Bonnor stars in d spacetime dimensions

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Bonnor stars are regular static compact configurations in equilibrium, composed of an extremal dust fluid, i.e., a charged dust fluid where the mass density is equal to the charge density in appropriate units and up to a sign, joined to a suitable exterior vacuum solution, both within Newtonian gravity and general relativity. In four dimensions, these configurations obey the Majumdar-Papapetrou system of equations: in one case, the system is a particular setup of Newtonian gravity coupled to Coulomb electricity and electrically charged matter or fluid, in the other case, the system is a particular setup of general relativity coupled to Maxwell electromagnetism and electrically charged matter or fluid, where the corresponding gravitational potential is a specially simple function of the electric potential field and the fluid, when there is one, is made of extremal dust. Since the Majumdar-Papapetrou system can be generalized to d spacetime dimensions, as has been previously done, and higher-dimensional scenarios can be important in gravitational physics, it is natural to study this type of Bonnor solutions in higher dimensions, $d \geq 4$. As a preparation, we analyze Newton-Coulomb theory with an electrically charged fluid in a Majumdar-Papapetrou context, in $d = n + 1$ spacetime dimensions, with n being the number of spatial dimensions. We show that within the Newtonian theory, in vacuum, the Majumdar-Papapetrou relation for the gravitational potential in terms of the electric potential, and its related Weyl relation, are equivalent, in contrast to general relativity where they are distinct. We study a class of spherically symmetric Bonnor stars within this theory. Under sufficient compactification they form point mass charged Newtonian singularities. We then study the analogue-type systems in the Einstein-Maxwell theory with an electrically charged fluid. Drawing on our previous work on the d-dimensional Majumdar-Papapetrou system, we restate some properties of this system. We obtain spherically symmetric Bonnor star solutions in $d = n + 1$ spacetime dimensions. We show that these stars, under sufficient compactification, form d-dimensional quasi-black holes. We also show that in the appropriate low gravity limit theses solutions turn into the solutions of Newtonian gravity, i.e., they are quasi-Newtonian Bonnor stars. In this connection, we note that the star solutions in Majumdar-Papapetrou Newtonian gravity, when contrasted to those solutions in Majumdar-Papapetrou general relativity, display clearly the branching off of the high density objects that may arise in the strong field regime of each theory, mild singularities in one theory, quasi-black holes in the other. Another important feature worth mentioning is that, whereas there are no solutions for Newtonian or relativistic stars supported by degenerate pressure in higher dimensions, higher-dimensional Bonnor stars, supported by electric repulsion, do indeed have solutions within Newtonian gravity and general relativity. So the existence of stars in higher dimensions depends on the number of dimensions itself, and on the underlying field content of those stars.

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I. INTRODUCTION

A. Definition

Extremal charged dust, or simply extremal dust, is understood as charged dust fluid, or matter, with the mass density being equal to the charge density, in appropriate units, which implies that for each such a dust particle, eventually composing a system, the gravitational attraction is precisely balanced by the electric repulsion, both within Newtonian gravity coupled to Coulomb electricity and an electrically charged fluid or matter, i.e., the Newton-Coulomb with charged fluid system, and within general relativity coupled to Maxwell electromagnetism and an electrically charged fluid, i.e., the Einstein-Maxwell with charged fluid system. Bonnor stars are then defined as regular static equilibrium configurations, in Newtonian and general relativity contexts, composed of extremal dust, with a finite boundary appropriately attached to an asymptotically flat regular extremal charged vacuum, and where the configuration of the matter dust can have any shape, a spherical symmetric shape being of special interest, due to the added symmetry and due to the fact that it can be joined to an asymptotically flat regular extremal outer Reissner-Nordström spacetime in the general relativistic case, with mass M equal to charge Q in appropriate units. Bonnor stars appear in $d = n + 1$ spacetime dimensions, where n is the number of the spatial dimensions. The

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initial studies were performed for $d = 4$. Bonnor stars have also been called Majumdar-Papapetrou stars, but here we reserve the name Majumdar-Papapetrou for the type of matter, and name the whole system, namely, Majumdar-Papapetrou matter plus vacuum plus junction, as a Bonnor star.

B. Four-dimensional analyses

1. Context

Such stars were studied mainly within general relativity, although with some incursions onto Newtonian gravity, by Bonnor [1–12] and in several other works; see, e.g., [13,14] for Bonnor stars properly said and [15] for a variant where there is no need for a junction. One striking property of these solutions found in [13–15] is that when they approach their gravitational radius in a static sequence of configurations, these stars do not form black holes, but rather quasi-black holes, where a quasi-black hole is an object indistinguishable to the exterior from a black hole but with different intrinsic properties. In [16,17] the properties of quasi-black holes formed from Bonnor stars were studied in detail. See also [18] for a further study on the properties of Bonnor stars.

2. Vacuum Majumdar-Papapetrou solutions

Within Newtonian gravity coupled to Coulomb electricity, i.e., the Newton-Coulomb system, electrically charged solutions in vacuum represent charged point masses. Within general relativity, electrically charged solutions, in vacuum, have to be analyzed through the Einstein-Maxwell system of equations, where one couples Einstein gravity to Maxwell electromagnetism. These solutions were found just after general relativity was formulated. On one hand, Reissner [19], then Nordström [20], then Jeffery [21], hit on the static vacuum charged spherically symmetric solution, the Reissner-Nordström solution, with its two parameters, the mass M and the charge Q. We now know that for $\sqrt{G}M < \epsilon Q$ (we put the speed of light $c = 1$ throughout), where G is Newton's gravitational constant in four spacetime dimensions ($G_4 \equiv G$) and $\epsilon = 1$ if the charge is positive and $\epsilon = -1$ if the charge is negative, one has a charged naked singularity, for $\sqrt{G}M > \epsilon Q$ one has a Reissner-Nordström black hole, and $\sqrt{G}M = \epsilon Q$ one has an extremal black hole (see [22] for an early discussion and [23] for a full discussion of this type of solutions).

On the other hand, with the purpose of seeking vacuum static solutions electrically charged, a different route was originated from Weyl [24], the route that will take us to the Bonnor stars [1–18]. Wanting to go a step further from spherical symmetry he sought axial symmetry. Define the metric component g_{00} as $g_{00} \equiv W^2$, where, depending on the situation, it can be more convenient to define $U \equiv W^{-1}$, i.e., $g_{00} \equiv U^{-2}$. Then, Weyl asked himself, within Einstein-Maxwell theory, what would happen if W^2, in a static axisymmetric vacuum electric system, is to have a functional dependence on the electric potential field φ, i.e., the Weyl ansatz $g_{00} = g_{00}(\varphi)$ or equivalently $W = W(\varphi)$. He found first what is now called the Weyl relation, i.e., $W^2 = (a_0 - \epsilon\sqrt{G}\varphi)^2 + b_0$, where a_0 and b_0 are constants of integration and G is Newton's gravitational constant, and second that the spatial components of the metric had to obey other specific differential equations. Majumdar [25] made several improvements on Weyl's work. He showed that the Weyl relation, if it existed, was independent of the symmetry, axial or otherwise. But further, he showed, still in vacuum, that if the relation was to be a perfect square, so that $W = a_0 - \epsilon\sqrt{G}\varphi$, then the spatial part of the metric could be put in a simple form, as $1/W^2$, i.e., U^2, times the flat spatial metric, and the Einstein-Maxwell system of equations would reduce to one single equation for W, i.e., for U, a Laplace equation in flat space. In this perfect square case, one can show that specializing to spherical symmetry, the mass M of the solution is equal to its charge Q, $\sqrt{G}M = \epsilon Q$. This makes contact with the Reissner-Nordström family of solutions through the extremal solution, $\sqrt{G}M = \epsilon Q$, although not through the other ones, since the Reissner-Nordström family generically does not admit a functional relationship between the metric and the electric potentials. These vacuum $\sqrt{G}M = \epsilon Q$ solutions were further commented by Papapetrou [26], who also understood that since the gravitational attraction is equal to the electric repulsion for such objects one could have many discrete such objects scattered at will in space, and it would also give a vacuum static configuration solution, with no symmetry whatsoever. The perfect square relation is usually called the Majumdar-Papapetrou relation, as we do here, although sometimes it is called, perhaps more appropriately, the Weyl-Majumdar relation. The complete understanding of a single extremal Reissner-Nordström solution, also a Majumdar-Papapetrou solution, was achieved by Carter [27], through a Carter-Penrose diagram, and the complete understanding of the vacuum Majumdar-Papapetrou solutions, with many extremal black holes scattered around, was performed by Hartle and Hawking [28], who have done the maximal analytical continuation in the molds of Carter [27].

3. Beyond vacuum: Dust Majumdar-Papapetrou solutions

Things become more interesting if one goes beyond vacuum and puts matter into the Newton-Coulomb system of equations and into the Einstein-Maxwell system of equations. Majumdar [25] and Papapetrou [26] understood this, and within general relativity showed that, for some special restrictions on the metric inspired from the vacuum case, such as the relation $W = a_0 - \epsilon\sqrt{G}\varphi$ (which in this case can be considered an ansatz), one could find that the system of equations yields a single equation that moreover reduces to a Poisson equation, and in which the mass density $\rho_{\rm m}$ times $\sqrt{G}$ is equal to the charge density $\rho_{\rm e}$, up to a sign, $\sqrt{G}\rho_{\rm m} = \epsilon\rho_{\rm e}$, with again $\epsilon = \pm 1$. That is, the matter is made of an extremal dust fluid. This is the Majumdar-Papapetrou condition. Note that we call the relation between the potentials the Majumdar-Papapetrou relation, whereas we call the relation between the densities the Majumdar-Papapetrou condition. As seen in [25,26], it is remarkable that a simple obvious fact in Newton-Coulomb theory with an electrically charged fluid, that if the mass density and charge density are equal [in appropriate (geometric) units where $G = 1$] then there is exact balancing of the gravitational and electric forces throughout the matter and so there is a static solution, also holds in Einstein-Maxwell theory with an electrically charged fluid, with no need for further stresses, such as pressure or tension. The basic feature of Majumdar-Papapetrou systems is that they describe static spacetimes filled either with extremal charged vacuum or extremal charged dust fluids, such that the metric and electromagnetic fields may be characterized by two scalar functions, namely, the redshift metric function W, i.e., U^{-1}, which plays the role of the gravitational potential, and the electric potential φ, which in turn obey the Majumdar-Papapetrou relation, $W = a_0 - \epsilon\sqrt{G}\varphi$. Further interesting developments were achieved by Das [29], and De and Raychaudhuri [30], who considered charged dust distributions in equilibrium, the way envisioned by Majumdar and Papapetrou, and showed some other conditions related to the functional form of the metric in terms of the electric potential and the equality between mass and charge densities. Das [29] revealed that the equality between the densities implies the functional form on the potentials, and De and Raychaudhuri [30] proved that given the functional form above, and provided there are no singularities in the distribution, the equality of mass and charge densities follows directly from the field equations. There are other interesting properties of Majumdar-Papapetrou systems in the context of conformal static charged solutions [31].

4. Bonnor stars: Junction of dust with vacuum Majumdar-Papapetrou solutions

When one joins smoothly, within Newtonian gravity as well as within general relativity, Majumdar-Papapetrou interior matter solutions to Majumdar-Papapetrou exterior vacuum solutions, i.e., the two types of solutions mentioned in the previous paragraphs, one obtains Bonnor stars [1–12] and their developments [13–18]. Bonnor stars could instead be called Majumdar-Papapetrou stars as was done in [14], but it is more proper to characterize the matter part as a Majumdar-Papapetrou system, and this combined with placing a boundary and the corresponding junction to a vacuum, bringing together a whole lot of new properties, as a Bonnor star. Throughout his works, Bonnor gradually improved the understanding of the properties of these stars.

