PHYSICAL REVIEW D **77,** 064008 (2008)

Perfect fluid spheres with cosmological constant

Christian G. Böhmer*

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom

Gyula Fodor†

KFKI Research Institute for Particle and Nuclear Physics, H-1525 Budapest, P.O. Box 49, Hungary

(Received 12 November 2007; published 6 March 2008)

We examine static perfect fluid spheres in the presence of a cosmological constant. Because of the cosmological constant, new classes of exact matter solutions are found. One class of solutions requires the Nariai metric in the vacuum region. Another class generalizes the Einstein static universe such that neither its energy density nor its pressure is constant throughout the spacetime. Using analytical techniques we derive conditions depending on the equation of state to locate the vanishing pressure surface. This surface can, in general, be located in regions where, going outwards, the area of the spheres associated with the group of spherical symmetry is decreasing. We use numerical methods to integrate the field equations for realistic equations of state and find consistent results.

DOI: 10.1103/PhysRevD.77.064008 PACS numbers: 04.20.Jb, 04.40.Dg

I. INTRODUCTION

Static and spherically symmetric perfect fluid solutions have always been a rich source of investigation in classical general relativity ever since the pioneering work of Schwarzschild in 1916. He solved the field equations for the interior region by assuming a perfect fluid of constant energy density, and also for the outside vacuum region, the famous Schwarzschild solutions.

The interior solution has the geometry of a three-sphere, which was noticed as early as 1919 by Weyl [1]. The pressure of the Schwarzschild interior solution always vanishes before the equator of the three-sphere; therefore, this geometrical picture is not of great importance. This situation changes significantly in the presence of the cosmological constant [2,3]. In some special cases the interior metric with cosmological constant was studied earlier; see e.g. [4,5]. However, the complete analysis was only completed [2,3] 85 years after Weyl noted the interesting geometrical structure of these solutions: With Λ, the pressure can vanish exactly at the equator of the three-sphere, in which case one has to join on the Nariai metric [6,7] as the exterior vacuum metric. We note that, although the spatial geometry of the interior Schwarzschild is a three-sphere, the four-metric is not homogeneous. There is a center of symmetry in the fluid region, surrounded with concentric spheres defined by the orbits of the group generating spherical symmetry. The spacetime metric determines an induced metric and thereby an area to each group orbit. Going outwards from the center this area is increasing until one reaches the equator, the sphere with the maximal area. It is furthermore possible that the pressure vanishes in a region where the area of the group orbits is decreasing.

Lastly, the matter can occupy the whole three-sphere having two regular centers. This generalizes the Einstein static universe [2,3,8]; see [9] for early results in that direction. The Einstein universe also emerges as a special case (vanishing expansion rate and vanishing vorticity) when considering homogeneous shear-free perfect fluids [10]. In recent years it has been generalized in various different theories, like brane world models [11], Einstein-Cartan theory [12], in modified gravity theories [13], or loop quantum gravity [14].

In this paper we are analyzing systematically the effects of a positive cosmological constant on perfect fluid spheres. We generalize known exact solutions of the field equations with cosmological term and discuss their new properties. The principal result of the analysis is the increase of the radial sizes of matter spheres. This naturally leads to questions regarding the physical picture applicable to these solutions. Analytical and numerical techniques are used to obtain a consistent picture of the underlying physics.

A positive cosmological constant can be regarded as an external force pulling matter apart. Therefore, a "large" positive cosmological constant can increase the radius of a known perfect fluid solution such that it occupies more than just "half" of the three-space. The effects of the actual cosmological constant are very small, and therefore one may ask why the investigation of solutions with "large" values of the cosmological term is beneficial. This can be answered from two points of view. From a mathematical point of view, we deepen our understanding about exact solutions of Einstein's field equations and the influence of the additional parameter Λ. However, also from a physical point of view, this study can be justified easily. In the bag model of hadrons [15–17] the bag is stabilized by a term of the form Bg_{ab}, which has the same form as a cosmological constant, though its numerical value is considerably bigger, $B_e^{1/4} \approx 8.91$ MeV, whereas

*c.boehmer@ucl.ac.uk
†gfodor@rmki.kfki.hu

 © 2008 The American Physical Society

we have $\Lambda^{1/4} \approx 1.78 \times 10^{-9}$ MeV. Hence, the key physical motivation is the possibility that effects within stellar models can be effectively described by a term that looks like a cosmological constant, which can indeed have large effects. Also, in the context of Boson stars [18,19] the matter energy-momentum tensor contains a part proportional to the metric which can be read as an effective cosmological term. Within the context of loop quantum gravity, it has recently been shown [20] that quantum gravity effects can be effectively of the form of a cosmological constant.

Apart from an equation of state relating the density ρ to the pressure p, a spherically symmetric perfect fluid has to satisfy only one condition, the pressure isotropy condition, requiring the equality of the radial and angular directional pressures. Since this condition does not involve the cosmological constant Λ, any solution with $\Lambda = 0$ having an equation of state $f(\rho, p) = 0$ can also be interpreted as a $\Lambda \neq 0$ cosmological solution with equation of state $f(\rho + \Lambda, p - \Lambda) = 0$. It is one of the main purposes of the present paper to reinvestigate known perfect fluid solutions with cosmological constant and to analyze whether these solutions for some special cosmological constant require the Nariai metric as the exterior vacuum spacetime. We also try to construct solutions with two regular centers, i.e. constructing more general Einstein static universes, having neither constant energy density nor constant pressure.

Since the static and spherically symmetric field equations with more realistic equations of state, in general, cannot be integrated analytically, we also use numerical methods to study perfect fluid spheres. In particular, we will show that the effect of a large cosmological constant on polytropic perfect fluids is such that the matter is pulled sufficiently apart so that it occupies much of the three-space. The same result is also found by considering the stiff matter equation of state and also the Hagedorn equation of state. However, realistic equations of state seem not to allow the presence of a second regular center. It is important to distinguish between coordinate and physical effects. For instance, one cannot numerically integrate the constant density solutions for large cosmological constants if the original radius is used as a variable. The code breaks down at the equator of the three-sphere. This is a pure coordinate effect, since the coordinate system does not cover the whole spacetime. In order to distinguish coordinate and physical effects, the Riemann curvature tensor is also considered.

Various other effects of the cosmological constant have been studied in the past, like the dynamical instability of perfect fluid spheres [21,22], possibly detectable effects of the cosmological constant within our solar system [23–26], and effects within astrophysical structures [27–31]. Recently, the bending of light with Λ has been discussed in [32,33].

The paper is outlined in the following manner: In Sec. II we analytically study the effect of the cosmological constant on the Whittaker and Tolman *IV* solution, and also discuss the matching of the matter and the vacuum solution. In Sec. III we derive conditions on the equation of state and the cosmological constant to characterize the different possible solutions. In Sec. IV we numerically integrate the field equations for realistic equations of state and find results consistent with our analytical results. We summarize and conclude our work in Sec. V.

II. THE WHITTAKER AND TOLMAN SOLUTIONS

In the present section the Whittaker solution [34] and the Tolman *IV* solution [35] are recalled. In both cases we first introduce a third angular coordinate α, so that the coordinate system covers the complete three-space and not just "half" of it like the usual radial coordinate r. Second, we introduce the cosmological constant in the solutions and analyze its effect (the "external" force due to Λ) on those solutions. We also explicitly show how to join the interior and the exterior solutions through the vanishing pressure surface.

