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Stability of squashed Kaluza-Klein black holes

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The stability of squashed Kaluza-Klein black holes is studied. The squashed Kaluza-Klein black hole looks like a five-dimensional black hole in the vicinity of horizon and looks like a four-dimensional Minkowski spacetime with a circle at infinity. In this sense, squashed Kaluza-Klein black holes can be regarded as black holes in the Kaluza-Klein spacetimes. Using the symmetry of squashed Kaluza-Klein black holes, $SU(2) \times U(1) \simeq U(2)$, we obtain master equations for a part of the metric perturbations relevant to the stability. The analysis based on the master equations gives strong evidence for the stability of squashed Kaluza-Klein black holes. Hence, the squashed Kaluza-Klein black holes deserve to be taken seriously as realistic black holes in the Kaluza-Klein spacetime.

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I. INTRODUCTION

Recently, higher dimensional black holes have attracted much attention. In particular, many exotic black holes in the asymptotically flat spacetime are found [1–7]. From a realistic point of view, however, the extra dimensions need to be compactified to reconcile the higher dimensional gravity theory with our apparently four-dimensional world. The higher dimensional spacetimes with compact extra dimensions are called Kaluza-Klein spacetimes. The black holes should reside not in the asymptotically flat spacetimes, but in the asymptotically Kaluza-Klein spacetimes. We call these "Kaluza-Klein black holes." It would be important to study Kaluza-Klein black holes in the general dimensions. In this paper, we will consider five-dimensional Kaluza-Klein black holes as a first step.

It is well known that the simplest five-dimensional Kaluza-Klein black hole is the black string which is the direct product of a four-dimensional Schwarzschild black hole and a circle [8]. The topology of the horizon of black strings is $S^2 \times S^1$. The stability analysis of black strings has been done, and it was shown that black strings are stable when the horizon radius is larger than the scale of compact extra dimensions [9]. Because of the stability, black strings are natural candidates for Kaluza-Klein black holes.

Interestingly, another possibility has been recognized [10], that of squashed Kaluza-Klein (SqKK) black holes which could also reside in the Kaluza-Klein spacetime. The topology of the horizon of SqKK black holes is S^3, while it looks like four-dimensional black holes with a circle as an internal space in the asymptotic region. SqKK black holes were originally derived as five-dimensional vacuum solutions in the context of Kaluza-Klein theory [11,12]. Recently, much effort has been devoted to reveal the properties of squashed Kaluza-Klein black holes [13–29]. Since the horizons of these black holes have the same nature as the five-dimensional black holes, Hawking radiation and quasinormal modes from SqKK black holes would be different from those seen in four-dimensional black holes even at low energy [22,27,28]. This means that the extra dimension can be observed through these squashed black holes. These are distinct properties from black strings for which we need to see the excitation of Kaluza-Klein modes to find the extra dimension. However, the stability of SqKK black holes is needed for these arguments to be meaningful.

Related to the stability problem, Bizon *et al.* [29] investigated the nonlinear perturbation of the Gross-Perry-Sorkin (GPS) monopole [30,31], which is the zero mass limit of the SqKK black hole. They showed that the GPS monopole is stable against small perturbations but unstable against large perturbations and collapses to a SqKK black hole. This suggests that the SqKK black hole is a final state of a gravitational collapse in the presence of the GPS monopole. Hence, SqKK black holes seem to be stable, although the stability has not yet been proven. The purpose of this paper is to study the stability of SqKK black holes directly.

To analyze the stability, it is important to obtain a set of single ordinary differential equations of motion, the so-called master equations. To achieve this aim, we focus on the symmetry of SqKK black holes, $SU(2) \times U(1) \simeq U(2)$. Since SqKK black holes have the same symmetry as five-dimensional Myers-Perry black holes with equal angular momenta, the analysis of field equations in the degenerate Myers-Perry spacetime [32] can be applicable to SqKK black holes. By doing so, we show that metric perturbations which are supposed to be relevant to the stability can

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be described by master equations. Using the master equations, we prove the stability of SqKK black holes under these perturbations.

The organization of this paper is as follows. In Sec. II, we present the SqKK black holes and discuss the symmetry of these black holes. In Sec. III, the formalism to classify metric perturbations is explained. First, we introduce Wigner functions, which are irreducible representation of $U(2)$. The tensor fields are expanded in terms of these Wigner functions in invariant forms. Using the classification based on the symmetry, we find an infinite number of master variables. In Sec. IV, we derive the master equations for master variables. By analyzing these equations, we give strong evidence of the stability of SqKK black holes. The final section is devoted to discussions.

II. SYMMETRY OF SQUASHED KALUZA-KLEIN BLACK HOLES

In this paper, we concentrate on the static SqKK black holes in vacuum whose metric is given by

$$ds^2 = -F(\rho)dt^2 + \frac{K(\rho)^2}{F(\rho)}d\rho^2 + \rho^2 K(\rho)^2[(\sigma^1)^2 + (\sigma^2)^2] + \frac{\rho_0(\rho_0+\rho_+)}{K(\rho)^2}(\sigma^3)^2. \tag{1}$$

Here, the functions $F(\rho)$ and $K(\rho)$ are given by

$$F(\rho) = 1 - \frac{\rho_+}{\rho}, \qquad K^2(\rho) = 1 + \frac{\rho_0}{\rho}, \tag{2}$$

where ρ_+ and ρ_0 are constant parameters. The invariant forms $\sigma^a (a = 1, 2, 3)$ of $SU(2)$ are given by

$$\sigma^1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi,$$
$$\sigma^2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \qquad \sigma^3 = d\psi + \cos\theta d\phi, \tag{3}$$

which satisfy $d\sigma^a = 1/2\epsilon^{abc}\sigma^b \wedge \sigma^c$, where ϵ^{abc} is the Levi-Civita symbol. The coordinate ranges are $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi \le 4\pi$.

The angular part of the space, on which the metric (1) is spanned by σ^a, is topologically S^3. The horizon is located at $\rho = \rho_+$, and then its topology is S^3. In fact, the radius of S^2 is $\sqrt{\rho_+(\rho_+ + \rho_0)}$ and the radius of the circle is $\sqrt{\rho_+\rho_0}$. Hence, the geometry is a squashed three-sphere. The asymptotic form of the metric at infinity becomes

$$ds^2 \sim -dt^2 + d\rho^2 + \rho^2 d\Omega_2^2 + \rho_0(\rho_0 + \rho_+)(d\psi + \cos\theta d\phi)^2, \tag{4}$$

where $d\Omega_2^2 = (\sigma^1)^2 + (\sigma^2)^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of S^2. From the metric (4), we see the asymptotic geometry has the structure of S^1 fibered over M^4. Therefore, the extra dimension of spacetime (1) is compactified at infinity, and the scale of compactification ℓ is given by

$$\ell = \sqrt{\rho_0(\rho_0 + \rho_+)}. \tag{5}$$

In this sense, the spacetime given by the metric (1), which has a squashed horizon, can be regarded as a kind of Kaluza-Klein black hole. Thus, the SqKK black hole has an S^3 horizon as a five-dimensional black hole and the asymptotic structure similar to that of a five-dimensional black string. It is well known that there exists Gregory-Laflamme instability [9] in the black string system. On the other hand, five-dimensional Schwarzschild black holes are stable [33,34]. Therefore, it is interesting to study the stability of squashed black holes.