In the first two papers [1,2] Bonnor worked out aspects of electric Majumdar-Papapetrou solutions in an axial symmetric vacuum and extended these results through dualities to magnetic solutions. In [3] a pre-Bonnor star is developed, and it is noted that for $\sqrt{G}\rho_{\rm m} = \epsilon\rho_e$, in a Majumdar-Papapetrou system, the gravitational mass of the system is equal to the matter mass because the negative gravitational self-energy of the distribution is balanced by the corresponding positive electrical self-energy, also pointing out that $\sqrt{G}M = \epsilon Q$ models can have various interests and applications. In [4,5] Bonnor understood for the first time that although delicate, the balance can exist, an atom stripped off of an electron immersed in about 10^{18} atoms is enough, and that the charge density can play an important part in the equilibrium of large bodies, further suggesting that it may halt gravitational collapse, at a time where large body studies were in vogue due to the appearance of quasars. It is also mentioned that bodies of arbitrary shape comprised of such extremal dust can exist, constructing explicitly a spherically symmetric solution, the first Bonnor star. The way it is constructed delineates a standard way of finding such types of solutions. Assuming a given form for the gravitational potential U one can find the density distribution, and one hopes that the assumption yields a physical distribution of charged dust matter. It is not a method for solving the differential equation of the Majumdar-Papapetrou problem, it is an art of correct guessing. In [6,7], both works in collaboration with Wickramasuriya, interesting physical properties of some Bonnor stars are discussed. In particular in [6] the name electrically counterpoised dust is coined for the first time for $\sqrt{G}\rho_{\rm m} = \epsilon\rho_{\rm e}$ dust, i.e., for what we call and will always call, less contrived perhaps, extremal charged dust, with the same acronym. Spherically symmetric exact solutions are studied with the virtue that even when the solutions are about to form a horizon the energy density $\rho_{\rm m}$ is finite. Prolate solutions are also studied showing that in the disk solution limit the energy density $\rho_{\rm m}$ is infinite, naturally. Also, in particular, in [7] several important attributes of the solutions are perceived. First, it is noticed that, although no doubt matter thus delicately balanced is rare, it is physically possible and easily understood. Second, it is shown that one can construct spheres of matter where infinite redshifts of light emanating from the surface are attainable, whereas in an interior Schwarzschild solution, say, only finite redshifts are possible. Third, it displays exact solutions for spheroidal configurations, and mentions that near the spheroidal horizon, nonspherically symmetric features are filtered out. Fourth, it is argued convincingly that these solutions are stable. In [8] the study is interesting with strange results. First, there is an incursion into solutions of the Newton-Coulomb with an electrically charged fluid theory, where it is shown that for a given Newtonian potential, call it V, there are equilibrium non-Bonnor star solutions, not obeying the Majumdar-

Papapetrou matter condition, although these are singular. It also shows the analogue of De and Raychaudhuri's theorem [30], i.e., that Newtonian systems which do not obey the Majumdar-Papapetrou condition, of the equality of mass and charge densities, and which have equipotential surfaces, are singular. Second, in turning into general relativity, with axial symmetry, spacetimes obeying the Majumdar-Papapetrou condition are found, one of them being of physical interest with positive energy density $\rho_{\rm m}$, and the others of less interest. In [9,10] it is understood, perhaps for the first time, that when the radius of the configuration r_0 approaches the horizon radius, i.e., $r_0 = M$, where r is the Schwarzschildean radial coordinate and M the mass of the configuration, the spacetime is somehow singular, being thus an idea precursor of the concept of what a quasi-black hole is. In these works the hoop conjecture is discussed and some lower bounds in connection to it are given. In [11], spheroidal bodies made of extremal charged dust are studied in connection still with the hoop conjecture and also with the isoperimetric conjecture, which says that under certain general conditions $M \geq (A/16\pi)^{1/2}$, where A is the area of the apparent horizon and M the mass of the configuration. In [12] it is reinforced that spherically symmetric configurations made of physically reasonable matter, though admittedly not widely available, i.e., made of extremal charged dust, yield solutions that can come arbitrarily close to the horizon of an extremal black hole, and a general class of such solutions is constructed by correct guessing.

Bonnor stars were studied further by other authors. In [13] a thick shell solution of Bonnor type was found. In [14] it was noted that Bonnor star solutions and gravitational magnetic monopole solutions have strikingly similar properties, and a comparison of both solutions was performed and discussed thoroughly. Previously, Lemos and Weinberg [15], seeing in Bonnor's wake that these stars, which are made of normal matter obeying the several important energy conditions, can probe deeply the spacetime structure, proposed new solutions, extended Bonnor star systems with a more sophisticated density distribution asymptotic to an extreme Reissner-Nordström solution, not needing any junction. Similar properties were found for Bonnor stars properly said as well as for extended Bonnor stars. Most notably, is the fact that at the threshold of the formation of an event horizon the system displays a very peculiar trait, instead of an extremal black hole one has a quasi-black hole, with the formation of a quasihorizon instead of the usual event horizon. Although to external observers the system looks like an extremal black hole, its internal properties are very different from what one could expect in the case of a standard black hole. These properties, along similar ones of gravitational magnetic monopoles and glued vacuum solutions with shells, have been analyzed in [16,17]. In [18] other attributes of these systems were explored.

5. Further connections

One can associate these Bonnor stars to several related topics. (i) Both astrophysically and physically, Bonnor stars are of interest. On one hand, they can be realized if a gravitating sphere, of neutral hydrogen which has lost a fraction 10^{-18} of its electrons, forms. On the other hand, they are supersymmetric solutions of $N = 2$ supergravity [32], so are of interest in an elementary particle context. (ii) A matter always of maximal interest is the stability of the systems one is considering, in this case, the Bonnor stars. Interestingly enough it was found, through different methods, that these stars are neutrally stable. First, Omote and Sato [33] found this stability criterion using both an energy method and a small adiabatic radial oscillation method, results which were later confirmed in [34,35]. (iii) When discussing static equilibrium configurations it is always important to discuss the Buchdahl limits, where, for instance for a perfect fluid sphere, the star cannot reach beyond $r_0 < 9/8 r_{\rm Schw}$, where r_0 is the star radius, $r_{\rm Schw}$ is the Schwarzschild radius, $r_{\rm Schw} = 2GM$ [36], and r is the Schwarzschildean radial coordinate. On the contrary, for Bonnor stars, stars made of extremal charged matter, the limits are precisely the horizon radius as was first noted by Bonnor [1–12], and then in subsequent works [13–17]; see also [37–41] for interesting discussions on the Buchdahl limits for charged stars. (iv) The hoop conjecture is relevant for these stars as was first noticed by Bonnor [9,10] (see also [11]). The conjecture states that a black hole forms when matter of mass M is compacted within a given definite hoop, in [42] taken to be $\sim 4\pi GM$. Later, it was shown that the hoop should be reduced for extremal charged matter to $\sim 2\pi GM$ [9,10]. However, it seems that systems like Bonnor stars violate it, since no black hole forms ever, only a quasi-black hole [16,17]. (v) A pertinent question, specially related to stars, is whether they can form from gravitational collapse or not. The issue of the collapse of extremal charged dust solutions has not been studied in detail; see, however, the interesting work of De [43]. (vi) Concerning the generalization of Bonnor stars to include pressure terms, and thus go beyond dust matter, there are some stimulating developments. For instance, still within the Majumdar-Papapetrou ansatz for the potential $W = a_0 - \epsilon\sqrt{G}\varphi$, systems with pressure were studied by Ida [44], where one finds, with an additional ansatz for the pressure, a Helmholtz-type equation which can be solved; in the case the pressure is zero see also [45]. These are thus extensions of Bonnor stars, stars that include charged matter, nonextremal, and pressure. Extensions to solutions with potentials different from the Majumdar-Papapetrou potential, and even different from Weyl's potential, were done in [46,47]. These solutions include pressure and have interesting structure. Charged stars with pressure were studied numerically in [48], a paper that has attracted some attention, where the limiting configuration is found to have mass equal to charge, in appropriate units, being thus a Bonnor star. In [49] a set of static charged solutions with pressure were studied and it was proposed that their gravitational collapse would lead to the formation of a charged Reissner-Nordström black hole. (vii) There are many other solutions of charged matter in various situations which have been discovered throughout the years. Many of them are interesting and would deserve a review, but there are too many to be quoted here; see [50] for a very partial list. (viii) Charged gravitating solutions have also been used to study Abraham-type models for the electron, with and without Poincaré stresses; see, e.g., [51–53] and also [3,17]. (ix) A related issue to Bonnor star solutions and quasi-black holes is the set of black holes devised by Bardeen [54], in which the interior to the horizon is nonsingular. These solutions are magnetically charged, instead of electrically charged, and have been further explored in [55]. The connection with the quasi-black holes is that there is a theorem by Borde [56] which says that if there is no singularity inside the event horizon then the regular solutions have different inside and outside topologies. Now, it is not possible to put extremal charged dust, with positive rest mass, inside an extremal black hole, *à la* Bardeen, a result first found in d-dimensional studies (see below). So physical (positive rest mass) Bonnor stars do not provide Bardeen-like solutions. On the other hand quasi-black holes are not true black holes, but have a weird topology and properties [16,17] approaching considerably the topology change of Borde. For further connections of Bonnor stars and quasi-black holes with other issues, such as no hair theorems, naked black holes, objects that mimic black holes, and the entropy issue, see [16,17].

C. Higher-dimensional analyses

1. Context

The possibility of the existence of extra dimensions arises in several theoretical schemes. In what is called a Kaluza-Klein unification model, the unification idea has emerged first as a way of unifying the gravitational and electromagnetic fields in five spacetime dimensions, and later the gravitational and Yang-Mills fields in seven spacetime dimensions. Within this idea the gravitational field in higher dimensions gives rise to the gravitational field itself and the other possible fields in four dimensions. Later, the Kaluza-Klein process was enforced in theories which start from the outset in higher dimensions, such as supergravity or string-M theory, which can have up to 11 dimensions. In the course of reducing the dimensions to four, a profusion of new fields materialize; see [57] for the original papers. These schemes require that the extra dimensions are compactified in Planck size manifolds, and so due to the lack of a properly accepted theory at these scales it is very hard to do physics on the extra dimensions. There has now appeared an idea that makes the higher dimensions large, when compared to the Planck scale, which means, if correct, it can have measurable consequences on current or near future experiments. By postulating that the gravitational field propagates also in at least extra three space dimensions, while electromagnetism and the standard model fields propagate only in our universe, the brane, it is possible to reduce the Planck scale to the electroweak scale and make the extra dimensions large, of the order of hundredths of a centimeter or a little less. The hierarchy problem, of understanding the huge differences in the gravity and electroweak scales, is now pushed into the acceptance of the large extra dimensions (see [58–60]; see also [61] for possible developments).

Now, since within this arrangement gravity is an electroweak scale phenomenon, so are black holes. Thus, for instance, by splashing electrically charged particles together black holes or other gravitational objects can be created in higher dimensions with the charge remaining in the brane. In scenarios with extra dimensions it is thus important to study charged solutions in connection to the formation of these tiny black holes because the charge and the solutions suggest that the charge may halt gravitational collapse. Solutions for charged objects in such a frame are certainly not spherically symmetric, thus not Reissner-Nordström, and at the moment have not been found. Nonetheless, it is certainly of interest to consider spherically symmetric electrically charged solutions in higher dimensions because, first, such a study can give an idea of how the existence of the charge influences the solution, and second, other charged fields, analogous in many respects to the electromagnetic field, may propagate in the higher dimensions, making Maxwell electrically charged solutions prototype solutions.

In addition, related to studies on the role played by the dimensionality of space on the laws of physics and its connection to the anthropic principle, it has been shown that there are no Newtonian solutions for stars supported by degenerate pressure in higher dimensions; i.e., a higher-dimensional self-gravitating Fermi gas either collapses into a black hole or evaporates. Indeed, interesting papers discussing degenerate stars, like white dwarfs and neutron stars, in higher dimensions have appeared [62,63]. In [62] a complete heuristic study, following the original work of Landau (see, e.g., [64]), was performed. Then in [63], the full study, following the original works of Chandrasekhar (see, e.g., [65]), was completed. The main conclusion is that there are no Newtonian solutions for degenerate stars in higher dimensions, thus no general relativistic solutions either, because the Fermi pressure energy cannot balance the gravitational energy. Of course this may not follow for other stars. Stars supported by classical gas pressure may perhaps exist in higher dimensions; no conclusive study has been presented so far. Thus, it is of interest to show whether stars, supported by electric repulsion, such as Bonnor stars, do have solutions within Newtonian gravity and general relativity. In case there are solutions, one shows by example that the existence of stars in higher dimensions depends both on the number of dimensions itself and on the underlying field content of the stars themselves.

It is thus important, for the reasons just raised, to study Bonnor stars in higher dimensions, prior to compactification of any sort.

2. Vacuum Majumdar-Papapetrou solutions

Electrically charged solutions in vacuum in d dimensions within Newton-Coulomb theory are a direct generalization from four dimensions. Within Einstein-Maxwell theory the d-dimensional solutions were found by Tangherlini [66], with a prescient discussion on the physical laws and their relationship to the three dimensionality of space. These solutions generalize the four-dimensional Reissner-Nordström solutions, and they also have the mass M and the charge Q, as the higher-dimensional parameters, such that for $\sqrt{G_d}M < \epsilon Q$ one has a charged naked singularity, for $\sqrt{G_d}M > \epsilon Q$ one has a Reissner-Nordström black hole, and for $\sqrt{G_d}M = \epsilon Q$ one has an extremal black hole, where $\epsilon = \pm 1$ depending on the sign of the charge. Here G_d is the d-dimensional Newton's gravitational constant, where in four spacetime dimensions we put $G_4 \equiv G$ (see Appendix A for more on this). If one takes Weyl and Majumdar's route into higher-dimensional Einstein-Maxwell theory, see now [67], and seeks the initial ansatz that the metric potential W, or its inverse $U = W^{-1}$, depends on the electric potential φ, i.e., $W(\varphi)$ with $g_{00} \equiv W^2$, one finds the relation $W^2 = (a_0 - \epsilon\sqrt{G_d}\varphi)^2 + b_0$, also independent of the symmetry. In the perfect square Majumdar-Papapetrou case, i.e., $W = a_0 - \epsilon\sqrt{G_d}\varphi$, one can also show that specializing to spherical symmetry, the mass M of the solution is equal in appropriate units to its charge Q, $\sqrt{G_d}M = \epsilon Q$. This makes contact with the Tangherlini black holes through the extremal solution $\sqrt{G_d}M = \epsilon Q$, although not through the other ones, since the Tangherlini family generically does not admit a functional relationship between the metric and the electric potentials. The complete understanding of a single extremal Reissner-Nordström solution can also be achieved through Carter-Penrose diagrams, and the complete understanding of the vacuum Majumdar-Papapetrou solutions, with many extremal black holes scattered around in d dimensions was performed in [68].

