minkowski-space picture of string-breaking via gluon pair production should somehow carry over to N-ality dependence for Wilson loops evaluated in the vacuum state at a fixed time.[3]

At present this is only an optimistic speculation, but the following observation may be relevant: It is possible to compute the ground state $\Psi_0[U]$ in strong-coupling Hamiltonian lattice gauge theory, and to identify the term in that ground state which is responsible for color screening. From the expansion of this term in powers of the lattice spacing, we can identify the leading correction to dimensional reduction. It turns out that this leading correction has the same form as the leading correction to dimensional reduction that is found in the proposed vacuum state $\Psi_0[A]$.

Denote the lattice vacuum state by $\Psi_0[U] = \exp[R(U)]$. A strong-coupling technique for calculating $R(U)$ in Hamiltonian lattice gauge theory was developed in Ref. [2]. In this expansion $R(U)$ is expressed as a sum over spacelike Wilson loops and products of loops on the lattice, as indicated schematically in Fig. 7. The coefficient c_i multiplying a contour constructed from (or filled by) n_P plaquettes is proportional to $(\beta^2)^{n_P}$. For SU(2) lattice gauge theory in $D = 2 + 1$ dimensions, the first few coefficients c_0, c_1, c_2, c_3 of the strong-coupling series for $R[U]$ were computed in Ref. [20]. The various terms in $R[U]$ can be expanded in a power series in the lattice

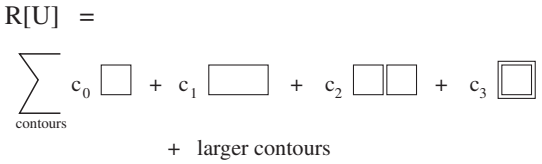

FIG. 7. The first few terms in the strong-coupling expansion of the lattice vacuum state $\Psi_0[U]$, with $R[U] = \log(\Psi_0[U])$.

spacing a, and for smoothly varying fields it is found that [20]

$$\Psi_0[U] = \exp\left[-\frac{2}{\beta}\int d^2x(a\kappa_0 B^2 - a^3\kappa_2 B(-D^2)B + \ldots)\right] \tag{74}$$

where

$$\kappa_0 = \tfrac{1}{2}c_0 + 2(c_1 + c_2 + c_3), \qquad \kappa_2 = \tfrac{1}{4}c_1 \tag{75}$$

and coefficient c_0 is $O(\beta^2)$, coefficients c_1, c_2, c_3 are $O(\beta^4)$.

There are several points to note, in connection with Eq. (74). First, dimensional reduction is associated with the term proportional to κ_0, which receives contributions from all four terms shown in Fig. 7, but the leading correction to dimensional reduction, in the term proportional to κ_2, comes from the 1×2 loop in $R[U]$ proportional to c_1. This is the contour which couples B (rather than B^2) terms in neighboring plaquettes. Secondly, it is not hard to see that the 1×2 loop in $R[U]$ gives rise to color screening. Consider evaluating a spacelike Wilson loop in the adjoint representation

$$W_{\text{adj}}[C] = \int DU\, \text{Tr}[U_{\text{adj}}(C)]\Psi_0^2[U]. \tag{76}$$

There is a nonzero contribution to the rhs of Eq. (76) which comes from lining the perimeter of the adjoint loop with overlapping 1×2 rectangular loops, as shown in Fig. 8, deriving from the power series expansion of $\Psi_0^2[U]$. For a rectangular loop of perimeter $P(C)$ this diagram gives a perimeter-law contribution

$$\left(\frac{c_1}{2}\right)^{P(C)-4} \tag{77}$$

to $W_{\text{adj}}(C)$.[4] Thus, the same term that gives the leading correction to dimensional reduction is also responsible for the screening of adjoint loops. Finally, we note that this leading correction, proportional to κ_2, comes in with a negative sign relative to the B^2 term.

[3]The transition from Casimir scaling to N-ality dependence, due to gluon string-breaking effects, is very likely to be associated with a vacuum center domain structure, as discussed recently in Ref. [19]. Gluon charge screening and vacuum center domains are simply two different descriptions, one in terms of particles, the other in terms of fields, of the same effect.

[4]Generalizing to an SU(N) theory, it is not hard to show (cf. Ref. [2]) that $c_0 \sim 1/g^4N$, $c_1 \sim 1/g^8N^3$, and that the perimeter-law contribution shown in Fig. 8 is down by an overall factor of $1/N^2$ relative to the leading area-law contribution, as it should be.