A. The Whittaker solution

The Whittaker solution is characterized by the relation $\rho + 3p = \rho_0$ between the energy density and the pressure, where ρ_0 is a positive constant. Similarly to the constant density case, this condition allows one to write down the solution to Einstein's field equations in terms of elementary functions. The metric of the Whittaker solution in the original Schwarzschild coordinates reads [34]

$$\begin{aligned} ds^2 = &-b\left[1 + B - \frac{B}{ar}\sqrt{1-a^2r^2}\arcsin(ar)\right]dt^2 \\ &+ \left[(1+B)(1-a^2r^2) \right.\\ &\left.- \frac{B}{ar}(1-a^2r^2)^{3/2}\arcsin(ar)\right]^{-1}dr^2 + r^2 d\Omega^2, \end{aligned} \quad (1)$$

where a, b, and B are constants. The special importance of the Whittaker solution lies in the fact that it is the nonrotating static limit of the Wahlquist solution [36], the most important rotating perfect fluid exact solution. The parameter κ of the Wahlquist solution is related to the parameter B of the Whittaker metric by $B = 1/\kappa^2$. Since the change of the parameter n corresponds merely to a rescaling of the coordinate t, we set $b = 1$. After introducing a new radial coordinate, the third angle α, by $r = (1/a) \times \sin\alpha$, metric (1) simplifies to

$$ds^2 = -f dt^2 + \frac{1}{a^2}\left(\frac{d\alpha^2}{f} + \sin^2\alpha\, d\Omega^2\right), \quad (2)$$

$$f = 1 + B(1 - \alpha\cot\alpha). \quad (3)$$

Although the introduced radial coordinate α is very similar to the third angle of the ellipsoid used for the interior Schwarzschild solution, the spatial metric is not ellipsoidal in this case because of the nonconstant nature of the metric component $g_{\alpha\alpha}$. On the other hand, it should be emphasized that the $r = \text{const}$ hypersurfaces are round two-spheres. Therefore, the topology of the three-space essentially depends on the function $g_{\alpha\alpha}$. In the above-mentioned constant density case, the three-space is in fact a three-sphere. When we discuss next the modified Tolman *IV* solution, the three-space will be ellipsoidal. The introduction of the new radial coordinate α is important in all cases where there is a group orbit with maximum area, since in these cases the usual radial coordinate only covers the region up to this maximum orbit. Henceforth we will refer to the coordinate α as the third angle.

It is possible to express the constant in the equation of state $\rho + 3p = \rho_0$ in terms of the constants in the metric and the cosmological constant by

$$\rho_0 = \frac{a^2 B + \Lambda}{4\pi}. \tag{4}$$

Positivity of the pressure and density implies $\rho_0 > 0$, which we require from now on, by assuming

$$\Lambda > -a^2 B. \tag{5}$$

If there is a spherical surface where the solution is matched to a Schwarzschild–de Sitter (or Schwarzschild–anti-de Sitter) exterior region, then p must go to zero at the surface, and the fluid density becomes ρ_0 there.

The pressure and energy density of the Whittaker solution are given by

$$p = \frac{\rho_0}{2} - \frac{a^2 f}{8\pi}, \tag{6}$$

$$\rho = \frac{3a^2 f}{8\pi} - \frac{\rho_0}{2}. \tag{7}$$

The central pressure and central energy density can be derived from Eqs. (6) and (7) by noting that $\lim_{\alpha\to 0} f = 1$, and read

$$p_c = \frac{\rho_0}{2} - \frac{a^2}{8\pi}, \qquad \rho_c = \frac{3a^2}{8\pi} - \frac{\rho_0}{2}. \tag{8}$$

Requiring both to be positive, we get $\frac{4\pi}{3}\rho_0 < a^2 < 4\pi\rho_0$. Since at the center $\frac{df}{d\alpha} = 0$ and $\frac{d^2 f}{d\alpha^2} = \frac{2B}{3}$, the pressure is maximal at the center if and only if $B > 0$. Since in the $B < 0$ case there is no zero pressure surface, we assume $B > 0$ in the present discussion.

The equator, where the area of the spheres of symmetry is maximal, is located at $\alpha = \pi/2$. There $f = 1 + B$, and for the equatorial pressure $p_{\rm eq}$ and density $\rho_{\rm eq}$, we get

$$p_{\rm eq} = \frac{1}{8\pi}(\Lambda - a^2), \qquad \rho_{\rm eq} = \frac{3}{8\pi}(a^2 - \Lambda) + \rho_0. \tag{9}$$

Let us now choose the cosmological constant such that the pressure vanishes at the equator, $p_{\rm eq} = 0$ in (9), which yields

$$\Lambda = a^2 =: \Lambda_N. \tag{10}$$

Hence for solutions which satisfy the relation $\Lambda < \Lambda_N$, the pressure vanishes before the equator, where the group orbits are still increasing.

One should check whether the choice $\Lambda = \Lambda_N$ is compatible with the positivity of pressure and energy density at the center, and also with the positivity of energy density at the equator (otherwise, these solutions would not be physical). Positivity of energy density at the surface is ensured by (5), which now takes the form $B > -1$. Using $\Lambda = a^2$ in (4) and (8) we get

$$p_c = \frac{\Lambda_N}{8\pi} B > 0, \tag{11}$$

$$\rho_c = \frac{\Lambda_N}{8\pi}(2 - B) > 0. \tag{12}$$

All three conditions are satisfied if $0 < B < 2$.

For larger cosmological constants, i.e. for $\Lambda > \Lambda_N$, the pressure vanishes after the equator of the ellipsoid, where the group orbits are decreasing. Since the pressure function $p \to -\infty$ as $\alpha \to \pi$, there always exists a zero pressure surface at which one joins on the Schwarzschild–de Sitter metric as an exterior vacuum spacetime. Therefore the Whittaker solution cannot have a second regular center. Since the derivative of the function f is positive for $0 < \alpha < \pi$ in the case $B > 0$, the function f remains positive; consequently, the coordinate system and the metric remain regular in the whole fluid region.

As already outlined in the Introduction we will explicitly show that the choice $\Lambda = \Lambda_N$ necessitates the Nariai metric as the exterior spacetime. Since the pressure in this case vanishes at the maximum of the area of the group orbits, the exterior spacetime must have constant area spheres of symmetry, which excludes the Schwarzschild–de Sitter or anti-de Sitter spacetimes. The only other static and spherically symmetric vacuum spacetime with cosmological constant is indeed the Nariai spacetime, and its group orbits have constant area.

B. Matching procedure

Here we review the necessary conditions for matching a spherically symmetric static perfect fluid solution to an exterior vacuum region. We write the metric in both regions in the form

$$ds^2 = -e^{2\nu}dt^2 + \frac{1}{y^2}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{13}$$

where ν, y, and R are functions of the coordinate r. A metric written in this form has a center where $R = 0$, and this center is regular, i.e. free of conical singularities, if the area of small spheres is proportional to their radius square, with the appropriate proportionality factor, $\frac{dR}{dr} = \pm\frac{1}{y}$. The regularity of the four-metric also requires a finite value for ν and $\frac{d\nu}{dr} = 0$ at the central point.

We assume that the matching is performed along the hypersurface described by $r = r_s$. The Darmois-Israel matching conditions [37,38] essentially state that the induced metric and the extrinsic curvature have to agree on the hypersurfaces used for joining the two solutions. The outward pointing normal vector to the symmetry surfaces has the components $n^a = (0, y, 0, 0)$. The induced metric h_{ab} can be expressed using the spacetime metric as $h_{ab} = g_{ab} - n_a n_b$, while the extrinsic curvature can be calculated as $K_{ab} = h_a^c \nabla_c n_b$, where ∇_c denotes the covariant derivative associated to the spacetime metric g_{ab}. Using the coordinate system (t, θ, ϕ) on the matching surface, the induced metric h_{ab} has the form

$$h_{ab} = \begin{pmatrix} -e^{2\nu} & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2\theta \end{pmatrix}, \tag{14}$$

while the components of the extrinsic curvature are

$$K_{ab} = \begin{pmatrix} -y e^{2\nu} \frac{d\nu}{dr} & 0 & 0 \\ 0 & yR\frac{dR}{dr} & 0 \\ 0 & 0 & yR\frac{dR}{dr}\sin^2\theta \end{pmatrix}. \tag{15}$$

From (14) it is apparent that the induced metric agrees if and only if the values of R and ν agree on the matching surfaces. Then the extrinsic curvatures are matched appropriately if and only if $y\frac{d\nu}{dr}$ and $y\frac{dR}{dr}$ agree. The agreement of R has the obvious physical meaning of equal area matching spheres, while the equality of ν can always be ensured by appropriately rescaling the time coordinate in the interior fluid domain. If we use Gauss coordinates in both domains, then $y = 1$ and the matching of the extrinsic curvature is equivalent to the continuity of the first derivative of ν and R. However, in general, it is also possible to give a coordinate system invariant meaning to these conditions.