Apparently, the metric (1) has the $SU(2)$ symmetry generated by Killing vectors ξ_α, $(\alpha = x, y, z)$:

$$\xi_x = \cos\phi\partial_\theta + \frac{\sin\phi}{\sin\theta}\partial_\psi - \cot\theta\sin\phi\partial_\phi,$$
$$\xi_y = -\sin\phi\partial_\theta + \frac{\cos\phi}{\sin\theta}\partial_\psi - \cot\theta\cos\phi\partial_\phi, \tag{6}$$
$$\xi_z = \partial_\phi.$$

The symmetry can be explicitly shown by using the relation $\mathcal{L}_{\xi_\alpha}\sigma^a = 0$, where $\mathcal{L}_{\xi_\alpha}$ is a Lie derivative with respect to ξ_α. The dual vectors to σ^a are given by

$$\boldsymbol{e}_1 = -\sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \cot\theta\cos\psi\partial_\psi,$$
$$\boldsymbol{e}_2 = \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \cot\theta\sin\psi\partial_\psi,$$
$$\boldsymbol{e}_3 = \partial_\psi, \tag{7}$$

and, by definition, they satisfy $\sigma^a_i \boldsymbol{e}^i_b = \delta^a_b$. Let us define the two kind of angular momentum operators:

$$L_\alpha = i\xi_\alpha, \qquad W_a = i\boldsymbol{e}_a, \tag{8}$$

where $\alpha, \beta, \cdots = x, y, z$ and $a, b, \cdots = 1, 2, 3$. They satisfy commutation relations

$$[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma}L_\gamma, \qquad [W_a, W_b] = -i\epsilon_{abc}W_c. \tag{9}$$

They commute each other, $[L_\alpha, W_a] = 0$. From the metric (1), we can also read off the additional $U(1)$ symmetry, which keeps the S^2 metric, $\sigma_1^2 + \sigma_2^2$, invariant. Thus, the spatial symmetry of SqKK black holes is $SU(2) \times U(1) \simeq U(2)$,[1] where $\boldsymbol{e}_3$ generates $U(1)$ and $\xi_\alpha (\alpha = x, y, z)$ generate $SU(2)$. As will be seen later, these symmetries yield the separability of equations for the metric perturbations.

It is convenient to define the new invariant forms,

$$\sigma^\pm = \tfrac{1}{2}(\sigma^1 \mp i\sigma^2). \tag{10}$$

Here, we note that

[1]The metric (1) also has time translation symmetry generated by $\partial/\partial t$.

$$\mathcal{L}_{W_3}\sigma^{\pm} = \pm\sigma^{\pm}, \qquad \mathcal{L}_{W_3}\sigma^3 = 0. \tag{11}$$

The dual vectors to $\sigma^{\pm}$ are

$$\boldsymbol{e}_{\pm} = \boldsymbol{e}_1 \pm i\boldsymbol{e}_2. \tag{12}$$

By use of $\sigma^{\pm}$, the metric (1) can be rewritten as

$$ds^2 = -F(\rho)dt^2 + \frac{K(\rho)^2}{F(\rho)}d\rho^2 + 4\rho^2 K(\rho)^2\sigma^+\sigma^- + \frac{\rho_0(\rho_0+\rho_+)}{K(\rho)^2}(\sigma^3)^2. \tag{13}$$

III. CLASSIFICATION OF THE METRIC PERTURBATIONS BASED ON THE SYMMETRY

Because the squashed black hole spacetime (1) has the $SU(2)\times U(1)$ symmetry, the metric perturbations can be expanded by the irreducible representation of $SU(2)\times U(1)$. We explain the formalism to obtain master equations for the metric perturbations [32,35].

Let us construct the representation of $U(2)\simeq SU(2)\times U(1)$. The eigenfunctions of $L^2 \equiv L_\alpha^2 = W_a^2$ are degenerate, but can be completely specified by eigenvalues of the operators L_z and W_3. The eigenfunctions are called Wigner functions, which are defined by

$$L^2 D^J_{KM} = J(J+1)D^J_{KM}, \qquad L_z D^J_{KM} = MD^J_{KM}, \qquad W_3 D^J_{KM} = KD^J_{KM}, \tag{14}$$

where J, K, M satisfy $J \geq 0$, $|K| \leq J$, $|M| \leq J$. From Eqs. (14), we see that D^J_{KM} form the irreducible representation of $SU(2)\times U(1)$. The Wigner functions $D^J_{KM}(x^i)$ are functions defined on S^3, i.e., $x^i = \theta, \phi, \psi$, which satisfy the orthonormal relation

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \sin\theta D^J_{KM}(x^i) D^{J'*}_{K'M'}(x^i) = \delta_{JJ'}\delta_{KK'}\delta_{MM'}. \tag{15}$$

Now, we consider metric perturbations $g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ is the background metric (13). The tensor field $h_{\mu\nu}$ can be classified into three parts, h_{AB}, h_{Ai}, h_{ij} ($A, B = t, \rho$) which behave as scalars, vectors, and a tensor under the coordinate transformation of θ, ϕ, ψ. The scalars h_{AB} can be expanded by the Wigner functions as

$$h_{AB} = \sum_K h^K_{AB}(x^A)D_K(x^i), \tag{16}$$

where we have omitted the indices J, M, because the metric perturbations with different J and M are decoupled trivially in the perturbed equations.