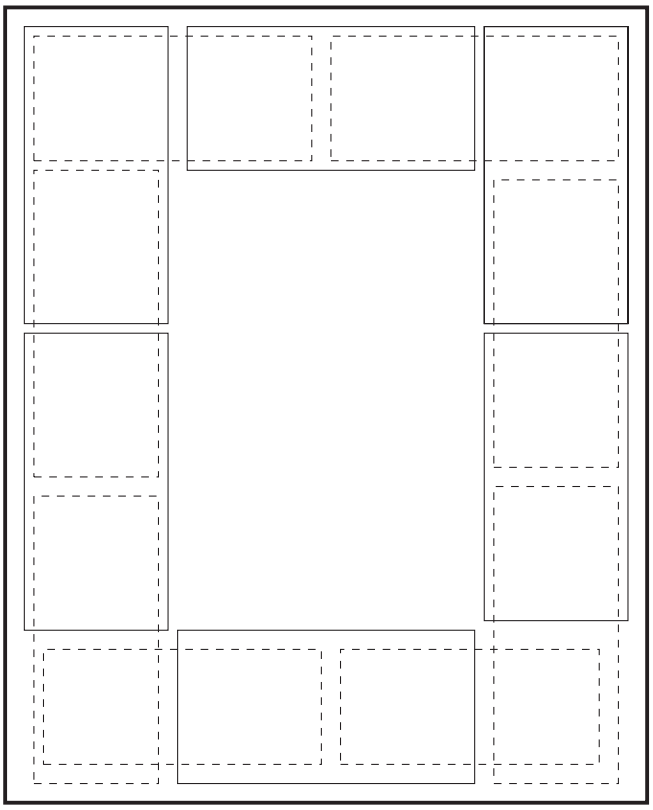

FIG. 8. How 1×2 rectangles in $R[U]$ screen an adjoint Wilson loop. The adjoint Wilson loop (in this case with extension 4×5 lattice spacings) is denoted by a heavy solid line. The overlapping 1×2 rectangles are indicated by (alternately) light solid and light dashed lines. The integration over lattice link variables yields a finite result, leading to a perimeter-law falloff [Eq. (77)] for large adjoint loops.

Now let us consider the leading correction to dimensional reduction in the proposed vacuum state $\Psi_0[A]$ of Eq. (1). The dimensional reduction term was given in Eq. (35), and is quadratic in $B^{\rm slow}$. The definition of $B^{\rm slow}$ in Eq. (34) involves a mode cutoff $n_{\rm max}$, chosen such that $\Delta\lambda \equiv \lambda_{n_{\rm max}} - \lambda_0 \ll m^2$, and the first correction to dimensional reduction comes from terms in the vacuum wave functional of order $(\lambda_n - \lambda_0)/m^2$, with $n < n_{\rm max}$. These are obtained from the $1/m^2$ expansion

$$\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} = \frac{1}{m}\left(1 - \frac{-D^2 - \lambda_0}{2m^2} + \ldots\right). \quad (78)$$

Taking the second term in the rhs into account, the part of the vacuum wave functional which is Gaussian in $B^{\rm slow}$ is

$$\exp\left[-\frac{1}{m}\int d^2x\left(B^{\rm slow}B^{\rm slow} - B^{\rm slow}\frac{-D^2 - \lambda_0}{2m^2}B^{\rm slow} + \ldots\right)\right] \quad (79)$$

where the ellipsis indicates higher powers of the covariant derivative. We note the similarity of Eq. (79) to the strong-coupling expression (74). In particular, there is in both cases a relative minus sign between the first and second terms.

The fact that the element responsible for color screening in $\Psi_0[U]$ generates, in a lattice spacing expansion, the $B(-D^2)B$ term coupling B fields in neighboring plaquettes, is a hint that it is this term which might be responsible for the color screening effect.[5] If so, the presence of a very similar correction to dimensional reduction, found in $\Psi_0[A]$, would presumably give rise to the same effect.

Of course, it is also possible that the vacuum state (1) is simply incomplete, and must be supplemented by some additional terms which are responsible for color screening. Cornwall [21] has recently conjectured that the dimensional reduction form (28) of the vacuum wave functional must be altered by the addition of a gauge-invariant mass term, implemented through the introduction of a group-valued auxiliary field $\Phi(x)$, i.e.

$$\Psi[A, \Phi] = \exp\left[-\int d^dx\{c_1\,\mathrm{Tr}[F_{ij}^2] + c_2\,\mathrm{Tr}[\Phi^{-1}D_k\Phi]^2\}\right] \quad (80)$$

The exponent of this state is stationary around center vortex solutions, suggesting a vacuum state dominated at large scales by center vortices. This would presumably solve the N-ality problem. At the moment, however, we lack any direct motivation from the Schrödinger wave functional equation for the existence of such a mass term.

