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Topological deformation of isolated horizons

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We show that the Gauss-Bonnet term can have physical effects in four dimensions. Specifically, the entropy of a black hole acquires a correction term that is proportional to the Euler characteristic of the cross sections of the horizon. While this term is constant for a single black hole, it will be a nontrivial function for a system with dynamical topologies such as black-hole mergers: it is shown that for certain values of the Gauss-Bonnet parameter, the second law of black-hole mechanics can be violated.

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The study of black-hole thermodynamics continues to be one of the most exciting areas in gravitational theory. The celebrated four laws of black-hole mechanics [1–3] have revealed a very deep and profound relationship between classical and quantum aspects of gravitational phenomena. Among these, the first law relates the small changes of energy to small changes of surface area of nearby equilibrium states of a black hole within the phase space of solutions. This leads to an identification of a multiple of the surface gravity κ on the horizon with the temperature $\mathcal{T}$ of the hole, and a multiple of the surface area A with the entropy $\mathcal{S}$. More precisely, the temperature and entropy are [1,2,4]

$$\mathcal{T} = \frac{\kappa}{2\pi} \quad \text{and} \quad \mathcal{S} = \frac{A}{4G}, \tag{1}$$

with G the Newton constant. Remarkably, this expression for the entropy is independent of other properties of the black hole, such as the electric (or Yang-Mills) charge or rotation.

A general analysis based on Noether charge methods [5–7] has revealed that modifications to the Bekenstein-Hawking entropy relation will only present themselves in cases when gravity is nonminimally coupled to matter, or when the action for gravity is supplemented with higher-curvature interactions. The presence of higher-curvature interactions is important within the context of string theory; the Kretchman scalar appears in the low-energy effective action from the heterotic string theory [8]. Of particular interest is the Gauss-Bonnet (GB) term, which is the only combination of curvature-squared interactions for which the effective action is ghost-free [9]. The complete action for gravity in D dimensions is then [9]

$$S = \frac{1}{2k_D}\int_{\mathcal{M}} d^Dx\sqrt{-g}(R - 2\Lambda + \alpha\mathcal{L}_{\rm GB})$$
$$\mathcal{L}_{\rm GB} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \tag{2}$$

In this expression, g is the determinant of the spacetime metric tensor g_{ab} $(a, b, \ldots \in \{0, \ldots, D-1\})$, R_{abcd} is the Riemann curvature tensor, $R_{ab} = R^c{}_{acb}$ is the Ricci tensor, $R = g^{ab}R_{ab}$ is the Ricci scalar, $k_D = 8\pi G_D$ with G_D the D-dimensional Newton constant is the D-dimensional coupling constant, Λ is the cosmological constant, and α is the GB parameter.

A common belief within the literature about the action (2) is that in four dimensions the GB term can be discarded because it is a topological invariant (the Euler characteristic), and only leads to nontrivial effects in $D \geq 5$ dimensions. However, variation of $\mathcal{L}_{\rm GB}$ in $D = 4$ dimensions gives a surface term; this can be discarded locally, but becomes an important contribution if the manifold has boundaries. So if we are to believe that the GB term is significant in $D \geq 5$ dimensions, then (for a bounded spacetime) it should be considered to be significant in $D = 4$ dimensions as well. As we will show, inclusion of the GB term in $D = 4$ dimensions has important implications for black-hole mechanics.

We will elaborate on the above point in a moment, in particular, how variation of $\mathcal{L}_{\rm GB}$ gives rise to a surface term in four dimensions. This will be done in the connection formulation of general relativity. However, because we are interested in a manifold with boundaries, we first introduce the boundary conditions; it will be shown that first variation leads to a well defined action principle. We consider a four-dimensional spacetime manifold $\mathcal{M}$ of topology $R \times M$ with the following properties: (a) $\mathcal{M}$ contains a three-dimensional null surface Δ as inner boundary (representing the event horizon); and (b) $\mathcal{M}$ is bounded by three-dimensional (partial) Cauchy surfaces $M^{\pm}$ that intersect Δ in two-surfaces $\mathscr{S}^{\pm}$ and extend to the (arbitrary) boundary at infinity $\mathscr{B}$. See Fig. 1.