Moreover, it is interesting to note that if instead of working in Einstein-Maxwell theory one works in string-M theory or in supergravity theory in 11 dimensions, there are Majumdar-Papapetrou–type solutions, in the sense that the attraction due to the gravitational field is counterbalanced by the repulsion of the charged field of the theory, see, e.g., [69], as well as [70], for reviews on this topic (see also [71] for a review on black hole and other solutions of higher-dimensional vacuum general relativity and higher-dimensional supergravity theories). In 11 dimensions in string-M theory, there are two bosonic fields, the metric and the A_3 form field which is a variant of the electromagnetic field with a corresponding charge, and one fermionic field. Thus the bosonic part is as simple as Einstein-Maxwell. One finds that for the solutions to be purely bosonic one has to have that the mass of the solution has to be equal to the A_3 charge, in appropriate units. Note that this is the analogue of the extremality bound for Reissner-Nordström black holes. Solutions with mass equal to charge are called Bogomolnyi-Prasad-Sommerfield (BPS) spacetimes. The solutions are not pointlike generically. They are branelike, and are called p-branes, or black p-branes, where a zero-brane is a zero dimensional object like a black hole, a one-brane is a string like a black string, a two-brane is a membrane, and so on. Indeed, in string-M theory, where supergravity in 11 dimensions is a low energy theory, there are the M2-brane (a membrane, i.e., a two-brane electrically charged under A_3), the M5-brane (a five-brane magnetically charged under A_3), the wave solution or Aichelburg-Sexl metric, and the Kaluza-Klein monopole. All of these are BPS, the last two having momentum which is a form of charge. These solutions are best found and studied in isotropic, also called harmonic, coordinates, as is the case of Majumdar-Papapetrou solutions in general relativity. One can then have, for instance, many M2-branes scattered around, as one can have many black holes in the Majumdar-Papapetrou case, since the charged field force still balances the gravitational force. One can, in addition, combine the solutions with different charge type, for instance a M2-brane with a M5-brane, with no analogue in Majumdar-Papapetrou since here there is only one charge. Through careful dimensional reduction, Kaluza-Klein or otherwise, these solutions are also solutions of the reduced theories. Usually the branes in 11 dimensions are nonsingular and considered as solitonic objects. But when one reduces to ten dimensions, singularities in the solutions appear, in which case it is better to consider the branes as coupled to extremal dust, in the place of the singularities (see, e.g., [69,70]), making thus the consideration of extremal dust solutions in higher dimensions a subject of interest. It is also worth commenting that in string-M theory in 11 dimensions one can also perform some brane engineering, by adding together solutions of the same type of charge. It is common practice to stack an array of M2 electrically charged branes, for instance, and then take the continuum limit, or smear, the array in the correct direction, yielding a new brane with a new dimension; see, e.g., [69,70]. Of course one can also do the same type of manipulation in Majumdar-Papapetrou general relativity. Draw an array of equally sparse extremal black holes on a line, smear them together correctly, and obtain a one dimensional extremal black string obeying the d-dimensional Majumdar-Papapetrou equations.

3. Beyond vacuum: Dust Majumdar-Papapetrou solutions

In d dimensions, as in four, things become more interesting if one goes beyond vacuum and puts an electrically charged fluid or matter into the Newtonian gravity or general relativistic systems of equations. Leaning on the general relativistic analysis of Majumdar [25], Lemos and Zanchin [67] showed, for the special relation, or ansatz in this context, on the metric inspired from the vacuum case, i.e., $W = a_0 - \epsilon\sqrt{G_d}\varphi$, that the whole system reduces to a single equation, a Poisson type equation, in which the mass density $\rho_{\rm m}$ is equal to the charge density $\rho_{\rm e}$ in appropriate units, $\sqrt{G_d}\rho_{\rm m} = \epsilon\rho_e$. Thus, a basic feature of such a system is that, although being a system containing charged matter, it is described by the metric function W, the redshift function. The electric potential φ can then be found implicitly through the Majumdar-Papapetrou relation. It is also possible to generalize to d dimensions the theorem, proved in four dimensions in general relativity in [30], that, provided the pressure is zero and there are no singularities in the distribution, the Majumdar-Papapetrou ansatz $W = a_0 - \epsilon\sqrt{G_d}\varphi$ and condition $\sqrt{G_d}\rho_{\rm m} = \epsilon\rho_{\rm e}$ follow [72]. One can then show that the d-dimensional Newtonian limit follows, with the four-dimensional situation studied in [8] being a particular case. Also theorems with nonzero pressure [47] can be render into d dimensions [72], namely, that for perfect fluid solutions satisfying the Majumdar-Papapetrou condition the pressure is related to redshift function, as in the four-dimensional case [47].

4. Bonnor stars: Junction of dust with vacuum Majumdar-Papapetrou solutions

In [67] it was proved that if the pressure is functionally related to the redshift function, which in turn obeys the Majumdar-Papapetrou relation for the potentials, then to have a surface with zero pressure, i.e., a star, one has to have the pressure equal to zero everywhere. This in turn means the star is a Bonnor star, with a d-dimensional Majumdar-Papapetrou interior and a d-dimensional extremal Reissner-Nordström exterior. This result is valid within both Newtonian gravity and general relativity. The aim of this paper is to discuss Bonnor star solutions in the spherically symmetric case. We show that spherically symmetric Bonnor stars in d dimensions have a number of interesting properties. In Newtonian theory their mass and radius may be arbitrary and the object with the highest compression is a point electric mass, i.e., a Newtonian singularity. In general relativity the stars can yield very large redshifts and their exteriors can be made arbitrarily near to the exterior of an extremal charged black hole. Even in these extremal situations, many of their characteristics remain finite and nontrivial. These extremal kinds of d-dimensional systems are the quasi-black holes, possessing quasihorizons, already mentioned.

5. Further connections

As in four dimensions, in d dimensions one can try to associate Bonnor stars to several related topics. (i) Bonnor stars, or something related, in higher dimensions are of interest in situations prior to compactification. Since astrophysically the world is already compactified to four spacetime dimensions, the main interest in these solutions is for high energy physics, for instance in a large extra dimension scenario, where charged configurations in higher dimensions can be of interest. It would be of interest to know whether d-dimensional Bonnor stars, for generic d, are supersymmetric solutions when embedded in some supergravity theory. (ii) Of course, the study of the stability of these higher-dimensional stars is important, although we do not do it here. (iii) Buchdahl limits in higher dimensions have not been found for either uncharged or for charged stars. We are preparing such a study. (iv) As far as we know, there is no discussion of the hoop conjecture for objects in d dimensions. (v) In d dimensions it is also important to understand if the configurations under study can form from gravitational collapse. Collapsing and static charged shells in d dimensions within Einstein-Maxwell theory with an electrically charged fluid have been analyzed in [73]. Static shells, with vanishing pressure, in this context are Majumdar-Papapetrou solutions. Collapsing charged shells in Lovelock theory have been studied in [74]. (vi) One can also use the Majumdar-Papapetrou relation for the potential $W = a_0 - \epsilon\sqrt{G_d}\varphi$, and study systems with pressure in much the same way as Ida [44]. We will not discuss this type of solutions in d dimensions. (vii) There are a few other, non-Majumdar-Papapetrou–type, solutions of charged matter, see, e.g., the interesting ones discussed in [75,76], where charged spheres with specific distributions of matter, charge, and pressure were found. (viii) Electron models in the molds of Abraham and Poincaré have not been studied in d dimensions. (ix) It would be interesting to study Bardeen models and Borde's theorem in d dimensions. An interesting result first derived in [73] is that for a shell in d dimensions with positive proper mass there is no static solution inside the event horizon, the result being valid in four dimensions as mentioned above, as well as in $d > 4$. This in some sense connects with Borde's theorem [56].

D. Layout

We start by analyzing, in Sec. II, the Newtonian theory for charged fluids in higher dimensions, looking for static solutions. We verify in Sec. II A that if the condition $\sqrt{G_d}\rho_{\rm m} = \epsilon\rho_{\rm e}$, with $\epsilon = \pm 1$, is to be satisfied, then there are equilibrium Bonnor star solutions in $d = (n + 1)$-dimensional spherically symmetric spacetimes, where n is the dimension of the space; see Sec. II B. Bonnor stars of Majumdar-Papapetrou general relativity are studied in Sec. III. In Sec. III A we write the basic equations and particularize them for spherical symmetry. Part of the section is devoted to review the main properties of d-dimensional spherically symmetric solutions. A solution of a d-dimensional Bonnor star is also analyzed in Sec. III B in some detail. Its generic properties are shown, as well as its quasi-black hole and its quasi-Newtonian limits. In Sec. IV we present final comments and conclusions.

II. NEWTON-COULOMB THEORY WITH AN ELECTRICALLY CHARGED FLUID IN d-DIMENSIONAL SPACETIMES ($d = n + 1$, n BEING THE NUMBER OF SPACE DIMENSIONS): WEYL AND MAJUMDAR-PAPAPETROU ANALYSIS AND BONNOR STAR SOLUTIONS

In d-dimensional Newtonian gravity coupled to both Coulomb electricity and a charged fluid matter, one can find solutions representing charged stars, where here d is the number of spacetime dimensions, with $d = n + 1$, n being the number of space dimensions. The dynamics of such a kind of system is governed by the Euler equation, where the gravitational and electric force fields are determined conjointly by Newtonian gravity and Coulomb electricity. The fluid can be in static equilibrium even with zero pressure and stresses, because the electric repulsion counterbalances the gravitational pull if the charge density of the fluid, ρ_e, equals its mass density ρ_m in appropriate units, i.e., $\sqrt{G_d}\rho_m = \epsilon\rho_e$, where G_d is Newton's gravitational constant in d dimensions (see Appendix A), and $\epsilon = \pm 1$. This condition makes it possible to build a distribution of charged dust with any shape in neutral equilibrium. Charged fluids with $\sqrt{G_d}\rho_m = \epsilon\rho_e$ are called extremal charge dust fluids. By introducing a convenient boundary one turns the solutions into stars. In this section we study some properties of these objects. One can also put some form of pressure, either positive or negative, into these systems and find solutions which of course do not obey the extremal condition. Solutions with pressure will not be considered here.

A. Gravitating Newtonian charged dust fluid and Weyl and Majumdar-Papapetrou–type analysis

1. Gravitating Newtonian charged dust fluid

We consider first the dynamics of a gravitating Newtonian charged fluid in a $n = (d - 1)$-dimensional Euclidean space according to the Euler description. A dust fluid is completely specified by its velocity vector, with components v_i, with $i = 1, \ldots, d - 1$ (Latin indices run through 1 to $n = d - 1$), and its matter density ρ_m, all being functions of the position vector represented by spatial coordinates r_i, and of the universal time t. Thus, $v_i = v_i(r_j, t)$ and $\rho_m = \rho_m(r_j, t)$. The basic equations governing the flow of a Newtonian fluid are the continuity equation, which expresses mass conservation, and the Euler equation, which expresses momentum conservation. Consider a fluid element with mass $dm = \rho_m d\mathcal{V}$, in the $(d - 1)$-dimensional space, $d\mathcal{V}$ being the $(d - 1)$-dimensional space volume element. Then, the continuity and the Euler equations may be written as

$$\frac{\partial \rho_m}{\partial t} + \nabla_i(\rho_m v^i) = 0, \tag{1}$$

$$\rho_m \frac{dv_i}{dt} = F_i, \tag{2}$$

respectively, where $d/dt \equiv \partial/\partial t + v^i\nabla_i$ is the convective temporal derivative, ∇_i is the $(d - 1)$-dimensional gradient operator, F_i is the volumetric external force acting upon the fluid element, and the sum convention on indices is adopted. The Newtonian systems we are interested in here are gravitating charged fluids distributions in static equilibrium. The fluid is then allowed to have some net electric charge, so that the charge of a fluid element is $dq = \rho_e d\mathcal{V}$, ρ_e being the electric charge density of the fluid. Thus, there are two independent forces acting on a fluid element, the gravitational and electrostatic forces. Both these forces may be derived from scalar potentials, V and ϕ, respectively, such that one has

$$F_i = -\rho_m \nabla_i V - \rho_e \nabla_i \phi. \tag{3}$$

The gravitational potential V is related to the mass density ρ_m by

$$\nabla^2 V = S_{d-2} G_d \rho_m, \tag{4}$$

while the electric potential ϕ is related to the charge density ρ_e by

$$\nabla^2 \phi = -S_{d-2}\rho_e, \tag{5}$$

where the operator ∇^2 is the Laplace operator in $d - 1$ space dimensions, S_{d-2} is the area of the unit sphere in a $(d - 1)$-dimensional space given by $S_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d - 1)/2)$, Γ is the usual gamma function, and G_d is Newton's gravitational constant in $d = n + 1$ dimensions (see Appendix A for the definition of G_d). S_{d-2} reduces to 4π in four spacetime dimensions and Eqs. (4) and (5) are the natural generalizations of the corresponding three-dimensional Poisson equations for the potentials V and ϕ to $(d - 1)$-dimensional space.