To decompose the vector part h_{Ai}, we construct vector harmonics as

$$\begin{aligned} D^+_{i,K} &= \sigma^+_i D_{K-1}, \qquad (|K-1| \leq J),\\ D^-_{i,K} &= \sigma^-_i D_{K+1}, \qquad (|K+1| \leq J),\\ D^3_{i,K} &= \sigma^3_i D_K, \qquad (|K| \leq J). \end{aligned} \tag{17}$$

One can check that

$$L^2 D^a_{i,K} = J(J+1)D^a_{i,K}, \qquad L_z D^a_{i,K} = MD^a_{i,K}, \qquad W_3 D^a_{i,K} = KD^a_{i,K}, \tag{18}$$

where $a = \pm, 3$ and operations are defined by Lie derivatives, that is, $W_a D^b_{i,K} \equiv \mathcal{L}_{W_a} D^b_{i,K}$ and $L_\alpha D^a_{i,K} \equiv \mathcal{L}_{L_\alpha} D^a_{i,K}$. In Eq. (17), taking the relation (11) into account, we have shifted the index K of Wigner functions so that $D^a_{i,K}$ have the same $U(1)$ charge K [32]. From Eqs. (18), we see that $D^a_{i,K}$ form the irreducible representation of $SU(2)\times U(1)$. Then, h_{Ai} can be expanded as

$$h_{Ai}(x^\mu) = \sum_K h^K_{Aa}(x^A) D^a_{i,K}(x^i). \tag{19}$$

Similarly, the expansion of the tensor part h_{ij} can be carried out as

$$h_{ij}(x^\mu) = \sum_K h^K_{ab}(x^A) D^{ab}_{ij,K}(x^i), \tag{20}$$

where tensor harmonics $D^{ab}_{ij,K}$ are defined by

$$\begin{aligned} D^{++}_{ij,K} &= \sigma^+_i\sigma^+_j D_{K-2} \qquad (|K-2| \leq J),\\ D^{+-}_{ij,K} &= \sigma^+_i\sigma^-_j D_K \qquad (|K| \leq J),\\ D^{+3}_{ij,K} &= \sigma^+_i\sigma^3_j D_{K-1} \qquad (|K-1| \leq J),\\ D^{--}_{ij,K} &= \sigma^-_i\sigma^-_j D_{K+2} \qquad (|K+2| \leq J),\\ D^{-3}_{ij,K} &= \sigma^-_i\sigma^3_j D_{K+1} \qquad (|K+1| \leq J),\\ D^{33}_{ij,K} &= \sigma^3_i\sigma^3_j D_K \qquad (|K| \leq J). \end{aligned} \tag{21}$$

We have shifted the eigenvalue K of Wigner functions so that the tensor harmonics $D^{ab}_{ij,K}$ satisfy

$$L^2 D^{ab}_{ij,K} = J(J+1)D^{ab}_{ij,K}, \qquad L_z D^{ab}_{ij,K} = MD^{ab}_{ij,K}, \qquad W_3 D^{ab}_{ij,K} = KD^{ab}_{ij,K}. \tag{22}$$

Equations (22) mean that $D^{ab}_{ij,K}$ form the irreducible representation of $SU(2)\times U(1)$.

Using the expansions (16), (19), and (20), we can obtain a set of equations for expansion coefficient fields labeled by J, M, K. Because of $SU(2)\times U(1)$ symmetry, no coupling appears between coefficients with different sets of indices (J, M, K).

Interestingly, without explicit calculation, we can reveal the structure of couplings between coefficients with the same (J, M, K). First, since the index K is shifted in the definition of vector and tensor harmonics, then the coefficients h^K_{AB}, h^K_{Aa}, and h^K_{ab} exist for K satisfying the inequality listed in the following table:

h_{++}	h_{A+}, h_{+3}	$h_{AB}, h_{A3}, h_{+-}, h_{33}$	h_{A-}, h_{-3}	h_{--}
$\lvert K-2\rvert \leq J$	$\lvert K-1\rvert \leq J$	$\lvert K\rvert \leq J$	$\lvert K+1\rvert \leq J$	$\lvert K+2\rvert \leq J$

Therefore, for $J=0$ modes, we can classify the coefficients by possible K as follows.

$J=0$:

h_{++}	h_{A+}, h_{+3}	$h_{AB}, h_{A3}, h_{+-}, h_{33}$	h_{A-}, h_{-3}	h_{--}
$K=2$				
	$K=1$			
		$K=0$		
			$K=-1$	
				$K=-2$

Apparently, for h_{++} and h_{--}, we can obtain the master equation for each variable. For other sets of components, (h_{A+}, h_{+3}), $(h_{AB}, h_{A3}, h_{+-}, h_{33})$, (h_{A-}, h_{-3}), they are coupled with each other in the same set. As we will see later, after fixing the gauge symmetry, we have the master equation for a single variable in each set. In total, there are five master equations, which matches the number of physical degrees of freedom of the gravitational perturbations.

For $J=1$ modes, we can classify the coefficients as follows.

$J=1$:

h_{++}	h_{A+}, h_{+3}	$h_{AB}, h_{A3}, h_{+-}, h_{33}$	h_{A-}, h_{-3}	h_{--}
$K=3$				
$K=2$	$K=2$			
$K=1$	$K=1$	$K=1$		
	$K=0$	$K=0$	$K=0$	
		$K=-1$	$K=-1$	$K=-1$
			$K=-2$	$K=-2$
				$K=-3$

We can see that h_{++} in $(J=1, M, K=3)$ modes and h_{--} in $(J=1, M, K=-3)$ modes are decoupled from other coefficients. It is easy to generalize this fact for arbitrary J, and we can also see that h_{++} in $(J, M, K=J+2)$ modes and h_{--} in $(J, M, K=-(J+2))$ modes are always decoupled. The perturbation equations for these modes can be reduced to the master equations for the single variables, respectively.

IV. STABILITY ANALYSIS OF SQUASHED KALUZA-KLEIN BLACK HOLES

The gravitational perturbation equation in vacuum is

$$\delta R_{\mu\nu} = \tfrac{1}{2}[\nabla^\rho \nabla_\mu h_{\nu\rho} + \nabla^\rho \nabla_\nu h_{\mu\rho} - \nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu h] = 0, \tag{23}$$

where ∇_μ denotes the covariant derivative with respect to the background metric $g_{\mu\nu}$ and $h = g^{\mu\nu} h_{\mu\nu}$. As is mentioned in the previous section, we can obtain master equations for variables in $(J=0, M=0, K=0, \pm1, \pm2)$ modes and $(J, M, K=\pm(J+2))$ modes. We derive these explicitly.

A. Zero mode perturbations ($J=0$)

In the case $J=0$, there are five physical degrees of freedom, namely, $K=\pm2, \pm1, 0$ modes. We treat these modes separately.