The invariant mass function in spherically symmetric cosmological spacetimes can be defined as

$$m = \frac{R}{2}(1 - g^{ab}R_{,a}R_{,b}) - \frac{\Lambda}{6}R^3. \tag{16}$$

For vanishing cosmological constant this gives back the usual mass definition [39]. For the metric form (13) we get

$$m = \frac{R}{2}\left[1 - y^2\left(\frac{dR}{dr}\right)^2\right] - \frac{\Lambda}{6}R^3. \tag{17}$$

From this we can see that if m agrees on the two matching surfaces of equal area, then $y\frac{dR}{dr}$ must also be the same. The other invariant quantity is the pressure at the matching surface, which for the metric (13) takes the form

$$p = \frac{y^2}{R}\frac{dR}{dr}\left(2\frac{d\nu}{dr} + \frac{1}{R}\frac{dR}{dr}\right) + \Lambda. \tag{18}$$

It is apparent that if R and $y\frac{dR}{dr}$ both agree on the matching surfaces and $\frac{dR}{dr} \neq 0$, then $y\frac{d\nu}{dr}$ will be the same if and only if the pressures are the same. Consequently, if $\frac{dR}{dr} \neq 0$, the matching of two static perfect fluid solutions can be done at two chosen spherical surfaces if and only if the surfaces have the same area, the mass functions have the same value, and the pressures agree as well. Obviously, if the exterior domain is a vacuum, then the fluid pressure at the surface must vanish. It is interesting that in case of a Nariai exterior $\frac{dR}{dr} = 0$ and the $p = 0$ condition is not enough to ensure the continuity of $y\frac{dR}{dr}$.

The quantity $y\frac{d\nu}{dr}$ is closely related to the acceleration of static nonrotating observers staying at constant radius r. In the coordinate system $x^a = (t, r, \theta, \phi)$ used in (13), these observers have the velocity vector $v^a = (e^{-\nu}, 0, 0, 0)$. The only nonvanishing component of their acceleration vector $a^a = u^b\nabla_b u^a$ is $a^r = y^2\frac{d\nu}{dr}$. The norm of the acceleration is $|a| = \sqrt{a^a a_a} = y|\frac{d\nu}{dr}|$. This shows that apart from a possible signature change the continuity of the magnitude of the acceleration implies the continuity of $y\frac{d\nu}{dr}$ in the matching condition.

C. Joining the interior and exterior solutions

For cosmological constants satisfying $\Lambda < \Lambda_N$ the area of the group orbits at the $p = 0$ surface is increasing and we join the Schwarzschild–de Sitter (or Schwarzschild–anti-de Sitter for $\Lambda < 0$) metric on as the exterior vacuum spacetime. Since the cosmological constant is fixed by the specific solution, it remains to choose the mass appropriately. In the Schwarzschild area coordinate R the mass is defined by $M = \int_0^{R_s} 4\pi R^2 \rho(R) dR$, where R_s is defined by $p(R_s) = 0$. By using Gauss coordinates relative to the $r =$ const hypersurfaces, the metric is C^1 at the boundary. If the energy density is nonvanishing at the boundary, this cannot be improved. After placing one object in the Schwarzschild–de Sitter spacetime, it still contains an infinite series of singularities. However, by placing a second object appropriately in that spacetime, it is possible to construct a singularity-free spacetime; see Fig. 1. This possibility has been discussed earlier in greater detail in [2,3].

For $\Lambda = \Lambda_N$ we explicitly show the matching of the interior perfect fluid spacetime with the exterior Nariai spacetime. We follow the generic discussion of matching two static and spherically symmetric regions presented in the previous subsection. We read off and compare at the matching surface the corresponding functions ν, y, and R in the general form of the line element (13).

Recall that the static form of the Nariai metric is given by [7]

$$ds^2 = -\cos^2\chi dt^2 + \frac{1}{\Lambda}(d\chi^2 + d\Omega^2). \tag{19}$$

We note that this form of the metric is not homogeneous, since the static observers described by constant (χ, θ, ϕ)

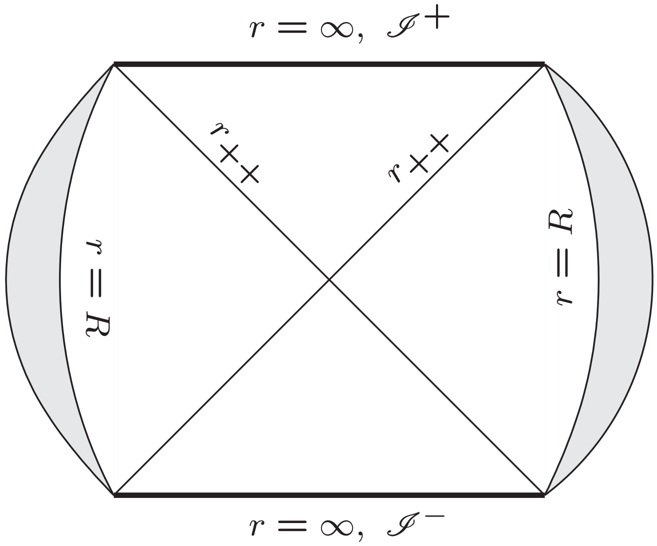

FIG. 1. Penrose-Carter diagram with two stellar objects separated by a Schwarzschild–de Sitter vacuum domain. The radii R of the stellar objects is between the radii of the black hole and cosmological horizons. Since the group orbits are increasing up to R, the vacuum part contains the cosmological event horizon r_{++}.

are not equivalent. The magnitude of their acceleration is $|a| = \sqrt{\Lambda}\tan\chi$. Somewhat surprisingly, this acceleration is towards the $\chi = 0$ surface, since the nonvanishing component of their acceleration, given by $a^r = -\Lambda\tan\chi$, is negative for $\chi > 0$.

The metric functions of the Nariai spacetime in the coordinate system (13) are given by

$$e^{2\nu_N} = \cos^2\chi, \qquad y_N^2 = \Lambda, \qquad R_N^2 = \frac{1}{\Lambda}. \tag{20}$$

The corresponding functions in the Whittaker fluid region are

$$e^{2\nu_W} = c^2 f, \qquad y_W^2 = a^2 f, \qquad R_W^2 = \frac{\sin^2\alpha}{a^2}, \tag{21}$$

where f is given by (3). The agreement of the induced metric, i.e. the matching of R and ν, implies

$$\frac{1}{\Lambda} = \frac{\sin^2\alpha}{a^2}, \qquad \cos^2\chi = c^2 f. \tag{22}$$

The condition $y_N \frac{dR_N}{d\chi} = y_W \frac{dR_W}{d\alpha}$ implies $\alpha = \pi/2$ for the matching surface in the fluid region. From this it follows that $\Lambda = a^2$, which is just the condition of vanishing pressure at the equator of the Whittaker solution. The remaining matching condition $y_N \frac{d\nu_N}{d\chi} = y_W \frac{d\nu_W}{d\alpha}$ yields

$$\tan\chi = -\frac{\pi B}{4\sqrt{1+B}} = \frac{\pi}{4}\left(1 - \frac{4\pi\rho_0}{\Lambda}\right)\sqrt{\frac{\Lambda}{4\pi\rho_0}}. \tag{23}$$

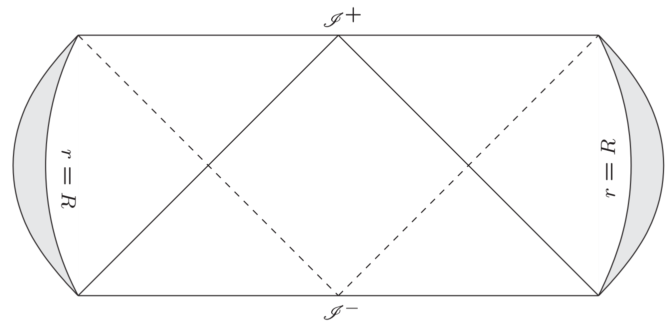

FIG. 2. Penrose-Carter diagram with two stellar objects having radii R, which require the Nariai spacetime to be the vacuum part of the global solution. The solid and dashed lines represent the future and past event horizons, respectively.

