Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling

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A Hamiltonian formulation is given for the gravitational dynamics of two spinning compact bodies to next-to-leading order (G/c^4 and G^2/c^4) in the spin-orbit interaction. We use a novel approach (valid to linear order in the spins) which starts from the second-post-Newtonian metric (in Arnowitt-Deser-Misner coordinates) generated by two spinless bodies and computes the next-to-leading order precession, in this metric, of suitably redefined "constant-magnitude" 3-dimensional spin vectors $\mathbf{S}_1$, $\mathbf{S}_2$. We prove the Poincaré invariance of our Hamiltonian by explicitly constructing 10 phase-space generators realizing the Poincaré algebra. A remarkable feature of our approach is that it allows one to derive the *orbital* equations of motion of spinning binaries to next-to-leading order in spin-orbit coupling without having to solve Einstein's field equations with a spin-dependent stress tensor. We show that our Hamiltonian (orbital and spin) dynamics is equivalent to the dynamics recently obtained by Faye, Blanchet, and Buonanno, by solving Einstein's equations in harmonic coordinates.

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I. INTRODUCTION

In view of the needs of upcoming gravitational-wave observations, it is crucial to be able to describe in detail the dynamics of *spinning* compact binaries. We think that this aim will be fulfilled by combining the knowledge acquired by analytical techniques with that obtained by numerical ones. The present paper is devoted to a new, Hamiltonian analytical treatment of the general relativistic dynamics of spinning binaries.

The dynamics of spinning bodies in general relativity is a rather complicated problem which has been the subject of many works over many years (starting from the pioneering contributions of Mathisson [1], Papapetrou [2], Pirani [3], Tulczyjew [4], and others). This paper focuses on (gravitational) spin-orbit effects, i.e. dynamical effects which are *linear* in the spins of a binary system. The spin-orbit interaction can be analytically obtained as a post-Newtonian (PN) expansion. The leading-order contribution of this expansion is proportional to G/c^2, while the next-to-leading order one contains two sorts of terms: G/c^4 and G^2/c^4 (here G denotes Newton's gravitational constant and c the speed of light). The first complete derivation of leading-order (LO) spin-orbit effects in comparable-mass binary systems is due to Barker and O'Connell [5,6]. These authors derived the spin-orbit interaction by considering the *quantum* scattering amplitude of two spin-$\frac{1}{2}$ particles. This curious fact prompted several authors to give purely *classical* derivations of LO spin-orbit effects (see, e.g., Refs. [7–9]). For a discussion of LO spin-orbit effects in coalescing binary systems see Refs. [10,11].

The next-to-leading order (NLO) spin-orbit interaction was analytically tackled only over the last few years. After a first incomplete attack due to Tagoshi, Ohashi, and Owen [12], complete results were obtained very recently by Faye, Blanchet, and Buonanno [13], and Blanchet, Buonanno, and Faye [14]. Reference [13] calculated the translational equations of motion, as well as the rotational equations of motion for compact spinning binaries to NLO (as here, only terms linear in spin were considered). For their derivation, Blanchet *et al.*, working in harmonic coordinates, introduced the pole-dipole energy-momentum tensor due to Tulczyjew [4] in the Einstein field equations. They also used the general-relativistic-covariant *spin supplementary condition* (SSC) of Tulczyjew [4] or, equivalently in the linear-in-spin approximation, of Pirani [3].

The new derivation of NLO spin-orbit interactions in the present paper is based on a novel approach, and is totally independent from the results of Refs. [13,14]. At the end, we shall be able to connect our results to those of [13,14], thereby giving us confidence in the correctness of both

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investigations. We do not use Tulczyjew's pole-dipole energy-momentum tensor. We do not either make use of the Papapetrou (or, more completely, Mathisson-Papapetrou-Pirani) translational equations of motion. Our starting point consists of the second post-Newtonian (2PN) metric generated by spinless point masses in Arnowitt-Deser-Misner (ADM) coordinates, say $g_{(2\mathrm{PN})\mathrm{o}}$. The crux of our approach then consists in noting that (to linear order in the spins) it is enough to compute the NLO spin precession equations in $g_{(2\mathrm{PN})\mathrm{o}}$ to derive the spin-orbit NLO contribution in the Hamiltonian, say $H_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2)$. Then, from $H_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, \mathbf{S}_1, \mathbf{S}_2)$ we can derive the NLO spin-dependent terms in the translational equations of motion (simply by using Hamilton's canonical evolution equations). Technically, we shall derive the spin precession equations by starting from the 4-dimensional parallel transport equation for the spin 4-vector (with covariant spin supplementary condition), and then by rewriting them in terms of a suitably defined 3-dimensional spin vector, having a constant Euclidean magnitude. (This method is essentially that used in Ref. [7] at the LO.) We shall then check the Poincaré invariance of our Hamiltonian by explicitly constructing 10 phase-space generators realizing the Poincaré algebra (similarly to the proof of the Poincaré invariance of the 3PN orbital Hamiltonian given in [15]). After our construction, we shall give the relation with the results obtained in Refs. [13,14] in the form of explicit transformation formulas.

We leave to a sequent paper a discussion of the physical consequences of our Hamiltonian formulation, and notably its use for improving the description of spin effects within the *effective one-body approach* [16].

II. 3-DIMENSIONAL EUCLIDEAN SPIN VECTOR IN CURVED SPACETIME, AND ITS ANGULAR VELOCITY

When working to linear order in the spin, the translational and rotational equations of motion of a spinning particle in curved space [1–4] (see also [13,17]) read[1]

$$m\frac{\mathrm{D}u_\mu}{\mathrm{d}\tau} = \frac{1}{2}\frac{\epsilon^{\alpha\beta\lambda\rho}}{\sqrt{-g}}\tilde{S}_\alpha u_\beta u_\nu R^\nu{}_{\mu\lambda\rho}, \tag{2.1}$$

$$\frac{\mathrm{D}\tilde{S}_\mu}{\mathrm{d}\tau} = 0, \tag{2.2}$$

where u^μ is the normalized 4-velocity of the spinning particle, $u^\mu u_\mu = -1$, m its conserved mass, and $\tilde{S}^\mu$ its 4-dimensional spin vector; in addition, τ denotes the proper time parameter, $\mathrm{d}x^\mu/\mathrm{d}\tau = cu^\mu$, D the 4-dimensional covariant derivative, $R^\mu{}_{\nu\lambda\rho}$ the Riemann curvature tensor, and g the determinant of the 4-dimensional metric $g_{\mu\nu}$.

An important feature of our approach is that we shall not need to consider the *translational* equations of motion (2.1). It will be enough to consider the *rotational* ones (2.2). One immediate consequence of (2.2) is that the 4-dimensional length of $\tilde{S}^\mu$ is preserved along the world line

$$g^{\mu\nu}\tilde{S}_\mu\tilde{S}_\nu = s^2, \qquad s^2 = \mathrm{const}, \tag{2.3}$$

where $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$. The constant scalar s measures the proper magnitude of the spin. The Eqs. (2.1) and (2.2), to linear order in spin, are compatible with the covariant SSC

$$\tilde{S}_\mu u^\mu = 0. \tag{2.4}$$

At the same approximation, this (Pirani [3]) SSC is equivalent to the Tulczyjew [4] one $S^{\mu\nu}p_\nu^{\mathrm{kin}} = 0$, where $p_\mu^{\mathrm{kin}} = mcu_\mu + O(s^2)$ is the *kinematical* momentum (which differs from the canonical momentum we shall use below), and where $S^{\mu\nu}$ is the antisymmetric spin tensor (see, e.g., [13]).

