Classical and quantum gravitational collapse in *d*-dimensional AdS spacetime: Classical solutions

Rakesh Tibrewala,[1,*] Sashideep Gutti,[1,†] T. P. Singh,[1,‡] and Cenalo Vaz[2,§]

[1]*Tata Insitute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India*

[2]*RWC and Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221-0011, USA*

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We study the collapse of a spherically symmetric dust distribution in *d*-dimensional AdS spacetime. We investigate the role of dimensionality, and the presence of a negative cosmological constant, in determining the formation of trapped surfaces and the end state of gravitational collapse. We obtain the self-similar solution for the case of zero cosmological constant, and show that one cannot construct a self-similar solution when a cosmological constant is included.

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I. INTRODUCTION

There are many models of spherical gravitational collapse in classical general relativity which exhibit the formation of black holes as well as naked singularities, starting from regular initial data [1,2]. The study of quantum effects in the vicinity of the gravitational singularity then becomes significant. Such studies can be divided into two classes : (i) quantum field theory in curved space, and (ii) quantum general relativistic treatment of gravitational collapse.

The earliest investigations of quantum field theory in the dynamical background of a collapsing spherical star were probably those due to Ford and Parker [3] and Hiscock *et al.* [4]. These works introduced important techniques, such as the calculation of the quantum flux in the geometric optics approximation, and the regularization of the 2d quantum stress tensor, which were used extensively in later studies. A systematic study of semiclassical effects in gravitational collapse was initiated by Vaz and Witten in [5,6] and pursued in a series of papers [7–13]. Typically, these studies showed an important and interesting difference in the nature of quantum particle creation between the two cases—one in which collapse ends in a black hole, and another in which it ends in a naked singularity. The formation of a black hole is accompanied by the emission of Hawking radiation, as expected. However, when the collapse ends in a (shell-focusing) naked singularity, there is no evidence of some universal behavior in the nature of quantum emission. It is typically found, though, that the emitted quantum flux diverges in the approach to the Cauchy horizon. This divergence disappears when the calculation of the quantum flux is terminated about a Planck time before the formation of the Cauchy horizon, when the semiclassical approximation breaks down. Instead of the divergence, one finds that only about a Planck unit of energy is emitted during the semiclassical phase, and a full quantum gravitational treatment of the physics of the singularity and the Cauchy horizon becomes unavoidable. These developments have been reviewed in [14].

A full quantum gravitational treatment of collapse can be performed via a midisuperspace quantization within the framework of quantum general relativity. The aims of such a program are manifold—to construct a quantum gravitational description of the black hole; to check if the gravitational singularity can be avoided in quantum gravity; to obtain a statistical derivation of the black hole entropy from quantum gravitational microstates; and to determine the role of quantum gravity in ascertaining the nature of quantum emission from a naked singularity. The midisuperspace quantization program has been carried out by us in a series of papers [15–21], and work along these lines is still in progress. It is fair to say that while some progress has been made on aspects related to quantum black holes and black hole entropy, issues related to singularity avoidance and the nature of quantized naked singularities have thus far proved difficult to address, largely because of problems relating to finding a suitable regularization scheme for the quantized Hamiltonian constraint in canonical general relativity. Also, we still do not have a definitive answer as to the nature of quantum gravitational corrections to the semiclassical spectrum of Hawking radiation. By this we mean the following: starting from a candidate theory of quantum gravity such as quantum general relativity, one can derive Hawking radiation in the semiclassical approximation using a suitable midisuperspace model. Going beyond the semiclassical approximation, it is expected that quantum gravity will induce (possibly nonthermal) corrections to Hawking spectrum, but this still remains to be worked out in its full generality. It is hoped, though, that some progress will be possible on these unsolved problems if one makes contact with the methods of loop quantum gravity.

All the classical and quantum studies mentioned so far have pertained to gravitational collapse in $3 + 1$ dimensions. Motivated by the desire to overcome some of the obstacles faced in $3 + 1$ physics, we turned attention to the investigation of $2 + 1$ gravitational collapse.

*rtibs@tifr.res.in
†sashideep@tifr.res.in
‡tpsingh@tifr.res.in
§Cenalo.Vaz@UC.Edu

Homogeneous dust collapse in 2 + 1 dimensions was first studied in [22] and for the case of collapsing shells in [23]. This lower dimensional model, though simpler in some aspects, throws up new fascinating issues of its own, which have been studied in the context of inhomogeneous dust collapse in [24–26]. Classical 2 + 1 collapse admits a naked singularity for some initial data, but there is no corresponding quantum particle creation. A black hole solution (the well-known BTZ black hole [27]) is possible in the presence of a negative cosmological constant, but the thermodynamics and statistics of the quantized BTZ black hole is completely different from that of the 4d Schwarzschild black hole.

These differences prompt us to the following question: in determining the nature of thermodynamics and statistics of the quantized black hole, and the nature of quantum emission from naked singularities, what is the role of the cosmological constant, and of the number of spatial dimensions? The present paper is the first in a series of three papers which addresses this question, by studying classical and quantum aspects of spherical dust collapse in an AdS spacetime with an arbitrary number of dimensions. In the current paper, we solve the Einstein equations for a collapsing dust ball in an asymptotically AdS spacetime, and examine the nature of the gravitational singularity. Quantization of this model will be taken up in two subsequent papers.

The plan of the paper is as follows. In Sec. II we give results for spherical gravitational collapse of dust in an asymptotically flat d-dimensional spacetime. While this problem has been studied earlier by various authors [20,28–30], we present here a simpler derivation of the occurrence of a locally naked singularity, and also obtain new results on the self-similar solution. More importantly, the results of this section serve as a prelude to the corresponding analysis presented in Sec. III, for collapse in an AdS spacetime with arbitrary number of dimensions. While gravitational collapse of dust in four-dimensional spacetime with a positive cosmological constant has been studied in [31] (see also [32]), and for a negative cosmological constant in [33], to the best of our knowledge dust collapse in a d-dimensional AdS spacetime has not been studied before.

One could question the introduction of a negative cosmological constant, as is done in this paper, when the observed universe has a cosmological constant which is perhaps positive, or at best zero, but certainly not negative. First, collapse physics in a de Sitter spacetime is complicated by the presence of a de Sitter event horizon, in addition to the black hole event horizon. It thus seems natural to first address the AdS case before moving on to the more realistic, and more difficult, de Sitter case. There are also reasons to believe that it would not make sense to directly construct a quantum black hole model in a higher dimensional space with a positive cosmological constant, because quantum gravity in such a spacetime may not exist nonperturbatively [34,35]. Pure quantum gravity with a positive cosmological constant may hence not exist as an exact theory, but only as a part of a larger system [35]. It is also a question of great interest as to whether studies of statistical properties of AdS black holes in canonical quantum general relativity can benefit from what is known about the AdS/conformal field theory (CFT) correspondence, as suggested recently in [36] for the 4d case.

II. HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC DUST COLLAPSE IN THE ABSENCE OF A COSMOLOGICAL CONSTANT

A. Solution

The metric for a spherically symmetric spacetime can be written in the form

$$ds^2 = -e^{\mu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + R^2(t,r)d\Omega^2, \qquad (1)$$

where

$$d\Omega^2 = d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \ldots + \sin^2\theta_{n-1}d\theta_n^2)). \qquad (2)$$

Here the number of spacetime dimensions is $(n+2)$ where $n \geq 1$ is the number of angular coordinates and the 2 designates one time dimension and one radial dimension. For the case where the cosmological constant $\Lambda = 0$, Einstein equations are

$$G_{\mu\nu} = kT_{\mu\nu}, \qquad (3)$$

where k is a constant related to Newton's constant of gravitation G (see Sec. II D) and $T_{\mu\nu}$ is the stress-energy tensor. For the case of nonrotating dust one can choose a synchronous and comoving coordinate system in which the only nonzero component of the stress-energy tensor is $T_{00} = \epsilon(t,r)$, where $\epsilon(t,r)$ is the energy density of the dust. Further, in comoving coordinates the g_{00} component of the metric can be chosen to be minus one. With this choice for the metric in (1) we get the following independent set of Einstein equations:

$$G_{00} = \frac{e^{-\lambda}}{R^2}\left[-\frac{n(n-1)}{2}R'^2 + \frac{n}{2}RR'\lambda' + \frac{n(n-1)}{2}e^{\lambda}(1+\dot{R}^2) + \frac{n}{2}(-2RR'' + e^{\lambda}R\dot{R}\dot{\lambda})\right] = k\epsilon(t,r), \qquad (4)$$

$$G_{01} = \frac{n}{2}\frac{(R'\dot{\lambda} - 2\dot{R}')}{R} = 0, \qquad (5)$$

$$G_{11} = \frac{1}{R^2}\left[\frac{n(n-1)}{2}(R'^2 - e^{\lambda}(1+\dot{R}^2)) - ne^{\lambda}R\ddot{R}\right] = 0, \tag{6}$$

$$\begin{aligned} G_{22} = &-\frac{1}{4}e^{-\lambda}[-2(n-2)(n-1)R'^2 + 2(n-1)RR'\lambda' \\ &+ 2(n-2)(n-1)e^{\lambda}(1+\dot{R}^2) \\ &- 2(n-1)(2RR'' - e^{\lambda}(R\dot{R}\dot{\lambda} + 2R\ddot{R}) \\ &+ e^{\lambda}R^2(\dot{\lambda}^2 + 2\ddot{\lambda})] = 0. \end{aligned} \tag{7}$$

Components G_{33}, G_{44}, etc. are given by expressions similar to that for G_{22} except for overall sine squared factor(s). The Ricci scalar is given by

$$\begin{aligned} \mathcal{R} = &\frac{e^{-\lambda}}{2R^2}[-2n(n-1)(R'^2 + e^{\lambda}(1+\dot{R}^2)) \\ &+ 2nR(R'\lambda' - 2R'' + e^{\lambda}(\dot{R}\dot{\lambda} + 2\ddot{R})) \\ &+ e^{\lambda}R^2(\dot{\lambda}^2 + 2\ddot{\lambda})]. \end{aligned} \tag{8}$$

Solving the equation for G_{01} we obtain

$$e^{\lambda} = \frac{R'^2}{1+f(r)}. \tag{9}$$

In the above expression $f(r)$ is an arbitrary function called the energy function. Integration of the equation for G_{11} after using Eq. (9) gives

$$\dot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}}. \tag{10}$$

Here $F(r)$ is another arbitrary function and is called the mass function. In what follows we will only consider the so-called marginally bound case for which $f(r) = 0$. In this case (10) can be integrated easily and after choosing the negative sign for the square root corresponding to infalling matter we get

$$t - t_c(r) = -\frac{2}{n+1}\frac{R^{(n+1)/2}}{\sqrt{F(r)}}, \tag{11}$$

where $t_c(r)$ is yet another arbitrary function which can be fixed by using the freedom in the choice of the r-coordinate. We relabel r such that at $t = 0$, $R = r$. With this choice we have

$$t_c(r) = \frac{2}{n+1}\frac{r^{(n+1)/2}}{\sqrt{F(r)}}. \tag{12}$$

From the above equations we see that at $t = t_c(r)$, $R(t, r) = 0$ and this implies singularity formation for the shell labeled r as indicated by the blowing up of the Ricci scalar in (8). Finally, substituting for λ from (9) in the equation for G_{00} we find that

$$k\epsilon(t, r) = \frac{n}{2}\frac{F'}{R^n R'}. \tag{13}$$

From this one can obtain an expression for the mass function

$$F(r) = \frac{2k}{n}\int \epsilon(0, r)r^n dr. \tag{14}$$

B. A simple derivation of the naked singularity

We now look at the nature of the $R = 0$ singularity formed at the center $r = 0$ of the dust cloud. For this we follow the method used in [37] and start by assuming that the initial density profile $\epsilon(0, r)$ has the following series expansion near the center $r = 0$ of the dust cloud

$$\epsilon(r) = \epsilon_0 + \epsilon_1 r + \frac{\epsilon_2}{2!}r^2 + \ldots \tag{15}$$

Using this in (14) we find that in this case the mass function can be written as

$$F(r) = F_{n+1}r^{n+1} + F_{n+2}r^{n+2} + F_{n+3}r^{n+3} + \ldots, \tag{16}$$

where it is to be noted that n is not a free index but, as before, refers to the number of angular dimensions and

$$F_{n+i} = \frac{2k}{n(n+i)}\frac{\epsilon_{i-1}}{(i-1)!} \tag{17}$$

and $i = 1, 2, 3 \ldots$ From (12) we know that the singularity curve is given by

$$t_s(r) = \frac{2}{n+1}\frac{r^{(n+1)/2}}{\sqrt{F(r)}}. \tag{18}$$

The central singularity at $r = 0$ forms at the time

$$t_0 = \frac{2}{n+1}\frac{1}{\sqrt{F_{n+1}}} = \frac{2}{n+1}\sqrt{\frac{n(n+1)}{2k\epsilon_0}}. \tag{19}$$

Here, as a special case, we note that when $\epsilon(r) = \epsilon_0$, a constant (Oppenheimer-Snyder collapse), $F(r) = F_{n+1}r^{n+1}$ and the singularity curve is given by $t_s = 2/(n+1)\sqrt{F_{n+1}}$ which is independent of r implying that all shells become singular at the same time as the central shell. Near $r = 0$ one can use the expansion for $F(r)$ as in (16) and approximate the singularity curve as

$$t_s(r) \approx t_0 - \frac{1}{(n+1)}\frac{F_{n+i}}{F_{n+1}^{3/2}}r^{i-1}. \tag{20}$$

In the above equation F_{n+i} is the first nonvanishing term beyond F_{n+1} in the expansion for $F(r)$.

One would like to know whether the singularity at $t = t_0$, $r = 0$ is naked or not, and for this we focus attention on radial null geodesics. We want to check if there are any outgoing radial null geodesics which terminate on the central singularity in the past. Assuming that there exist such geodesics we assume their form near $r = 0$ to be

$$t = t_0 + ar^{\alpha}. \tag{21}$$

Comparing this with (20) we conclude that for the null geodesic to lie in the spacetime one must have $\alpha \geq i - 1$ and if $\alpha = i - 1$ then

$$a < -\frac{F_{n+i}}{(n+1)F_{n+1}^{3/2}}. \tag{22}$$

(This is because F_{n+i} is negative, which will be the case if we demand that $\epsilon(0, r)$ be a decreasing function of r).