So, if Gauss coordinates are used, i.e. $y = 1$, we explicitly showed the matching of the interior and the exterior metric to be of degree C^1 (ν, R, ν', and R' all agree on the zero pressure surface). This differentiability condition cannot be improved when the equation of state of the fluid constrains the energy density at the boundary to be positive. In this case the energy-momentum tensor jumps at the boundary and the metric is at most C^1. Figure 2 shows the Penrose diagram of this spacetime.

Lastly, for cosmological constants satisfying $\Lambda > \Lambda_N$ the group orbits at the $p = 0$ surfaces are *decreasing* and we join the Schwarzschild–de Sitter metric on as the exterior vacuum spacetime. Since the cosmological constant is fixed by the specific solution, it remains to choose the mass appropriately; see the discussion above. By using Gauss coordinates the metric is C^1 at the boundary. If the energy density is nonvanishing at the boundary, this cannot be improved. It is important to note that the vacuum part of

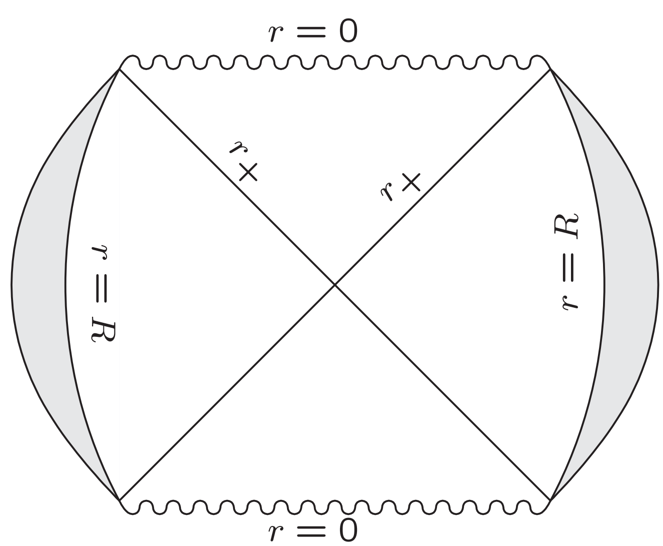

FIG. 3. Penrose-Carter diagram with two stellar objects connected by a Schwarzschild–de Sitter vacuum domain. Since the group orbits are decreasing at the matching surface, the vacuum part contains the black hole horizon and the $r = 0$ singularity.

this spacetime contains the singularity at the origin $r = 0$ and that the matter occupies the "outer" region of the spacetime; see the Penrose diagram 3, where a second object was inserted in the spacetime to remove the infinite sequence of singularities present in the Schwarzschild–de Sitter diagram.

D. Cosmological Tolman solutions

In principle, the above discussion can now be repeated for all 127 candidate solutions presented in [40] (of which only 60% are isotropic and regular at the center). For all of these one could check whether the inclusion of the cosmological constant can pull the matter up to or beyond the equator of the corresponding ellipsoid. We will, however, only repeat this analysis for the Tolman IV solution. This particular choice is motivated by the simplicity of the solution which also allows one to analytically express its equation of state.

We now take a fresh look at the Tolman IV solution [35]. Its metric reads

$$ds^2 = -B^2(1 + r^2/A^2)dt^2 + \frac{1 + 2r^2/A^2}{(1 - r^2/R^2)(1 + r^2/A^2)}dr^2 + r^2 d\Omega^2, \tag{24}$$

where A and R are positive constants. Introducing the third angle by $r = R\sin\alpha$ yields

$$ds^2 = -B^2(1 + (R^2/A^2)\sin^2\alpha)dt^2 + R^2\left[\frac{1 + 2(R^2/A^2)\sin^2\alpha}{1 + (R^2/A^2)\sin^2\alpha}d\alpha^2 + \sin^2\alpha d\Omega^2\right]. \tag{25}$$

Pressure and energy density, respectively, are given by

$$8\pi p(\alpha) = \frac{R^2 - 3R^2\sin^2\alpha - A^2}{R^2(A^2 + 2R^2\sin^2\alpha)} + \Lambda, \tag{26}$$

$$8\pi\rho(\alpha) = \frac{R^2 + 3R^2\sin^2\alpha + 3A^2}{R^2(A^2 + 2R^2\sin^2\alpha)} + \frac{2A^2\cos^2\alpha}{(A^2 + 2R^2\sin^2\alpha)^2} - \Lambda. \tag{27}$$

Since all quantities depend on α only through $\sin^2\alpha$ and $\cos^2\alpha$, the solution is symmetric to the equator $\alpha = \pi/2$.

Eliminating the variable α from Eqs. (26) and (27) leads to the following equation of state:

$$\rho = c_0 + c_1 p + c_2 p^2, \tag{28}$$

where the three constants c_i are given by

$$\begin{aligned} c_0 &= \frac{(3 - 2\Lambda R^2)(R^2 + A^2(2 - \Lambda R^2))}{4\pi R^2(A^2 + 2R^2)}, \\ c_1 &= \frac{2R^2 + A^2(13 - 8\Lambda R^2)}{A^2 + 2R^2}, \\ c_2 &= \frac{32\pi A^2 R^2}{A^2 + 2R^2}. \end{aligned} \tag{29}$$

As in the previous discussion let us now compute the value of the pressure and density at the first center ($\alpha = 0$), at the equator of the ellipsoid ($\alpha = \pi/2$), and at the second possible center ($\alpha = \pi$), which yields

$$8\pi p(0) = \frac{1}{A^2} - \frac{1}{R^2} + \Lambda, \qquad 8\pi\rho(0) = \frac{3}{A^2} + \frac{3}{R^2} - \Lambda, \tag{30}$$

$$\begin{aligned} 8\pi p(\pi/2) &= -\frac{1}{R^2} + \Lambda, \\ 8\pi\rho(\pi/2) &= \frac{3A^2 + 4R^2}{R^2(A^2 + 2R^2)} - \Lambda, \end{aligned} \tag{31}$$

$$8\pi p(\pi) = \frac{1}{A^2} - \frac{1}{R^2} + \Lambda, \qquad 8\pi\rho(\pi) = \frac{3}{A^2} + \frac{3}{R^2} - \Lambda. \tag{32}$$

Similarly to the Whittaker case, there exists a cosmological constant such that the pressure vanishes at the equator,

$$\Lambda = \frac{1}{R^2} =: \Lambda_N, \tag{33}$$

where the energy density is positive. In this case one has to match the Nariai metric. For smaller cosmological constants, i.e. $\Lambda < \Lambda_N$, the pressure vanishes before the equator and one has to join on the Schwarzschild–de Sitter (or Schwarzschild–anti-de Sitter) metric as the exterior vacuum spacetime. However, the pressure cannot vanish after the equator, as can be seen from the mirror symmetry to the equator of the solution. Therefore, solutions with $\Lambda > \Lambda_N$ have two centers. One easily verifies that both centers are regular by checking that the derivatives of pressure and energy density vanish at both centers. Hence, as a side result we already found a new generalization of the Einstein static universe.

The Einstein universe is characterized by its constant energy and constant pressure (originally Einstein assumed a pressureless universe) throughout the three-sphere. By a generalization of the Einstein static universe we mean a globally regular solution of the field equations with cosmological constant where the spatial part of the metric is a closed three-space and where either the energy density or the pressure, or both, is varying.