1. $K=\pm2$ modes

In $K=\pm2$ modes, there exist two coefficients, h_{++} and h_{--}. We note that these are gauge invariant. We consider only h_{++} because $\bar{h}_{++} = h_{--}$, where the bar denotes the complex conjugate. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}(x^\mu) dx^\mu dx^\nu = h_{++}(\rho) e^{-i\omega t} \sigma^+ \sigma^+. \tag{24}$$

Substituting Eq. (24) into Eq. (23), we get the equation of motion for h_{++} as

$$\begin{aligned}\delta R_{++} = {}& \frac{h_{++}}{2\rho^2 \rho_0 (\rho+\rho_0)^3 (\rho_+ + \rho_0)} [4\rho^5 + 16\rho^4 \rho_0 \\ & - 4\rho^3(\rho_+ - 5\rho_0)\rho_0 + \rho_+ \rho_0^3 (\rho_+ + \rho_0) \\ & + \rho\rho_0^2(3\rho_+^2 + \rho_+\rho_0 + 2\rho_0^2) + 4\rho^2 \rho_0 (\rho_+^2 + 3\rho_0^2)] \\ & - \frac{-2\rho^2 + 3\rho\rho_+ + \rho_+\rho_0}{2\rho(\rho+\rho_0)^2} \frac{dh_{++}}{d\rho} \\ & - \frac{\rho - \rho_+}{2(\rho+\rho_0)} \frac{d^2 h_{++}}{d\rho^2} - \frac{\rho}{2(\rho-\rho_+)} \omega^2 h_{++} = 0.\end{aligned} \tag{25}$$

In order to rewrite the equation in the Schrödinger form, we introduce the new variable

$$\Phi_2(\rho) \equiv \frac{1}{\rho^{1/4}(\rho + \rho_0)^{3/4}} h_{++}(\rho), \tag{26}$$

and the tortoise coordinate ρ_* defined by

$$\frac{d\rho_*}{d\rho} = \frac{K(\rho)}{F(\rho)}. \tag{27}$$

Then, the final form of the equation becomes

$$-\frac{d^2}{d\rho_*^2}\Phi_2 + V_2(\rho)\Phi_2 = \omega^2\Phi_2, \tag{28}$$

where the potential $V_2(\rho)$ is defined by

$$V_2(\rho) = \frac{\rho - \rho_+}{16\rho^3\rho_0(\rho_+ + \rho_0)(\rho + \rho_0)^3}[4\rho_+(\rho_+ + \rho_0)^2(16\rho_+^2 + 28\rho_+\rho_0 + 11\rho_0^2) + (320\rho_+^4 + 960\rho_+^3\rho_0 + 996\rho_+^2\rho_0^2 + 391\rho_+\rho_0^3 + 35\rho_0^4)(\rho - \rho_+) + 8(80\rho_+^3 + 182\rho_+^2\rho_0 + 127\rho_+\rho_0^2 + 25\rho_0^3)(\rho - \rho_+)^2 + 32(20\rho_+^2 + 31\rho_+\rho_0 + 11\rho_0^2)(\rho - \rho_+)^3 + 64(5\rho_+ + 4\rho_0)(\rho - \rho_+)^4 + 64(\rho - \rho_+)^5]. \tag{29}$$

From this expression, we can see $V_2 > 0$ in the region $\rho_+ < \rho < \infty$, explicitly. Typical profiles of the potential V_2 are plotted in Fig. 1.

We consider that Φ is square integrable in the region $-\infty < \rho_* < \infty$. Then, ω^2 is real. Multiplying both sides of Eq. (28) by $\bar{\Phi}_2$, we have

$$-\bar{\Phi}_2\frac{d^2}{d\rho_*^2}\Phi_2 + V_2(\rho)\bar{\Phi}_2\Phi_2 = \omega^2\bar{\Phi}_2\Phi_2. \tag{30}$$

Adding Eq. (30) and its complex conjugate equation, and integrating it, we obtain

$$\int d\rho_*\left[\left|\frac{d\Phi_2}{d\rho_*}\right|^2 + V_2|\Phi_2|^2\right] - \frac{1}{2}\left[\bar{\Phi}_2\frac{d}{d\rho_*}\Phi_2 + \Phi_2\frac{d}{d\rho_*}\bar{\Phi}_2\right]_{\rho_*=-\infty}^{\rho_*=\infty} = \omega^2\int d\rho_*|\Phi_2|^2. \tag{31}$$

Because the boundary term vanishes, the positivity of V_2 means $\omega^2 > 0$. Therefore, we have proved that the background metric is stable against the $K = \pm 2$ perturbations.

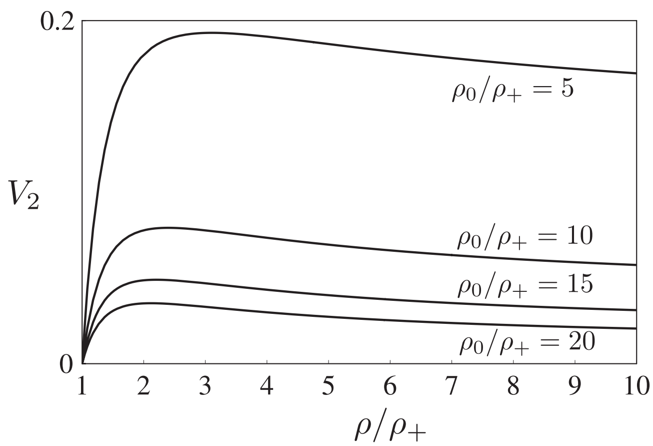

FIG. 1. The effective potential V_2 for the $K = \pm 2$ mode.

2. $K = \pm 1$ modes

Because of the relations $\bar{h}_{A+} = h_{A-}$ and $\bar{h}_{+3} = h_{-3}$, we consider only h_{A+} and h_{3+}. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}dx^\mu dx^\nu = 2h_{A+}(\rho)e^{-i\omega t}dx^A\sigma^+ + 2h_{+3}(\rho)e^{-i\omega t}\sigma^+\sigma^3. \tag{32}$$

There are three components in Eq. (32). The gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ for these variables are given by

$$h_{t+} \to h_{t+} - i\omega\xi_+, \tag{33}$$

$$h_{\rho+} \to h_{\rho+} - \frac{2\rho + \rho_0}{\rho(\rho + \rho_0)}\xi_+ + \frac{d\xi_+}{d\rho}, \tag{34}$$

$$h_{3+} \to h_{3+} - \frac{i(\rho^2 + 2\rho\rho_0 - \rho_+\rho_0)}{(\rho + \rho_0)^2}\xi_+, \tag{35}$$

where we set $\xi_\mu dx^\mu$ as

$$\xi_\mu dx^\mu = \xi_+(\rho)e^{-i\omega t}\sigma^+. \tag{36}$$

So we can choose the gauge condition[2]

$$h_{+3} = 0, \tag{37}$$

which completely fixes the gauge freedom. Substituting Eqs. (32) and (37) into $\delta R_{A+} = 0$ and $\delta R_{+3} = 0$, we obtain

[2]Note that we cannot choose this gauge condition in the case of the five-dimensional Schwarzschild black hole limit.