For a discussion of the N-ality problem in the context of the KKN approach, see Ref. [22].

Another type of contribution which is expected to exist in the static quark potential is the Lüscher $-\pi(D-2)/24R$ term. We have no insight, at present, as to whether or not this term can be generated by the proposed vacuum state of Eq. (1).

IX. CONCLUSIONS

Our proposal for the ground state of quantized Yang-Mills theory, in $D = 2 + 1$ dimensions, has a number of virtues. Apart from agreeing with the ground state of the free theory in the appropriate limit, which is a natural starting point for any investigation of this type, we also find agreement in a highly nontrivial limit, where the Yang-Mills Schrödinger equation is truncated to the zero modes of the gauge field. In addition we find, surprisingly, that our vacuum state is amenable to numerical investigation, despite its very nonlocal character.

[5] In fact, apart from an overall sign, the $B(-D^2)B$ term looks like the kinetic term of a scalar field in the color adjoint representation in two Euclidean dimensions. Matter fields of that type can, of course, screen adjoint Wilson loops.

We believe that this vacuum state may provide some insight into the origins of confinement in a non-Abelian theory, and the precise relationship between the mass gap and the string tension. Confinement arises here via dimensional reduction, as proposed long ago in Ref. [1], and this reduction is obtained if the mass parameter m in the vacuum wave functional is nonzero. We have seen that $m \neq 0$ is likely to lower the vacuum energy, in $2+1$ dimensions, and this is related to the fact that in a non-Abelian gauge theory the lowest eigenvalue λ_0 of the covariant Laplacian is nonzero. The relation between m and the asymptotic string tension in $2+1$ dimensions is simple, i.e. $\sigma = 3m/4\beta$, and if the parameter m is chosen to produce the string tension known from earlier lattice Monte Carlo studies [16], then we find that the mass gap extracted from an appropriate correlator yields a value within 6% of the mass gap obtained by standard lattice Monte Carlo methods.

The most important unresolved question concerns higher representation string tensions. At issue is whether corrections to the simple dimensional reduction limit will convert Casimir scaling to N-ality dependence, as we have speculated in the previous section, or whether some additional terms (such as a gauge-invariant mass term [21]) are required. It would also be worthwhile to extend our considerations to $3+1$ dimensions, and to excited-state (glueball and flux-tube) wave functionals. These possibilities are currently under investigation.

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APPENDIX A: 3 + 1 DIMENSIONS

Although in this article we are mainly interested in the $2+1$ dimensional case, it is worth pointing out that the discussion in section II can be extended to $3+1$ dimensions. Define

$$\begin{aligned} S_3 &= A_1 \cdot A_1 + A_2 \cdot A_2 + A_3 \cdot A_3 \\ C_3 &= (A_1 \times A_2) \cdot (A_1 \times A_2) + (A_2 \times A_3) \cdot (A_2 \times A_3) \\ &\quad + (A_3 \times A_1) \cdot (A_3 \times A_1) \\ D_3 &= [A_1 \cdot (A_2 \times A_3)]^2. \end{aligned} \tag{A1}$$

The zero-mode Yang-Mills Hamiltonian is

$$H = -\frac{1}{2}\frac{1}{V}\frac{\partial^2}{\partial A_k^a \partial A_k^a} + \frac{1}{2} g^2 V C_3. \tag{A2}$$

Again we express Ψ_0 as in Eq. (9), and try to solve $H\Psi_0 = E_0\Psi_0$ to leading order in V. This time, with

$$R_0 = \frac{1}{2} g \frac{C_3}{\sqrt{S_3}} \tag{A3}$$

we find

$$T_0 = V\Big[-\frac{\partial R_0}{\partial A_k^a}\frac{\partial R_0}{\partial A_k^a} + g^2 C_3 \Big] = 0 + g^2 V\Big(\frac{7C_3^2}{4S_3^2} - \frac{3D_3}{S_3}\Big). \tag{A4}$$

In the large volume limit, the ground-state wave function will only be non-negligible in the region of the "Abelian valley," where the zero-mode components A_1, A_2, A_3 are nearly aligned, or antialigned, in color space. For definiteness, take the large color component (denoted by uppercase A) of the color 3-vectors to all lie in the color 3-direction; i.e.