It is expected that other known perfect fluid solutions, e.g. those given in Ref. [40], will show similar properties. Therefore, following the above procedure, we can explicitly show the existence of a wide class of generalized

Einstein static universes and an even wider class of static and spherically symmetric perfect fluid solutions, for which the pressure vanishes in regions where the group orbits are decreasing.

III. ANALYTIC CONSIDERATIONS

In the previous sections we analyzed perfect fluid solutions which may extend through the equator of the ellipsoid that describes the global geometry of the spatial hypersurfaces. The present section supplements the explicit and later numerical results by presenting some general statements. It should also be mentioned that the existence and uniqueness of perfect fluid solutions was proved in [41]. The restrictions on the equations of state could be weakened in Refs. [42,43]. The existence and uniqueness proof of [41] could be extended to include cosmological constants satisfying $\Lambda < 4\pi\rho(p = 0)$ in [2,44]. Let us now consider the static and spherically symmetric line element in Gauss coordinates relative to the $r = \text{const}$ hypersurfaces,

$$ds^2 = -e^{2\nu(r)}dt^2 + dr^2 + R^2(r)d\Omega^2. \tag{34}$$

The resulting field equations $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$ are given by

$$\frac{1 - R'^2 - 2RR''}{R^2} - \Lambda = 8\pi\rho, \tag{35}$$

$$\frac{R'^2 - 1 + 2RR'\nu'}{R^2} + \Lambda = 8\pi p, \tag{36}$$

$$\frac{\nu' R' + R(\nu'^2 + \nu'') + R''}{R} + \Lambda = 8\pi p, \tag{37}$$

which imply the conservation of the energy-momentum tensor

$$p' + \nu'(p + \rho) = 0, \tag{38}$$

where the prime denotes differentiation with respect to r. The maximum of the area of the group orbits is located at the maximum of the function $R(r)$, which means $R'(r_m) = 0$, where r_m is the location of the maximum. We henceforth assume that such a maximum exists. [To be precise, we only assume that the function $R(r)$ has an extremum, and it will turn out that this extremum is a maximum if we require the energy density to be positive at r_m. For $r = r_m$ the mass definition (17) implies $1 - 2m(r_m)/R(r_m) - \Lambda/3R(r_m)^3 = 0$.]

Note that mass (17) and energy density ρ are related by

$$\frac{dm}{dR} = \frac{m'}{R'} = 4\pi\rho R^2. \tag{39}$$

Eliminating the function ν' from the first two field equations (35) and (36) and the conservation equation (38) yields the Tolman-Oppenheimer-Volkoff (TOV) [35,45] equation

$$\frac{dp}{dR} = -R\frac{(p + \rho)(4\pi p + m/R^3 - \Lambda/3)}{1 - \frac{2m}{R} - \frac{\Lambda}{3}R^3}, \tag{40}$$

where we used that $dp/dr = (dp/dR)R'$. At the maximum r_m the TOV equation is ill defined since the denominator tends to zero. However, this is not a physical singularity, as can easily be seen by considering the derivative of the second field equation (36) evaluated at $R' = 0$, which reads

$$8\pi p'(r_m) = 2\nu'(r_m)\frac{R''(r_m)}{R(r_m)}, \tag{41}$$

and, moreover, all Riemann tensor components (A1) are well defined at $r = r_m$. Furthermore, one can express the sum of the energy density and the pressure in terms of the Riemann tensor,

$$4\pi(\rho + p) = R^{r\theta}{}_{r\theta} - R^{\theta t}{}_{\theta t} = R^{r\phi}{}_{r\phi} - R^{\phi t}{}_{\phi t}, \tag{42}$$

so that this sum is well defined if the spacetime is nonsingular. Let us furthermore evaluate the field equations at r_m, which yields

$$\frac{1 - 2R(r_m)R''(r_m)}{R^2(r_m)} - \Lambda = 8\pi\rho(r_m), \tag{43}$$

$$\frac{-1}{R^2(r_m)} + \Lambda = 8\pi p(r_m), \tag{44}$$

$$\frac{R(r_m)(\nu'(r_m)^2 + \nu''(r_m)) + R''(r_m)}{R(r_m)} + \Lambda = 8\pi p(r_m). \tag{45}$$

Next, from Eq. (44) we find that the pressure is positive at the equator if the cosmological constant is large enough, that is, if

$$\Lambda > \frac{1}{R^2(r_m)} =: \Lambda_N. \tag{46}$$

For the special case $\Lambda = \Lambda_N$ the pressure vanishes at the equator, i.e. the maximum of the area of the group orbits. Putting this particular value of the cosmological constant into the first field equations yields

$$\frac{1 - 2R(r_m)R''(r_m)}{R^2(r_m)} - \Lambda_N = -2\frac{R''(r_m)}{R(r_m)} = 8\pi\rho(r_m), \tag{47}$$

from which we conclude that $R''(r_m) < 0$ to have a physically meaningful perfect fluid solution. Indeed, this condition simply states that the equator is a local maximum of the group orbits' area. This fact was not assumed explicitly and is a direct consequence of assuming physical solutions: positivity of the energy density.

The above analysis clarifies under which conditions static and spherically symmetric perfect fluid ellipsoids may extend through the equator and have a zero pressure surface in regions where the area of the group orbits is decreasing. However, condition (46) depends on the function $R(r)$ and therefore essentially depends on the solution. Given an equation of state, a central pressure, and a cosmological constant, such that the pressure (and by the equation of state the energy density) is decreasing near the center, the above considerations are not sufficient to decide whether the pressure vanishes before the equator or not. Therefore one should find a condition depending solely on the equation of state and on the initial conditions (pressure at the center and the cosmological constant) so that one controls the location of the zero pressure surface.

The TOV equation (40) seems to be ill defined at r_m, since its denominator vanishes. On the other hand, the field equations imply that the spacetime is regular where $R'(r_m) = 0$. Therefore we can conclude that the limit $\lim_{r\to r_m} p'(r)$ must exist, so that we write $\lim_{r\to r_m} p'(r) = a$. Existence of $p'(r_m)$ can be put back into (40), and yields

$$a = \frac{p(r_m) + \rho(r_m)}{R(r_m)^2} \lim_{r\to r_m} \frac{(4\pi p(r)R(r)^3 + m(r) - \Lambda/3R(r)^3)}{R'(r)}. \tag{48}$$

Since $\lim_{r\to r_m} R'(r) = 0$, the numerator also must vanish as $r \to r_m$,

$$\lim_{r\to r_m} (4\pi p(r) + m(r)/R(r)^3 - \Lambda/3) = 4\pi p(r_m) + m(r_m)/R(r_m)^3 - \Lambda/3 = 0, \tag{49}$$

which can be written more conveniently (for the present purpose),

$$4\pi p(r_m) = \frac{1}{R(r_m)^3}\left(\frac{\Lambda}{3} R(r_m)^3 - m(r_m)\right). \tag{50}$$

Before exploiting the latter equation (50), we note that it is easy to show that the pressure in the TOV equation (40) is decreasing near the center if

$$\Lambda < 4\pi\rho(p_c) + 12\pi p_c, \tag{51}$$

a condition that only depends on the initial values and the equation of state.

According to Eq. (50) the signature of the pressure at the equator is determined by the signature of the quantity

$$\frac{\Lambda}{3} R_m^3 - m(R_m) = \int_0^{R_m} [\Lambda - 4\pi\rho(R)]R^2 dR, \tag{52}$$

where we used the definition of mass; see (39). Since pressure and by a monotonic equation of state also the energy density are decreasing functions, $\rho(p = p_c) \geq \rho(r_m)$, a sufficient condition to satisfy the inequality $p(r_m) > 0$ is

$$4\pi\rho(p = p_c) < \Lambda. \tag{53}$$

Hence, if the cosmological term is large enough, compared to the central density, then the pressure does not vanish before the equator of the interior spacetime.