We will consider only static systems, so all quantities are functions of the $d - 1$ space coordinates only, and the fluid's velocity can be put equal to zero, $v_i = 0$. Then the Euler equation (2) for the charged fluid in static equilibrium reads

$$\rho_m \nabla_i V + \rho_e \nabla_i \phi = 0. \tag{6}$$

Equations (4)–(6) are the important equations for the problem. Equations (4) and (5) are the field equations that determine the gravitational and the electric potentials once the mass and charge densities are given, while Eq. (6) is the equilibrium equation for the system.

2. Weyl and Majumdar-Papapetrou–type analysis

In vacuum, doing for Newtonian gravity what Weyl did for general relativity [24], assume now an ansatz, i.e., a functional relation, between the gravitational and the electric potential,

$$V = V(\phi). \tag{7}$$

Equation (7) is the Weyl ansatz which implies that V and ϕ have the same equipotential surfaces. With this ansatz, Weyl originally worked out the Einstein-Maxwell vacuum equations that would follow and found that the relativistic potential is a quadratic function of the electric potential. Doing the same here in Newtonian gravity, we find that the ansatz (7), in vacuum, $\rho_{\rm m} = 0$ and $\rho_{\rm e} = 0$, when put into Eqs. (4) and (5), gives the following equation, $(V')^2\nabla^2\phi + V'V''(\nabla_i\phi)^2 = 0$, where the prime stands for the derivative with respect to ϕ. Thus, since $\nabla^2\phi = 0$ in vacuum, and $(\nabla_i\phi)^2 \neq 0$, it follows that $V'' = 0$, i.e., $V(\phi) = a_0 + \text{const} \times \phi$, where a_0 is an arbitrary constant, that without loss of generality can be put to zero. In addition, with our choice of units one has that const $= -\epsilon\sqrt{G_d}$. Thus,

$$V(\phi) = a_0 - \epsilon\sqrt{G_d}\phi. \tag{8}$$

This is the Weyl relation for the Newton-Coulomb theory in vacuum. Following Majumdar's [25] and Papapetrou's [26] lead in general relativity, it is interesting to investigate the consequences of the linear relation between electric and Newtonian potentials, $V = a_0 + \text{const} \times \phi$, and see what happens in the presence of matter. Such a relation is a particular case of Weyl's general ansatz (7), and is the same as in Eq. (8), i.e., it is the same as Weyl's relation for vacuum. It is remarkable that the Majumdar-Papapetrou relation is equivalent to Weyl's relation in Newton-Coulomb theory in vacuum, while it is not so in general relativity. We use this relation (8) to also treat solutions with matter, as has been done in general relativity [67].

In matter, we work out the basic equations using the general form (8) and so generalize to higher dimensions the analysis in four dimensions done by Bonnor [8]. We show for $d - 1$ space dimensions the interesting result that the equality of mass and charge densities follows from the field equations as long as there are no singularities within the charged matter distribution (see Bonnor [8] for Newtonian systems, and De and Raychaudhuri [30] for general relativistic systems in four dimensions). The basic equations (4)–(6) can be rewritten by taking the Weyl ansatz (7) into account. To begin with, it is convenient to consider first Eq. (6), which now reads $(\rho_{\rm m}V' + \rho_{\rm e})\nabla_i\phi = 0$. So, the two fields V and ϕ have the same equipotential surfaces. Since we consider $\nabla_i\phi \neq 0$, Eq. (6) is then equivalent to $\rho_{\rm m}V' + \rho_{\rm e} = 0$, where again the prime stands for the derivative with respect to ϕ. By substituting $\rho_{\rm m}$ from the previous equation into Eq. (4), one finds $(V')^2\nabla^2\phi + V'V''(\nabla_i\phi)^2 = -S_{d-2}\rho_{\rm e}$, where we made use of the assumption $V = V(\phi)$. Then, with the help of Eq. (5) one finds $\nabla_i(\sqrt{Z}\nabla^i\phi) = 0$, where Z is defined as $Z \equiv G_d - V'^2$. Now, in order to have a nonsingular solution with closed boundary it is required that $Z = 0$, or equivalently, $(V')^2 = G_d$. All equilibrium solutions with $(V')^2 \neq G_d$ with a closed equipotential hypersurface S have a singularity within S. The most physically interesting solutions are then those for which $(V')^2 = G_d$. Therefore, the Majumdar-Papapetrou relation for the Newton-Coulomb theory with a charged dust fluid is $V(\phi) = a_0 - \epsilon\sqrt{G_d}\phi$, where $\epsilon \equiv \pm 1$, implying, after Eq. (8), that the const appearing before the potential ϕ is indeed $-\epsilon\sqrt{G_d}$. Thus, for the Weyl relation or Majumdar-Papapetrou relation (they are the same here), Eq. (8), with the equations $\rho_{\rm m}V' + \rho_{\rm e} = 0$ and $V(\phi) = a_0 - \epsilon\sqrt{G_d}\phi$ derived above, gives $\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}$. This last equation is the Majumdar-Papapetrou condition in Newtonian gravity. Observe that the relation between the potentials we call Majumdar-Papapetrou relation and the relation between the densities we call Majumdar-Papapetrou condition. In the case the Majumdar-Papapetrou condition holds, distributions of charged dust of any shape can be put in equilibrium. All the quantities can now be given. Once the mass density $\rho_{\rm m}$ is given, the gravitational potential is determined by the Poisson equation (9) and all the other quantities, including the electrogravitational Newtonian spacetime structure, and possible singularity structure, follow from V and $\rho_{\rm m}$. The resulting system of equations can be put in the form

$$\nabla^2 V = S_{d-2}G_d\rho_{\rm m}, \tag{9}$$

$$\phi = -\frac{\epsilon}{\sqrt{G_d}}V, \tag{10}$$

$$\rho_{\rm e} = \epsilon\sqrt{G_d}\rho_{\rm m}, \tag{11}$$

where the zero points of the potentials are suitably chosen.

B. Spherical d spacetime (n space) dimensional Newtonian Bonnor star solutions

1. Equations in spherical coordinates

We now assume the mass distribution is spherically symmetric, in which case all the dynamical variables and fields depend only on the radial coordinate r in $(d-1)$-dimensional space. Our interest here is in spherical solutions to Eqs. (9)–(11). First we define the mass $m(r)$ and the electric charge $q(r)$ inside a sphere of radius r, respectively, as

$$m(r) = S_{d-2}\int_0^r \rho_{\rm m}(r)r^{d-2}dr, \tag{12}$$

$$q(r) = S_{d-2} \int_0^r \rho_e(r) r^{d-2} dr. \tag{13}$$

Equations (9)–(11) are then conveniently written explicitly in terms of the radial coordinate as

$$\frac{dV(r)}{dr} = G_d \frac{m(r)}{r^{d-2}}, \tag{14}$$

$$\frac{d\phi(r)}{dr} = -\frac{q(r)}{r^{d-2}}, \quad \text{or} \quad \phi(r) = -\frac{\epsilon}{\sqrt{G_d}} V(r), \tag{15}$$

$$q(r) = \epsilon \sqrt{G_d} m(r), \tag{16}$$

where the zero points of the potentials were suitably chosen. In Eq. (14) there is also a term C_0/r^{d-2} which we have put to zero, without loss of generality, i.e., $C_0 = 0$. This term can be included in the term $G_d m(r)/r^{d-2}$ by an appropriate choice of the function $m(r)$.

2. Solutions

(a) Electrovacuum solutions in $d = n + 1$ spacetime dimensions

The solution to Eq. (14) in vacuum is

$$V = -\frac{1}{d-3} \frac{G_d M}{r^{d-3}}, \tag{17}$$

$$M = \text{const}, \tag{18}$$

with M representing the Newtonian mass of the source. To complete the solution one must give the electric potential ϕ, which is obtained from Eq. (15), $\phi = (d-3)^{-1} Q/r^{d-3}$, with Q being the total charge of the source, which in turn satisfies Eq. (16), $Q = \epsilon\sqrt{G_d}M$. These two equations, together with Eqs. (17) and (18), form the set of equations corresponding to the solution of a Newtonian Majumdar-Papapetrou vacuum system in $n = d - 1$ space dimensions. Such a set of solutions also follows from the Poisson equations in which the mass and charge densities are Dirac delta functions, $\rho_m(r) = M\delta(r)$ and $\rho_e(r) = Q\delta(r)$, and $Q = \epsilon\sqrt{G_d}M$.

(b) Newtonian Bonnor star solutions in $d = n + 1$ spacetime dimensions

Now we find a class of solutions to the Newton-Coulomb system with electrically charged fluid matter, considering the Majumdar-Papapetrou relation (8) (which in the Newtonian case is also Weyl's relation) and a spherically symmetric distribution of matter. Upon joining this class of solutions to an external vacuum we obtain Bonnor stars in the Newtonian theory. The relevant equations are the ones presented in system (12)–(16).

Let us call r_0 the radius of the star. Physical conditions require the mass density to be a continuous function with a finite value at the center of the star. One can choose a mass density function $\rho(r)$ and the remaining functions are then obtained by integrating the appropriate equations. For instance, one can give $\rho_0(r/r_0)^\alpha$, for $0 \le r \le r_0$, and make it zero in all the exterior region for $r > r_0$. Integration of the Poisson equation (14) gives a power law function for the potential, $a_1 r^{\alpha+2} + a_2/r^{d-2} + a_3$, where the constant a_2 is made equal to zero in order to avoid a singularity at $r = 0$, and the constant a_3 is fixed by the matching conditions at $r = r_0$. Alternatively, instead of giving $\rho_m(r)$, one can choose a potential $V(r)$ satisfying reasonable boundary conditions, and then obtain the other functions from it. This is the simplest route, the one we follow here. We can choose the following interesting potential, for the interior $V_i(r)$, given by $V_i(r) = c_0 + c_1(r/r_0)^\alpha + c_2(r/r_0)^\beta$, for $r \le r_0$, where α and β are arbitrary constant parameters, possibly satisfying some restrictions. The other constants, c_0, c_1, and c_2, are fixed by imposing appropriate matching conditions at the surface of the star, $r = r_0$. One can impose that the potentials are C^1 functions at r_0, which is usually done in order to simplify the calculations. This means continuity of the potential and continuity of the gravitational field strength. Then, in this case, the density has a finite discontinuity at the boundary, falling from some finite value just inside matter to zero just outside. Also through the junction conditions above, one can find the constants c_0, c_1, and c_2, with one of them arbitrary, c_1 say. Here we want to go a step further and impose that the potentials are C^2 functions at r_0, i.e., continuity of the potential, continuity of its first derivative, and continuity of its second derivative. Continuity of the first derivative of the potential means that the gravitational field strength is continuous, and continuity of the second derivative means that the mass density at the surface of the star is continuous with zero value. Continuity of the potential gives $V_i(r_0) = V_e(r_0) = -(d-3)^{-1} G_d M/r_0^{d-3}$, where $V_e(r) = -(d-3)^{-1} G_d M/r^{d-3}$ is the Newtonian potential in the exterior region, $r \ge r_0$, with V_e zero at infinity, and M is the total mass of the star. Continuity of the gravitational field strength gives $V_i'(r_0) = V_e'(r_0) = G_d M/r^{d-2}$. Continuity of the mass density at the surface of the star gives $\rho_m(r_0) = 0$. With these choices, the spherical Newtonian star is described by the following potential

$$V = \begin{cases} V_i = -\frac{G_d}{d-3} \frac{M}{r_0^{d-3}} \left(1 + \frac{(d-3)(\beta+d-3)}{\alpha(\beta-\alpha)} [1 - (\frac{r}{r_0})^\alpha] - \frac{(d-3)(\alpha+d-3)}{\beta(\beta-\alpha)} [1 - (\frac{r}{r_0})^\beta]\right) & r \le r_0 \\ V_e = -\frac{G_d}{d-3} \frac{M}{r^{d-3}} & r > r_0, \end{cases} \tag{19}$$

and by the following mass density

$$\rho_{\mathrm{m}} = \begin{cases} \frac{(\alpha+d-3)(\beta+d-3)}{(d-3)(\beta-\alpha)} \frac{M}{S_{d-2} r_0^{d-1}} \left[\left(\frac{r}{r_0}\right)^{\alpha-2} - \left(\frac{r}{r_0}\right)^{\beta-2} \right] & r \leq r_0 \\ 0 & r > r_0. \end{cases} \tag{20}$$

In these equations M is the mass and r_0 is the radius of the star, with M being obtained from Eq. (12) with $r = r_0$, i.e., $M = m(r_0)$, and α and β are arbitrary constant parameters satisfying the restrictions $\alpha \geq 2$ and $\beta \geq 2$. In addition, the parameters α and β must be different from each other, $\alpha \neq \beta$, in order that the mass density be finite at $r = 0$, and the other functions that follow from it be also finite there. Such conditions ensure also the positivity of the mass density. Note that the quantity $M/(S_{d-2} r_0^{d-1})$ appears naturally, indeed in Newtonian theory one can define the mean density of the matter by $\bar{\rho}_{\mathrm{m}} = (d-1)M/(S_{d-2} r_0^{d-1})$. The other quantities, ϕ and ρ_{e}, are obtained by substituting the expression for the gravitational potential and for the mass density given in Eqs. (19) and (20) into Eqs. (10) and (11), respectively.