A three-dimensional null hypersurface Δ (with topology $R \times \mathscr{S}$) together with a degenerate metric q_{ab} of signature $0++$ and a null normal ℓ_a is said to be a nonexpanding horizon if (a) the expansion $\theta_{(\ell)}$ of ℓ_a vanishes on Δ, (b) the field equations hold on Δ, and (c) the matter stress-energy

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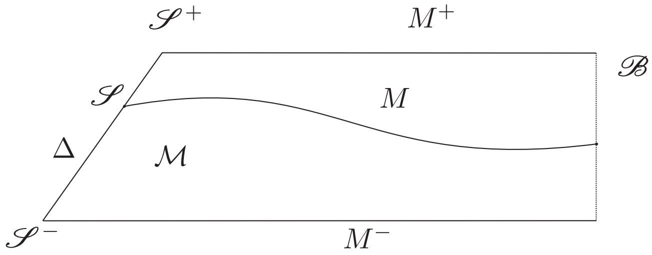

FIG. 1. The region of the four-dimensional spacetime $\mathcal{M}$ being considered has an internal boundary Δ representing the event horizon, and is bounded by two (partial) Cauchy surfaces $M^\pm$ which intersect Δ in two-surfaces $\mathscr{S}^\pm$ and extend to the boundary at infinity $\mathscr{B}$.

tensor is such that $-T^a{}_b\ell^b$ is a future-directed causal vector. Condition (a) implies that the rotation tensor is zero. Condition (c) is the dominant energy condition imposed on any matter fields that may be present in the neighborhood of the horizon. These conditions along with the Raychaudhuri equation imply that the shear tensor also vanishes. In turn, this implies that $\nabla_{\underleftarrow{a}}\ell_b \approx \omega_a\ell_b$. (The underarrow indicates pullback to $\Delta \subset \mathcal{M}$; "$\approx$" denotes equality restricted to Δ.) Thus the one-form ω is the natural connection (in the normal bundle) induced on the horizon.

The "time independence" of ω on Δ captures the notion of weak isolation. That is, a nonexpanding horizon together with an equivalence class of null normals $[\ell]$ becomes a weakly isolated horizon if $\pounds_\ell\omega_a = 0$ for all $\ell \in [\ell]$ (where $\ell' \sim \ell$ if $\ell' = c\ell$ for some constant c). This condition is a restriction on the rescaling freedom of ℓ. It turns out that this condition is enough to establish the zeroth law: the surface gravity $\kappa_{(\ell)} = \ell^a\omega_a$ is constant over the surface Δ of a weakly isolated horizon. This form of the zeroth law was first established in [10].

Let us now look at the action principle, and the implications of the boundary conditions on the first variation. This is most transparent in the connection formulation of general relativity, where the action (2) becomes

$$S = \frac{1}{2k_4}\int_{\mathcal{M}} \Sigma_{IJ} \wedge \Omega^{IJ} - 2\Lambda\boldsymbol{\epsilon} + \alpha\epsilon_{IJKL}\Omega^{IJ} \wedge \Omega^{KL}. \tag{3}$$

This action depends on the coframe e^I and the connection $A^I{}_J$. The coframe determines the metric $g_{ab} = \eta_{IJ}e_a{}^I \otimes e_b{}^J$, two-form $\Sigma_{IJ} = (1/2)\epsilon_{IJKL}e^K \wedge e^L$, and spacetime volume four-form $\boldsymbol{\epsilon} = e^0 \wedge e^1 \wedge e^2 \wedge e^3$, where ϵ_{IJKL} is the totally antisymmetric Levi-Civita tensor. The connection determines the curvature two-form

$$\Omega^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J = \tfrac{1}{2}R^I{}_{JKL}e^K \wedge e^L, \tag{4}$$

with $R^I{}_{JKL}$ as the Riemann tensor. Internal indices $I, J, \ldots \in \{0, \ldots, 3\}$ are raised and lowered using the Minkowski metric $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$. The gauge covariant derivative $\mathscr{D}$ acts on generic fields Ψ_{IJ} such that

$$\mathscr{D}\Psi^I{}_J = d\Psi^I{}_J + A^I{}_K \wedge \Psi^K{}_J - A^K{}_J \wedge \Psi^I{}_K. \tag{5}$$

In general, the equations of motion are given by $\delta S = 0$, where δ is the first variation; i.e. the stationary points of the action. In the present case, the equations of motion are obtained from independently varying the action (3) with respect to the coframe and connection. Denoting the pair (e, A) collectively as a generic field variable Ψ, the first variation gives

$$\delta S = \frac{1}{2k_4}\int_{\mathcal{M}} E[\Psi]\delta\Psi - \frac{1}{2k_4}\int_{\partial\mathcal{M}} J[\Psi, \delta\Psi]. \tag{6}$$

Here $E[\Psi] = 0$ symbolically denotes the equations of motion. Specifically, these are

$$\frac{\delta S}{\delta A} \to \mathscr{D}(\Sigma_{IJ} + 2\alpha\epsilon_{IJKL}\Omega^{KL}) = 0, \tag{7}$$

$$\frac{\delta S}{\delta e} \to \epsilon_{IJKL}e^J \wedge (\Omega^{KL} - 2\Lambda e^K \wedge e^L) = 0. \tag{8}$$