Next we find a necessary condition for a positive pressure at the equator. Let us assume $p(r_m) \geq 0$. Then the integral of $\Lambda - 4\pi\rho(R)$ is non-negative. Since going outwards p and ρ are monotonically decreasing, it follows that $\Lambda - 4\pi\rho(R_m) \geq 0$. But because the zero pressure surface is after the equator, by the monotonicity condition we have $\rho(R_m) \geq \rho(p = 0)$, and consequently $\Lambda \geq 4\pi\rho(p = 0)$. On the other hand, this means that if

$$\Lambda < 4\pi\rho(p = 0) \tag{54}$$

then necessarily $p(r_m) < 0$. This condition is in agreement with previous results; see e.g. [2,3]. Hence, if the given equation of state and the cosmological constant satisfy the condition (54), then the pressure vanishes before the equator of the ellipsoid.

Next, let us assume that the pressure vanishes at the maximum of the area of the group orbits, so that the equator is also the zero pressure surface. In this case, as we already showed in the previous sections, one has to join on the Nariai metric. Putting $p(r_m) = p(r_b) = 0$ into (50) leads to

$$m(r_b) = M = \frac{\Lambda}{3} R_b^3, \qquad \int_0^{R_b} 4\pi\rho(R)dR = \frac{\Lambda}{3} R_b^3, \tag{55}$$

which relates the mass of the solution to the cosmological constant. Unfortunately this condition cannot be written in a form such that the equation of state suffices to choose the initial values for such a solution.

These three observations can be summarized as follows:

$$\begin{aligned} &\Lambda < 4\pi\rho(p = 0) && \text{pressure vanishes before the equator,} \\ &4\pi\rho(p = 0) < \Lambda < 4\pi\rho(p = p_c) && \text{no analytical control,} \\ &4\pi\rho(p = p_c) < \Lambda && \text{pressure can vanish only after the equator,} \end{aligned}$$

which only depend on the equation of state and the initial conditions. However, these conditions only yield quite general conclusions. They do not suffice to decide whether the solution can have a second regular center, i.e. perfect fluid solutions which occupy the whole ellipsoid. From Eq. (51) we can conclude that the pressure is increasing near the first regular center if we assume $\Lambda > 4\pi\rho(p_c) + 12\pi p_c$; however, we cannot control the further behavior of the solution and may obtain a singular solution where the pressure diverges, or a solution with a second center having a conical singularity.

Let us furthermore discuss the consequences of having a regular center. Regularity of the solution at the center, in particular, fixes some of the coefficients in the power series expansion of the function $R(r)$ (see e.g. [46]), which are given by

$$R(r_c) = 0, \qquad R'(r_c) = 1, \qquad R''(r_c) = 0. \tag{56}$$

However, the part $R'''(r_c)$ is also determined by the initial conditions, as can be seen from the first field equation (35),

$$8\pi\rho(p_c) + \Lambda = \lim_{r\to r_c} \frac{1 - R'^2}{R^2} - \lim_{r\to r_c} \frac{2R''}{R}. \tag{57}$$

After applying the rule of L'Hopital and using the relations (56), we arrive at

$$8\pi\rho(p_c) + \Lambda = -3R'''(r_c), \tag{58}$$

so that the third derivative is also fixed by the initial conditions. In case there exists a second regular center r_{c_2}, we have

$$R(r_{c_2}) = 0, \qquad R'(r_{c_2}) = -1, \qquad R''(r_{c_2}) = 0 \tag{59}$$

so that $R'''(r_{c_2})$ is given by

$$8\pi\rho(p_{c_2}) + \Lambda = 3R'''(r_{c_2}). \tag{60}$$

The conditions (56) and (58) and also (59) and (60) must be satisfied independently by any solution admitting two regular centers. While one prescribes initial conditions at the first center, the regularity of the second center is by no means warranted since it actually depends on the solution of the function R together with the equation of state that also enter (60).

In the next section we will analyze the field equations numerically for given equations of state. It will turn out that none of the equations of state considered allows a second regular center. Therefore, all solutions that have an increasing pressure near the first center will have either a divergent pressure or a second center with a conical singularity, and are hence not of physical interest in general.

IV. NUMERICAL CONSIDERATIONS

In Secs. II A and II D we generalized two known solutions to include the cosmological constant. For large cosmological constant we found that new properties arise like the possibility of having a second regular center. In the previous analytic section we presented some arguments in favor of the existence of solutions with cosmological constant that may occupy more than "half" the three-space. This result can also be read in the following way: For sufficiently large cosmological constants the pressure cannot vanish before the equator of the three-space, so that some new physical properties may arise. However, it was also argued that, in general, a second regular center does not exist.

It is the aim of the present section to solve the field equations (35)–(37) numerically for a given equation of state $\rho = \rho(p)$. In particular, we are interested in solutions where the pressure vanishes after the group orbit's maximum of the three-space and in solutions that possibly have two regular centers. We concentrate on these two classes of solutions, since the others are well known already.

The most natural starting point for the numerical analysis is the polytropic equations of state

$$p(\rho) = K\rho^{(n+1)/n}, \qquad \rho(p) = \left(\frac{p}{K}\right)^{n/(n+1)}, \tag{61}$$

where K is some constant and n is the polytropic index. In the Newtonian case stellar models are finite if $1 < n < 5$ and do not have a finite radius for $n \geq 5$, where also $\Lambda = 0$ is assumed. We slightly modify the polytropic equations of state, in order to allow a nonvanishing boundary density $\rho_b = \rho(p = 0)$. Hence, we consider the following equation of state:

$$\rho(p) = \left(\frac{p}{K}\right)^{n/(n+1)} + \rho_b, \tag{62}$$

where the boundary density is a new free parameter that we must specify. We chose $K = 1$, the polytropic index $n = 3$, and $\rho_b = 0.5$. For two different cosmological constants ("small" and "large") we obtain the two following solutions, Figs. 4(a) and 4(b).

As expected, for small cosmological constant [see Fig. 4(a)] the pressure vanishes before the maximum of the group orbits. At the vanishing pressure surface one can join on the Schwarzschild–de Sitter metric C^1 as the exterior spacetime, with the methods described in Sec. II B. These solutions are represented by the Carter-Penrose diagram 1 discussed earlier. For large cosmological constant, however [see Fig. 4(b)], the pressure vanishes after the maximum, in a region where the group orbits are decreasing. This would necessarily yield a global solution represented by the Carter-Penrose diagram 3, where the exterior spacetime contains the singularity. Since the numerical solutions vary smoothly in the cosmological constant, it is evident that a fine-tuned cosmological constant can be chosen such that the pressure vanishes exactly at the maximum of the group orbits, in which case one has to join the Nariai metric as the exterior spacetime.

Next, let us analyze the solutions for the stiff matter equation of state (the $n \to \infty$ limit of the polytropic equation of state)

$$\rho(p) = p + \rho_b, \tag{63}$$

which we, as before, supplemented by a boundary density term ρ_b. For the stiff matter case we again take two differ-

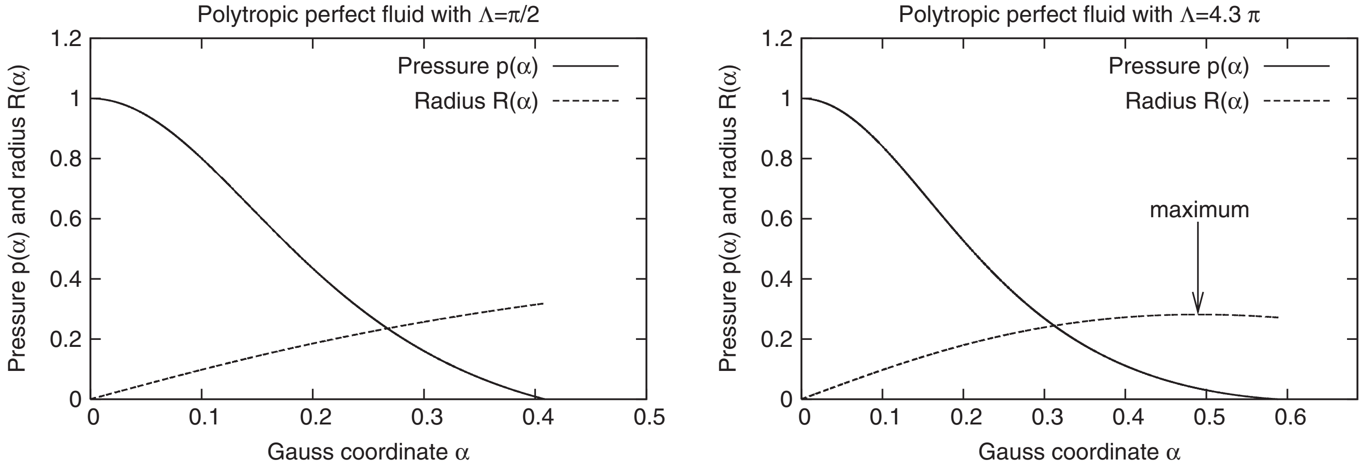

FIG. 4. Pressure function and radius for the polytropic equations of state, with $K = 1$, $n = 3$, and $\rho_b = 0.5$. Initial conditions are $p_c = 1.0$ ($\rho_c = 1.5$) and $\Lambda = \pi/2$ (left panel), and $\Lambda = 4.3\pi$ (right panel).