So the class of Bonnor stars is defined essentially by Eqs. (19) and (20), through the parameters G_d, M, r_0, d, α, and β. In the analysis of these Bonnor stars, an important parameter appears, the d-dimensional generalization of the mass to radius ratio of the star,

$$a = \frac{G_d}{d-3} \frac{M}{r_0^{d-3}}. \tag{21}$$

It measures how compact the star is, and is a free parameter in the model. Taking M as a fixed parameter, different stars are parametrized by different values of a, which means different values of the radius r_0. There are no constraints on a for Newtonian stars, it can vary from 0, a highly dispersed star, to ∞, a point mass, i.e., the limiting configuration here is a Newtonian singularity at $r = 0$ obeying the Majumdar-Papapetrou condition $Q = M$. As we will see, in the relativistic case a cannot be larger than unity (see also [13–15]).

The relevant functions $V(r) + 1$, $\rho_{\mathrm{m}}(r)$, $\phi(r)$, and $\rho_{\mathrm{e}}(r)$, given in terms of the coordinate r follow from the above relations. They are dependent on the variable r, and also depend on two other arbitrary parameters, the mass and the radius of the star, M and r_0, respectively. Instead of writing the explicit form of such functions, it is more convenient to plot them for several choices of parameters. In the calculations we normalized the coordinate r to the mass parameter $\mu = (G_d M/(d-3))^{1/(d-3)}$ which was kept fixed. In fact, the important parameter to this end is the mass to radius ratio a, given by Eq. (21). The function $V(r) + 1$: The behavior of the rescaled potential $V(r) + 1$ as a function of the rescaled coordinate r/μ, for four different values of a ($a = 0.1$, $a = 0.4$, $a = 0.7$, and $a = 1$), and in four different spacetime dimensions ($d = 4, 5, 6, 7$) is shown in Fig. 1. We plot the rescaled function $V(r) + 1$ instead of simply $V(r)$ for direct comparison with the relativistic case studied later. Now, the parameters α and β in the solution (19) and (20) are free parameters. We have chosen them so that $\beta = 3\alpha/2 = 3(d-3)$, which is a convenient choice when one studies the counterparts of these solutions in general relativity. With this choice, the form of the curves depends on the number of spacetime dimensions d and on the parameter a alone. Note that all the interior functions $V_{\mathrm{i}}(r, a)$ match the exterior solution $V_{\mathrm{e}}(r) + 1 = 1 - G_d M/((d-3) r^{d-3})$, each one at a different value of r_0. The reason for that is because the change of

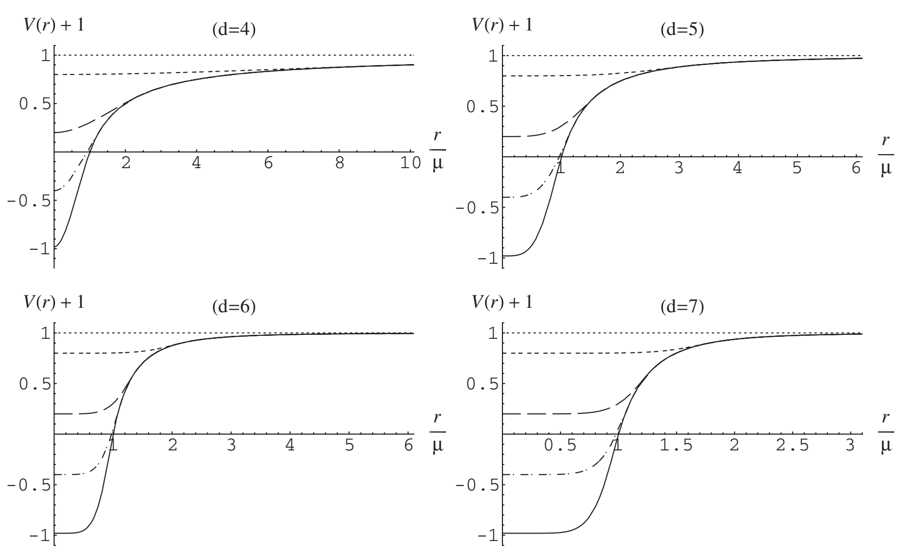

FIG. 1. The rescaled Newtonian potential $V(r) + 1$ as a function of r/μ, where $\mu \equiv (G_d M/(d-3))^{1/(d-3)}$, for four spacetime dimensions, $d = n + 1$, $d = 4$ (top left panel), $d = 5$ (top right panel), $d = 6$ (bottom left panel), and $d = 7$ (bottom right panel), and for four different values of the parameter a. The lowest, solid, curve is for $a = 1$, the dot-dashed line is for $a = 0.7$, the dashed line is for $a = 0.4$, and the dotted line for $a = 0.1$.

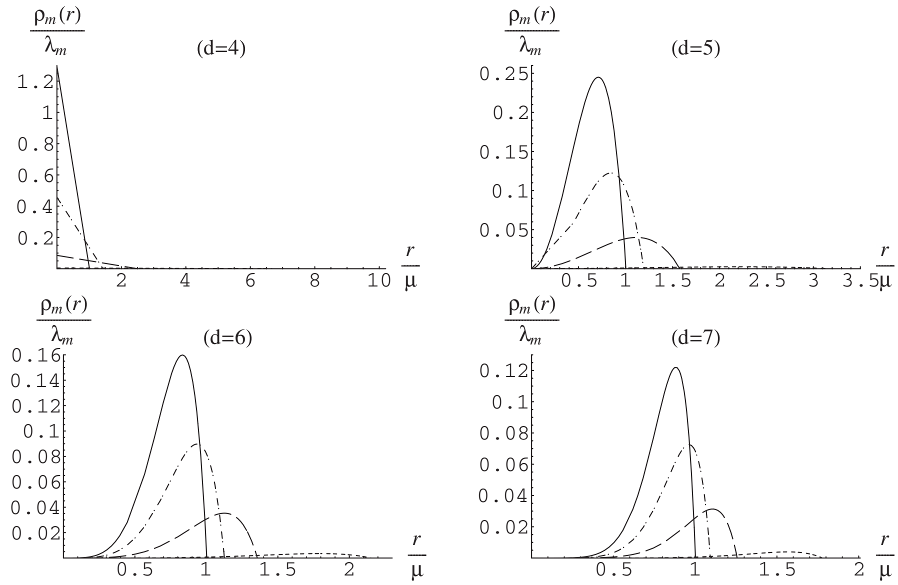

FIG. 2. The normalized Newtonian mass density $\rho_{\rm m}(r)/\lambda_{\rm m}$ as a function of r/μ, where $\lambda_{\rm m} = \frac{12}{d-1}\bar{\rho}_{\rm m}$, $\bar{\rho}_{\rm m}$ being the average density (see text), and $\mu \equiv (G_d M/(d-3))^{1/(d-3)}$, for four different spacetime dimensions ($d = n + 1 = 4, 5, 6, 7$, as indicated) and for four different values of the parameter a. The solid line is for $a = 1$, the dot-dashed curve is for $a = 0.7$, the dashed line is for $a = 0.4$, and the (lowest) dotted line is for $a = 0.1$. The normalized Newtonian mass density $\rho_{\rm m}(r)/\lambda_{\rm m}$ goes to zero at the surface of the star, defining thus the radius r_0 in each plotted case.

a is made by keeping the mass of the star fixed, while r_0 varies accordingly. The function $\rho_{\rm m}(r)$: Another quantity of interest is the mass density $\rho_{\rm m}(r)$. In Fig. 2 we plot $\rho_{\rm m}(r)/\lambda_{\rm m}$ as a function of the normalized radial coordinate r/μ. The density $\lambda_{\rm m}$ is defined as $\lambda_{\rm m} = \frac{(\alpha+d-3)(\beta+d-3)}{(d-1)(d-3)(\beta-\alpha)}\bar{\rho}_{\rm m}$, where the mean density $\bar{\rho}_{\rm m}$ is given by $\bar{\rho}_{\rm m} = (d-1)M/(S_{d-2}r_0^{d-1})$. For our choice of parameters, $\beta = 3\alpha/2 = 3(d-3)$, one has $\lambda_{\rm m} = \frac{12}{d-1}\bar{\rho}_{\rm m}$. It is seen that $\rho_{\rm m}(r)$ is finite at $r = 0$. In fact, with our choice, $\rho_{\rm m}$ vanishes at $r = 0$ for all $d > 4$. In addition it goes to zero at the surface of the star, defining thus the radius r_0 in each plotted case. The behavior of the potential $\phi(r)$ is simply given by $\phi(r) = -(\epsilon/\sqrt{G_d})V(r)$, and it is not plotted. The behavior of the charge density is $\rho_{\rm e}(r) = \epsilon\sqrt{G_d}\rho_{\rm m}$, and it is not plotted. Note that the potentials V and ϕ are C^2 functions of r, so that the corresponding field strengths are continuous (C^1 functions, in fact) through the surface of the star. The mass and charge densities, $\rho_{\rm m}$ and $\rho_{\rm e}$, are C^0 functions vanishing at $r = r_0$. When $r_0 \longrightarrow 0$ one obtains a point charge with a central Newtonian, mild, singularity. It is mild because it is not a nasty spacetime singularity, it is a matter singularity only.

III. EINSTEIN-MAXWELL THEORY WITH AN ELECTRICALLY CHARGED FLUID IN d SPACETIME DIMENSIONS ($d = n + 1$): WEYL AND MAJUMDAR-PAPAPETROU ANALYSIS AND BONNOR STAR SOLUTIONS

In d-dimensional general relativity coupled to both Maxwell electromagnetism and a charged fluid matter one can also find solutions representing charged stars. The fluid can be in static relativistic equilibrium if it is made of extremal matter, where the electric repulsion from the charge density of the fluid, $\rho_{\rm e}$, counterbalances the gravitational pull from its mass density, $\rho_{\rm m}$, in appropriate units, i.e., $\sqrt{G_d}\rho_{\rm m} = \epsilon\rho_{\rm e}$. Thus, relativistic Bonnor stars in d dimensions can also be constructed. Within general relativity the behavior and properties of these solutions are much richer, allowing the possibility of quasi-black-hole behavior, for a sufficient compact object, rather than the pointlike dull singularity of Newtonian objects. In this section we study some properties of relativistic charged fluids in the context of a Majumdar-Papapetrou analysis and the corresponding Bonnor stars.

A. Gravitating relativistic charged dust fluid and Weyl and Majumdar-Papapetrou analysis

1. Relativistic gravitating charged dust fluid

With the aim of finding exact solutions for d-dimensional Bonnor stars we first write the basic equations for the Majumdar-Papapetrou systems and analyze their general properties in brief. In the following sections we particularize for spherically symmetric spacetimes, show a particular solution, and study it in some detail.

The general relativistic analog of the Newtonian charged fluid discussed in the preceding section was considered in [67]. Such a relativistic system is described by the d-dimensional Einstein-Maxwell with an electrically charged fluid system of equations which read (we use units such that $c = 1$),

$$G\mu\nu = \frac{d-2}{d-3} S_{d-2} G_d (E_{\mu\nu} + T_{\mu\nu}), \tag{22}$$

$$\nabla_\nu F^{\mu\nu} = S_{d-2} J^\mu, \tag{23}$$

with $G_{\mu\nu}$ being the Einstein tensor, such that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, $R_{\mu\nu}$ being the Ricci tensor and R the Ricci scalar. The right-hand side of Eq. (22) bears a universal constant G_d, which in four dimensions corresponds to Newton's gravitational constant (see Appendix A for the definition of G_d). $S_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$, where Γ is the usual gamma function, and the whole factor $(d-2) G_d S_{d-2}/(d-3)$ corresponds to the $8\pi G$ term in four dimensions. The electromagnetic energy-momentum tensor, $E_{\mu\nu}$, is given by

$$E_{\mu\nu} = \frac{1}{S_{d-2}} \left(F^\rho_\mu F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \tag{24}$$

where $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$, A_μ is the electromagnetic gauge field, with ∇_μ being the covariant derivative. J_μ, in Eq. (23), is the current density

$$J_\mu = \rho_e u_\mu, \tag{25}$$

where ρ_e is the charge density and u_μ is the velocity of the fluid in the d-dimensional spacetime with $g_{\mu\nu} u^\mu u^\nu = -1$. Finally, $T_{\mu\nu}$ is the matter energy-momentum tensor for dust given by

$$T_{\mu\nu} = \rho_m u_\mu u_\nu, \tag{26}$$

with ρ_m being the energy density of the fluid. In all the above definitions, Greek indices μ, ν, etc., run from 0 to $d-1$, where 0 represents the time, and the other $d-1$ coordinates are spacelike.