The first of these reduces to $\mathscr{D}e = 0$ by virtue of the Bianchi identity. The surface term J is given by

$$J[\Psi, \delta\Psi] = \tilde{\Sigma}_{IJ} \wedge \delta A^{IJ}, \qquad \tilde{\Sigma}_{IJ} \equiv \Sigma_{IJ} + 2\alpha\epsilon_{IJKL}\Omega^{KL}. \tag{9}$$

If the integral of J on the boundary $\partial\mathcal{M}$ vanishes, then the action principle is said to be differentiable. We must show that this is the case. Because the fields are held fixed at $M^\pm$ and at $\mathscr{B}$, J vanishes there. So we only need to show that J vanishes at the inner boundary Δ. To show that this is true we need to find an expression for J in terms of A and $\tilde{\Sigma}$ pulled back to Δ. This is accomplished by fixing an internal Newman-Penrose basis consisting of the null vectors $(\ell, n, m, \bar{m})$ such that $\ell = e_0$, $n = e_1$, $m = (e_2 + ie_3)/\sqrt{2}$, and $\bar{m} = (e_2 - ie_3)/\sqrt{2}$; normalizations are such that $\ell \cdot n = -1$, $m \cdot \bar{m} = 1$, and all other contractions are zero. The pullback of A can be expressed as

$$A_{\underleftarrow{a}IJ} \approx -2\ell_{[I}n_{J]}\omega_a + X_a\ell_{[I}m_{J]} + Y_a\ell_{[I}\bar{m}_{J]} + Z_a m_{[I}\bar{m}_{J]}, \tag{10}$$

where X_a, Y_a, and Z_a are one-forms in the cotangent bundle $T^*(\Delta)$. It follows that the variation of (10) is

$$\delta A_{\underleftarrow{a}IJ} \approx -2\ell_{[I}n_{J]}\delta\omega_a + \delta X_a\ell_{[I}m_{J]} + \delta Y_a\ell_{[I}\bar{m}_{J]} + \delta Z_a m_{[I}\bar{m}_{J]}. \tag{11}$$

To find the pullback to Δ of $\tilde{\Sigma}$, we use the decompositions

$$e_a{}^I = -\ell^I n_a - n^I\ell_a + m^I\bar{m}_a + \bar{m}^I m_a, \tag{12}$$

$$\epsilon_{IJKL} = i\ell_I \wedge n_J \wedge m_K \wedge \bar{m}_L. \tag{13}$$

The pullback of Σ is [10]

$$\underset{\leftarrow}{\Sigma}_{IJ} \approx 2\ell_{[I}n_{J]}\tilde{\epsilon} + 2n \wedge (im\ell_{[I}\bar{m}_{J]} - i\bar{m}\ell_{[I}m_{J]}). \quad (14)$$

Here we have defined the area form $\tilde{\epsilon} = im \wedge \bar{m}$. To calculate the pullback of the curvature we use the definition

$$\Omega_{abIJ} = R_{IJKL}e_{[a}{}^{K}e_{b]}{}^{L}, \quad (15)$$

whence

$$\underset{\leftarrow}{\Omega}_{abIJ} \approx 2R_{IJKL}[\ell^K m^L(\bar{m} \wedge n) + \ell^K \bar{m}^L(m \wedge n) + m^K \bar{m}^L(\bar{m} \wedge m)]. \quad (16)$$

Now, we note that $\underset{\leftarrow}{\Sigma}_{IJ} \wedge \delta \underset{\leftarrow}{A}^{IJ} \approx 2\tilde{\epsilon} \wedge \delta\omega$. Using this together with the expressions (11), (14), and (15), we find that the surface term (9) becomes

$$J[\Psi, \delta\Psi] \approx [\tilde{\epsilon} + 2i\alpha R_{IJKL}m^I\bar{m}^J e^K \wedge e^L] \wedge \delta\omega - \frac{i\alpha}{2}R_{IJKL}\ell^I[m^J\delta X + \bar{m}^J\delta Y - n^J\delta Z] \wedge e^K \wedge e^L. \quad (17)$$