Since one is interested in the region close to $r = 0$, we expand (11) to leading order in r to obtain

$$R \approx r\left[1 - \frac{(n+1)}{2}\sqrt{F_{n+1}}\left(1 + \frac{1}{2}\frac{F_{n+i}}{F_{n+1}}r^{i-1}\right)t\right]^{2/(n+1)}. \tag{23}$$

From the metric one finds that for null geodesics $dt/dr|_{NG} = R'$. Differentiating (23) with respect to (w.r.t.) r we get

$$R' = \left[1 - \frac{(n+1)}{2}\sqrt{F_{n+1}}\left(1 + \frac{1}{2}\frac{F_{n+i}}{F_{n+1}}r^{i-1}\right)t\right]^{-((n-1)/(n+1))} \times \left[1 - \frac{(n+1)}{2}\sqrt{F_{n+1}}t - \frac{(n+2i-1)}{4}\frac{F_{n+i}}{\sqrt{F_{n+1}}}r^{i-1}t\right]. \tag{24}$$

Along the assumed geodesic, t is given by (21). Substituting this in (24) and equating it with the derivative of (21), i.e. $dt/dr = \alpha a r^{\alpha-1}$ gives

$$\alpha a r^{\alpha-1} = \left[1 - \frac{(n+1)}{2}\sqrt{F_{n+1}}\left(1 + \frac{1}{2}\frac{F_{n+i}}{F_{n+1}}r^{i-1}\right) \times \left(t_0 + ar^{\alpha}\right)\right]^{-((n-1)/(n+1))} \times \left[1 - \frac{(n+1)}{2}\sqrt{F_{n+1}}(t_0 + ar^{\alpha}) - \frac{(n+2i-1)}{4}\frac{F_{n+i}}{\sqrt{F_{n+1}}}r^{i-1}(t_0 + ar^{\alpha})\right]. \tag{25}$$

This is the main equation. If it admits a self-consistent solution then the singularity will be naked, otherwise not. To simplify this we note that $\sqrt{F_{n+1}}t_0 = 2/(n+1)$, as follows from (19).

We first consider the case $\alpha > i - 1$. To leading order this gives

$$\alpha a r^{\alpha-1} = \left(-\frac{F_{n+i}}{2F_{n+1}}\right)^{2/(n+1)}\left(\frac{n+2i-1}{n+1}\right)r^{(2(i-1)/(n+1))}. \tag{26}$$

This equation implies

$$\alpha = \frac{n+2i-1}{n+1}; \qquad a = \left(-\frac{F_{n+i}}{2F_{n+1}}\right)^{2/(n+1)}. \tag{27}$$

Since F_{n+i} is the first nonvanishing term beyond F_{n+1}, we have the condition $i > 1$. Also, for consistency we require $\alpha = (n+2i-1)/(n+1) > i - 1$, which together with the previous condition on i implies $1 < i < 2n/(n-1)$.

This implies that in 4 dimensions, where $n = 2$, we have $1 < i < 4$, which means that $i = 2$, 3 are the allowed values. That is, models for which either $\epsilon_1 < 0$ (corresponding to $i = 2$) or $\epsilon_1 = 0$, $\epsilon_2 < 0$ (corresponding to $i = 3$) will have a naked singularity.

Similarly, in 5 dimensions, where $n = 3$, we find that $1 < i < 3$ implying $i = 2$, i.e. only for $\epsilon_1 < 0$ we get naked singularity. In 6 dimensions, $n = 4$ and we have $1 < i < 8/3$, implying $i = 2$ as the only allowed value, i.e. the singularity is naked only if $\epsilon_1 < 0$. One notes that for all higher dimensions $2 < 2n/(n-1) < 3$ and therefore only $i = 2$, i.e. $\epsilon_1 < 0$ gives naked singularity.

As another special case we note that for $n = 1$, that is in $(2 + 1)$ dimensions, $\alpha = (n+2i-1)/(n+1) = i$ and therefore the condition $\alpha > i - 1$ is always satisfied, implying that in this case we always have a naked singularity, which is in agreement with what has been observed in earlier work on $(2 + 1)$-dimensional dust collapse [24].

We next consider the case where $\alpha = i - 1$. Here (25) gives

$$(i-1)ar^{i-2} = \left(-\frac{(n+1)}{2}\sqrt{F_{n+1}}a - \frac{F_{n+i}}{2F_{n+1}}\right)^{-((n-1)/(n+1))} \times \left(-\frac{(n+1)}{2}\sqrt{F_{n+1}}a - \frac{(n+2i-1)}{2(n+1)}\frac{F_{n+i}}{F_{n+1}}\right) \times r^{(2(i-1))/(n+1)}, \tag{28}$$

which implies $i = 2n/(n-1)$. Now the conditions on i are that it be an integer greater than 1. These two conditions are met only for 2 and $n = 3$, that is, in $(3 + 1)$ dimensions and in $(4 + 1)$ dimensions, respectively. For $n = 2$, $i = 4$ (which corresponds to $\epsilon_3 < 0$) and for $n = 3$, $i = 3$ (corresponding to $\epsilon_2 < 0$). Since the 4-dimensional case, corresponding to $n = 2$, is already reported in the literature [38] we focus attention on the 5-dimensional case corresponding to $n = 3$.

Substituting $n = 3$, $i = 3$ in (28) we obtain

$$8\sqrt{F_4}a^3 + \left(\frac{2F_6}{F_4} + 4F_4\right)a^2 + \frac{4F_6}{\sqrt{F_4}}a + \frac{F_6^2}{F_4^2} = 0. \tag{29}$$

The above cubic for a has to be solved subject to the constraint $0 < a < -F_6/4F_4^{3/2}$ as mentioned earlier. By defining $a = \sqrt{F_4}b$ and $F_6 = F_4^2\xi$ the above equation is simplified to

$$2b^2(4b + \xi) + (2b + \xi)^2 = 0 \tag{30}$$

and the constraint on a results in a constraint on b given by $0 < b < -\xi/4$. By defining $-b/\xi = Y$ and $-1/\xi = \eta$, the above cubic is further simplified to

$$2Y^2(4Y - 1) + \eta(2Y - 1)^2 = 0. \tag{31}$$

For a naked singularity to form this equation for Y should have a positive root subject to the constraint $0 < Y < 1/4$.

Now for a general cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0, \tag{32}$$

if we define $H \equiv a_0a_2 - a_1^2$ and $G \equiv a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$, we have the following conditions on the roots of the cubic [39]:

(1) $G^2 + 4H^3 < 0$, the roots of the cubic are all real.
(2) $G^2 + 4H^3 > 0$, the cubic has two imaginary roots.
(3) $G^2 + 4H^3 = 0$, two roots of the cubic are equal.
(4) $G = 0$ and $H = 0$, all three roots of the cubic are equal.

Using these we can find the conditions on η for which the cubic in (31) has at least one real root in the desired range. Here it should be noted that η as defined above has to be positive. It is found that for $0 < \eta \leq (-11 + 5\sqrt{5})/4$ all the three roots are real and at least one of these satisfies $0 < Y < 1/4$. For $\eta > (-11 + 5\sqrt{5})/4$ the real root is negative. The range of η found above implies that for $\xi \leq 4/(11 - 5\sqrt{5})$ one gets a naked singularity.

We also note that for the Oppenheimer-Snyder collapse mentioned earlier, no naked singularity is formed since all shells become singular at the same time.