It is assumed the spacetime is static, in which case the metric can be written in the form

$$ds^2 = -W^2 dt^2 + \frac{1}{W^{2/(d-3)}} h_{ij} dx^i dx^j, \tag{27}$$

where Latin indices run from 1 to $d-1$, h_{ij} is the metric in $(d-1)$-dimensional space, and W is a function of the spacelike coordinates x^i only. The four-velocity and the gauge field are then given, respectively, by

$$u_\mu = W \delta^0_\mu, \tag{28}$$

and

$$A_\mu = -\varphi \delta^0_\mu, \tag{29}$$

where the electric potential φ is also a function of the space coordinates alone. (Note that in the definition of A_μ we have put a minus sign in front of φ. Although not the usual choice, this is the useful choice to compare with the Newtonian case.)

From Einstein-Maxwell with charged dust fluid equations one obtains the following equations for W and φ:

$$\nabla^2 W - \frac{1}{W} (\nabla_i W)^2 = \frac{G_d}{W} (\nabla_i \varphi)^2 + S_{d-2} G_d W^{(d-5)/(\mathrm{d}-3)} \rho_m, \tag{30}$$

$$\nabla^2 \varphi = 2 \frac{\nabla_i W}{W} \nabla^i \varphi - S_{d-2} W^{(d-5)/(\mathrm{d}-3)} \rho_e, \tag{31}$$

where ∇_i stands for the covariant derivative with respect to the space metric h_{ij}. Making now the connection to the Newton-Coulomb theory with a charged dust fluid, one may say that Einstein-Maxwell with charged dust fluid equations, Eqs. (30) and (31), correspond to the Poisson equations for the gravitational and electric potentials, Eqs. (4) and (5), respectively. Moreover, continuity and Euler equations (1) and (2) are, in a certain sense, analogous to the relativistic conservation equations $\nabla_\mu E^{\mu\nu} + \nabla_\mu T^{\mu\nu} = 0$, which in turn also follow from the general relativity equations. In the present case one has

$$\rho_m \nabla_i W + \rho_e \nabla_i \varphi = 0, \tag{32}$$

for the conservation equation, which has the same form as Eq. (6).

2. Weyl and Majumdar-Papapetrou analysis

In vacuum, the generalization of the Majumdar-Papapetrou system to d-dimensional spacetimes was done in [67] and, for completeness, we summarize the main properties of such systems here. Following the lines of that work but changing the strategy in order to compare the present analysis to the Newton-Coulomb case of previous sections, we assume there is a Weyl-type functional relation between the metric potential W and the relativistic electric potential φ

$$W = W(\varphi). \tag{33}$$

This is the relativistic Weyl's ansatz. For the sake of comparison to the Newtonian case, let us review here the main consequences of the last equation. For the vacuum case, $\rho_m = \rho_e = 0$. So, Eqs. (30) and (31) can be combined to yield $(\nabla_i \varphi)^2 (W W'' + W'^2 - W G_d) = 0$. Since $(\nabla_i \varphi)^2 \neq 0$, this equation implies $W W'' + W'^2 - W G_d = 0$, which integrates to $W^2 = (a_0 - \epsilon \sqrt{G_d} \varphi)^2 + b_0$, where a_0 and b_0 are integration constants. This form of the metric potential W is known as the Weyl potential or, in our context, the Weyl relation. Moreover, in the particular case where $b_0 = 0$, W^2 assumes the form of a perfect square so that

$$W = a_0 - \epsilon \sqrt{G_d} \varphi, \tag{34}$$

where $\epsilon = \pm 1$, and without loss of generality we kept the plus sign when taking the square root of W^2. In general relativity, this form of W is known as the Majumdar-

Papapetrou potential, and one usually refers to Eq. (34) as the Majumdar-Papapetrou relation.

In matter, we now render into d dimensions De and Raychaudhuri's theorem [30] (see [72] for the generalization of it for systems with pressure). To begin with, one substitutes Eq. (33) into the conservation equation (32) and finds $(\rho_m W' + \rho_e)\nabla_i \varphi = 0$ which, for $\nabla_i \varphi \neq 0$, is then equivalent to

$$\rho_m W' + \rho_e = 0, \tag{35}$$

where the prime denotes derivative with respect to φ. This is the general relativistic analog to the equilibrium equation of Newtonian theory, cf. the equation $\rho_m V' + \rho_e = 0$ derived in Sec. II A 2. Using Eq. (35), it is also readily seen that, together with Eqs. (30) and (31), it implies $\nabla_i(\sqrt{Z}\nabla^i \varphi / W) = 0$ where here $Z = G_d - W'^2$. This equation is to be compared to its Newtonian analog and, in fact, has the same form. So it is possible to generalize the theorem by De and Raychaudhuri [30] to higher dimensions (see also [14]). According to such a theorem, in order to have charged dust solutions satisfying Weyl hypothesis without singularities, the quantity Z must vanish. This implies $W'^2 = G_d$, as in the Newton-Coulomb with electric matter theory, so that the result is the Majumdar-Papapetrou relation, the same as in the relativistic vacuum case [see Eq. (34)], $W = a_0 - \epsilon\sqrt{G_d}\varphi$, with $\epsilon = \pm 1$, and a_0 being an integration constant. Then, substituting W from the latter equation into Eq. (35) gives $\rho_e = \epsilon\sqrt{G_d}\rho_m$ as in the Newtonian case. To summarize, let us write the resulting equations for the important functions W, φ, ρ_m, and ρ_e. In order to get a field equation similar to Poisson equation (9), it is convenient to introduce a new potential U such that

$$U = \frac{1}{W}. \tag{36}$$

The relevant equations are then

$$\nabla^2 U = -S_{d-2} G_d U^{(d-1)/(d-3)} \rho_m, \tag{37}$$

$$\varphi = \epsilon \frac{1}{\sqrt{G_d}}\left(1 - \frac{1}{U}\right), \tag{38}$$

$$\rho_e = \epsilon\sqrt{G_d}\rho_m, \tag{39}$$

where an arbitrary constant in the potentials was adjusted to unity. Some special solutions to these types of systems are going to be analyzed in the next sections. Equation (39) is the Majumdar-Papapetrou condition. Note that these equations can be compared to the Newton-Coulomb with an electrically charged fluid case. In fact, taking the Newtonian limit in which $U \simeq 1 - V$, with $|V| \ll 1$, one sees that Eqs. (37)–(39) reduce exactly to Eqs. (9)–(11), respectively.

B. Spherical d spacetime dimensional relativistic Bonnor star solutions

1. Equations in spherical coordinates

In what follows we confine attention to spherically symmetric static spacetimes and write the foregoing equations in isotropic and Schwarzschild spherical coordinates.

Equations in isotropic coordinates: The starting point is the metric (27), which with Eq. (36) now reads

$$ds^2 = -U^{-2}dt^2 + U^{2/(d-3)}(dR^2 + R^2 d\Omega_{d-2}^2), \tag{40}$$

with $d\Omega_{d-2}^2$ being the metric on the unit $(d-2)$-dimensional sphere S^{d-2}. U is now a function of the radial coordinate R only, and it obeys

$$\frac{d}{dR}\left(R^{d-2}\frac{dU}{dR}\right) = -S_{d-2}G_d \rho_m R^{d-2} U^{(d-1)/(d-3)}, \tag{41}$$

which is obtained from Eq. (37). The matter and charge densities are also functions of R only, $\rho_m = \rho_m(R)$ and $\rho_e = \rho_e(R)$, and they are related to each other through Eq. (39). From Eq. (41), and in analogy with the Newtonian theory, define the mass function $m(R)$ and the charge function $q(R)$ (see other mass function definitions in Appendix B) as

$$m(R) = S_{d-2}\int_0^R \rho_m(R) U(R)^{(d-1)/(d-3)} R^{d-2} dR, \tag{42}$$

$$q(R) = S_{d-2}\int_0^R \rho_e(R) U(R)^{(d-1)/(d-3)} R^{d-2} dR. \tag{43}$$

Equations (37)–(39) may then be written as

$$\frac{dU(R)}{dR} = -G_d \frac{m(R)}{R^{d-2}}, \tag{44}$$

$$\frac{d\varphi(R)}{dR} = -U(R)^{-2}\frac{q(R)}{R^{d-2}}, \quad \text{or} \quad \varphi(R) = \frac{\epsilon}{\sqrt{G_d}}\left(1 - \frac{1}{U(R)}\right), \tag{45}$$

$$q(R) = \epsilon\sqrt{G_d}\, m(R), \tag{46}$$

where arbitrary constants in the potentials were set to one. In addition, terms of the form const$/R^{d-2}$ in Eqs. (44) and (45) were not written explicitly since they are implicitly absorbed in those equations, and moreover they should be put to zero as the fields shall be regular functions of the radial coordinate R. Equations (44)–(46) can then be compared to the Newton-Coulomb with an electrically charged fluid case. In fact, taking the Newtonian limit in which $U \simeq 1 - V$, with $|V| \ll 1$, and $R \simeq r$, one sees that Eqs. (44)–(46) reduce exactly to Eqs. (14)–(16), respectively. For future reference we write here the Kretschmann ($\mathcal{K}$) and Ricci ($\mathcal{R}$) scalars for the metric (40):

$$\mathcal{K} = \frac{d-1}{d-3}\frac{4U''^2}{U^{2(d-1)/(d-3)}} + \left(8 + \frac{(3d-8)(4d-11)}{(d-3)^3}\right) \times \frac{2U'^4}{U^{4(d-2)/(d-3)}} - \frac{(d-2)(2d-5)}{(d-3)^2}\frac{4U'^2U''}{U^{(3d-5)/(d-3)}} + \frac{8S_{d-2}}{d-3}G_d\frac{\rho_{\rm m}}{U^{(d-1)/(d-2)}} \times \left(U'' - \frac{U'^2}{U} + \frac{S_{d-2}}{2}G_d\frac{d-2}{d-3}\rho_{\rm m}U^{(d-1)/(d-3)}\right), \tag{47}$$

$$\mathcal{R} = \frac{2S_{d-2}}{d-3}G_d\rho_{\rm m} - \frac{d-4}{d-3}\frac{U'^2}{U^{2(d-2)/(d-3)}}, \tag{48}$$

where the prime stands for the derivative with respect to R. From this it is seen that spacetime singularities occur at points where $U = 0$, as long as the derivatives of U do not vanish at the same points as U does. Although the field equations are easily written and solved by working in harmonic coordinates, the physical interpretation of the solutions is clearer if one uses Schwarzschild coordinates.