$$\delta R_{t+} = \frac{\rho^4 + \rho(3\rho^2 - 2\rho_+^2)\rho_0 + (\rho - \rho_+)(3\rho + \rho_+)\rho_0^2 + (\rho - \rho_+)\rho_0^3}{2\rho^2\rho_0(\rho + \rho_0)^2(\rho_+ + \rho_0)} h_{t+} - \frac{i(\rho - \rho_+)\omega}{\rho(\rho + \rho_0)} h_{\rho+} - \frac{(\rho - \rho_+)\rho_0}{2\rho(\rho + \rho_0)^2}\frac{dh_{t+}}{d\rho} - \frac{i(\rho - \rho_+)\omega}{2(\rho + \rho_0)}\frac{dh_{\rho+}}{d\rho} - \frac{(\rho - \rho_+)}{2(\rho + \rho_0)}\frac{d^2h_{t+}}{d\rho^2} = 0, \tag{38}$$

$$\delta R_{\rho+} = -\frac{i\omega(2\rho + \rho_0)}{2(\rho - \rho_+)(\rho + \rho_0)} h_{t+} - \frac{h_{\rho+}}{2\rho(\rho - \rho_+)\rho_0(\rho + \rho_0)^3(\rho_+ + \rho_0)}[-\rho^4(\rho - \rho_+) + \rho^2(-4\rho^2 + 6\rho\rho_+ - 2\rho_+^2 + \rho^3\rho_+\omega^2)\rho_0 + ((-2\rho + \rho_+)^2(-\rho + \rho_+) + \rho^4(\rho + 3\rho_+)\omega^2)\rho_0^2 + 3\rho^3(\rho + \rho_+)\omega^2\rho_0^3 + \rho^2(3\rho + \rho_+)\omega^2\rho_0^4 + \rho^2\omega^2\rho_0^5] + \frac{i\rho\omega}{2(\rho - \rho_+)}\frac{dh_{t+}}{d\rho} = 0, \tag{39}$$

$$\delta R_{3+} = \frac{i\rho\omega(\rho^2 + 2\rho\rho_0 - \rho_+\rho_0)}{2(\rho - \rho_+)(\rho + \rho_0)^2} h_{t+} + \frac{2\rho^3 - \rho^2(\rho_+ - 5\rho_0) + 6\rho\rho_0^2 - \rho_+\rho_0(3\rho_+ + 5\rho_0)}{2(\rho + \rho_0)^4} h_{\rho+} + \frac{(\rho - \rho_+)(\rho^2 + 2\rho\rho_0 - \rho_+\rho_0)}{2(\rho + \rho_0)^3}\frac{dh_{\rho+}}{d\rho} = 0. \tag{40}$$

Eliminating h_{t+} from these equations, we get the master equation for the $K = 1$ mode. Defining a new variable

$$\Phi_1(\rho) \equiv \frac{4(\rho - \rho_+)(\rho_+\rho_0 - \rho(2\rho_0 + \rho))}{\rho^{3/4}(\rho + \rho_0)^{9/4}} h_{\rho+}(\rho), \tag{41}$$

we have the master equation in Schrödinger form:

$$-\frac{d^2}{d\rho_*^2}\Phi_1 + V_1(\rho)\Phi_1 = \omega^2\Phi_1. \tag{42}$$

The potential V_1 reads

$$V_1(\rho) = \frac{\rho - \rho_+}{16\rho_0(\rho_+ + \rho_0)\rho^3(\rho_0 + \rho)^3(\rho_+\rho_0 - \rho(2\rho_0 + \rho))^2}[4\rho_+^3(\rho_+ + \rho_0)^4(4\rho_+^2 - 8\rho_+\rho_0 - 11\rho_0^2) + \rho_+^2(\rho_+ + \rho_0)^3 \times (144\rho_+^3 + 48\rho_+^2\rho_0 - 68\rho_+\rho_0^2 + 31\rho_0^3)(\rho - \rho_+) + 4\rho_+(\rho_+ + \rho_0)^3(144\rho_+^3 + 152\rho_+^2\rho_0 + 152\rho_+\rho_0^2 + 75\rho_0^3) \times (\rho - \rho_+)^2 + 2(\rho_+ + \rho_0)^2(672\rho_+^4 + 1520\rho_+^3\rho_0 + 1548\rho_+^2\rho_0^2 + 781\rho_+\rho_0^3 + 126\rho_0^4)(\rho - \rho_+)^3 + 4(\rho_+ + \rho_0)^2 \times (504\rho_+^3 + 1032\rho_+^2\rho_0 + 757\rho_+\rho_0^2 + 191\rho_0^3)(\rho - \rho_+)^4 + (2016\rho_+^4 + 7072\rho_+^3\rho_0 + 9164\rho_+^2\rho_0^2 + 5211\rho_+\rho_0^3 + 1103\rho_0^4)(\rho - \rho_+)^5 + 8(168\rho_+^3 + 460\rho_+^2\rho_0 + 411\rho_+\rho_0^2 + 119\rho_0^3)(\rho - \rho_+)^6 + 96(6\rho_+^2 + 11\rho_+\rho_0 + 5\rho_0^2) \times (\rho - \rho_+)^7 + 16(9\rho_+ + 8\rho_0)(\rho - \rho_+)^8 + 16(\rho - \rho_+)^9]. \tag{43}$$

Typical profiles of the potential V_1 are shown in Fig. 2.

From Fig. 2, we see that this potential V_1 contains a negative region. Hence, we hardly show the stability from this form of potential. However, we can overcome this difficulty by using a transformation of the coordinate. We introduce a new radial coordinate y as

$$\frac{d}{dy} = \frac{1}{\beta(\rho)}\frac{d}{d\rho_*}, \tag{44}$$

where $\beta(\rho)$ is some real function and must be nonsingular outside of the horizon, $\rho_+ \leq \rho < \infty$. Then, the master equation becomes

$$-\frac{d^2}{dy^2}\Phi_1 - \frac{1}{\beta}\frac{d\beta}{dy}\frac{d}{dy}\Phi_1 + \frac{V_1}{\beta^2}\Phi_1 = \frac{\omega^2}{\beta^2}\Phi_1. \tag{45}$$

Multiplying both sides of the equation by $\bar{\Phi}_1$, we obtain

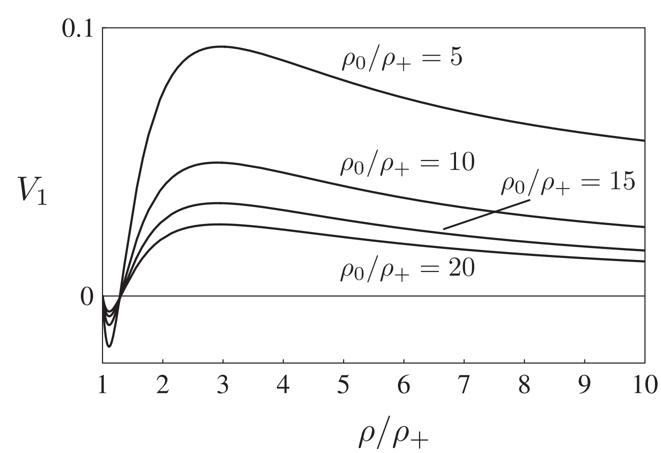

FIG. 2. The effective potential V_1 for the $K = \pm 1$ mode.