More explicitly, Eq. (2.2) reads, when expressed in terms of the coordinate time $t \equiv x^0/c$,

$$\frac{\mathrm{d}\tilde{S}_\mu}{\mathrm{d}t} = c\Gamma^\nu{}_{\mu\rho}\tilde{S}_\nu v^\rho, \tag{2.5}$$

where $\Gamma^\nu{}_{\mu\rho}$ are the Christoffel symbols and $v^\mu \equiv c^{-1}\mathrm{d}x^\mu/\mathrm{d}t = u^\mu/u^0 = (1, v^i)$. Note that, in this paper, we normalize the "velocity" $v^i \equiv c^{-1}\mathrm{d}x^i/\mathrm{d}t$ so that it is dimensionless.

In addition, we can use Eq. (2.4) to compute the covariant time component of the spin vector in terms of its (covariant) spatial components:

$$\tilde{S}_0 = -\tilde{S}_i v^i. \tag{2.6}$$

Substituting this result into Eq. (2.3) one finds that the constancy of the 4-dimensional spin magnitude takes the 3-dimensional form

$$G^{ij}\tilde{S}_i\tilde{S}_j = s^2, \tag{2.7}$$

where G^{ij} is the symmetric matrix:

$$G^{ij} \equiv g^{ij} - g^{0i}v^j - g^{0j}v^i + g^{00}v^i v^j. \tag{2.8}$$

Now a technically very useful fact is that a *positive-definite symmetric* matrix such as the one just defined, G^{ij}, admits a *unique positive-definite symmetric square root*, say $H^{ij} = H^{ji}$, such that

$$G^{ij} = H^{ik}H^{kj}. \tag{2.9}$$

This uniqueness result (in some given coordinate system)

[1]In this paper, Greek indices run over the numbers 0, 1, 2, 3; Latin indices over 1, 2, 3; $\epsilon^{\lambda\rho\alpha\beta}$ is the completely antisymmetric (flat-spacetime) Levi-Civita symbol with $\epsilon^{0123} = 1$. Note that Ref. [17] uses an opposite sign convention for $\epsilon^{\lambda\rho\alpha\beta}$, which leads to an opposite sign on the right-hand-side of (2.1).

then naturally leads us to *defining* a constant-in-magnitude *3-dimensional Euclidean spin vector* $S_i \equiv S^i$ as[2]

$$S_i \equiv H^{ij}\tilde{S}_j, \qquad S_i S_i = s^2. \tag{2.10}$$

Upon further use of the spin supplementary condition (2.6), the spatial covariant component of the rotational equation of motion (2.5) yields

$$\frac{\mathrm{d}\tilde{S}_i}{\mathrm{d}t} = \tilde{V}^{ij}\tilde{S}_j, \tag{2.11}$$

where

$$\tilde{V}^{ij} \equiv c(\Gamma^j{}_{i0} + \Gamma^j{}_{ik}v^k - \Gamma^0{}_{i0}v^j - \Gamma^0{}_{ik}v^j v^k). \tag{2.12}$$

Making use of Eqs. (2.10) and (2.11) one can now easily derive an evolution equation for the constant-magnitude 3-dimensional spin vector S_i (dot means differentiation with respect to the coordinate time t):

$$\dot{S}_i = V^{ij}S_j, \qquad V^{ij} \equiv \dot{H}^{ik}(H^{-1})^{kj} + H^{ik}\tilde{V}^{kl}(H^{-1})^{lj}. \tag{2.13}$$

The constancy of the Euclidean magnitude of S_i implies that the matrix V^{ij} determining the "rotational velocity" of $S_i = S^i$ is *antisymmetric*: $V^{ij} = -V^{ji}$ (a result which is easily checked to hold for the explicit expression of V^{ij} given above). It is then convenient to "dualize" V^{ij} and to replace it by the 3-dimensional Euclidean (pseudo-)vector

$$\Omega_i \equiv -\tfrac{1}{2}\varepsilon_{ijk}V^{jk}. \tag{2.14}$$

With this notation the rotational equation of motion (2.13) reads

$$\dot{S}_i = +\varepsilon_{ijk}\Omega_j S_k. \tag{2.15}$$

In other words, we get a Newtonian looking *spin precession equation* $\dot{\mathbf{S}} = \mathbf{\Omega} \times \mathbf{S}$.

In summary, the angular velocity of rotation $\mathbf{\Omega}$ of the constant-magnitude spin 3-vector (2.10) is directly computable from the spacetime metric (and its Christoffel symbols) by using the explicit formulas (2.12), (2.13), and (2.14). (For the self-gravitating spinning particles we are considering, one will need, as usual, to regularize the self-interaction terms hidden in the formal results written above. See below.) Note that $\mathbf{\Omega}$ depends, in general, both on the positions *and the velocities* of all the particles in the system. Indeed, from the explicit formulas above, one sees that $\mathbf{\Omega}$ depends on the velocity of the considered spinning particle. Moreover, the metric and Christoffel symbols at the location of some particle will depend on the positions and velocities of the *other* particles.

III. DERIVING THE SPIN-ORBIT INTERACTION HAMILTONIAN FROM THE ANGULAR VELOCITY OF THE EUCLIDEAN SPIN 3-VECTOR

Let us now show how the knowledge of the just discussed spin angular velocity vector $\mathbf{\Omega}$ allows one to derive the spin-orbit interaction Hamiltonian H_{so}, i.e. the part of the Hamiltonian which is linear in the spin variables.

Let us first recall that a basic result in Hamiltonian dynamics is Darboux's theorem which says that any (nonsingular) symplectic form ω on an even-dimensional manifold can always be (locally) rewritten (after a suitable change of phase-space coordinates) in the canonical form $\omega = \sum_A \mathrm{d}q^A \wedge \mathrm{d}p_A$. When considering N (interacting) spinning particles, the dimension of phase space is $N(3+3+2) = 8N$, because the description of each particle requires: 3 spatial coordinates, 3 momenta, and *2 spin degrees of freedom*, such as two angles θ, ϕ needed to parametrize the direction of the (constant-magnitude) spin 3-vector S_i. Darboux's theorem then means, in this case, that it is always possible to redefine phase-space coordinates such that the symplectic form takes the form

$$\omega = \sum_a \Big(\sum_i \mathrm{d}q_a^i \wedge \mathrm{d}p_{ai} + s_a \mathrm{d}(-\cos\theta_a) \wedge \mathrm{d}\phi_a \Big).$$

Here $a = 1, \ldots, N$ labels the various particles (with $N = 2$ in our case), while $i = 1, 2, 3$ labels the spatial dimensions. We have written ω in the form it is known to take in special relativity [18,19]. In the latter case (and, say for simplicity, in the case of free particles), the spin-dependent term in ω was shown to take (globally) the form indicated, with s_a denoting the magnitude of the conserved spin of the ath particle, in the sense of (2.3), and with θ_a and ϕ_a denoting the polar angles of the flat-space limit of the above-introduced constant-magnitude Euclidean spin vector S_a^i, (2.10). When considering the *interacting* case (i.e. turning on a nonzero value of G/c^2), and when keeping, for simplicity, only the terms linear in spin (so that one can expand the dynamics in powers of both G and s_a), it is easily checked (by a perturbation analysis[3]) that it is always possible to construct Darboux-type canonical coordinates where the spin degrees of freedom are simply the polar angles (in a local orthonormal frame) of the above-introduced constant-magnitude Euclidean spin vector S_a^i.[4]

[2]A slightly more geometrical way of phrasing this definition would consist in saying that, starting from a given coordinate system, we are constructing a well-defined orthonormal "repère mobile" (or "vierbein") along the worldline of a spinning particle, with respect to which the covariant spin 4-vector has components $(0, S^i)$. By definition, the spatial components of the metric in this local orthonormal frame take the standard Euclidean values δ_{ij}, so that we can trivially raise or lower indices on our spin 3-vector.

[3]For example, by considering general coordinate changes of the form $q' = q + O(s)$, $p' = p + O(s)$ and working linearly in the spins s.

[4]As we shall discuss below, we can still modify S_a^i by a rather general local rotation, but the important point is that our definition of S_a^i, (2.10), is a smooth deformation of the correct flat-spacetime limit.