C. Formation of trapped surfaces

We now consider the formation of trapped surfaces. For this consider a congruence of outgoing radial null geodesics with tangent vector $K^i = dx^i/dk$ where k is a parameter along the geodesic and $i = (0, 1)$ [40]. The expansion for these geodesics is given by

$$\theta = K^i_{;i} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}K^i). \tag{33}$$

From this one finds that

$$\theta = \frac{nR'}{R}\left(1 - \sqrt{\frac{F(r)}{R^{n-1}}}\right)K^r. \tag{34}$$

Trapping occurs when $\theta = 0$ and the above equation with $R' > 0$, $R > 0$, and $K^r > 0$ implies that this condition is met for

$$\frac{F(r)}{R^{n-1}} = 1. \tag{35}$$

In 4 dimensions where $n = 2$ we get the well-known result

$$\frac{F(r)}{R} = 1. \tag{36}$$

For the general case one finds that the time at which trapping occurs t_{tr} is given by

$$t_{tr}(r) = \frac{2}{n+1}\left(\frac{r^{(n+1)/2}}{\sqrt{F(r)}} - F(r)^{1/(n-1)}\right), \tag{37}$$

which means that the central shell is trapped at $t_{tr}(r) = 2/(n+1)\sqrt{F_{n+1}}$, that is, at the same time as the formation of the central singularity. For the outer shells trapping occurs before those shells become singular.

D. Exterior solution and matching with the interior

We take the metric in the exterior to be independent of time and given by

$$ds^2 = -f(x)dT^2 + g(x)dx^2 + x^2d\Omega^2, \tag{38}$$

where $(T, x, \theta_1, \theta_2 \ldots)$ are the coordinates in the spacetime exterior to the dust cloud. The components of the Einstein tensor corresponding to the above metric are

$$G_{00} = \frac{-n(n-1)fg + n(n-1)fg^2 + nxfg'}{2x^2g^2}, \tag{39}$$

$$G11 = \frac{n(n-1) - n(n-1)fg + nxf'}{2x^2f}, \tag{40}$$

$$G_{22} = \frac{-x^2gf'^2 + 2(n-1)(n-2)f^2g(1-g) - 2(n-1)xf^2g' + xf(-xf'g' + 2(n-1)gf' + 2xgf'')}{4f^2g^2}, \tag{41}$$

and G_{33}, G_{44}, etc. are related to G_{22} as in the interior. Solving the vacuum Einstein equations $G_{\mu\nu} = 0$ one finds

$$g(x) = \left(1 - \frac{C}{x^{n-1}}\right)^{-1}, \tag{42}$$

$$f(x) = 1 - \frac{C}{x^{n-1}}. \tag{43}$$

Here C is a constant of integration. Thus the exterior metric is the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{C}{x^{n-1}}\right)dT^2 + \left(1 - \frac{C}{x^{n-1}}\right)^{-1}dx^2 + x^2d\Omega^2. \tag{44}$$

For this to be a valid solution in the exterior we need to match the metric coefficients as well as their first derivatives (extrinsic curvature) in the exterior with the corresponding quantities in the interior at the boundary of the dust cloud $r = r_s$ say, [41,42]. This will also determine the only unknown quantity C in the Schwarzschild solution. At the surface the exterior coordinates will be some functions $x = x(t, r_s) \equiv x_s(t)$ and $T = T(t, r_s) \equiv T_s(t)$ of the interior coordinates. These relations imply $dT = \dot{T}_sdt$ and

$dx = \dot{x}_s dt$. Therefore at the surface (where $dr = 0$)

$$(ds^2)_{\rm surf} = \left[-\left(1 - \frac{C}{x_s^{n-1}}\right)\dot{T}_s^2 + \frac{\dot{x}_s^2}{1 - \frac{C}{x_s^{n-1}}}\right]dt^2 + x_s^2 d\Omega^2 = -dt^2 + R_s^2(t)d\Omega^2. \tag{45}$$

Matching the metric coefficients for $d\Omega^2$ gives $x_s(t) = R_s(t)$ and matching the metric coefficients for dt^2 then implies

$$\left(1 - \frac{C}{R_s^{n-1}}\right)\dot{T}_s^2 - \frac{\dot{R}_s^2}{1 - \frac{C}{R_s^{n-1}}} = 1. \tag{46}$$

To match the extrinsic curvature (second fundamental form) we need the normal to the surface. In the interior coordinates the components of the normal are found to be $n^i_\mu = (0, R', 0\ldots0)$. Similarly in the exterior coordinates the normal is given by $n^e_\mu = (-\dot{R}, \dot{T}_{ss}, 0\ldots0)$, where the relation $dx - \dot{R}_s dt = 0$ was used. The extrinsic curvature is given by $K_{ab} = n_{\mu;\nu}e^\mu_a e^\nu_b$, where $e^\mu_a = \partial x^\mu/\partial y^a$ with x^μ being the coordinates of the $(n+2)$-dimensional manifold and y^a being the coordinates on the boundary of the manifold. Since there is only one undetermined constant C, we match only the $K_{\theta_1\theta_1}$ component of the extrinsic curvature. It can be easily checked that the other components do not give anything new. We find that at the surface the extrinsic curvature in the interior coordinates is given by $K^i_{\theta_1\theta_1} = R_s$. Similarly in the exterior coordinates we have $K^e_{\theta_1\theta_1} = R_s(1 - C/R_s^{n-1})\dot{T}_s$. Equating these two expressions for $K_{\theta_1\theta_1}$ gives

$$R_s = R_s\left(1 - \frac{C}{R_s^{n-1}}\right)\dot{T}_s. \tag{47}$$

Using (46) and $\dot{R}_s^2 = F(r_s)/R_s^{n-1}$ (see (10)) the above equation gives $C = F(r_s)$, where from (14) it is clear that $F(r_s)$ is proportional to the total mass of the dust cloud. Thus we find that for the metric coefficients and their first derivatives to be continuous across the boundary the metric in the exterior is given by

$$ds^2 = -\left(1 - \frac{F_s}{x^{n-1}}\right)dT^2 + \left(1 - \frac{F_s}{x^{n-1}}\right)^{-1}dx^2 + x^2 d\Omega^2. \tag{48}$$

Now $F(r_s) = (2k/n)\int_0^{r_s}\epsilon(0,r)r^n dr$ and we know that mass of the dust cloud is given by $M = \int_0^{r_s}\epsilon(0,r)dV$ where dV is the volume element of a spherical shell lying between r and $r + dr$ in $(n+1)$ space dimensions. This volume element is given by

$$dV = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}r^n dr. \tag{49}$$

Therefore

$$M = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}\int_0^{r_s}\epsilon(0,r)r^n dr. \tag{50}$$

This implies

$$\int_0^{r_s}\epsilon(0,r)r^n dr = \frac{M\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}}. \tag{51}$$

Using this we find that the mass function can be written as

$$F(r_s) = C = \frac{2k}{n}\frac{M\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}}. \tag{52}$$