$$A_1 = \begin{bmatrix} a_1^1 \\ a_1^2 \\ A_1^3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a_2^1 \\ a_2^2 \\ A_2^3 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} a_3^1 \\ a_3^2 \\ A_3^3 \end{bmatrix} \tag{A5}$$

and lowercase a denotes the small components. With $a \ll A$ and $VR_0 \sim O(1)$ it follows that, in the Abelian valley,

$$a \sim \frac{1}{\sqrt{gAV}} \tag{A6}$$

where A and a denote the magnitudes of the large (color 3-direction) and transverse field components, respectively. Since C_3^2 and D_3 are both $O(a^4)$, the nonzero terms contributing to T_0 in Eq. (14) are at most of order $1/V^2$ and can be neglected. Therefore $\Psi_0 = \exp[-VR_0]$, with R_0 as given in Eq. (A3), solves the zero-mode Yang-Mills Schrödinger equation to leading order in V, in the Abelian valley region away from the origin ($A_k = 0 \Rightarrow S_3 = 0$) of field space.

The generalization of Eq. (1) to $3+1$ dimensions is

$$\begin{aligned} \Psi_0[A] &= \exp[-Q] \\ &= \exp\Big[-\frac{1}{4}\int d^3x d^3y F_{ij}^a(x) \\ &\quad \times \Big(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\Big)_{xy}^{ab} F_{ij}^b(y)\Big]. \end{aligned} \tag{A7}$$

Again we consider a corner of configuration space in which only the nonzero modes make a significant contribution to the wave functional, and $|gA|^2 \gg m_0^2$. Then

$$(-D^2)_{xy}^{ab} = g^2 \delta^2(x-y) M^{ab} \tag{A8}$$

as before, with

$$M^{ab} = S_3\delta^{ab} - A_k^a A_k^b. \tag{A9}$$

For a configuration in the Abelian valley, with large components in the color 3 direction as shown in Eq. (A5), we find

$$\begin{aligned} Q &= \frac{1}{4} gV(A_i \times A_j)^a (M^{-1/2})^{ab} (A_i \times A_j)^b \\ &\approx \frac{1}{4} gV(A_i \times A_j)^a \left(\frac{\delta^{ab}}{\sqrt{S_3}} - \frac{\delta^{a3}\delta^{b3}}{\sqrt{S_3}} + \frac{\delta^{a3}\delta^{b3}}{m} \right) (A_i \times A_j)^b \end{aligned} \tag{A10}$$

Neglecting the overall coupling and volume factors, the relative orders of magnitude of each of the three contributions to Q are as follows:

$$\begin{aligned} \kappa_1 &= \frac{(A_i \times A_j) \cdot (A_i \times A_j)}{\sqrt{S_3}} \sim Aa^2 \\ \kappa_2 &= \frac{(A_i \times A_j)^3 (A_i \times A_j)^3}{\sqrt{S_3}} \sim \frac{a^4}{A} \\ \kappa_3 &= \frac{(A_i \times A_j)^3 (A_i \times A_j)^3}{m} \sim \frac{a^4}{m}. \end{aligned} \tag{A11}$$

Assume that $\kappa_2, \kappa_3 \ll \kappa_1$. Then we would have

$$Q = \frac{1}{2} gV \frac{C_3}{\sqrt{S_3}} \tag{A12}$$

and $\Psi_0[A]$, evaluated for large zero-mode gauge field configurations, would agree with the ground state solution of the zero-mode Yang-Mills Schrödinger equation in $D = 3 + 1$ dimensions, at least in the neighborhood of the Abelian valley. But we have already seen that for the solution of the zero-mode equation, the magnitude a of the small components is related to the magnitude A of the large components according to (A6). From this it follows that the assumption $\kappa_{2,3} \ll \kappa_1$ in the Abelian valley is self-consistent, and justified at large V for $m \neq 0$.

APPENDIX B: THE SPIRAL GAUGE

Since $\Psi_0[A]$ in temporal gauge and $2 + 1$ dimensions is gauge-invariant under two-dimensional gauge-transformations, then it is legitimate to carry out a further gauge-fixing in the two-dimensional plane when evaluating expectation values

$$\langle \Psi_0 | Q | \Psi_0 \rangle = \int DAQ[A]\Psi_0^2. \tag{B1}$$

In particular, with a complete axial gauge fixing, it is possible to change variables from A to field-strength B without introducing any further constraints or field-dependent Jacobian factors, i.e.

$$DA_1 DA_2 \rightarrow \text{const} \times DB. \tag{B2}$$

In higher dimensions, as Halpern has shown [3], this change of variables would be accompanied by a delta function enforcing the Bianchi constraints, but in two dimensions these constraints are absent.