Finally, we can transcribe this result in the language of Poisson brackets (instead of that of a symplectic form), by stating that there exist phase-space variables $\mathbf{x} = (x_a^i)$, $\mathbf{p} = (p_i^a)$, and $\mathbf{S} = (S_i^a)$ (with $a = 1, \ldots, N$, and $i = 1, 2, 3$), where S_i^a are, say, the constant-magnitude vectors (2.10) such that the usual (Newtonian-like) Poisson brackets

$$\begin{aligned} &\{x_a^i, p_j^b\} = \delta_a^b \delta_j^i, \\ &\{S_i^a, S_j^b\} = \delta^{ab} \varepsilon_{ijk} S_k^a, \qquad \text{zero otherwise,} \end{aligned} \tag{3.1}$$

apply to the case of a general-relativistically interacting sytem of N spinning particles.

Note, however, that this result is essentially *kinematical*, and has nearly no *dynamical* content. To describe the dynamics of interacting spinning particles, we need to know the expression of the Hamiltonian in terms of the canonical variables: $H = H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$. As we work linearly in the spins, we look for a Hamiltonian of the general form:

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\rm o}(\mathbf{x}_a, \mathbf{p}_a) + H_{\rm so}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a). \tag{3.2}$$

Here, $H_{\rm o}$ denotes the orbital part of H, while $H_{\rm so}$ contains all the linear-in-spin terms, and can be called the "spin-orbit part." The orbital Hamiltonian $H_{\rm o}$ is explicitly known up to the 3PN order [15,20]. Our aim here is to compute the spin-orbit Hamiltonian $H_{\rm so}$ to NLO. Because $H_{\rm so}$ is, by definition, linear in the spins we can always write it in the general form

$$H_{\rm so}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a \mathbf{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b) \cdot \mathbf{S}_a, \tag{3.3}$$

where $\mathbf{\Omega}_a = (\Omega_a^i)$ depends on (all) the orbital degrees of freedom $(\mathbf{x}_b, \mathbf{p}_b)$, but does not depend on the spins $\mathbf{S}_b$. The scalar product in the Eq. (3.3) is the usual Euclidean one.

In Eq. (3.3) $\mathbf{\Omega}_a$ is *a priori* just a notation for the coefficient of $\mathbf{S}_a$ in $H_{\rm so}$. But let us now show that it is equal to the quantity computed in the previous section, i.e. the angular velocity with which the ath spin vector $\mathbf{S}_a$ precesses. Indeed, the general principles of Hamiltonian dynamics, together with the canonical Poisson brackets (3.1) and the form (3.3), yield

$$\dot{\mathbf{S}}_a = \{\mathbf{S}_a, H_{\rm so}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)\} = \mathbf{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b) \times \mathbf{S}_a. \tag{3.4}$$

The only difference between (3.4) and the previous result (2.15) is that, in (3.4), $\mathbf{\Omega}_a$ is expressed in terms of canonical positions and momenta, while in (2.15) $\mathbf{\Omega}$ was computed in terms of (say ADM) coordinates and coordinate velocities. Because we are working only to linear order in the spin, and because (as was explained above) the canonical phase-space coordinates appearing in (3.3) and (3.4) differ from the usual ADM-type coordinates used to express the metric [and thereby to compute the angular velocity $\mathbf{\Omega}_a(\mathbf{x}_b^{\rm ADM}, \mathbf{v}_b^{\rm ADM})$ by means of (2.12), (2.13), and (2.15)] only by terms proportional to the spins, it suffices to use the known [15,20] spinless link between ADM momenta and ADM velocities to compute $\mathbf{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b)$ from $\mathbf{\Omega}_a(\mathbf{x}_b^{\rm ADM}, \mathbf{v}_b^{\rm ADM})$.

In the previous section we introduced a specific, well-defined "conserved" spin 3-vector S_i to parametrize the 2 degrees of freedom of a spinning particle. Our choice had the nice features of being universally associated to the choice of a coordinate system, and of reducing to the choice made in the flat-spacetime limit [19]. However, it was by no means physically unique.

Let us now show that the freedom in the choice of conserved spin vector is simply a "gauge freedom" (local rotation group) which does not change the physical results one can deduce from the Hamiltonian. Indeed, the condition $S_i S_i = s^2$ leaves as ambiguity in the definition of the conserved spin variable S_i a local 3-dimensional Euclidean rotation $S_i \to S_i'$, with

$$S_i' = R_{ij} S_j, \tag{3.5}$$

where R is an arbitrary rotation matrix. It is sufficient to consider the case of an infinitesimal rotation, say

$$R_{ij} = \delta_{ij} - \theta_{ij}, \tag{3.6}$$

where θ_{ij} is a small antisymmetric matrix. This leads to an infinitesimal change

$$\delta\mathbf{S} = \boldsymbol{\theta} \times \mathbf{S}, \tag{3.7}$$

where we introduced the dual vector $\boldsymbol{\theta}$ such that $\theta_{ij} = \varepsilon_{ijk}\theta_k$.

Let us show that such a change can be considered as being induced by an infinitesimal canonical transformation g in the full phase-space $(\mathbf{x}, \mathbf{p}, \mathbf{S})$. (Canonical transformations are symmetries of Hamiltonian dynamics. In particular they preserve the basic Poisson brackets written above.) We recall that such a canonical transformation acts on any phase-space function f according to

$$\delta f = \{f, g\}. \tag{3.8}$$

It is then easily checked that a transformation of the form

$$g(\mathbf{x}, \mathbf{p}, \mathbf{S}) = \boldsymbol{\theta}(\mathbf{x}, \mathbf{p}) \cdot \mathbf{S} \tag{3.9}$$

transforms the spin vector according to

$$\delta\mathbf{S} = \{\mathbf{S}, g\} = \boldsymbol{\theta} \times \mathbf{S}, \tag{3.10}$$

which exactly reproduces the effect of an infinitesimal local rotation written above. However, we have learned that such a local rotation must be accompanied by a corresponding transformation of the orbital degrees of freedom $(\mathbf{x}, \mathbf{p})$ of the form $\delta\mathbf{x} = \{\mathbf{x}, g\}$, $\delta\mathbf{p} = \{\mathbf{p}, g\}$. Then, under the simultaneous changes of $\mathbf{x}$, $\mathbf{p}$, $\mathbf{S}$ induced by the canonical transformation g (and the corresponding change of the spin angular velocity $\mathbf{\Omega}' \simeq \mathbf{\Omega} + \mathrm{d}\boldsymbol{\theta}/\mathrm{d}t$) one finds that the numerical value (evaluated at corresponding phase-space points) of the Hamiltonian is invariant.

We have therefore shown that the arbitrariness in the "rotational state" of the conserved spin is simply (as expected) a "gauge symmetry" [under a local SO(3) group].

IV. DERIVATION OF THE SPIN-ORBIT HAMILTONIAN IN ADM COORDINATES

Let us now sketch the computation of the NLO angular velocity Ω_i in ADM coordinates [which will then give us the NLO spin-orbit Hamiltonian according to Eq. (3.3)].