One can also find the constant C in the Schwarzschild solution using the weak field limit. For this we assume that Newton's law for gravity holds for any number of dimensions, i.e. $\nabla\cdot g = 4\pi G\epsilon(r)$ and $g = -\nabla\phi(r)$. Here g is the gravitational field strength and ϕ is the gravitational potential (note: Newton's gravitational constant G being dimensionful will be different in different dimensions, however, this does not affect the form of the equations). Using this we find that in $(n+1)$ spatial dimensions the gravitational potential is given by

$$\phi(r) = \frac{4\pi GM\Gamma(\frac{n+1}{2})}{2(n-1)\pi^{(n+1)/2}r^{n-1}}, \tag{53}$$

where $n > 1$ (potential has a logarithmic dependence on r in $2+1$ dimensions). In the weak field limit the Schwarzschild solution is $g_{00} \to -(1 - \frac{C}{r^{n-1}})$ and $g_{11} \to (1 + \frac{C}{r^{n-1}})$. Also, using the geodesic equation we find that generically, in the weak-static field limit $g_{00} = -(1 - 2\phi)$ and $g_{11} = (1 + 2\phi)$. Comparing the two expressions for g_{00} (or for g_{11}) one finds that $C = 2\phi r^{n-1}$ and using the expression for $\phi(r)$ as found above one gets

$$C = \frac{4\pi GM\Gamma(\frac{n+1}{2})}{(n-1)\pi^{(n+1)/2}}. \tag{54}$$

This expression for C will be the same as that found above from matching if the constant in Einstein's equations is chosen to be $k = 4n\pi G/(n-1)$. For $n = 2$ this reduces to the value $8\pi G$ as used in 4-dimensional theory and which when used in (48) results in the familiar Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{x}\right)dT^2 + \left(1 - \frac{2GM}{x}\right)^{-1}dx^2 + x^2 d\Omega^2. \tag{55}$$

E. The self-similar solution

To see the effect of dimensions on the nature of quantum particle flux (which will be described in a work subsequent to this), we would like to have a globally naked singularity. It is known that a locally naked self-similar solution is also globally naked [43], where self-similar spacetimes are defined by the existence of a homothetic Killing vector field. Therefore here we look at the dependence on dimensions of the self-similar dust model.

In a self-similar collapse any dimensionless quantity made from the metric functions, has to be a function only of t/r. This can be seen by starting from the definition of a homothetic Killing vector field ξ,

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 2g_{\alpha\beta} \tag{56}$$

(we emphasize here that we are dealing only with self-similarity of the first kind, which is defined by the above equation). This condition implies that

$$\mathcal{L}_{\xi} G_{\mu\nu} = 0, \tag{57}$$

where $G_{\mu\nu}$ is the Einstein tensor for the spacetime [44,45]. It follows then that the energy-momentum tensor should also satisfy the equation

$$\mathcal{L}_{\xi} T_{\mu\nu} = 0. \tag{58}$$

For a perfect fluid the energy-momentum tensor is $T_{\mu\nu} = (p+\mu)u_\mu u_\nu + pg_{\mu\nu}$. The condition (58) implies the following:

$$\mathcal{L}_{\xi} u^\mu = -u^\mu, \tag{59}$$

$$\mathcal{L}_{\xi} \mu = -2\mu, \tag{60}$$

$$\mathcal{L}_{\xi} p = -2p. \tag{61}$$

Based on the above equations, the existence of a homothetic vector implies that the metric components in a comoving coordinate system will be of the form such that the dimensionless quantities become functions of r/t, [46]. Following [46] we assume a general spherically symmetric ansatz of the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Psi}dr^2 + R^2 d\Omega^2 \tag{62}$$

and also we have

$$\mathcal{L}_{\xi} u^\mu = -u^\mu. \tag{63}$$

If we assume the vector ξ has only r and t components given by $\xi^\mu = \alpha\delta^\mu_r + \beta\delta^\mu_t$ and expand (56) we get partial differential equations for the vector components. The condition of the comoving metric implies the condition that $\alpha_{,t} = 0$ and $\beta_{,r} = 0$. After redefining the independent variables to $\bar{r}_{,r} = r/\alpha$ and $\bar{t}_{,t} = t/\beta$ we now redefine the dependent variables by

$$\bar{\Psi} = \Psi + \log(\alpha) - \log(\bar{r}), \tag{64}$$

$$\bar{\Phi} = \Phi + \log(\beta) - \log(\bar{r}). \tag{65}$$

Under the above change of variables Eqs. (56) become

$$\bar{r}R_{\bar{r}} + \bar{t}R_{\bar{t}} = R, \tag{66}$$

$$\bar{r}\bar{\Psi}_{\bar{r}} + \bar{t}\bar{\Psi}_{\bar{t}} = 0, \tag{67}$$

$$\bar{r}\bar{\Phi}_{\bar{r}} + \bar{t}\bar{\Phi}_{\bar{t}} = 0. \tag{68}$$

So this shows that we can go to a coordinate system in which the metric functions are functions of $\bar{r}/\bar{t}$. So $\bar{\Psi}$ and $\bar{\Phi}$ are functions of z and R is $r\mathcal{R}(z)$.

It can now be shown that for spherically symmetric, self-similar dust collapse, the mass function is given by $F = \lambda r^{n-1}$, where λ is a constant. For this we start with the G_{01} component of the Einstein equation

$$G_{01} = \frac{n}{2}\frac{(R'\dot{\lambda} - 2\dot{R}')}{R} = 0. \tag{69}$$

Defining the self-similarity parameter $z = r/t$ and writing $R \equiv r\tilde{R}$, where $\tilde{R}$ and λ being dimensionless are functions only of z, the above equation can be written as

$$-\tilde{R}\frac{d\lambda}{dz} + 4\frac{d\tilde{R}}{dz} - z\frac{d\lambda}{dz}\frac{d\tilde{R}}{dz} + 2z\frac{d^2\tilde{R}}{dz^2} = 0, \tag{70}$$

where we have used $R' = \tilde{R} + z d\tilde{R}/dz$ and $\dot{R} = -z^2 d\tilde{R}/dz$. The above equation is solved easily to obtain

$$e^\lambda = c\left(\tilde{R} + z\frac{d\tilde{R}}{dz}\right)^2, \tag{71}$$

where c is a constant of integration and equals one for the marginally bound case (and will therefore be ignored in what follows). Similarly the G_{11} component of the Einstein equation gives

$$\frac{n(n-1)}{2}\left(\tilde{R} + z\frac{d\tilde{R}}{dz}\right)^2 - \frac{n(n-1)}{2}\left(\tilde{R} + z\frac{d\tilde{R}}{dz}\right)^2 \times\left(1 + z^4\left(\frac{d\tilde{R}}{dz}\right)^2\right) - nz\tilde{R}\left(\tilde{R} + z\frac{d\tilde{R}}{dz}\right)^2 \times\left(2z^2\frac{d\tilde{R}}{dz} + z^3\frac{d^2\tilde{R}}{dz^2}\right) = 0. \tag{72}$$

Solving this we obtain

$$\tilde{R}^{n-1}\tilde{R}'^2 = \frac{c}{z^4}, \tag{73}$$

where c is a constant of integration.

Now the mass function is given by $F = \dot{R}^2 R^{n-1}$ and using $\dot{R} = -z^2 d\tilde{R}/dz$ this can be written as

$$F = z^4 r^{n-1}\tilde{R}^{n-1}\tilde{R}'^2. \tag{74}$$

Using (73) in the above equation we obtain $F = cr^{n-1}$ which is the desired result.