$$-\bar{\Phi}_1 \frac{d^2}{dy^2}\Phi_1 - \frac{1}{\beta}\frac{d\beta}{dy}\bar{\Phi}_1\frac{d}{dy}\Phi_1 + \frac{V_1}{\beta^2}\bar{\Phi}_1\Phi_1 = \frac{\omega^2}{\beta^2}\bar{\Phi}_1\Phi_1. \tag{46}$$

Adding Eq. (46) and its complex conjugate equation, and integrating it, we obtain the equation

$$\int dy\left[\left|\frac{d\Phi_1}{dy}\right|^2 + \frac{\tilde{V}_1}{\beta^2}|\Phi_1|^2\right] - \frac{1}{2}\left[\bar{\Phi}_1\frac{d}{dy}\Phi_1 + \Phi_1\frac{d}{dy}\bar{\Phi}_1 + \frac{1}{\beta}\frac{d\beta}{dy}|\Phi_1|^2\right]_{\rho_*=-\infty}^{\rho_*=\infty} = \omega^2\int dy\frac{|\Phi_1|^2}{\beta^2}, \tag{47}$$

where

$$\tilde{V}_1 = V_1 + \frac{1}{2}\beta^2\frac{d}{dy}\left(\frac{1}{\beta}\frac{d\beta}{dy}\right). \tag{48}$$

The boundary terms in (47) vanish because Φ_1 is square integrable. Therefore, if the deformed effective potential $\tilde{V}$ is positive everywhere, there is no $\omega^2 < 0$ mode. Now, we choose β as

$$\beta^2 = \frac{15}{K(\rho)^2}. \tag{49}$$

Then the potential becomes

$$\begin{aligned}\tilde{V}_1 = {}& \frac{\rho-\rho_+}{16\rho_0(\rho_++\rho_0)\rho^3(\rho_0+\rho)^3(\rho_+\rho_0-\rho(2\rho_0+\rho))^2}[16\rho_+^3(\rho_+-\rho_0)^2(\rho_++\rho_0)^4 + \rho_+^2(\rho_++\rho_0)^3\\ &\times(144\rho_+^3+48\rho_+^2\rho_0+112\rho_+\rho_0^2+211\rho_0^3)(\rho-\rho_+) + 4\rho_+(\rho_++\rho_0)^3(144\rho_+^3+152\rho_+^2\rho_0+152\rho_+\rho_0^2+75\rho_0^3)\\ &\times(\rho-\rho_+)^2 + 2(\rho_++\rho_0)^2(672\rho_+^4+1520\rho_+^3\rho_0+1248\rho_+^2\rho_0^2+361\rho_+\rho_0^3+6\rho_0^4)(\rho-\rho_+)^3 + 4(\rho_++\rho_0)^2\\ &\times(504\rho_+^3+1032\rho_+^2\rho_0+532\rho_+\rho_0^2+11\rho_0^3)(\rho-\rho_+)^4 + (\rho_++\rho_0)(2016\rho_+^3+5056\rho_+^2\rho_0+3568\rho_+\rho_0^2+563\rho_0^3)\\ &\times(\rho-\rho_+)^5 + 32(\rho_++\rho_0)(2\rho_++\rho_0)(21\rho_++26\rho_0)(\rho-\rho_+)^6 + 96(\rho_++\rho_0)(6\rho_++5\rho_0)(\rho-\rho_+)^7\\ &+ 16(9\rho_++8\rho_0)(\rho-\rho_+)^8 + 16(\rho-\rho_+)^9].\end{aligned} \tag{50}$$

We can see that $\tilde{V}_1 > 0$ from the above expression. Thus, we have proved the stability for $K = \pm 1$ modes.

3. $K = 0$ mode

For the $K = 0$ mode, we have h_{AB}, h_{A3}, h_{33}, h_{+-}. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}dx^\mu dx^\nu = h_{AB}(\rho)e^{-i\omega t}dx^A dx^B + 2h_{A3}(\rho)e^{-i\omega t}dx^A\sigma^3 + 2h_{+-}(\rho)e^{-i\omega t}\sigma^+\sigma^- + h_{33}(\rho)e^{-i\omega t}\sigma^3\sigma^3. \tag{51}$$

The gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ for these variables are given by

$$h_{tt} \to h_{tt} - 2i\omega\xi_t - \frac{\rho_+(\rho-\rho_+)}{\rho^2(\rho+\rho_0)}\xi_\rho, \tag{52}$$

$$h_{t\rho} \to h_{t\rho} - \frac{\rho_+}{\rho(\rho-\rho+)}\xi_t - i\omega\xi_\rho + \frac{d\xi_t}{d\rho}, \tag{53}$$

$$h_{\rho\rho} \to h_{\rho\rho} + \frac{\rho_++\rho_0}{(\rho-\rho_+)(\rho+\rho_0)}\xi_\rho + 2\frac{d\xi_\rho}{d\rho}, \tag{54}$$

$$h_{t3} \to h_{t3} - i\omega\xi_3, \tag{55}$$

$$h_{\rho 3} \to h_{\rho 3} - \frac{\rho_0}{\rho(\rho+\rho_0)}\xi_3 + \frac{d\xi_3}{d\rho}, \tag{56}$$

$$h_{+-} \to h_{+-} + \frac{2(\rho-\rho_+)(2\rho+\rho_0)}{\rho+\rho_0}\xi_\rho, \tag{57}$$

$$h_{33} \to h_{33} + \frac{(\rho-\rho_+)\rho_0^2(\rho_++\rho_0)}{(\rho+\rho_0)^3}\xi_\rho, \tag{58}$$

where we set $\xi_\mu dx^\mu$ as

$$\xi_\mu dx^\mu = \xi_A(\rho)e^{-i\omega t}dx^A + \xi_3(\rho)e^{-i\omega t}\sigma^3. \tag{59}$$

So we can choose the gauge conditions[3]

$$h_{+-} = 0, \qquad h_{tt} = 0, \qquad h_{t3} = 0. \tag{60}$$

Substituting Eqs. (51) and (60) into $\delta R_{AB} = 0$, $\delta R_{33} = 0$, and $\delta R_{+-} = 0$, we have

[3]Note that, for static perturbation, we cannot choose this gauge condition.