As usual we split the four-dimensional metric $g_{\mu\nu}$ into three-dimensional objects $(\alpha, \beta_i, \gamma_{ij})$, where

$$\alpha \equiv (-g^{00})^{-1/2}, \qquad \beta_i \equiv g_{0i}, \qquad \gamma_{ij} \equiv g_{ij}. \tag{4.1}$$

One can show, using the definitions $\beta^i = \gamma^{ij}\beta_j$, $\gamma^{ij}\gamma_{jk} = \delta_{ik}$, that the following exact formulas hold:

$$\Gamma^0{}_{0i} = \frac{1}{\alpha}(\alpha_{,i} + K_{ij}\beta^j), \tag{4.2a}$$

$$\Gamma^0{}_{ij} = \frac{1}{\alpha}K_{ij}, \tag{4.2b}$$

$$\Gamma^i{}_{j0} = \frac{1}{2}\gamma^{ik}\gamma_{kj,0} - \frac{1}{\alpha}\beta^i\alpha_{,j} + \frac{1}{2}\gamma^{ik}(\beta_{k,j} - \beta_{j,k}) - \frac{1}{\alpha}\beta^i\beta^k K_{kj}, \tag{4.2c}$$

$$\Gamma^i{}_{jk} = {}^3\Gamma^i{}_{jk} - \frac{1}{\alpha}\beta^i K_{jk}, \tag{4.2d}$$

where K_{ij} is the extrinsic curvature of the constant time slice. Note that, for convenience, we use the $K_{ij} \sim +\dot{\gamma}_{ij}$ sign convention (instead of the $-\dot{\gamma}_{ij}$ convention used e.g. in Ref. [17]). In terms of the field momenta π^{ij} it reads,

$$K_{ij} = \frac{16\pi G}{c^3}\frac{1}{\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{kl}, \tag{4.3}$$

where $\gamma = \det(\gamma_{ij})$. The Christoffel symbols related with the 3-metric γ_{ij} are denoted by ${}^3\Gamma^i{}_{jk}$. Let us also note that the dimensionless coordinate velocity v^i can be expressed in terms of the *bare kinematical* linear momenta $p_i^{\rm bare} = mcu_i$, in full generality, as follows:

$$v^i = \frac{\alpha\gamma^{ij}p_j^{\rm bare}}{(m^2c^2 + \gamma^{kl}p_k^{\rm bare}p_l^{\rm bare})^{1/2}} - \gamma^{ij}\beta_j. \tag{4.4}$$

Note, however, that the latter result applies to the *canonical* momentum only modulo corrections proportional to the spin.

We employ the ADMTT (ADM transverse-traceless) coordinate conditions [21]

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4\delta_{ij} + h_{ij}^{\rm TT}, \qquad \pi^{ii} = 0, \tag{4.5}$$

and recall that

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\rm TT} \tag{4.6}$$

with $\pi^{ij}_{\rm TT}$ being of the order $1/c^5$ [22].

Let us now expand all quantities in a post-Newtonian expansion. Here and below the subscript (n) indicates the part of a quantity which is of the nth post-Newtonian order, i.e. which is proportional to $(1/c^2)^n$. For instance we decompose

$$\Omega_i = \Omega_{(2)i} + \Omega_{(4)i} + \mathcal{O}(c^{-6}). \tag{4.7}$$

Here $\Omega_{(2)i} \propto G/c^2$ is the well-known LO contribution [5,7–9], while $\Omega_{(4)i} \propto G/c^4 + G^2/c^4$ is the NLO contribution that we wish to compute. These contributions are more explicitly given in terms of the "precession velocity" $\tilde{V}^{ij}$ of the "coordinate spin vector" $\tilde{S}_i$, which entered Eq. (2.11). Inserting in Eq. (2.12) the Christoffel symbols (4.2), and then inserting the result in Eq. (2.13) [where H^{ij} is computed from Eqs. (2.8), (2.9), (4.1), (4.3), (4.5), and (4.6)], we obtain the following more explicit formulas for the 3-vectors $\Omega_{(2)i}$ and $\Omega_{(4)i}$ from Eq. (4.7):

$$\Omega_{(2)i}/c = \tfrac{1}{2}\varepsilon_{ijk}(\beta_{(3)j,k} + (\alpha_{(2),j} - \tfrac{1}{2}\phi_{(2),j})v^k), \tag{4.8a}$$

$$\begin{aligned}\Omega_{(4)i}/c = \frac{1}{2}\varepsilon_{ijk}\Big(&\beta_{(5)j,k} + \beta_{(3)k}\alpha_{(2),j} - \frac{1}{2}\phi_{(2)}\beta_{(3)j,k} \\ &+ \frac{1}{16}\phi_{(2)}\phi_{(2),j}v^k - \frac{1}{2}\phi_{(4),j}v^k - h^{\rm TT}_{(4)kl,j}v^l \\ &+ (\alpha_{(4),j} - \alpha_{(2)}\alpha_{(2),j})v^k + \tilde{\pi}^{jl}_{(3)}v^k v^l \\ &- \frac{1}{2}\alpha_{(2),k}v^j v^l v^l + \frac{1}{4}\frac{\dot{v}^j}{c}v^k v^l v^l\Big).\end{aligned} \tag{4.8b}$$

At this point, it only remains to implement three technical steps: (i) to insert the explicit form of the 2PN-accurate metric describing two spinless particles in ADMTT coordinates (from [23,24]), (ii) to replace the velocities v^i by their 1PN-accurate expression in terms of the canonical momenta p_i, and, finally, (iii) to regularize the self-interaction terms that arise when evaluating Eqs. (4.8).

The explicit expressions for the metric functions entering Eqs. (4.8) can be found e.g. in Appendix A of Ref. [22] [where the functions $\phi_{(2)}$, $\phi_{(4)}$, $\tilde{\pi}^{ij}_{(3)}$, and $h^{\rm TT}_{(4)ij}$ can be found] and in Ref. [23] [where the functions $\alpha_{(2)}$, $\alpha_{(4)}$ and $\beta_{(3)i}$, $\beta_{(5)i}$ are given].

As for reexpressing the velocities in terms of momenta, it yields a further PN expansion of the form $v_a^i = v_{a(1)}^i + v_{a(3)}^i + O(1/c^5)$, where[5] $v_{a(1)}^i$ is the coordinate velocity of the ath particle expressed in terms of the canonical variables $\mathbf{x}_a$ and $\mathbf{p}_a$ at the Newtonian accuracy, i.e., $v_{a(1)}^i = p_{ai}/(m_a c)$, and $v_{a(3)}^i$ is the 1PN correction to $p_{ai}/(m_a c)$.

[5]Here and below a, $b = 1, 2$ are the particles' labels, so m_a, $\mathbf{x}_a = (x_a^i)$, and $\mathbf{p}_a = (p_{ai})$ denote, respectively, the mass parameter, the position vector, and the linear momentum vector of the ath body; for $a \neq b$ we also define $\mathbf{r}_{ab} \equiv \mathbf{x}_a - \mathbf{x}_b$, $r_{ab} \equiv |\mathbf{r}_{ab}|$, $\mathbf{n}_{ab} \equiv \mathbf{r}_{ab}/r_{ab}$; $|\cdot|$ stands here for the Euclidean length of a 3-vector.

The latter 1PN correction explicitly reads

$$v^i_{1(3)} = \frac{G(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2c^3 r_{12}} n^i_{12} + \left(-\frac{\mathbf{p}_1^2}{2m_1^3 c^3} - \frac{3Gm_2}{m_1 c^3 r_{12}}\right) p_{1i} + \frac{7G}{2c^3 r_{12}} p_{2i}, \tag{4.9}$$

the expression for $v^i_{2(3)}$ can be obtained from the above by exchanging the particles' labels.