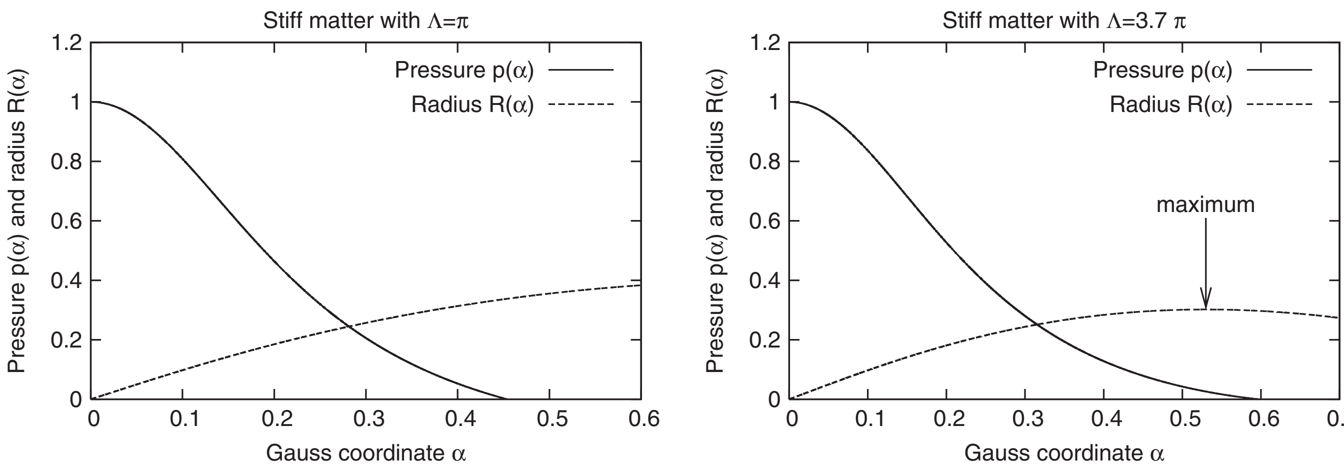

FIG. 5. Pressure function and radius for the stiff matter equation of state, with $\rho_b = 0.5$. Initial conditions are $p_c = 1.0$ ($\rho_c = 1.5$) and $\Lambda = \pi$ (left panel), and $\Lambda = 3.7\pi$ (right panel).

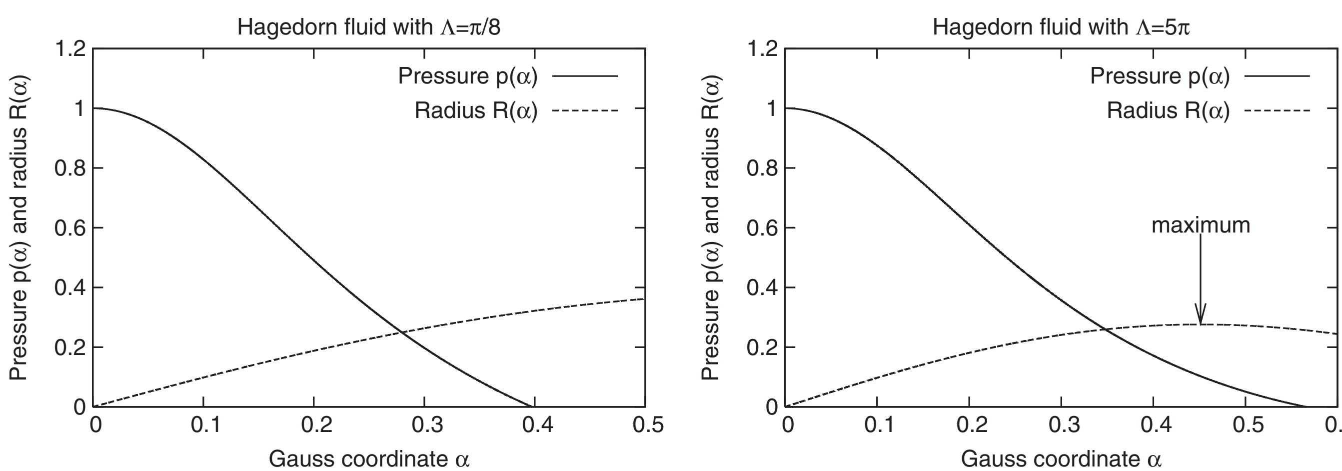

FIG. 6. Pressure function and radius for the Hagedorn equation of state, with $\rho^* = 2.0$. Initial conditions are $p_c = 1.0$ ($\rho_c \approx 1.21$) and $\Lambda = \pi/8$ (left panel), and $\Lambda = 5\pi$ (right panel).

ent values of the cosmological constant. The results are similar to those discussed already; see Figs. 5(a) and 5(b).

Finally let us consider the Hagedorn equation of state

$$\rho(p) = \rho^* \exp\left(\frac{p}{\rho^*} - 1\right), \tag{64}$$

where the free parameter ρ^* is related to the boundary density by $\rho_b = \rho(p = 0) = \rho^*/e$. As in the previously discussed cases, we find that for small cosmological constants the pressure vanishes before the maximum of the group orbits, whereas large values of the cosmological constant allow the pressure to vanish after the maximum; see Figs. 6(a) and 6(b).

Apart from the incompressible perfect fluid case we could not find numerically any other configuration with a second regular center. This indicates that the existence of the second center requires a very specific choice of the fluid's equation of state and carefully chosen initial conditions.

V. SUMMARY AND CONCLUSIONS

We have used analytical and numerical techniques to analyze the static and spherically perfect fluid field equations of general relativity in the presence of a cosmological constant. A positive cosmological term can be viewed as an external force having the effect of pulling matter apart. Hence, one can expect that the radial size of matter spheres is increased due to Λ. This naturally yields questions regarding the physical picture applicable to these solutions. It turns out that the effects of the cosmological constant lead to various different configurations, many of which have not been discussed previously.

By using Gauss coordinates relative to the $r =$ constant hypersurfaces, we analyzed geometrically the properties of the vanishing pressure surface that determines the boundary of the perfect fluid sphere. In the absence of the cosmological constant, going outwards, the area of the respective group orbits are always increasing close to the zero pressure surface. This situation changes drastically if Λ is allowed to be relatively large in comparison with the matter density. It is possible for the pressure to vanish exactly at the maximum of the group orbits or even vanish where the group orbits are decreasing. In the first case one has to join on the Nariai solution to get the metric C^1 at the boundary. In the latter case one matches the part of the Schwarzschild–de Sitter solution containing the black hole horizon and the singularity. This is in contrast to the small Λ situation where the vacuum region contains the cosmological horizon. Lastly, one is led to ask whether the matter can occupy the whole spacetime resulting in two regular centers corresponding to a fully generalized Einstein static universe where neither the energy density nor the pressure are constant.