$$\delta R_{tt} = \frac{\rho_+(-\rho+\rho_+)\rho_0 + 2\rho^3(\rho+\rho_0)^2\omega^2}{4\rho^4\rho_0(\rho+\rho_0)(\rho_++\rho_0)} h_{33} - \frac{i(4\rho-3\rho_+)\omega}{2\rho(\rho+\rho_0)} h_{t\rho} + \frac{(\rho-\rho_+)(-\rho_+(\rho_++\rho_0)+2\rho^2(\rho+\rho_0)^2\omega^2)}{4\rho^2(\rho+\rho_0)^3} h_{\rho\rho} + \frac{(\rho-\rho_+)\rho_+}{4\rho^3\rho_0(\rho_++\rho_0)}\frac{dh_{33}}{d\rho} - \frac{i(\rho-\rho_+)\omega}{\rho+\rho_0}\frac{dh_{t\rho}}{d\rho} + \frac{(\rho-\rho_+)\rho_+(-\rho+\rho_+)}{4\rho^2(\rho+\rho_0)^2}\frac{dh_{\rho\rho}}{d\rho} = 0, \tag{61}$$

$$\delta R_{t\rho} = -\frac{i\omega}{4\rho(\rho-\rho_+)\rho_0} h_{33} + \frac{i(-\rho+\rho_+)(4\rho+3\rho_0)\omega}{4\rho(\rho+\rho_0)^2} h_{\rho\rho} + \frac{i(\rho+\rho_0)\omega}{2\rho\rho_0(\rho_++\rho_0)}\frac{dh_{33}}{d\rho} = 0, \tag{62}$$

$$\delta R_{\rho\rho} = \frac{(-\rho+\rho_+)(4\rho^2-2\rho_+\rho_0+\rho(-5\rho_++\rho_0))}{4\rho^3(\rho-\rho_+)^2(\rho+\rho_0)(\rho_++\rho_0)} h_{33} + \frac{i\rho(\rho_++\rho_0)\omega}{2(\rho-\rho_+)^2(\rho+\rho_0)} h_{t\rho} - \frac{(\rho_++\rho_0)(-4\rho^2+3\rho(\rho_+-\rho_0)+2\rho_+\rho_0)+2\rho^2(\rho+\rho_0)^3\omega^2}{4\rho(\rho-\rho_+)(\rho+\rho_0)^3} h_{\rho\rho} + \frac{-2\rho_+\rho_0+\rho(-\rho_++\rho_0)}{4\rho^2(\rho-\rho_+)\rho_0(\rho_++\rho_0)}\frac{dh_{33}}{d\rho} + \frac{i\rho\omega}{\rho-\rho_+}\frac{dh_{t\rho}}{d\rho} + \frac{4\rho^2-3\rho\rho_++3\rho\rho_0-2\rho_+\rho_0}{4\rho(\rho+\rho_0)^2}\frac{dh_{\rho\rho}}{d\rho} - \frac{\rho+\rho_0}{2\rho\rho_0(\rho_++\rho_0)}\frac{d^2h_{33}}{d\rho^2} = 0, \tag{63}$$

$$\delta R_{t3} = \frac{-i(\rho-\rho_+)\omega}{\rho(\rho+\rho_0)} h_{\rho 3} - \frac{i(\rho-\rho_+)\omega}{2(\rho+\rho_0)}\frac{dh_{\rho 3}}{d\rho} = 0, \tag{64}$$

$$\delta R_{\rho 3} = -\frac{\rho\omega^2}{2(\rho-\rho_+)} h_{\rho 3} = 0, \tag{65}$$

$$\delta R_{+-} = \frac{2\rho(\rho-2\rho_+)-(\rho+\rho_+)\rho_0}{2\rho^2(\rho+\rho_0)(\rho_++\rho_0)} h_{33} + \frac{i\rho(2\rho+\rho_0)\omega}{\rho+\rho_0} h_{t\rho} + \frac{(\rho-\rho_+)(4\rho^2+\rho_0(-\rho_++3\rho_0)+2\rho(\rho_++5\rho_0))}{2(\rho+\rho_0)^3} h_{\rho\rho} - \frac{(\rho-\rho_+)(2\rho+\rho_0)}{2\rho\rho_0(\rho_++\rho_0)}\frac{dh_{33}}{d\rho} + \frac{(\rho-\rho_+)^2(2\rho+\rho_0)}{2(\rho+\rho_0)^2}\frac{dh_{\rho\rho}}{d\rho} = 0, \tag{66}$$

$$\delta R_{33} = \frac{(\rho-\rho_+)\rho_0(4\rho\rho_++3\rho\rho_0+\rho_+\rho_0)-2\rho^3(\rho+\rho_0)^3\omega^2}{4\rho^2(\rho-\rho_+)(\rho+\rho_0)^3} h_{33} + \frac{i\rho\rho_0^2(\rho_++\rho_0)\omega}{2(\rho+\rho_0)^3} h_{t\rho} + \frac{3(\rho-\rho_+)\rho_0^2(\rho_++\rho_0)^2}{4(\rho+\rho_0)^5} h_{\rho\rho} - \frac{4\rho^2-2\rho\rho_++\rho\rho_0+\rho_+\rho_0}{4\rho(\rho+\rho_0)^2}\frac{dh_{33}}{d\rho} + \frac{(\rho-\rho_+)^2\rho_0^2(\rho_++\rho_0)}{4(\rho+\rho_0)^4}\frac{dh_{\rho\rho}}{d\rho} - \frac{\rho-\rho_+}{2(\rho+\rho_0)}\frac{d^2h_{33}}{d\rho^2} = 0. \tag{67}$$

Because of the gauge symmetry and constraint equations, there remains only one physical degree of freedom. In fact, introducing the new variable

$$\Phi_0(\rho) \equiv \frac{(\rho+\rho_0)^{5/4}(2\rho+\rho_0)}{\rho^{1/4}(4\rho+3\rho_0)} h_{33}(\rho), \tag{68}$$

we get the master equation

$$-\frac{d^2}{d\rho_*^2}\Phi_0 + V_0(\rho)\Phi_0 = \omega^2\Phi_0, \tag{69}$$

where the potential V_0 is defined by

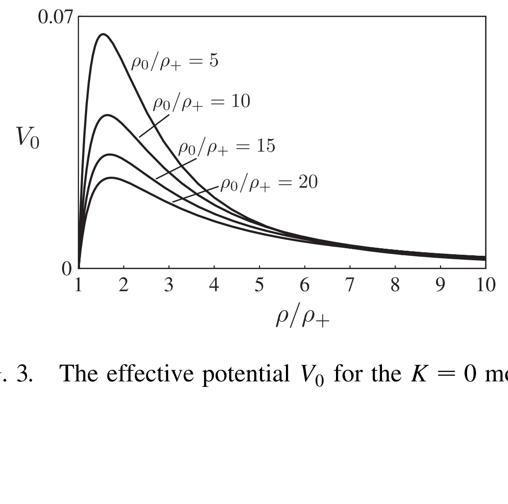

FIG. 3. The effective potential V_0 for the $K = 0$ mode.

$$V_0(\rho) = \frac{\rho - \rho_+}{16\rho^3(\rho+\rho_0)^3(4\rho+3\rho_0)^2}[4\rho_+(64\rho_+^4 + 304\rho_+^3\rho_0 + 516\rho_+^2\rho_0^2 + 375\rho_+\rho_0^3 + 99\rho_0^4) + (1024\rho_+^4 + 3776\rho_+^3\rho_0 + 4656\rho_+^2\rho_0^2 + 2220\rho_+\rho_0^3 + 315\rho_0^4)(\rho-\rho_+) + 48(32\rho_+^3 + 84\rho_+^2\rho_0 + 65\rho_+\rho_0^2 + 15\rho_0^3) \times (\rho-\rho_+)^2 + 16(64\rho_+^2 + 100\rho_+\rho_0 + 33\rho_0^2)(\rho-\rho_+)^3 + 128(2\rho_+ + \rho_0)(\rho-\rho_+)^4]. \tag{70}$$

This expression explicitly shows $V_0 > 0$ outside the horizon. Then, we see the stability for the $K = 0$ mode. Typical profiles of V_0 are shown in Fig. 3.