Equations in Schwarzschild coordinates: In Schwarzschild coordinates the line element reads

$$ds^2 = -B^2dt^2 + A^2dr^2 + r^2d\Omega^2_{d-2}, \tag{49}$$

where $B = B(r)$ and $A = A(r)$, r being the Schwarzschildean radial coordinate. By comparing Eq. (40) to Eq. (49), we see that the radial coordinates in the two systems are related by

$$r^{d-3} = UR^{d-3}, \tag{50}$$

and that the metric potentials are related by

$$B = \frac{1}{U}, \tag{51}$$

and

$$A = 1 - \frac{1}{d-3}\frac{r}{U}\frac{dU}{dr}. \tag{52}$$

Equation (50) gives r as a function of R. Although this implicitly determines R as a function of r, it is only in special cases that this relation can be worked out explicitly. For the sake of completeness, we present here the Schwarzschild coordinate form of the field equations. With the metric in the form of Eq. (49), Eq. (37) turns into

$$\frac{1}{A}\frac{d}{dr}\left(r^{d-2}\frac{1}{AB}\frac{dB}{dr}\right) = S_{d-2}G_dr^{d-2}\rho_m. \tag{53}$$

This is, in fact, the equation for the potential B, since A is not independent of B. Namely, Eqs. (51) and (52) give

$$A = 1 + \frac{1}{d-3}\frac{r}{B}\frac{dB}{dr}, \tag{54}$$

which is a consequence of the Majumdar-Papapetrou condition in a fluid with vanishing stresses. At last, the electric functions are expressed in Schwarzschild coordinates. No effort is needed to obtain the electric charge density since it is proportional to the mass density. The electric potential $\varphi(r)$ comes after Eqs. (38) and (51), i.e.,

$$\varphi = \frac{\epsilon}{\sqrt{G_d}}(1-B), \tag{55}$$

where, as usual, the arbitrary constant was set to unity. Now, defining $\mathcal{M}(r)$ and $\mathcal{Q}(r)$ (see for comparison other mass definitions in Appendix B) as

$$\mathcal{M}(r) = S_{d-2}\int_0^r \rho_m(r)A(r)r^{d-2}dr, \tag{56}$$

$$\mathcal{Q}(r) = S_{d-2}\int_0^r \rho_{\rm e}(r)A(r)r^{d-2}dr, \tag{57}$$

Eqs. (37)–(39) may be written as

$$\frac{dB(r)}{dr} = G_dA(r)B(r)\frac{\mathcal{M}(r)}{r^{d-2}}, \tag{58}$$

$$\frac{d\varphi(r)}{dr} = -A(r)B(r)\frac{\mathcal{Q}(r)}{r^{d-2}}, \quad \text{or} \quad \varphi(r) = \frac{\epsilon}{\sqrt{G_d}}(1-B(r)), \tag{59}$$

$$\mathcal{Q}(r) = \epsilon\sqrt{G_d}\mathcal{M}(r), \tag{60}$$

and $A(r)$ is given in terms of $B(r)$ by Eq. (54). These are the fundamental equations in Schwarzschild coordinates. The Newtonian limit is obtained by noticing that for weak gravity fields one has that the metric functions $B(r)$ and $A(r)$ are close to unity, $B(r) = 1 + \delta B(r)$, and $A(r) = 1 + \delta A(r)$, with δ indicating small quantities. Hence, to the first order approximation, the above equations reduce, respectively, to Eqs. (14)–(16).

2. Solutions

(a) Electrovacuum solutions in d spacetime dimensions

As a first example and to set up notation let us report here on the case of d-dimensional vacuum Majumdar-Papapetrou solutions. These are nothing but the extreme Reissner-Nordström spacetimes generalized to higher dimensions that were first studied in Ref. [66]. The general solution of Eq. (44) in vacuum is usually written in the form

$$U = 1 + \frac{G_d}{d-3}\frac{M}{R^{d-3}}, \tag{61}$$

$$M = \text{const}, \tag{62}$$

where M is an integration constant equal to the total mass of the source. The electric potential follows from Eq. (45), $\phi = \epsilon(1 - 1/U)/\sqrt{G_d}$, and the electric charge is related to the total mass of the source by $Q = \epsilon\sqrt{G_d}M$, as required by the Majumdar-Papapetrou condition and in agreement with Eq. (46). The corresponding spacetime metric is

$$ds^2 = -\left(1 + \frac{G_d}{d-3}\frac{M}{R^{d-3}}\right)^{-2} dt^2 + \left(1 + \frac{G_d}{d-3}\frac{M}{R^{d-3}}\right)^{2/(d-3)} (dR^2 + R^2 d\Omega_{d-2}^2). \quad (63)$$

Using Eqs. (50)–(52) we find the relation between r and R, given by

$$r^{d-3} = R^{d-3} + \frac{G_d}{d-3} M. \quad (64)$$

One also finds that $B = \frac{1}{A} = (\frac{R^{d-3}}{R^{d-3}+[G_d/(d-3)]M}) = (1 - \frac{G_d}{d-3}\frac{M}{r^{d-3}})$ which leads to the metric for an extreme Reissner-Nordström black hole with mass and charge equal to M, and holds for all $d \geq 4$,

$$ds^2 = -\left(1 - \frac{G_d}{d-3}\frac{M}{r^{d-3}}\right)^2 dt^2 + \frac{dr^2}{(1 - \frac{G_d}{d-3}\frac{M}{r^{d-3}})^2} + r^2 d\Omega_{d-2}^2. \quad (65)$$

The coordinate r can be extended until $r = 0$, which is in fact a singularity. This is seen from the Ricci and Kretschmann scalars which are, respectively,

$$\mathcal{K} = 4\frac{(d-1)(d-2)^2}{d-3}\frac{G_d^2 M^2}{r^{2(d-1)}} + 2\left(8 + \frac{(3d-8)(4d-11)}{(d-3)^3}\right)\frac{G_d^4 M^4}{r^{4(d-2)}} - 4\frac{(2d-5)(d-2)^2}{(d-3)^2}\frac{G_d^3 M^3}{r^{(3d-5)}}, \quad (66)$$

$$\mathcal{R} = -\frac{d-4}{d-3}\frac{G_d^2 M^2}{r^{2(d-2)}}, \quad (67)$$

where we used Eqs. (47), (48), (50), and (64). The region of the spacetime which in Schwarzschild coordinates corresponds to $0 < r \leq (G_d M/(d-3))^{1/(d-3)}$ is not covered by the isotropic coordinates. The maximal analytical extension of these vacuum solutions representing extremal black holes can then be found following the usual methods.

(b) Relativistic Bonnor star solutions in d spacetime dimensions

Interesting exact solutions in the context of Majumdar-Papapetrou relativistic systems are the Bonnor stars, see now specifically [6,7,12], which are spherically symmetric distributions of a charged dust fluid satisfying the Einstein-Maxwell with matter equations in four-dimensional spacetimes. The d-dimensional version of this kind of stars is the solution to Eq. (42), or Eq. (53), with appropriate boundary and matching conditions. We look for solutions using the equations in harmonic coordinates, and then do the analysis in Schwarzschild coordinates. In order to find solutions to Eq. (42), a first, possible, procedure is to provide the mass density as a function of the radial coordinate, $\rho_m = \rho_m(R)$. This is the procedure usually adopted because it furnishes by construction physically acceptable mass distribution for the star. In the present case, however, such a strategy is not advisable because it results in a second order nonlinear differential equation for $U(R)$, whose solutions can be found just after fixing the number of dimensions of the spacetime. A second procedure, of no interest in the Newtonian case, but valuable here, is to choose the energy density profile in such a way to transform Eq. (42) into an equation whose solutions are known, such as the case of sine-Gordon equation used in Ref. [45], or transforming it into a linear equation, so that one can use the well-known methods to solve ordinary linear second order differential equations to find solutions. A third alternative procedure is to fix *a priori* the metric potential $U = U(R)$, and then determining the other physical quantities that follow from it. This is the strategy we follow here, it allows us to write the solutions in closed form, and it is the same strategy as the one opted for in the Newtonian Bonnor stars studied above.

(i) Solutions with smooth boundary conditions and some special solutions:—First we make the analysis in isotropic coordinates. We consider the general relativistic analog of the one studied in Sec. II B 2 b (see also [6,7,12]). We then choose

$$U = \begin{cases} U_i = c_0 + c_1 R^\alpha + c_2 R^\beta & R \leq R_0 \\ U_e = 1 + \frac{1}{d-3}\frac{G_d M}{R^{d-3}} & R > R_0, \end{cases} \quad (68)$$

where α and β are real numbers and R_0 shall be identified as the surface of the star. The arbitrary constants c_0, c_1, and c_2 are fixed in such a way to guarantee the matching conditions at the surface of the star, $R = R_0$. Bonnor [6,7,12] imposed U to be a C^1 function and the energy density to be a step function at the boundary. In this case one can verify that the constants are given by $c_0 = 1 + \frac{G_d}{d-3}\frac{M}{R_0^{d-3}}(\frac{\beta+d-3}{\beta}) + c_1\frac{\alpha-\beta}{\beta}R_0^\alpha$, c_1 one can take as arbitrary, and $c_2 = -\frac{G_d}{\beta}\frac{M}{R_0^{\beta+d-3}} - \frac{\alpha}{\beta}c_1 R_0^{\alpha-\beta}$. To reproduce Bonnor's choice for U [12] one puts $d = 4$, $c_1 = 0$, and $\beta = n$ (where n was the letter chosen for the exponent in [12]). Of course, if one wishes, one can choose U to be of any degree of differentiability at the boundary. Since it is interesting to test whether this choice of differentiability has any important influence on the properties of the star, one can, still in the spirit of Bonnor, go a step further and instead of choosing U as a C^1 function impose U to be a C^2 function of R. As a bonus, one gets in addition that the energy density is a C^0 function, i.e., continuous at the boundary R_0, indeed zero, which is more in accord with the usual properties of stars. For a C^2 choice for U there are no free constants and one finds

$$c_0 = 1 + \frac{1}{d-3}\frac{G_d M}{R_0^{d-3}}\left[1 + \frac{d-3}{\beta-\alpha}\left(\frac{\beta+d-3}{\alpha} - \frac{\alpha+d-3}{\beta}\right)\right], \tag{69}$$

$$c_1 = -\frac{(\beta+d-3)}{\alpha(\beta-\alpha)}\frac{G_d M}{R_0^{\alpha+d-3}}, \tag{70}$$

$$c_2 = -\frac{(\alpha+d-3)}{\beta(\alpha-\beta)}\frac{G_d M}{R_0^{\beta+d-3}}. \tag{71}$$

It then follows the potentials $U_{\rm i}$ and $U_{\rm e}$ are

$$U = \begin{cases} U_{\rm i} = 1 + \frac{G_d}{d-3}\frac{M}{R_0^{d-3}}\left(1 + \frac{(d-3)(\beta+d-3)}{\alpha(\beta-\alpha)}[1 - (\frac{R}{R_0})^\alpha] - \frac{(d-3)(\alpha+d-3)}{\beta(\beta-\alpha)}[1-(\frac{R}{R_0})^\beta]\right) & R \le R_0 \\ U_{\rm e} = 1 + \frac{1}{d-3}\frac{G_d M}{R^{d-3}} & R > R_0. \end{cases} \tag{72}$$

Equation (37) then gives the mass density

$$\rho_{\rm m} = \begin{cases} \frac{(\alpha+d-3)(\beta+d-3)}{(d-3)(\beta-\alpha)}\frac{M}{S_{d-2}R_0^{d-1}}[(\frac{R}{R_0})^{\alpha-2} - (\frac{R}{R_0})^{\beta-2}]\frac{1}{U^{(d-1)/(d-3)}} & R \le R_0 \\ 0 & R > R_0. \end{cases} \tag{73}$$

In the region outside the mass distribution, the solution takes the extreme Reissner-Nordström form (63), as expected. Since U is a C^2 function, the spacetime metric satisfies the Israel matching conditions at $R = R_0$. In order that $\rho_{\rm m}$ be a well-defined function and everywhere nonnegative, we must have $\alpha, \beta \ge 2$. The quantity $M/(S_{d-2}R_0^{d-1})$ appears naturally with units of mass density. Note that the electric potential φ and the electric density $\rho_{\rm e}$ can be found directly from Eqs. (45) and (46). The function φ is a continuous C^2 function through the surface of the star, which means the field strength is C^1 and the charge density is C^0. Moreover, using Eqs. (42) and (73) one finds that indeed $M = m(R_0)$, making the whole procedure a consistent one. This Bonnor star solution looks like the Newtonian star studied in Sec. II B 2. In fact, the resulting mass density, Eq. (73), resembles the function given by Eq. (20).