With this result (11) becomes

$$R^{(n+1)/2} = \frac{n+1}{2}\sqrt{\lambda r^{n-1}}(\theta r - t). \tag{75}$$

Here $\theta = t_c/r$ is a constant. This is because in self-similar collapse any dimensionless quantity has to be a function only of t/r whereas t_c/r, being a function only of r (see (12)) has to be a constant. We are interested in finding the behavior of density $\epsilon = nF'/2\kappa R'R^n$ in the neighborhood of the center $r = 0$. Using (75) and $F' = \lambda(n-1)r^{n-2}$ in the expression for density we find

$$\epsilon = \frac{4n}{\kappa(n+1)}\left[\left(\frac{n+1}{n-1}\right)\theta^2 r^2 - \frac{2n}{n-1}\theta \mathrm{tr} + t^2\right]^{-1}. \quad (76)$$

For $r \approx 0$ we neglect the second order term in the above equation and obtain

$$\epsilon = \frac{4n}{\kappa(n+1)t^2}\left[1 - \frac{2n}{n-1}\frac{r}{t}\right]^{-1}. \quad (77)$$

Also for $r \to 0$, $R^{n+1} = \frac{(n+1)^2}{4}\lambda t^2 r^{n-1}$, which implies

$$r = \left(\frac{2}{(n+1)t\sqrt{\lambda}}\right)^{2/(n-1)} R^{(n+1)/(n-1)}. \quad (78)$$

Substituting this in (77) we get

$$\epsilon = \frac{4n}{\kappa(n+1)t^2}\left[1 + \frac{2n\theta}{n-1}a_n^{2/(n-1)} z^{(n+1)/(n-1)} + \left(\frac{2n\theta}{n-1}\right)^2 a_n^{4/(n-1)} z^{(2(n+1))/(n-1)} \ldots\right], \quad (79)$$

where $a_n \equiv 2/(n+1)\sqrt{\lambda}$ and $z \equiv R/t$. This shows how the density profile should depend on the number of dimensions to obtain a self-similar solution. The above form for density profile implies that in 4 dimensions ($n = 2$) and in 5 dimensions ($n = 3$) the self-similar solution corresponds to an analytic density profile whereas in higher dimensions the density profile is no longer analytic.

Here we also note that for $n = 1$, that is in $2 + 1$ dimensions, $F = \lambda$ and is thus independent of r and therefore one requires that energy density ϵ should be zero. Thus self-similarity in $2 + 1$ dimensions is inconsistent with the presence of matter.

III. SPHERICALLY SYMMETRIC INHOMOGENEOUS DUST COLLAPSE IN THE PRESENCE OF A NEGATIVE COSMOLOGICAL CONSTANT

A. Solution

In the presence of a cosmological constant Λ, Einstein equations are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}. \quad (80)$$

For the case $\Lambda < 0$ we take $\Lambda \to -\Lambda$ in which case the Einstein equations become $G_{\mu\nu} - \Lambda g_{\mu\nu} = kT_{\mu\nu}$, where now $\Lambda > 0$. The expressions for the components of Einstein tensor are still the same as in the $\Lambda = 0$ case. In particular since $g_{01} = 0$, therefore we again have

$$G_{01} = \frac{n}{2}\frac{R'\dot{\lambda} - 2\dot{R}'}{R} = 0. \quad (81)$$

The solution of this equation is again given by (9) and we again consider the marginally bound case so that $f(r) = 0$. The 1-1 component of Einstein equations is

$$\frac{1}{R^2}\left[\frac{n(n-1)}{2}(R'^2 - e^{\lambda}(1 + \dot{R}^2)) - ne^{\lambda}R\ddot{R}\right] - \Lambda R'^2 = 0. \quad (82)$$

Integration of this equation gives

$$\dot{R}^2 = -\frac{2\Lambda}{n(n+1)}R^2 + \frac{F(r)}{R^{n-1}}, \quad (83)$$

where as before $F(r)$ is the mass function. Integrating this equation after taking the negative sign for the square root (to account for in-falling matter) we get

$$t - t_c(r) = -\frac{2\sin^{-1}\sqrt{\frac{2\Lambda}{n(n+1)}\frac{R^{n+1}}{F}}}{(n+1)\sqrt{\frac{2\Lambda}{n(n+1)}}}. \quad (84)$$

Relabeling the r coordinate as in the previous case so that at $t = 0$, $R = r$ we get

$$t_c(r) = \frac{2\sin^{-1}\sqrt{\frac{2\Lambda}{n(n+1)}\frac{r^{n+1}}{F}}}{(n+1)\sqrt{\frac{2\Lambda}{n(n+1)}}}. \quad (85)$$

For $t = t_c(r)$ we again get $R(t, r) = 0$ corresponding to the singularity formation for shell labeled r. From $G_{00} - \Lambda g_{00} = k\epsilon(t, r)$ we obtain an expression for $F(r)$ which is again given by (14).

B. A simple derivation of the naked singularity

As before we want to see if null geodesics can come out of the singularity. For this we proceed as before assuming that the density profile near the center is given by

$$\epsilon(r) = \epsilon_0 + \epsilon_1 r + \epsilon_2\frac{r^2}{2!} + \ldots \quad (86)$$

From the form of the mass function $F = \frac{2k}{n}\int \epsilon(0, r)r^n dr$ we have

$$F(r) = F_{n+1}r^{n+1} + F_{n+2}r^{n+2} + \ldots, \quad (87)$$

where $F_{n+i} = \frac{2k}{n(n+i)}\frac{\epsilon_{i-1}}{(i-1)!}$. From (84) and (85) we see that the singularity curve is given by

$$t_s(r) = \frac{2}{n+1}\frac{\sin^{-1}\sqrt{\frac{2\Lambda}{n(n+1)}\frac{r^{n+1}}{F(r)}}}{\sqrt{\frac{2\Lambda}{n(n+1)}}}. \quad (88)$$

This implies that the central singularity at $r = 0$ forms at time

$$t_0 = \sqrt{\frac{2n}{(n+1)\Lambda}}\sin^{-1}\sqrt{\frac{2\Lambda}{n(n+1)F_{n+1}}}. \quad (89)$$

We again note that, as in the $\Lambda = 0$ case, when ϵ is a constant all shells become singular at the same time as the central shell.

We now rewrite the expression for the singularity curve as

$$\sin\left[\sqrt{\frac{(n+1)\Lambda}{2n}}t_s(r)\right] = \sqrt{\frac{2\Lambda}{n(n+1)}\frac{r^{n+1}}{F(r)}}. \tag{90}$$

It is reasonable to assume that for shells near $r = 0$ the time for singularity formation is close to the time for the central shell to become singular, i.e. $t_s(r) \approx t_0$ and we can therefore write $t_s(r) = \Delta t_s(r) + t_0$ where because of the assumption made $\Delta t_s(r) \approx 0$. Using this we expand the left-hand side of the above equation using the addition formula for sines and make use of $\lim_{x\to 0}\sin(x) = x$ and $\lim_{x\to 0}\cos(x) = 1$ to get

$$\sqrt{\frac{(n+1)\Lambda}{2n}}\cos\left[\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right]\Delta t_s(r) = -\sin\left[\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right]\frac{F_{n+i}}{2F_{n+1}}r^{i-1}. \tag{91}$$

Here F_{n+i} is the first nonzero term beyond F_{n+1} and is negative since we assume a decreasing density profile. Using $\Delta t_s(r) = t_s(r) - t_0$ in the above equation we can finally write the expression for singularity curve for shells near the center as

$$t_s(r) = t_0 - \sqrt{\frac{2n}{(n+1)\Lambda}}\tan\left[\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right]\frac{F_{n+i}}{2F_{n+1}}r^{i-1}. \tag{92}$$