B. $K = \pm(J+2)$ modes perturbations

As noted in the previous section, the highest modes, h_{++} and h_{--}, are always decoupled for arbitrary J. Since these are gauge invariant, it is straightforward to get the equation of motion for h_{++} as

$$\delta R_{++} = \frac{h_{++}}{2\rho^2\rho_0(\rho+\rho_0)^3(\rho_+ + \rho_0)}[4\rho^5 + 16\rho^4\rho_0 - 4\rho^3(\rho_+ - 5\rho_0)\rho_0 + \rho_+\rho_0^3(\rho_+ + \rho_0) + \rho\rho_0^2(3\rho_+^2 + \rho_+\rho_0 + 2\rho_0^2) + 4\rho^2\rho_0(\rho_+^2 + 3\rho_0^2) + J\rho(\rho+\rho_0)^2(4\rho^2 + 8\rho\rho_0 + \rho_0(-\rho_+ + 3\rho_0)) + J^2\rho(\rho+\rho_0)^4] - \frac{-2\rho^2 + 3\rho\rho_+ + \rho_+\rho_0}{2\rho(\rho+\rho_0)^2}\frac{dh_{++}}{d\rho} - \frac{\rho-\rho_+}{2(\rho+\rho_0)}\frac{d^2h_{++}}{d\rho^2} - \frac{\rho}{2(\rho-\rho_+)}\omega^2 h_{++} = 0. \tag{71}$$

Defining a new variable

$$\Phi_J(\rho) \equiv \frac{1}{\rho^{1/4}(\rho+\rho_0)^{3/4}}h_{++}(\rho), \tag{72}$$

we obtain the master equation

$$-\frac{d^2}{d\rho_*^2}\Phi_J + V_J(\rho)\Phi_J = \omega^2\Phi_J, \tag{73}$$

where the potential $V_J(\rho)$ is defined by

$$V_J(\rho) = \frac{\rho-\rho_+}{16\rho^3\rho_0(\rho_+ + \rho_0)(\rho+\rho_0)^3}[4\rho_+(\rho_+ + \rho_0)^2(16\rho_+^2 + 28\rho_+\rho_0 + 11\rho_0^2) + (\rho_+ + \rho_0)(320\rho_+^3 + 640\rho_+^2\rho_0 + 356\rho_+\rho_0^2 + 35\rho_0^3)(\rho-\rho_+) + 8(\rho_+ + \rho_0)(80\rho_+^2 + 102\rho_+\rho_0 + 25\rho_0^2)(\rho-\rho_+)^2 + 32(\rho_+ + \rho_0) \times (20\rho_+ + 11\rho_0)(\rho-\rho_+)^3 + 64(5\rho_+ + 4\rho_0)(\rho-\rho_+)^4 + 64(\rho-\rho_+)^5 + J[16\rho_+(\rho_+ + \rho_0)^3(4\rho_+ + 3\rho_0) + 16(\rho_+ + \rho_0)^2(20\rho_+^2 + 21\rho_+\rho_0 + 3\rho_0^2)(\rho-\rho_+) + 16(\rho_+ + \rho_0)(40\rho_+^2 + 53\rho_+\rho_0 + 14\rho_0^2)(\rho-\rho_+)^2 + 16(\rho_+ + \rho_0)(40\rho_+ + 23\rho_0)(\rho-\rho_+)^3 + 64(5\rho_+ + 4\rho_0)(\rho-\rho_+)^4 + 64(\rho-\rho_+)^5] + J^2[16\rho_+(\rho_+ + \rho_0)^4 + 16(\rho_+ + \rho_0)^3(5\rho_+ + \rho_0)(\rho-\rho_+) + 32(\rho_+ + \rho_0)^2(5\rho_+ + 2\rho_0)(\rho-\rho_+)^2 + 32(\rho_+ + \rho_0)(5\rho_+ + 3\rho_0) \times (\rho-\rho_+)^3 + (80\rho_+ + 64\rho_0)(\rho-\rho_+)^4 + 16(\rho-\rho_+)^5]]. \tag{74}$$

Clearly, the potential V_J is positive. Then, we confirm the stability against all $K = \pm(J+2)$ modes.

V. SUMMARY AND DISCUSSION

We have studied the stability of SqKK black holes. By utilizing the symmetry $U(2)$ of the SqKK black holes, we have obtained the master equations for the metric perturbations labeled by $(J = 0, M = 0, K = 0, \pm1, \pm2)$ and $(J, M, K = \pm(J+2))$. We have proved the stability of SqKK black holes for these perturbations. Strictly speaking, we have not shown the stability of SqKK black holes completely, because we have analyzed the restricted modes. Empirically, the instability appears in the lower modes. For example, Gregory-Laflamme instability appears in an s wave. Therefore, our result for $(J = 0, M = 0, K = 0, \pm1, \pm2)$ modes gives strong evidence for stability of the SqKK black holes.

Our stability analysis suggests that the SqKK black holes deserve to be taken seriously as realistic black holes in the Kaluza-Klein spacetime. Because of the stability, the SqKK black holes could be created in colliders or in the cosmic history. If so, we can observe the extra dimension through Hawking radiation or quasinormal modes [22,28]. Namely, the SqKK black holes could be a window to the extra dimension.

There are several directions to be studied. Our method can be applicable to other $U(2)$ symmetric spacetimes such as five-dimensional Myers-Perry black holes with equal angular momenta[4] [37]. The rotating SqKK black holes [16] also have the symmetry $U(2)$. It is known that the rotation of black holes induces the superradiant instability for massive scalar fields. Since Kaluza-Klein modes of gravitational perturbation are regarded as massive fields from the four-dimensional point of view, the rotating SqKK black holes may show the superradiant instability. It is interesting to study if it occurs or not by using our formalism. As in another direction, it is intriguing to study squashed black holes in higher dimensions.

[4]In the case of odd dimensions greater than 5, gravitational perturbations of Myers-Perry black holes with equal angular momenta are discussed in [36].

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