[A factor of 2 has been absorbed into the coefficient outside the integral in (6).] For an isolated horizon, the Riemann tensor is severely restricted. This results in considerable simplification of (17). Details of these simplifications are worked out in the appendix in [11] for multidimensional weakly isolated and nonrotating horizons; here we just state the results and refer the reader to that article for more details. In particular, the pullback to Δ of the Riemann tensor is equivalent to the Riemann tensor $\mathcal{R}_{IJKL}$ of the two-dimensional cross sections of Δ. That is,

$$\tilde{q}_a{}^e\tilde{q}_b{}^f\tilde{q}_c{}^g\tilde{q}_d{}^h R_{efgh} = \mathcal{R}_{abcd}. \quad (18)$$

The $\tilde{q}$ in this expression is the projection tensor onto $\mathscr{S}$ defined by $\tilde{q}_a{}^b = q_a{}^b + \ell_a n^b$. Further simplification occurs if the horizon is nonrotating, in which case we have that $\omega_a = -\kappa_{(\ell)}n_a$. Using this with the fact that the expansion, rotation, and shear are all zero on Δ implies that $R_{\underset{\leftarrow}{ab}}{}^c{}_d\ell^d = 0$; with these considerations, it turns out that the only nonvanishing contribution in (17) is $\mathcal{R}_{IJKL}m^I\bar{m}^J m^K\bar{m}^L \approx \mathcal{R}$, with $\mathcal{R}$ the Ricci scalar of the cross sections $\mathscr{S}$ of the horizon. Hence the current (17) becomes

$$J[\Psi, \delta\Psi] \approx \tilde{\epsilon}(1 + 2\alpha\mathcal{R}) \wedge \delta\omega. \quad (19)$$

The final step is to note that $\delta\ell \propto \ell$ for some ℓ fixed in $[\ell]$, and this together with $\pounds_\ell\omega = 0$ implies that $\pounds_\ell\delta\omega = 0$. However, ω is held fixed on $M^\pm$, which means that $\delta\omega = 0$ on the initial and final cross sections of Δ (i.e. on $M^- \cap \Delta$ and on $M^+ \cap \Delta$), and because $\delta\omega$ is Lie dragged on Δ, it follows that $J \approx 0$. Therefore the surface term $J|_{\partial\mathcal{M}} = 0$ for four-dimensional gravity with GB term, and we conclude that the equations of motion $E[\Psi] = 0$ follow from the action principle $\delta S = 0$.

The expression (19) for the current pulled back to Δ is the same as the one that was obtained for a multidimensional horizon [11]. The calculation presented in this paper may seem like a simple recalculation of J that was presented in [11], with the dimensionality restricted to $D = 4$. However, we believe that the calculation presented here is a necessary one because the phase space of the horizon in four dimensions differs from the phase space of the corresponding horizon in $D \geq 5$ dimensions. Specifically, the GB density in four dimensions is $\mathcal{L}_{\mathrm{GB}} \sim \epsilon_{IJKL}\Omega^{IJ} \wedge \Omega^{KL}$ which only depends on the connection. In $D \geq 5$ dimensions the GB density becomes $\mathcal{L}_{\mathrm{GB}} \sim \Sigma_{IJKL} \wedge \Omega^{IJ} \wedge \Omega^{KL}$, with Σ defined by $\Sigma_{I_1\ldots I_m} = \epsilon_{I_1\ldots I_m I_{m+1}\ldots I_D}e^{I_{m+1}} \wedge \cdots \wedge e^{I_D}$. In addition to the connection, this term also depends on the coframe through Σ. As a result, the equations of motion are more complicated and physically different from their four-dimensional counterparts. Among other consequences, the equation of motion for the connection does not constrain the torsion two-form to vanish in higher dimensions.

The calculation of the first law from the surface term is now essentially the same as in [11]. For an appropriate normalization of some time evolution vector field t that points in the direction of ℓ, and defining the surface gravity $\kappa_{(t)} = t \cdot \omega$, the first law for the horizon with energy $\mathcal{E}_\Delta$ is

$$\delta\mathcal{E}_\Delta = \frac{\kappa_{(t)}}{k_4}\delta\oint_{\mathscr{S}}\tilde{\epsilon}(1 + 2\alpha\mathcal{R}). \quad (20)$$

In its standard form, the first law of thermodynamics (for a quasistatic process) is $\delta\mathcal{E} = \mathcal{T}\delta\mathcal{S} +$ (work terms). Here, the temperature is $\mathcal{T} = \kappa_{(t)}/2\pi$, whence the entropy of the horizon is

$$\mathcal{S} = \frac{1}{4G}\oint_{\mathscr{S}}\tilde{\epsilon}(1 + 2\alpha\mathcal{R}). \quad (21)$$

This differs from the Bekenstein-Hawking expression (1). Therefore, the GB term gives rise to a correction even though it is a topological invariant of the manifold and does not show up in the equations of motion. This happens because the GB term contributes a surface term which cannot be discarded in the covariant phase space.