The final step then consists in evaluating [note that the meaning of $\Omega^i_{a(2)}$ and $\Omega^i_{a(4)}$ is now slightly different because of the reexpansion of velocities in a PN expansion]

$$\Omega^i_{a(2)}/c \equiv \tfrac{1}{2}\varepsilon_{ijk}\mathrm{Reg}_a(\beta_{(3)j,k} + (\alpha_{(2),j} - \tfrac{1}{2}\phi_{(2),j})v^k_{a(1)}), \tag{4.10a}$$

$$\begin{aligned}\Omega^i_{a(4)}/c \equiv \frac{1}{2}\varepsilon_{ijk}\mathrm{Reg}_a\Big(&\beta_{(5)j,k} + \beta_{(3)k}\alpha_{(2),j} - \frac{1}{2}\phi_{(2)}\beta_{(3)j,k} \\ &+ \frac{1}{16}\phi_{(2)}\phi_{(2),j}v^k_{a(1)} - \frac{1}{2}\phi_{(4),j}v^k_{a(1)} - h^{\mathrm{TT}}_{(4)kl,j}v^l_{a(1)} \\ &+ (\alpha_{(4),j} - \alpha_{(2)}\alpha_{(2),j})v^k_{a(1)} + \tilde{\pi}^{jl}_{(3)}v^k_{a(1)}v^l_{a(1)} \\ &- \frac{1}{2}\alpha_{(2),k}v^j_{a(1)}v^l_{a(1)}v^l_{a(1)} + \frac{1}{4}\frac{\dot{v}^j_{a(1)}}{c}v^k_{a(1)}v^l_{a(1)}v^l_{a(1)} \\ &+ \left(\alpha_{(2),j} - \frac{1}{2}\phi_{(2),j}\right)v^k_{a(3)}\Big),\end{aligned} \tag{4.10b}$$

where $\mathrm{Reg}_a(f(\mathbf{x}))$ indicates that one must regularize the limit $\mathbf{x} \to \mathbf{x}_a$. At the level at which we are working, this regularization is not ambiguous and can, for instance, be simply performed by using Hadamard's "partie finie" regularization (as explained, e.g., in Appendix B of Ref. [22]). The final results we got read

$$\mathbf{\Omega}_{1(2)} = \frac{G}{c^2 r_{12}^2}\left(\frac{3m_2}{2m_1}\mathbf{n}_{12}\times\mathbf{p}_1 - 2\mathbf{n}_{12}\times\mathbf{p}_2\right), \tag{4.11a}$$

$$\begin{aligned}\mathbf{\Omega}_{1(4)} = \frac{G^2}{c^4 r_{12}^3}&\left(\left(-\frac{11}{2}m_2 - 5\frac{m_2^2}{m_1}\right)\mathbf{n}_{12}\times\mathbf{p}_1 + \left(6m_1 + \frac{15}{2}m_2\right)\mathbf{n}_{12}\times\mathbf{p}_2\right) \\ &+ \frac{G}{c^4 r_{12}^2}\left(\left(-\frac{5m_2\mathbf{p}_1^2}{8m_1^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2} + \frac{3\mathbf{p}_2^2}{4m_1m_2} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{4m_1^2} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{2m_1m_2}\right)\mathbf{n}_{12}\times\mathbf{p}_1\right. \\ &+ \left(\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\mathbf{n}_{12}\times\mathbf{p}_2 \\ &\left.+ \left(\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{4m_1^2} - \frac{2(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\mathbf{p}_1\times\mathbf{p}_2\right).\end{aligned} \tag{4.11b}$$

The expressions for $\mathbf{\Omega}_{2(2)}$ and $\mathbf{\Omega}_{2(4)}$ can be obtained from the above formulas by exchanging the particles' labels.

From these results we can then explicitly write the spin-orbit Hamiltonian to leading and next-to-leading PN orders. Indeed,

$$\begin{aligned}H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) &= \sum_a \mathbf{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b)\cdot\mathbf{S}_a \\ &= \sum_a (\mathbf{\Omega}_{a(2)}(\mathbf{x}_b, \mathbf{p}_b) + \mathbf{\Omega}_{a(4)}(\mathbf{x}_b, \mathbf{p}_b))\cdot\mathbf{S}_a.\end{aligned} \tag{4.12}$$

More explicitly, the separate LO and NLO contributions in the PN expansion of the spin-orbit interaction term,

$$H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \frac{1}{c^2}H^{\mathrm{LO}}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \frac{1}{c^4}H^{\mathrm{NLO}}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.13}$$

read,

$$H^{\mathrm{LO}}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = c^2\sum_a \mathbf{\Omega}_{a(2)}(\mathbf{x}_b, \mathbf{p}_b)\cdot\mathbf{S}_a, \tag{4.14a}$$

$$H^{\mathrm{NLO}}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = c^4\sum_a \mathbf{\Omega}_{a(4)}(\mathbf{x}_b, \mathbf{p}_b)\cdot\mathbf{S}_a. \tag{4.14b}$$

Finally, note a remarkable feature of our Hamiltonian approach to spin-orbit effects: the sole computation of the *rotational* velocity of the (conserved) spin vector (given by parallel transport in the 2PN-accurate metric of N *spinless* bodies) determines the NLO *spin-dependent* terms in the *translational* equations of motion of N *spinning* particles. Indeed, the sole knowledge of $\mathbf{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b)$ yields that of the total spin-dependent Hamiltonian (3.2) with (3.3), so that the general principles of Hamiltonian dynamics (with canonical Poisson brackets) yield

$$\dot{\mathbf{x}}_a = +\frac{\partial H(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)}{\partial\mathbf{p}_a}, \qquad \dot{\mathbf{p}}_a = -\frac{\partial H(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)}{\partial\mathbf{x}_a}. \tag{4.15}$$

In view of the availability of algebraic manipulation programs, there is no need to write down explicitly the translational equations of motion (4.15), with NLO accuracy in spin-orbit terms (and 3PN accuracy in spin-independent terms [15,20]). We shall verify below that the Hamiltonian, ADM-coordinate translational equations of motion (4.15) are equivalent to the harmonic-coordinate ones recently derived in [13,14] by a more complex calculation which involved the explicit consideration of spin-dependent contributions in the metric.

V. POINCARÉ INVARIANCE

The general relativistic dynamics of an *isolated* N-body system should admit the full Poincaré group as a *global* symmetry (because it is a symmetry which preserves asymptotic flatness). On the other hand, this symmetry is *not manifest* in the Hamiltonian ADM approach to the N-body dynamics because it splits space and time, and uses non-Lorentz-covariant coordinate conditions. In a previous paper [15], treating nonspinning particles, the authors showed how to bypass this technical mismatch: the basic idea is that, in the Hamiltonian formalism, the global Poincaré symmetry is realized in phase space in a nonlinear manner. However, one can efficiently detect the presence of this symmetry by proving the existence of 10 phase-space generators $H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$, $P_i(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$, $J_i(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$, $G_i(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$ (depending on all phase-space variables) whose Poisson brackets reproduce the standard Poincaré algebra. In the case of nonspinning particles, Ref. [15] constructed the 10 generators of the Poincaré group at the 3PN level of approximation. We shall show here how to extend this construction to the more involved case of a system of spinning particles.

Let us first recall the explicit Poisson-bracket form of the Poincaré algebra that should be realized:

$$\{P_i, P_j\} = 0, \qquad \{P_i, H\} = 0, \qquad \{J_i, H\} = 0, \tag{5.1a}$$

$$\{J_i, P_j\} = \varepsilon_{ijk}P_k, \qquad \{J_i, J_j\} = \varepsilon_{ijk}J_k, \tag{5.1b}$$

$$\{J_i, G_j\} = \varepsilon_{ijk}G_k, \tag{5.1c}$$

$$\{G_i, H\} = P_i, \tag{5.1d}$$

$$\{G_i, P_j\} = \frac{1}{c^2}H\delta_{ij}, \tag{5.1e}$$

$$\{G_i, G_j\} = -\frac{1}{c^2}\varepsilon_{ijk}J_k. \tag{5.1f}$$

The translation, P_i, and rotation, J_i, generators are simply realized as

$$P_i(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a p_{ai}, \tag{5.2a}$$

$$J_i(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a (\varepsilon_{ik\ell}x_a^k p_{a\ell} + S_{ai}). \tag{5.2b}$$

Note the very simple, additive, form of these generators, and, in particular, how our Hamiltonian "conserved spin" variables appear as Newtonian-like (but relativistically correct) contributions.