To know whether the central singularity at $t = t_0$, $r = 0$ is naked or not we focus attention on radial null geodesics and check if there are any outgoing radial null geodesics which terminate on the central singularity in the past. We proceed as in the earlier case, assuming that there exist such geodesics and take their form near $r = 0$ to be

$$t = t_0 + ar^{\alpha}, \tag{93}$$

where, comparing with (92), we see that $\alpha \geq i - 1$ and if $\alpha = i - 1$ then

$$a < -\sqrt{\frac{2n}{(n+1)\Lambda}}\tan\left[\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right]\frac{F_{n+i}}{2F_{n+1}} \tag{94}$$

for the assumed geodesic to lie in the spacetime. We use (85) and (87) (retaining only the first two nonzero terms in the latter in the $r \approx 0$ approximation) in (84) to get

$$R^{(n+1)/2} = \sqrt{\frac{n(n+1)}{2\Lambda}}rF_{n+1}\left(1 + \frac{F_{n+i}}{F_{n+1}}r^{i-1}\right) \times \sin\left[\sqrt{\frac{(n+1)\Lambda}{2n}}(t_0 - t) - \tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right)\frac{F_{n+i}}{F_{n+1}}r^{i-1}\right]. \tag{95}$$

Near $r = 0$, the time t appearing in the geodesic equation satisfies $t \approx t_0$ and therefore the argument of the sine function in (95) is close to zero and we use the approximation $\sin x \approx x$ obtaining

$$R = r\left[\frac{(n+1)}{2}\sqrt{F_{n+1}}t_0 - \sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right) \times \frac{F_{n+i}}{2\sqrt{F_{n+1}}}r^{i-1} - \frac{(n+1)}{2}\sqrt{F_{n+1}}t + \frac{(n+1)}{4} \times \frac{F_{n+i}}{\sqrt{F_{n+1}}}t_0 r^{i-1} - \frac{(n+1)}{4}\frac{F_{n+i}}{\sqrt{F_{n+1}}}r^{i-1}t\right]^{2/(n+1)}. \tag{96}$$

From the form of the metric we know that the radial null geodesics satisfy $dt/dr|_{NG} = R'$. We take the spatial derivative of the above equation, substitute for t from (93), and equate the result to the derivative of (93)

$$\alpha ar^{\alpha-1} = \left[-\frac{(n+2i-1)}{(n+1)}\sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right)\frac{F_{n+i}}{2\sqrt{F_{n+1}}}r^{i-1} - \frac{(n+1)}{2}\sqrt{F_{n+1}}ar^{\alpha} - \frac{(n+2i-1)}{4}\frac{F_{n+i}}{\sqrt{F_{n+1}}}ar^{\alpha+i-1}\right]\left[-\sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right)\frac{F_{n+i}}{2\sqrt{F_{n+1}}}r^{i-1} - \frac{(n+1)}{2}\left(\sqrt{F_{n+1}}ar^{\alpha} - \frac{F_{n+i}}{2\sqrt{F_{n+1}}}ar^{\alpha+i-1}\right)\right]^{-((n-1)/(n+1))}. \tag{97}$$

Consider first the case $\alpha > i - 1$. Keeping terms only to lowest order in r we get

$$\alpha ar^{\alpha-1} = \left(\frac{n+2i-1}{n+1}\right)\left[-\sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right) \times \frac{F_{n+i}}{2\sqrt{F_{n+1}}}\right]^{2/(n+1)} r^{(2(i-1))/(n+1)}. \tag{98}$$

From this we have

$$\alpha = \frac{n+2i-1}{n+1}; \qquad a = \left(-\frac{F_{n+i}}{2\sqrt{F_{n+1}}}\sqrt{\frac{n(n+1)}{n(n+1)F_{n+1} - 2\Lambda}}\right)^{2/(n+1)}, \tag{99}$$

where we have substituted for t_0 in the argument of tan.

Since the form of α is exactly the same as in the $\Lambda = 0$ case we find that the conditions for naked singularity formation are also the same as mentioned after (26). That is, in 4 dimensions ($n = 2$), $1 < i < 4$ implying $i = 2, 3$ are the allowed values so that we get naked singularity for $\epsilon_1 < 0$ or for $\epsilon_1 = 0$, $\epsilon_2 < 0$. Similarly in 5 dimensions ($n = 3$), $1 < i < 3$ implying that only $i = 2$ is allowed so that we get naked singularity only for $\epsilon_1 < 0$. In all higher dimensions we get naked singularity only if $\epsilon_1 < 0$.

Again $n = 2$, $i = 4$ and $n = 3$, $i = 3$ are critical cases satisfying $\alpha = i - 1$. To analyze these we proceed as in the $\Lambda = 0$ case. For $\alpha = i - 1$ (97) becomes

$$(i-1)ar^{i-2} = \left[-\frac{(n+2i-1)}{(n+1)}\sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right)\frac{F_{n+i}}{2\sqrt{F_{n+1}}}r^{i-1} - \frac{(n+1)}{2}\sqrt{F_{n+1}}ar^{i-1}\right] \times \left[-\sqrt{\frac{n(n+1)}{2\Lambda}}\tan\left(\sqrt{\frac{(n+1)\Lambda}{2n}}t_0\right)\frac{F_{n+i}}{2\sqrt{F_{n+1}}}r^{i-1} - \frac{(n+1)}{2}\sqrt{F_{n+1}}ar^{i-1}\right]^{-((n-1)/(n+1))}, \tag{100}$$

which after substituting for t_0 gives

$$(i-1)ar^{i-2} = \left[-\frac{F_{n+i}}{2\sqrt{F_{n+1}}}\sqrt{\frac{n(n+1)}{n(n+1)F_{n+1} - 2\Lambda}} - \frac{(n+1)}{2}\sqrt{F_{n+1}}a\right]^{-((n-1)/(n+1))} \times \left[-\frac{(n+2i-1)}{(n+1)}\frac{F_{n+i}}{2\sqrt{F_{n+1}}}\sqrt{\frac{n(n+1)}{n(n+1)F_{n+1} - 2\Lambda}} - \frac{(n+1)}{2}\sqrt{F_{n+1}}a\right]r^{(2(i-1))/n+1}. \tag{101}$$

Equating the power of r on the two sides gives $i = 2n/(n-1)$ (as in the $\Lambda = 0$ case). Since i should be an integer greater than one we find that these conditions are satisfied only for $n = 2$ ($i = 4$) and for $n = 3$ ($i = 3$).

Consider $n = 3$; in this case the above equation can be written as

$$8\sqrt{F_4}a^3 + \left(2\sqrt{\frac{6}{6F_4 - \Lambda}}\frac{F_6}{\sqrt{F_4}} + 4F_4\right)a^2 + 4\sqrt{\frac{6}{6F_4 - \Lambda}}F_6 a + \left(\frac{6}{6F_4 - \Lambda}\right)\frac{F_6^2}{F_4} = 0 \tag{102}$$

with the constraint that

$$0 < a < -\sqrt{\frac{3}{12F_4 - 2\Lambda}}\frac{F_6}{2F_4}. \tag{103}$$

If we define $a = \sqrt{F_4}b$ and $F_6 = F_4^{3/2}\sqrt{6F_4 - \Lambda}\xi$ the equation can be written in the simplified form

$$2b^2(4b + \sqrt{6}\xi) + (2b + \sqrt{6}\xi)^2 = 0 \tag{104}$$

with the requirement that $0 < b < -\sqrt{3/8}\xi$. If we further define $Y = -b/\xi$ and $\eta = -1/\xi$ the equation becomes

$$2Y^2(4Y - \sqrt{6}) + \eta(2Y - \sqrt{6})^2 = 0. \tag{105}$$

For a naked singularity to form this equation for Y should have a solution subject to the constraint $0 < Y < \sqrt{3/8}$ and $\eta > 0$. Using the conditions, as mentioned earlier, for the roots of a general cubic we can find the conditions on η for which the above cubic has at least one real root in the desired range. It is found that for $0 < \eta \leq \sqrt{6}(-11 + 5\sqrt{5})/4$ all the three roots are real and at least one of these satisfies $0 < Y < \sqrt{3/8}$. For $\eta > \sqrt{6}(-11 + 5\sqrt{5})/4$ the real root is negative. The range of η found above implies that for $\xi \leq 4/\sqrt{6}(11 - 5\sqrt{5})$ one gets a naked singularity. This shows that the critical case is also similar to the $\Lambda = 0$ case except that the allowed range for ξ has shifted.