The simplest approach is to set $A_1(n_1, n_2) = 0$ everywhere, where (n_1, n_2) are lattice site coordinates, and invert the discretized version of $B^a = \partial_1 A_2^a$ to determine A_2 from B. The problem with this is that setting $A_1 = 0$ everywhere on a finite, periodic lattice is more than a gauge choice. Gauge transformations cannot, in general, set the A field to zero everywhere on a closed loop, and lines parallel to the x-axis are closed by periodicity. Thus $A_1 = 0$ everywhere is a boundary condition, as well as a gauge choice. Although boundary conditions should be unimportant at sufficiently large lattice volumes, we would still like to keep such artificial conditions to a minimum, while retaining the simplicity of inverting $B^a = \partial_1 A_2^a$. A compromise is what we will call the "spiral gauge," in which we set $A = 0$ (or link variables $U = I_2$) along all links in a spiral around the toroidal lattice. An example, on a 10×10 lattice, is shown in Fig. 9. Along the straight sections of the spiral, parallel to the x-axis, we have

$$A_2^a(n_1 + 1, n_2) = B^a(n_1, n_2) + A_2^a(n_1, n_2). \tag{B3}$$

For the bent sections, its slightly different. Referring, e.g., to the bent section in Fig. 9 starting at $n_1 = 9$, $n_2 = 1$, we have

$$\begin{aligned} A_1^a(9, 2) &= -B^a(9, 1) - A_2^a(9, 1) \\ A_2^a(10, 2) &= B^a(9, 2) - A_1^a(9, 2). \end{aligned} \tag{B4}$$

Now suppose we start out with setting $A_2^a(1, 1) = 0$. Applying the above rules all around the spiral we can get

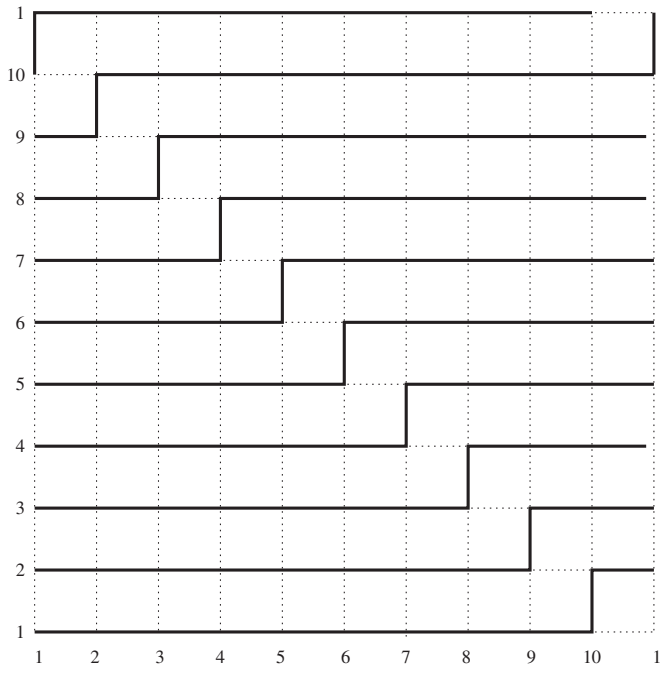

FIG. 9. The spiral gauge. Link variables on the solid lines are set equal to the identity.

all of the nonzero A-field variables from the B-field variables, but in order to come back to where we started, with $A_2(1, 1) = 0$, we have to require that

$$\sum_{n_1,n_2} B^a(n_1, n_2) = 0. \tag{B5}$$

To enforce this condition, we first generate the B-field without constraint, compute the sum

$$S^a = \sum_{n_1,n_2} B^a(n_1, n_2) \tag{B6}$$

and then make the readjustment

$$B^a(n_1, n_2) \to B^a(n_1, n_2) - \frac{S^a}{L^2}. \tag{B7}$$

So we have done two things beyond just fixing the gauge. First, the A-field has been set to zero on a single closed spiral around the toroidal lattice. Second, by setting in addition $A_2(1, 1) = 0$, we have imposed a restriction that the B-field on the lattice averages to zero in any given configuration. These conditions have been imposed for calculational simplicity; they are not as drastic as setting $A_1 = 0$ on all links (which sets all Polyakov lines in the x-direction equal to unity), and ought to be harmless at sufficiently large lattice volumes.

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