As for the Hamiltonian H, we already know that (in our linear-in-spin approximation), it is a sum of an orbital part, H_{o}, and of the above-determined spin-orbit part, H_{so}, Eqs. (4.12), (4.13), and (4.14):

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) + H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a). \tag{5.3}$$

The orbital Hamiltonian H_{o} (including the rest-mass contribution) is explicitly known up to the 3PN order [15,20]:

$$H_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a m_a c^2 + H_{\mathrm{oN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^2}H_{\mathrm{o1PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4}H_{\mathrm{o2PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^6}H_{\mathrm{o3PN}}(\mathbf{x}_a, \mathbf{p}_a) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{5.4}$$

The most delicate generator to consider is the boost (or center-of-mass) vector $\mathbf{G}$. It can be represented as a sum of "orbital" and "spin-orbit" parts

$$\mathbf{G}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \mathbf{G}_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) + \mathbf{G}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a), \tag{5.5}$$

where, as everywhere in this paper, we call "spin-orbit" the part which is linear in the spin variables. The orbital part, $\mathbf{G}_{\mathrm{o}}$, was explicitly determined up to the the 3PN order in Ref. [15]:

$$\mathbf{G}_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a m_a \mathbf{x}_a + \frac{1}{c^2}\mathbf{G}_{\mathrm{o1PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4}\mathbf{G}_{\mathrm{o2PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^6}\mathbf{G}_{\mathrm{o3PN}}(\mathbf{x}_a, \mathbf{p}_a) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{5.6}$$

The spin-orbit part can be decomposed in leading-order, next-to-leading order, and further contributions:

$$\mathbf{G}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \frac{1}{c^2}\mathbf{G}_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \frac{1}{c^4}\mathbf{G}_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{5.7}$$

The leading-order term in (5.7) is known from the special-relativistic limit (by replacing the special-relativistic energy in the results of, e.g., Refs. [19,25], by the rest-mass contribution)

$$\mathbf{G}_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = -\frac{\mathbf{S}_1 \times \mathbf{p}_1}{2m_1} + (1 \leftrightarrow 2), \tag{5.8}$$

where the operation "$+(1 \leftrightarrow 2)$" denotes the addition to each displayed term of another one obtained by exchanging the particles' labels.

The real difficulty lies in constructing the NLO contribution to the boost generator (and in proving that it satisfies the correct Poincaré algebra displayed above). We solved this problem by using (as in our previous work [15]) the method of undetermined coefficients. The most general form of $\mathbf{G}_{\mathrm{so}}^{\mathrm{NLO}}$ can *a priori* depend on 8 unknown dimensionless numerical coefficients $g_1, \ldots, g_8$:

$$\begin{aligned}\mathbf{G}_{\mathrm{so}}^{\mathrm{NLO}} = {}& \frac{\mathbf{p}_1^2}{8m_1^3}\mathbf{S}_1 \times \mathbf{p}_1 + \frac{Gm_2}{r_{12}}\Bigg(g_1\frac{\mathbf{S}_1 \times \mathbf{p}_1}{m_1} + g_2\frac{\mathbf{S}_1 \times \mathbf{p}_2}{m_2} \\ &+ \left(g_3\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1} + g_4\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\right)\mathbf{n}_{12} \times \mathbf{S}_1 \\ &+ \left(g_5\frac{(S_1, n_{12}, p_1)}{m_1} + g_6\frac{(S_1, n_{12}, p_2)}{m_2}\right)\mathbf{n}_{12}\Bigg) \\ &+ \frac{Gm_2}{r_{12}^2}\left(g_7\frac{(S_1, n_{12}, p_1)}{m_1} + g_8\frac{(S_1, n_{12}, p_2)}{m_2}\right)\mathbf{x}_1 \\ &+ (1 \leftrightarrow 2),\end{aligned} \tag{5.9}$$

where we have introduced the following notation for the Euclidean mixed product of 3-vectors: $(V_1, V_2, V_3) \equiv \mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3) = \varepsilon_{ijk} V_1^i V_2^j V_3^k$. Note that the coefficient of the first term on the right-hand side is determined by considering the special-relativistic limit [18,19,25]. We have also used some structural information coming from a conceivable field-theory computation of $\mathbf{G}$ (say as the space integral of the $0i$ component of some effective stress-energy tensor). Indeed, such a computation could be thought of in terms of some Feynman-like diagrams, where the interaction terms (i.e. those containing a power of G) would all be proportional to some basic "source" term involving either S_1 and m_2 (connected by a propagator, and possibly some power of the velocities, $v_1 \sim p_1/m_1$ or $v_2 \sim p_2/m_2$), or similar terms involving S_2 and m_1. The main point being that pure "self-interaction" terms (say proportional to S_1 and m_1) cannot appear.

Let us now consider the explicit Poincaré algebra requirements of Eqs. (5.1a)–(5.1f). It is easily verified that the generators P_i, J_i, H, and G_i, in the forms given above, exactly satisfy the relations (5.1a)–(5.1c). We now consider whether the center-of-mass vector $\mathbf{G}$ with the 2PN spin-orbit part given by Eq. (5.9) can satisfy the three relations (5.1d)–(5.1f). This requirement yields many equations that have to be satisfied by the unknown coefficients $g_1, \ldots, g_8$. We have first found that there exist *unique* values of the coefficients $g_1, \ldots, g_8$ ensuring the fulfillment of the sole relation (5.1d). These values are

$$\begin{aligned} &g_1 = \tfrac{5}{4}, \qquad g_2 = -\tfrac{3}{2}, \qquad g_3 = 0, \qquad g_4 = -\tfrac{1}{2}, \\ &g_5 = -\tfrac{1}{4}, \qquad g_6 = 1, \qquad g_7 = \tfrac{3}{2}, \qquad g_8 = -2. \end{aligned} \tag{5.10}$$

Then we have checked that the solution (5.10) also guarantees the fulfillment of the remaining relations (5.1e) and (5.1f).

In summary, we succeeded in proving the Poincaré invariance of the above-defined NLO spin-orbit interaction [determined by Eqs. (4.12), (4.13), and (4.14)] by explicitly constructing 10 phase-space generators satisfying the Poincaré algebra brackets of Eqs. (5.1a)–(5.1f).

VI. COMPARISON WITH HARMONIC-COORDINATE-BASED RESULTS

References [13,14] recently computed, by means of two separate calculations and in *harmonic coordinates*, both the NLO spin-dependent contributions in the translational equations of motion of two spinning particles, and the corresponding NLO terms in the spin precessional equations of motion. In the present section we shall prove that our results are physically equivalent to the results of Refs. [13,14] by finding the explicit form of the transformation that match the ADM variables used by us with the harmonic variables used in Refs. [13,14]. Let us start by warning the reader that in the whole paper [13] and in most of the paper [14] Blanchet *et al.* chose to express their results in terms of some "nonconserved" spin variables $\mathbf{S}_a^{\rm BBF}$, i.e. variables whose Euclidean magnitudes are not conserved in time. It is only in Sec. VII of [14] that redefined spin variables with conserved Euclidean lengths, say $\mathbf{S}_a^{\rm cBBF}$, are introduced and used.

Our task here will be to exhibit the explicit transformation between the "ADM variables" $(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$ used in our work, and the "harmonic variables" $(\mathbf{y}_a, \mathbf{v}_a \equiv \dot{\mathbf{y}}_a, \mathbf{S}_a^{\rm cBBF})$ used in [13,14], and to prove that this transformation maps the two sets of results into each other. (It is more convenient for us to exhibit the link with the conserved version of the harmonic spin variable used by Blanchet *et al.* The relation between their two spin variables, $\mathbf{S}_a^{\rm BBF}$ and $\mathbf{S}_a^{\rm cBBF}$, is given in Eq. (7.4) of [14].)