A similar analysis can be carried out for the case where $n = 2$ and $i = 4$. By defining $a = F_3 b$ and $F_6 = F_3^2\xi\sqrt{3F_3 - \Lambda}$ one gets a fourth order equation in b. If one subsequently defines $Y \equiv -b/\xi$ and $\eta \equiv -1/\xi$ one gets the equation

$$4Y^3(3Y - \sqrt{3}) - \eta(Y - \sqrt{3})^3 = 0 \tag{106}$$

with the consistency conditions $0 < Y < 1/\sqrt{3}$ and $\eta > 0$. It is found that the above conditions are satisfied for $0 < \eta < (1590 - 918\sqrt{3})/(-9 + 5\sqrt{3}) \approx 0.066\,642$ or in terms of conditions on ξ we get $\xi \leq (9 - 5\sqrt{3})/(1590 - 918\sqrt{3}) \approx -15.0056$.

C. Formation of trapped surfaces

As in the $\Lambda = 0$ case, we now look at the formation of trapped surfaces. Considering the expansion of outgoing radial null geodesics as in (33) we find that

$$\theta = \frac{nR'}{R}\left(1 - \sqrt{\frac{F(r)}{R^{n-1}} - \frac{2\Lambda R^2}{n(n+1)}}\right)K^r. \tag{107}$$

From this it is seen that the condition for trapping, $\theta = 0$, is met when

$$\frac{F(r)}{R^{n-1}} - \frac{2\Lambda R^2}{n(n+1)} = 1, \tag{108}$$

which for $n = 2$ (4 dimensions) reduces to the well-known result

$$\frac{F(r)}{R} - \frac{\Lambda R^2}{3} = 1. \tag{109}$$

It is easy to see that, as in the $\Lambda = 0$ case, for the central shell, trapping coincides with singularity formation.

D. Exterior solution with a negative cosmological constant

As before we take the metric in the exterior to be

$$ds^2 = -f(x)dT^2 + g(x)dx^2 + x^2 d\Omega^2. \tag{110}$$

The components of the Einstein tensor are the same as in (39)–(41). Solving the vacuum Einstein equations $G_{\mu\nu} - \Lambda g_{\mu\nu} = 0$ we find

$$g(x) = \frac{n(n+1)x^{n-1}}{2\Lambda x^{n+1} + n(n+1)x^{n-1} + Cn(n+1)}, \tag{111}$$

$$f(x) = 1 + \frac{C}{x^{n-1}} + \frac{2\Lambda x^2}{n(n+1)}. \tag{112}$$

With this the metric in the exterior becomes

$$ds^2 = -\left(1 + \frac{C}{x^{n-1}} + \frac{2\Lambda x^2}{n(n+1)}\right)dT^2 + \left(1 + \frac{C}{x^{n-1}} + \frac{2\Lambda x^2}{n(n+1)}\right)^{-1} dx^2 + x^2 d\Omega^2. \tag{113}$$

Here C is a constant of integration which is fixed by matching the exterior solution to the interior solution at the boundary in exactly the same way as for the $\Lambda = 0$ case and the result is $C = -F(r_s)$ where r_s is the boundary of the dust cloud. Thus in 4 dimensions the exterior is

$$ds^2 = -\left(1 - \frac{2GM}{x} + \frac{\Lambda x^2}{3}\right)dT^2 + \left(1 - \frac{2GM}{x} + \frac{\Lambda x^2}{3}\right)^{-1} dx^2 + x^2 d\Omega^2. \tag{114}$$

E. The absence of a self-similar solution in the presence of a Λ

It is interesting to note that it is not possible to have a self-similar solution in the presence of a cosmological constant. To see this we begin by noting that the condition that dimensionless functions made from the metric are functions only of t/r continues to hold. This follows because the cosmological constant term in the Einstein equations can be absorbed into the energy-momentum tensor in the right-hand side, by taking the Λ-term as a perfect fluid with equation of state $p = -\rho$. Equation (58) then continues to hold, with the understanding that the contribution of the cosmological constant is included in the energy-momentum tensor. The remaining argument, leading to the conclusion that $\tilde{R}$ is a function of z, then follows.

Now if we have a self-similar solution then we can write $R = r\tilde{R}$ with $\tilde{R}$ being dimensionless. If we define $\frac{r}{t} \equiv z$ then the condition of self-similarity implies that $\tilde{R}$ being dimensionless should be a function only of z. With this if we now consider the equation $G_{11} - \Lambda g_{11} = \kappa T_{11}$ we get

$$-\frac{n(n-1)}{2} z^4 \left(\frac{d\tilde{R}}{dz}\right)^2 - nz^4 \tilde{R}\frac{d^2\tilde{R}}{dz^2} - 2nz^3\tilde{R}\frac{d\tilde{R}}{dz} - \Lambda r^2 \tilde{R}^2 = 0. \tag{115}$$

The explicit presence of r in the above equation implies that $\tilde{R}$ cannot be expressed as a function of z alone and thus we do not have a self-similar solution in the presence of Λ.

The same conclusion also follows from Eq. (61). With dust matter, the only contribution to pressure is coming from the cosmological constant, and this pressure is constant. The Lie derivative on the left-hand side is thus zero, whereas on the right-hand side the pressure is nonzero, leading to a contradiction and showing that such a Killing vector field cannot exist. Physically speaking, the presence of a cosmological constant introduces a length scale which prevents self-similarity.

IV. CONCLUSIONS

We have studied the collapse of inhomogeneous spherically symmetric dust distribution in an arbitrary number of space dimensions both in the absence and in the presence of a cosmological constant. From the analysis presented we see that even though naked singularity is allowed in all dimensions there is more freedom on initial conditions for obtaining naked singularity in $2 + 1$, $3 + 1$, and $4 + 1$ dimensions, both in the absence as well as the presence of a negative cosmological constant. We have also seen that the formation of trapped surfaces is similar in all dimensions with the central shell getting trapped at the same time when it becomes singular. For outer shells trapping occurs before those shells become singular. We also saw explicitly that in the absence of a cosmological constant, globally naked self-similar models can be constructed in all dimensions, whereas in the presence of a cosmological constant such a solution cannot be constructed.

In the second paper in this series, we will study quantum field theory on the curved background provided by the classical solutions presented here, including the emission of Hawking radiation from an n-dimensional AdS black hole. In a third paper we will carry out a canonical quantization of this model, and also address the issue of black hole entropy, following the methods of [36].

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