Here, the spaces $\mathscr{S}$ are two-dimensional: the *correction term is (a multiple of) the Euler characteristic $\chi(\mathscr{S})$ of the cross sections of the horizon*. This is consistent with the conclusions in [12], but much more general because we did not specify any properties of the space $\mathscr{S}$. By the GB theorem, we have that $\oint_{\mathscr{S}}\tilde{\epsilon}\mathcal{R} = 4\pi\chi(\mathscr{S})$. The entropy (21) is therefore

$$\mathcal{S} = \frac{1}{4G}[A + 8\pi\alpha\chi(\mathscr{S})]. \quad (22)$$

For example, if Λ is zero, then by Hawking's topology theorem $\mathscr{S}$ has to be a sphere [13]. In this case $\chi(\mathscr{S}) = 2$

and the entropy becomes $\mathcal{S} = (A + 16\pi\alpha)/(4G)$. If the cosmological constant is negative, then physical black holes can have spherical, flat, or even toroidal as well as higher-genus horizon topologies [14]. For a torus, $\chi(\mathscr{S}) = 0$ and the Bekenstein-Hawking entropy $\mathcal{S} = A/(4G)$ is recovered.

For a single black hole, the correction is a constant. However, this will not be the case for a system with dynamical topologies such as black-hole mergers [15]. This is a form of topology change, which for a space with a degenerate metric is unavoidable even in classical general relativity [16]. As an example, let us consider the merging of two black holes—one with mass m_1 and entropy $\mathcal{S}_1 = [A_1 + 8\pi\alpha\chi(\mathscr{S}_1)]/4G$, the other with mass m_2 and entropy $\mathcal{S}_2 = [A_2 + 8\pi\alpha\chi(\mathscr{S}_2)]/4G$. Before the black holes merge, the total entropy is

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 = \frac{1}{4G}[A_1 + A_2 + 8\pi\alpha(\chi(\mathscr{S}_1) + \chi(\mathscr{S}_2))]. \quad (23)$$

After the black holes merge, the total entropy of the resulting black hole is

$$\mathcal{S}' = \frac{1}{4G}[A' + 8\pi\alpha\chi(\mathscr{S}')]. \quad (24)$$

Without knowing the specific details of the black holes in question, we cannot say anything further about $\mathcal{S}$ and $\mathcal{S}'$. Let us therefore consider for concreteness the simplest case—the merging of two Schwarzschild black holes in an asymptotically flat spacetime. In this case the cross sections of the horizons can only be spheres and therefore $\chi(\mathscr{S}_1) = \chi(\mathscr{S}_2) = \chi(\mathscr{S}') = 2$. This, together with the fact that the area of a Schwarzschild black hole is related to its mass via $A = 16\pi m^2$, implies that the entropies $\mathcal{S}$ and $\mathcal{S}'$ are given by

$$\mathcal{S} = \frac{4\pi}{G}[m_1^2 + m_2^2 + 2\alpha], \quad (25)$$

$$\mathcal{S}' = \frac{4\pi}{G}[(m_1 + m_2 - \gamma)^2 + \alpha]. \quad (26)$$

Here we included a small mass parameter $\gamma \geq 0$ for the surface area of the final black-hole state that corresponds to any mass that may be carried away by gravitational radiation during merging. The expressions (25) and (26) imply that $\mathcal{S}' > \mathcal{S}$ if

$$\alpha < 2m_1m_2 - \gamma[2(m_1 + m_2) - \gamma]. \quad (27)$$

Therefore the second law will be violated if α is greater than twice the product of the masses of the black holes before merging minus a correction due to gravitational radiation.

To summarize, we explored the role that the Gauss-Bonnet term can play in four-dimensional general relativity. In particular, we constructed a covariant phase space for an isolated horizon and calculated the first law. This led to an expression for the entropy that is given by the area of the horizon plus a correction term that is given by the Euler characteristic of the cross sections of the horizon. As was shown, the correction term can have some interesting effects during the merging of two black holes, as the second law can be violated for certain values of the GB parameter. Therefore we have shown that the GB term can have nontrivial physical effects in four dimensions, contrary to the common assumption that the term is only significant in spacetimes with five or more dimensions.

It would be interesting to investigate the quantum geometry of these "topological" isolated horizons by using the methods that were developed in [17–19]. Quantization of toroidal and higher-genus horizons in Einstein gravity with negative cosmological constant has been recently considered by Kloster *et al.* [20]. Interestingly it was found that the toroidal horizon is the only one for which the quantum entropy does not acquire any logarithmic corrections.

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