We write the transformation of variables in the general form[6]

$$\mathbf{y}_a(t) = \mathbf{Y}_a(\mathbf{x}_b(t), \mathbf{p}_b(t), \mathbf{S}_b(t)), \tag{6.1a}$$

$$\mathbf{S}_a^{\rm cBBF}(t) = \mathbf{\Sigma}_a(\mathbf{x}_b(t), \mathbf{p}_b(t), \mathbf{S}_b(t)). \tag{6.1b}$$

Let us first find the transformation $\mathbf{\Sigma}_a$ between spin variables. Section VII of Ref. [14] gives [see Eq. (7.6) there] the explicit result for the angular velocity vector $\mathbf{\Omega}_a^{\rm BBF}$ of their conserved harmonic spin variable, yielding a spin precessional equation of motion of the form

$$\frac{\mathrm{d}\mathbf{S}_a^{\rm cBBF}}{\mathrm{d}t} = \mathbf{\Omega}_a^{\rm BBF} \times \mathbf{S}_a^{\rm cBBF}, \qquad a = 1, 2. \tag{6.2}$$

They give the NLO expression of $\mathbf{\Omega}_a^{\rm BBF}$ in terms of the harmonic orbital coordinates $(\mathbf{y}_a, \mathbf{v}_a)$. We have reexpressed $\mathbf{\Omega}_a^{\rm BBF}(\mathbf{y}_b, \mathbf{v}_b)$ in terms of ADM coordinates and momenta, to 1PN accuracy (using the well-known link between the two sets of variables[7]). We then compared the result with our results (4.11). We have found

$$\mathbf{\Omega}_{a(2)}^{\rm BBF}(\mathbf{y}_b, \mathbf{v}_b) = \mathbf{\Omega}_{a(2)}(\mathbf{x}_b, \mathbf{p}_b), \tag{6.3a}$$

$$\mathbf{\Omega}_{a(4)}^{\rm BBF}(\mathbf{y}_b, \mathbf{v}_b) = \mathbf{\Omega}_{a(4)}(\mathbf{x}_b, \mathbf{p}_b) + \frac{\mathrm{d}\boldsymbol{\theta}_a}{\mathrm{d}t}, \tag{6.3b}$$

where

$$\begin{aligned} \boldsymbol{\theta}_1 = \frac{G}{c^4 r_{12}}\bigg(&-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{4m_1}\mathbf{n}_{12}\times\mathbf{p}_1 + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\mathbf{n}_{12}\times\mathbf{p}_2 \\ &- \frac{9}{4m_1}\mathbf{p}_1\times\mathbf{p}_2\bigg). \end{aligned} \tag{6.4}$$

[6]Here, both sides refer to the same numerical value of their respective coordinate times.

[7]We recall that harmonic and ADM *coordinates* coincide at 1PN, but that one must transform velocities into momenta by means of the 1PN transformation Eq. (4.9). See [24] for the 3PN-accurate version of this transformation.

From the results (6.3) and (6.4) it is easy to deduce that the two sets of spin precession equations of motion are physically equivalent,[8] and that the two sets of spin variables are related as in the general transformation links written above with a spin transformation $\mathbf{\Sigma}_a$ of the explicit form:

$$\mathbf{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{S}_a + \boldsymbol{\theta}_a(\mathbf{x}_b, \mathbf{p}_b) \times \mathbf{S}_a. \tag{6.5}$$

In other words, our conserved spin variable differs from the conserved spin variable defined in Eq. (7.4) of [14] by a small (time-dependent) rotation of angle $\boldsymbol{\theta}_a(\mathbf{x}_b, \mathbf{p}_b)$. Such a difference was *a priori* to be expected because constant-magnitude spin vectors are not uniquely defined. We have shown above that to each choice of coordinate system is canonically associated a particular choice of local orthonormal frame (along the worldline of a spinning particle), and thereby a particular choice of conserved spin 3-vector. We have investigated whether the conserved spin 3-vector defined by Blanchet *et al.* does correspond to applying our general definition to the case of harmonic coordinates. The answer is "no." We found that if Blanchet *et al.* had used our general definition (2.10) in their harmonic-coordinate system, the angular velocity $\mathbf{\Omega}_a$ that they would have obtained would differ from our ADM spin vector by a rotation vector $\boldsymbol{\theta}_a$ differing from the result above by having the factor 9 replaced by 1 in the last term of Eq. (6.4). There is nothing surprising in such a difference as the spin redefinition used by Blanchet *et al.* was somewhat arbitrary. Anyway, as already mentioned above physical results will not depend on such "gauge choices."

Let us now turn to the determination of the transformation $\mathbf{Y}_a$ between ADM and harmonic *orbital* degrees of freedom. As usual we can decompose $\mathbf{Y}_a$ into spin-independent $\mathbf{Y}_a^{\text{o}}$ and spin-dependent (and linear-in-spin) $\mathbf{Y}_a^{\text{so}}$ terms:

$$\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{x}_a + \mathbf{Y}_a^{\text{o}}(\mathbf{x}_b, \mathbf{p}_b) + \mathbf{Y}_a^{\text{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \tag{6.6}$$

where the spin-dependent term is of the form

$$\mathbf{Y}_a^{\text{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{Y}_{a(2)}^{\text{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) + \mathbf{Y}_{a(4)}^{\text{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) + \mathcal{O}(c^{-6}). \tag{6.7}$$

The spin-independent part of the transformation was explicitly given, up to the 3PN order, in Ref. [24]. The leading-order spin-dependent part has been known for many years (see, e.g., Ref. [26]), and equals

$$\mathbf{Y}_{a(2)}^{\text{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \frac{\mathbf{S}_a \times \mathbf{p}_a}{2m_a^2 c^2}. \tag{6.8}$$

We have determined the next-to-leading order spin-dependent part, $\mathbf{Y}_{a(4)}^{\text{so}}$, by using again the method of undetermined coefficients. We have considered the most general template for $\mathbf{Y}_{a(4)}^{\text{so}}$ which depends (after using the special-relativistic limit to determine the $1/c^4$ term which remains in the $G \to 0$ limit, and structural information of the same type as that explained above in the case of $\mathbf{G}_{\text{so}}$)[9] on 12 unknown coefficients. It reads

$$\begin{aligned}\mathbf{Y}_{1(4)}^{\text{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = {}& -\frac{\mathbf{p}_1^2}{8c^4 m_1^4}\mathbf{S}_1 \times \mathbf{p}_1 + \frac{Gm_2}{c^4 r_{12}}\frac{1}{m_1}\bigg(a_1 \frac{\mathbf{S}_1 \times \mathbf{p}_1}{m_1} + a_2 \frac{\mathbf{S}_1 \times \mathbf{p}_2}{m_2} + \bigg(a_3 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1} + a_4 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\bigg)\mathbf{n}_{12} \times \mathbf{S}_1 \\ & + \bigg(a_5 \frac{(S_1, n_{12}, p_1)}{m_1} + a_6 \frac{(S_1, n_{12}, p_2)}{m_2}\bigg)\mathbf{n}_{12}\bigg) + \frac{G}{c^4 r_{12}}\bigg(b_1 \frac{\mathbf{S}_2 \times \mathbf{p}_1}{m_1} + b_2 \frac{\mathbf{S}_2 \times \mathbf{p}_2}{m_2} \\ & + \bigg(b_3 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1} + b_4 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\bigg)\mathbf{n}_{12} \times \mathbf{S}_2 + \bigg(b_5 \frac{(S_2, n_{12}, p_1)}{m_1} + b_6 \frac{(S_2, n_{12}, p_2)}{m_2}\bigg)\mathbf{n}_{12}\bigg).\end{aligned} \tag{6.9}$$

One can now think of two different ways of determining whether there exists a set of coefficients $a_1, \ldots, a_6$; $b_1, \ldots, b_6$ such that our translational Hamiltonian equations of motion (with NLO spin-dependent terms), Eq. (4.15), are physically equivalent to the corresponding translational harmonic equations of motion derived in [13]. (1) A first way would consist of inserting the putative general transformation $\mathbf{Y}_a(a_1, \ldots, a_6; b_1, \ldots, b_6)$ directly into the translational equations of motion derived in [13] (using the fact that we have already determined how their spin variables are linked to ours), and to compare the result to the explicit form of our translational Hamiltonian equations of motion, Eq. (4.15). This approach is, however, computationally heavy. (2) Therefore, we have instead used a simpler approach consisting in comparing the 10 conserved quantities derived in harmonic coordinates in Ref. [13], namely, the energy $E(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\text{BBF}})$, the total linear momentum $\mathbf{P}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\text{BBF}})$, the total angular momentum $\mathbf{J}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\text{BBF}})$, and the center-of-mass vector

[8]As a further check, we have also explicitly verified that the Hamiltonian time derivative (computed with our dynamics, namely, $\{\mathbf{S}_a^{\text{BBF}}, H\}$) of the originally defined (nonconserved) spin vector $\mathbf{S}_a^{\text{BBF}}$ of [13] coincides with the NLO spin precession law given by Eqs. (6.1)–(6.3) there. To do this calculation we defined the phase-space quantity $\mathbf{S}_a^{\text{BBF}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)$ by inserting Eqs. (6.3) and (6.4) into Eq. (7.6) of [14].