We showed that the Whittaker solution can have its vanishing pressure surface where the group orbits are decreasing; however, a second center is not possible. On the other hand, the Tolman *IV* solution does allow for a second regular center, a solution that might be named the Tolman *IV* Einstein universe. By numerically integrating the field equations for physically motivated equations of state, we showed that, in general, the pressure can vanish where the group orbits are decreasing and consequently also at the maximum for sufficiently fine-tuned initial conditions. We also obtained analytical bounds that the cosmological constant has to satisfy to allow for such situations. However, we were not able to show that, in general, solutions with two regular centers for a given equation of state exist. This observation has analytical support since the conditions under which the solutions have two regular centers are very restrictive. Note, however, that the special Tolman *IV* equation of state (28) admits solutions with a second regular center.

Ever since the first exact matter solutions have been obtained, static and spherically symmetric perfect fluid spacetimes have remained a subject of great interest. The presence of matter that effectively acts like a perfect fluid with unusual equations of state, such as $p/\rho = -1$, drastically changes the geometry of known solutions.

ACKNOWLEDGMENTS

We thank Péter Forgács and Roy Maartens for the useful discussions. G. F. has been supported by OTKA Grant No. K67942, No. K61636, and No. NI68228, and would like to thank the Bolyai Foundation for financial support.

APPENDIX: FIELD EQUATIONS AND RIEMANN TENSOR

The nonvanishing Riemann tensor components are

$$R^{rt}{}_{rt} = -\nu'^2\nu'', \qquad R^{\theta\phi}{}_{\theta\phi} = \frac{1 - R'^2}{R^2},$$
$$R^{r\theta}{}_{r\theta} = R^{r\phi}{}_{r\phi} = -\frac{R''}{R}, \qquad R^{\theta t}{}_{\theta t} = R^{\phi t}{}_{\phi t} = -\nu'\frac{R'}{R}. \tag{A1}$$

One can rewrite the field equations (35)–(37) to get

$$\frac{R''}{R} = -4\pi\rho - \frac{\Lambda}{3} + \frac{m}{R^3}, \tag{A2}$$

$$\nu'\frac{R'}{R} = 4\pi p - \frac{\Lambda}{3} + \frac{m}{R^3}, \tag{A3}$$

$$\nu'^2 + \nu'' = 4\pi(\rho + p) - \frac{2m}{R^3} - \frac{\Lambda}{3}, \tag{A4}$$

and hence the Riemann tensor in terms of physical quantities,

$$R^{rt}{}_{rt} = -4\pi(\rho + p) + \frac{2m}{R^3} + \frac{\Lambda}{3}, \tag{A5}$$

$$R^{\theta\phi}{}_{\theta\phi} = \frac{2m}{R^3} + \frac{\Lambda}{3}, \tag{A6}$$

$$R^{r\theta}{}_{r\theta} = R^{r\phi}{}_{r\phi} = 4\pi\rho + \frac{\Lambda}{3} - \frac{m}{R^3}, \tag{A7}$$

$$R^{\theta t}{}_{\theta t} = R^{\phi t}{}_{\phi t} = -4\pi p + \frac{\Lambda}{3} - \frac{m}{R^3}. \tag{A8}$$

[1] H. Weyl, Phys. Z. **20**, 31 (1919).
[2] C. G. Böhmer, Diploma thesis, Potsdam University, 2002, arXiv:gr-qc/0308057.
[3] C. G. Böhmer, Gen. Relativ. Gravit. **36**, 1039 (2004).
[4] C.-m. Xu, X.-j. Wu, and Z. Huang, Report No. iC-86-392, 1986.
[5] Z. Stuchlík, Acta Phys. Slovaca **50**, 219 (2000).
[6] H. Nariai, Gen. Relativ. Gravit. **31**, 963 (1999); *The Science Reports of the Tohoku University*, Series I, Vol. XXXV, No. 1 (1951), pp. 46–57.
[7] H. Nariai, Gen. Relativ. Gravit. **31**, 951 (1999); *The Science Reports of the Tohoku University*, Series I, Vol. XXXIV, No. 3 (1950), pp. 160–167.
[8] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. (1917), pp. 142–152.
[9] A. Ibrahim and Y. Nutku, Gen. Relativ. Gravit. **7**, 949 (1976).
[10] C. B. Collins, Gen. Relativ. Gravit. **20**, 847 (1988).
[11] L. A. Gergely and R. Maartens, Classical Quantum Gravity **19**, 213 (2002).
[12] C. G. Böhmer, Classical Quantum Gravity **21**, 1119 (2004).
[13] C. G. Böhmer, L. Hollenstein, and F. S. N. Lobo, Phys. Rev. D **76**, 084005 (2007).
[14] L. Parisi, M. Bruni, R. Maartens, and K. Vandersloot, Classical Quantum Gravity **24**, 6243 (2007).
[15] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D **9**, 3471 (1974).
[16] A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, Phys. Rev. D **10**, 2599 (1974).
[17] T. A. DeGrand, R. L. Jaffe, K. Johnson, and J. E. Kiskis, Phys. Rev. D **12**, 2060 (1975).
[18] D. J. Kaup, Phys. Rev. **172**, 1331 (1968).
[19] R. Ruffini and S. Bonazzola, Phys. Rev. **187**, 1767 (1969).
[20] C. G. Böhmer and K. Vandersloot, Phys. Rev. D **76**, 104030 (2007).
[21] C. G. Böhmer and T. Harko, Phys. Rev. D **71**, 084026 (2005).
[22] S. Hledik, Z. Stuchlik, and K. Mrazova, arXiv:gr-qc/0701051.
[23] L. Iorio, Int. J. Mod. Phys. D **15**, 473 (2006).
[24] M. Sereno and P. Jetzer, Phys. Rev. D **73**, 063004 (2006).
[25] P. Jetzer and M. Sereno, Phys. Rev. D **73**, 044015 (2006).
[26] V. Kagramanova, J. Kunz, and C. Lammerzahl, Phys. Lett. B **634**, 465 (2006).
[27] A. Balaguera-Antolínez, C. G. Böhmer, and M. Nowakowski, Int. J. Mod. Phys. D **14**, 1507 (2005).
[28] A. Balaguera-Antolínez, C. G. Böhmer, and M. Nowakowski, Classical Quantum Gravity **23**, 485 (2006).
[29] A. Balaguera-Antolínez and M. Nowakowski, Astron. Astrophys. **441**, 23 (2005).
[30] A. Balaguera-Antolínez, D. F. Mota, and M. Nowakowski, arXiv:0708.2980.
[31] A. Balaguera-Antolínez and M. Nowakowski, Classical Quantum Gravity **24**, 2677 (2007).
[32] W. Rindler and M. Ishak, Phys. Rev. D**76**, 043006 (2007).
[33] M. Sereno, arXiv:0711.1802 [Phys. Rev. D (to be published)].
[34] J. M. Whittaker, Proc. R. Soc. A **306**, 1 (1968).
[35] R. C. Tolman, Phys. Rev. **55**, 364 (1939).
[36] H. D. Wahlquist, Phys. Rev. **172**, 1291 (1968).
[37] G. Darmois, *Mémorial de Sciences Mathématiques*, Vol. XXV (Gauthier-Villars, Paris, 1927).
[38] W. Israel, Nuovo Cimento B **44S10**, 1 (1966).
[39] T. Zannias, Phys. Rev. D **41**, 3252 (1990).
[40] M. S. R. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395 (1998).
[41] A. D. Rendall and B. G. Schmidt, Classical Quantum Gravity **8**, 985 (1991).
[42] T. W. Baumgarte and A. D. Rendall, Classical Quantum Gravity **10**, 327 (1993).
[43] M. Mars, M. M. Martin-Prats, and J. M. M. Senovilla, Phys. Lett. A **218**, 147 (1996).
[44] C. G. Böhmer, Ukr. Phys. J. **50**, 1219 (2005).
[45] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).
[46] G. Fodor, arXiv:gr-qc/0011040.