[9]More specifically we required that, say, $m_1 \mathbf{Y}_1^{\text{so}}$ be proportional (modulo some velocity-dependent factors involving $v_a \sim p_a/m_a$) either to $m_1 S_2$ or to $m_2 S_1$.

$\mathbf{G}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\rm BBF})$, with the 10 phase-space Poincaré generators constructed above within our Hamiltonian formalism. To do this comparison explicitly, we first need to perform two replacements: (i) to replace the nonconserved spin variable $\mathbf{S}_a^{\rm BBF}$ used in [13] in terms of the conserved one $\mathbf{S}_a^{\rm cBBF}$ introduced in [14], thereby obtaining new expressions $E(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\rm cBBF})$, $\mathbf{P}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\rm cBBF})$, $\mathbf{J}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\rm cBBF})$, $\mathbf{G}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\rm cBBF})$ for the 10 conserved quantities, and then (ii) to replace the harmonic-coordinate velocities $\mathbf{v}_a = \mathrm{d}\mathbf{y}_a/\mathrm{d}t$ in terms of Hamiltonian time-derivatives, namely, $\mathbf{V}_a = \{\mathbf{Y}_a, H\}$. Finally, the values of the coefficients $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$ must fulfill the equations

$$E(\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a), \tag{6.10a}$$

$$\mathbf{P}(\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \sum_a \mathbf{p}_a, \tag{6.10b}$$

$$\mathbf{J}(\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \sum_a (\mathbf{x}_a \times \mathbf{p}_a + \mathbf{S}_a), \tag{6.10c}$$

$$\mathbf{G}(\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \mathbf{G}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a). \tag{6.10d}$$

By considering the first three of these equations (i.e. by comparing the two expressions for the energy, the total linear momentum, and the total angular momentum), we obtained a unique set of values for all the unknown coefficients $a_1, \ldots, a_6$; $b_1, \ldots, b_6$. We then verified that these values satisfy also the fourth of Eqs. (6.10) (thereby giving us confidence in the correctness of our Hamiltonian, and providing many nontrivial checks of the previous results [13,14]).

Our unique solution for the spin-dependent transformation of orbital coordinates $\mathbf{Y}_{a(2)}^{\rm so} + \mathbf{Y}_{a(4)}^{\rm so}$ reads

$$\begin{aligned}
&\mathbf{Y}_{1(2)}^{\rm so}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \mathbf{Y}_{1(4)}^{\rm so}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) \\
&= \frac{\mathbf{S}_1 \times \mathbf{p}_1}{2c^2 m_1^2} - \frac{\mathbf{S}_1 \times \mathbf{p}_1}{c^4 m_1^2}\left(\frac{\mathbf{p}_1^2}{8m_1^2} + \frac{Gm_2}{r_{12}}\right) + \frac{G}{2c^4 m_2 r_{12}} \\
&\quad \times (3\mathbf{S}_2 \times \mathbf{p}_2 + 2(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{n}_{12} \times \mathbf{S}_2 + (S_2, n_{12}, p_2)\mathbf{n}_{12}).
\end{aligned} \tag{6.11}$$

Note that the first three terms on the right-hand side of Eq. (6.11) (i.e. the terms proportional to $\mathbf{S}_1$) have the same structure as the exact special-relativistic value [18,19,25] for the shift $\mathbf{Y}_1^{\rm so}$ between the canonical[10] orbital coordinate $\mathbf{x}_1$ and the usual Lorentz-covariant (harmonic) orbital coordinate $\mathbf{y}_1$, namely,

$$\mathbf{y}_1 = \mathbf{x}_1 + \frac{\mathbf{S}_1 \times \mathbf{p}_1}{m_1(m_1c^2 + E_1)}, \tag{6.12}$$

where $E_1 = \sqrt{(m_1c^2)^2 + (\mathbf{p}_1 c)^2}$ is the relativistic energy (including the rest-mass contribution). The $\mathbf{p}_1$-dependent terms in the first three terms of Eq. (6.11) correspond to the NLO expansion of the special-relativistic result, while the additional G-dependent contribution can be roughly understood as a gravitational addition to the special-relativistic energy E_1 (though it does not have the correct coefficient to be really interpreted so simply).

By contrast, the terms proportional to $\mathbf{S}_2$ in Eq. (6.11) do not have correspondents in the special-relativistic (i.e. $G \to 0$) limit. As was to be expected they vanish in the limit where the second body (of mass m_2) is heavy and fixed ($\mathbf{p}_2/m_2 \to 0$). (Indeed, if we consider a nonspinning test particle, m_1, $S_1 = 0$, moving in the background of a fixed, heavy spinning mass, m_2, S_2, the harmonic-coordinate geodesic action of m_1 will already yield a canonical Hamiltonian action.) We leave to future work a direct derivation of these terms from the perturbative construction of canonical coordinates [of the type $q = y + O(s)$, $p = p^{\rm bare} + O(s)$] alluded to above.

Finally, as a further check on the algebra, we have also used the "direct" method (1) mentioned above (the first method we could have used to determine the values of the coefficients $a_1, \ldots, a_6$; $b_1, \ldots, b_6$). More explicitly, we started from the harmonic-coordinate translational equations of motion with NLO spin-orbit effects given in Eqs. (5.3) of Ref. [13]. We then replaced in these equations the nonconserved spin vector $\mathbf{S}_a^{\rm BBF}$ by its expression (as given in Eq. (7.4) of [14]) in terms of their conserved spin vector $\mathbf{S}_a^{\rm cBBF}$. This yields 2PN-accurate translational equations of motion of the form

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} &= \mathbf{A}_{\rm oa}^{\rm N}(\mathbf{y}_b, \mathbf{v}_b) + \frac{1}{c^2}(\mathbf{A}_{\rm oa}^{\rm 1PN}(\mathbf{y}_b, \mathbf{v}_b) \\
&\quad + \mathbf{A}_{\rm soa}^{\rm LO}(\mathbf{y}_b, \mathbf{v}_b, \mathbf{S}_b^{\rm cBBF})) + \frac{1}{c^4}(\mathbf{A}_{\rm oa}^{\rm 2PN}(\mathbf{y}_b, \mathbf{v}_b) \\
&\quad + \mathbf{A}_{\rm soa}^{\rm NLO}(\mathbf{y}_b, \mathbf{v}_b, \mathbf{S}_b^{\rm cBBF})) + \mathcal{O}(c^{-6}).
\end{aligned} \tag{6.13}$$

We then compared the right-hand side of Eq. (6.13), let us denote it by $\mathbf{A}_a = \mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b, \mathbf{S}_b^{\rm cBBF})$, to its direct Hamiltonian recomputation by means of our Hamiltonian flow, i.e.

$$\mathbf{A}_a = \{\mathbf{V}_a, H\} = \{\{\mathbf{Y}_a, H\}, H\}, \tag{6.14}$$

together with the needed transformations (6.1) (determined

[10]The classical canonical variables, here denoted $\mathbf{x}_a$, $\mathbf{p}_a$, $\mathbf{S}_a$, correspond, at the quantum level, to the so-called Pryce-Newton-Wigner variables.

above) between harmonic and canonical variables. Again, this verification worked perfectly and (together with the similar direct verification of the NLO spin precession equation mentioned above) gives us confidence that *both* sets of results (harmonic and Hamiltonian) are correct.

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