Second, we make the analysis in Schwarzschild coordinates. Schwarzschild coordinates are interesting to analyze the physical properties of the spherical solutions found above. Equations (50) and (72) establish the relation between the harmonic radial coordinate R and the Schwarzschild radial coordinate r

$$r^{d-3} = \begin{cases} c_0 R^{d-3} + c_1 R^{\alpha+d-3} + c_2 R^{\beta+d-3} & R \le R_0 \\ R^{d-3} + \frac{1}{d-3}G_d M & R > R_0. \end{cases} \tag{74}$$

These relations furnish R as a function of r, $R = f(r)$, which is in fact defined by two functions. Let us call them, respectively, $f_{\rm i}(r)$ for the internal region and $f_{\rm e}(r)$ for the external region. The surface of the star, defined by $R = R_0$, is obtained in terms of the Schwarzschild coordinates, by imposing the continuity of the function $r(R)$ through such a surface, i.e.,

$$r_0^{d-3} = R_0^{d-3}U_{\rm i}(R_0) = R_0^{d-3}U_{\rm e}(R_0) = R_0^{d-3} + \frac{1}{d-3}G_d M, \tag{75}$$

where $U_{\rm i}$ and $U_{\rm e}$ are defined by Eq. (72). The aim now is to find the metric potentials B and A as functions of r. In order to do that one needs to find the functions $f_{\rm i}(r)$ and $f_{\rm e}(r)$, which is done by solving Eqs. (74) for R. For $r \le r_0$ one has

$$B_{\rm i}(r) = \frac{1}{U_{\rm i}(r)} = (c_0 + c_1 f_{\rm i}^\alpha + c_2 f_{\rm i}^\beta)^{-1}, \tag{76}$$

$$A_{\rm i}(r) = 1 + \frac{r}{f_{\rm i}}\frac{df_{\rm i}}{dr}(\alpha c_1 f_{\rm i}^\alpha + \beta c_2 f_{\rm i}^\beta) \times (c_0 + c_1 f_{\rm i}^\alpha + c_2 f_{\rm i}^\beta)^{-1}, \tag{77}$$

with $f_{\rm i} = f_{\rm i}(r)$ being a suitable solution of the following algebraic equation:

$$c_2 f_{\rm i}^{\beta+d-3} + c_1 f_{\rm i}^{\alpha+d-3} + c_0 f_{\rm i}^{d-3} - r^{d-3} = 0. \tag{78}$$

The constants c_0, c_1, and c_2 are now to be written in terms of r_0 instead of in terms of R_0. The corresponding expressions are obtained by substituting $R_0 = (r_0^{d-3} - G_d M/(d-3))^{1/(d-3)}$ into Eqs. (69)–(71). For $r \ge r_0$ one has

$$B_{\rm e}(r) = \frac{1}{A_{\rm e}(r)} = 1 - \frac{1}{d-3}\frac{G_d M}{r^{d-3}}. \tag{79}$$

Third, we find some special solutions with a simple algebraic structure. Generally, the only way of finding the solutions to Eq. (78) is by specifying the values of the parameters α and β and the number of spacetime dimensions d. Even in that case, in general, only numerical solutions are possible to find and we do not perform such an analysis here. There are, however, some special values of α and β for which Eq. (78) can be solved exactly for $f_i(r)$. Thus, in order to investigate some more properties of d-dimensional Bonnor stars, we consider a particular case that can be dealt with algebraically. For instance, one may choose

$$\beta = \frac{3}{2}\alpha = 3(d-3), \tag{80}$$

so that one finds a fourth degree polynomial equation to solve for R^{d-3}:

$$c_2(R^{d-3})^4 + c_1(R^{d-3})^3 + c_0 R^{d-3} - r^{d-3} = 0, \tag{81}$$

where now the coefficients c_0, c_1, and c_3 are simplified to

$$c_0 = 1 + \frac{2G_d}{d-3}\frac{M}{R_0^{d-3}}, \qquad c_1 = -\frac{2G_d}{d-3}\frac{M}{R_0^{3(d-3)}}, \qquad c_2 = \frac{G_d}{d-3}\frac{M}{R_0^{4(d-3)}}. \tag{82}$$

This polynomial equation can be solved in terms of radicals, and the physical quantities can then be expressed explicitly in terms of the coordinate r. In order to condense expressions, we first define the parameter a by

$$a = \frac{G_d}{d-3}\frac{M}{r_0^{d-3}}, \tag{83}$$

with $0 \leq a \leq 1$. As in the case of Newtonian stars [see Eq. (21)] this parameter measures how compact the star is and it is useful to parametrize the numerical solutions. Further, we define

$$\begin{aligned} b(r) &= \frac{1}{16}\left(1+\frac{1}{a}\right)^2 - \frac{1}{4a}\left(\frac{r}{r_0}\right)^{d-3},\\ c(r) &= \frac{1}{6} + \frac{1}{6a} - \frac{1}{3a}\left(\frac{r}{r_0}\right)^{d-3},\\ e(r) &= (b(r) + \sqrt{[b(r)]^2 - [c(r)]^3})^{1/3},\\ s(r) &= \sqrt{1 + 2e(r) + 2\frac{c(r)}{e(r)}}, \end{aligned} \tag{84}$$

where we have used the relation $R_0^{d-3} = r_0^{d-3} - \frac{G_d}{d-3}M$. Then, the solution for $R(r)$ is

$$R(r)^{d-3} = \begin{cases} \left(\frac{1}{2} - \frac{s(r)}{2} + \frac{1}{2}\sqrt{2 + 2s(r) + 2\frac{c(r)}{e(r)} - \frac{2}{as(r)}}\right)\left(r_0^{d-3} - \frac{G_d}{d-3}M\right) & r \leq r_0\\ r^{d-3} - \frac{G_d}{d-3}M & r > r_0. \end{cases} \tag{85}$$

Fourth, the relevant functions $B(r)$, $A(r)$, $\rho_m(r)$, $\varphi(r)$, and $\rho_e(r)$, given in terms of the Schwarzschild coordinates, follow from the above relations. They are dependent on the variable r, and also depend on two other arbitrary parameters, the mass and the radius of the star, M and r_0, respectively. Instead of writing the explicit form of such functions, which are cumbersome, it is more convenient to plot them for several choices of parameters. In the calculations we normalized the coordinate r to the mass parameter $\mu = (G_d M/(d-3))^{1/(d-3)}$ which was kept fixed. In fact, the important parameter to this end is the mass to radius ratio a, given by Eq. (83), which measures how relativistic the system is. Here we have the constraint $0 < a < 1$, and for small a the system is Newtonian, while for a close to unity it is fully relativistic. The function $B(r)$: The simplest function to be found in Schwarzschild coordinates is the metric potential $B(r)$, which is immediately obtained through the relation $B(r) = 1/U(r)$. Figure 3 shows $B(r)$ as a function of r/μ in $d = 4, 5, 6, 7$, as indicated. It is also seen in that figure the behavior of $B(r, a)$ as a function of a, for different values of the parameter a, as shown by the four curves in each graph. All the interior functions $B_i(r, a)$ match the exterior extreme Reissner-Nordström solution $B_e(r) = 1 - (\mu/r)^{d-3}$, each one at a different value of r_0. The reason for that is because the change of a is made by keeping the mass of the star fixed, while r_0 varies accordingly. Notice also that for $a \to 1$ the function $B_i(r)$ approaches zero in the whole region interior to $r = r_0$, meaning that the redshift with respect to infinity is infinite. For the extreme value ($a = 1$) the mass and the charge of the charged star are concentrated inside a quasi-horizon at $r = r_0$. In this limit, the spacetime solution is a quasi-black hole, similar to what was found for four-dimensional spacetimes (see [13–17]). There are no singularities inside r_0, the curvature is finite, so are the mass and charge densities of the charged dust [see also item (ii) below]. It can also be seen the Newtonian limit of the solution by comparing the curves for the smaller values of a in Fig. 3 with the corresponding curves for the Newtonian potential, Fig. 1 [see also item (iii) below]. The function $A(r)$: The behavior of the other metric potential $A(r)$ is seen in Fig. 4, where we plot $1/A$ against r/μ for the same values of d and a as in Fig. 3. The quasi-black-hole formation is seen in this case as $1/A(r)$ going to zero at $r = r_0$ for $a \to 1$. It appears in the figure as the sharp elbow in the solid line (lowest) curve shown in the graph. The exterior function is $A_e(r) = 1/B_e(r)$, and all the inner functions $A_i(r, a)$ for different a match $A_e(r)$ at a particular value of r_0. The function $\rho_m(r)$: Another quantity of interest is the mass density $\rho_m(r)$. In Fig. 5 we plot the normalized mass density $\rho_m(r)/\lambda_m$ as a function of the normalized radial coordinate r/μ. Here λ_m is defined as $\lambda_m = \frac{(\alpha+d-3)(\beta+d-3)}{(d-1)(d-3)(\beta-\alpha)}\bar{\rho}_m$, where $\bar{\rho}_m$, a kind of average density, here is given by $\bar{\rho}_m = (d-1)M/(S_{d-2}r_0^{d-1})$. For our

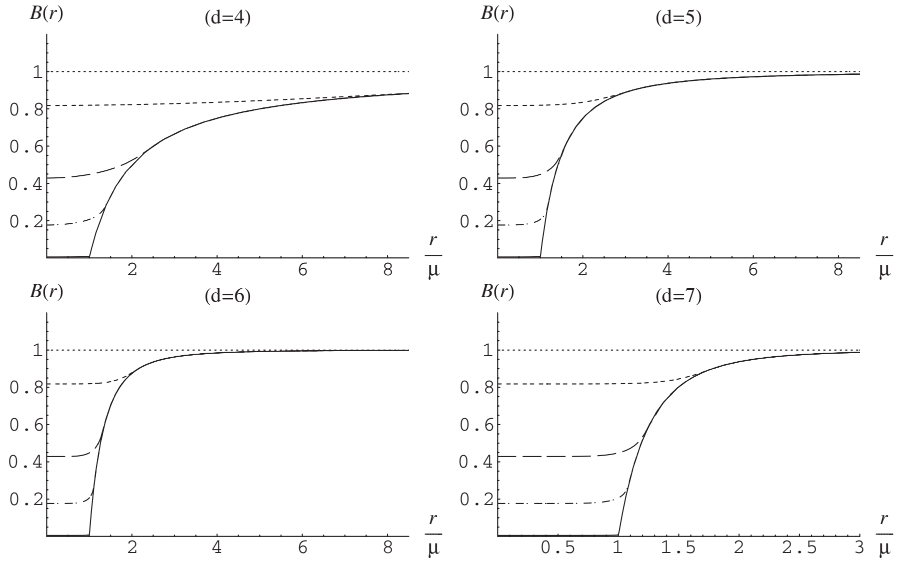

FIG. 3. The metric potential $B(r)$ as a function of r/μ, where $\mu \equiv (G_d M/(d-3))^{1/(d-3)}$, for $d = 4, 5, 6, 7$, and for four values of a in each graph (from top to bottom: $a = 0.1$, $a = 0.4$, $a = 0.7$, and $a = 1$).

choice of parameters, see Eq. (80), one has $\lambda_{\rm m} = \frac{12}{d-1}\bar{\rho}_{\rm m}$. We plot $\rho_{\rm m}(r)/\lambda_{\rm m}$ against r/μ for the same values of d and a as in Figs. 3 and 4. Notice that for $d > 4$ the general properties of this function do not depend upon the specific value of d. It is clearly seen that $\rho_{\rm m}(r)$ is finite at $r = 0$. In fact, with our choice, $\rho_{\rm m}$ vanishes at $r = 0$ for all $d > 4$. In addition it goes to zero at the surface of the star, defining thus the radius r_0 in each plotted case. Moreover, the mass density is everywhere well defined even in the quasi-black-hole limit. The comparison to the Newtonian case can be done considering the curves for small a in Fig. 5, and comparing the corresponding curves in Fig. 2 [see below item (iii)]. The functions $\varphi(r)$ and $\rho_{\rm e}(r)$: The other two functions, the electric potential $\varphi(r)$ and the electric charge density $\rho_{\rm e}(r)$, are so closely related to the respective gravitational quantities $B(r)$ and $\rho_{\rm m}(r)$ that no plot needs to be drawn for them. In fact, they are promptly obtained from their relations to the functions studied above [see Eqs. (55) and (39)]; namely, $\varphi(r) = \epsilon(B(r) - 1)/\sqrt{G_d}$ and $\rho_{\rm e}(r) = \epsilon\sqrt{G_d}\rho_{\rm m}(r)$.

(ii) Quasi-black-hole limit:—For the full relativistic limit, $a = 1 - \varepsilon$, with $\varepsilon \ll 1$, it is clear from the previous

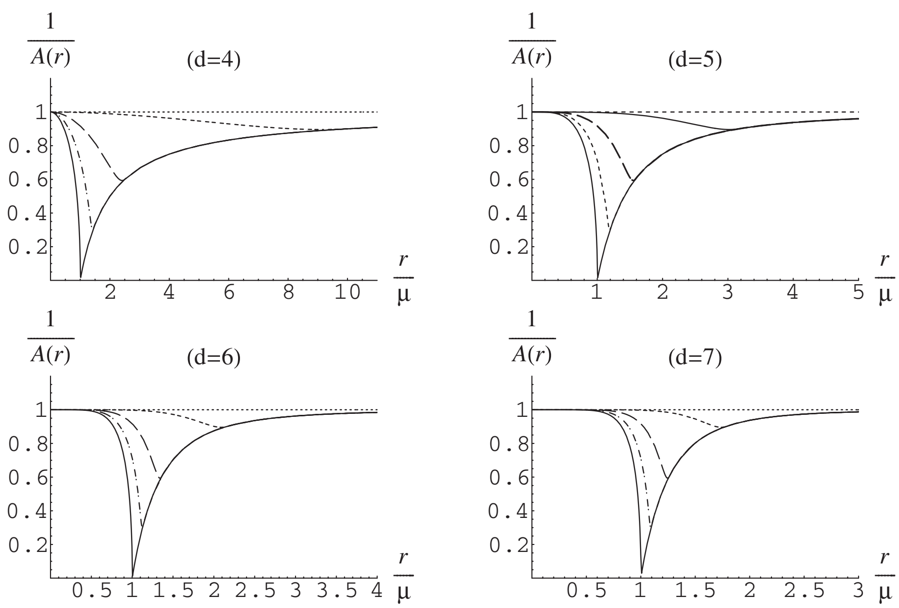

FIG. 4. The metric potential $1/A(r)$ as a function of r/μ, for $d = 4, 5, 6, 7$, and for four values of a in each graph (from top to bottom: $a = 0.1$, $a = 0.4$, $a = 0.7$, and $a = 1$).

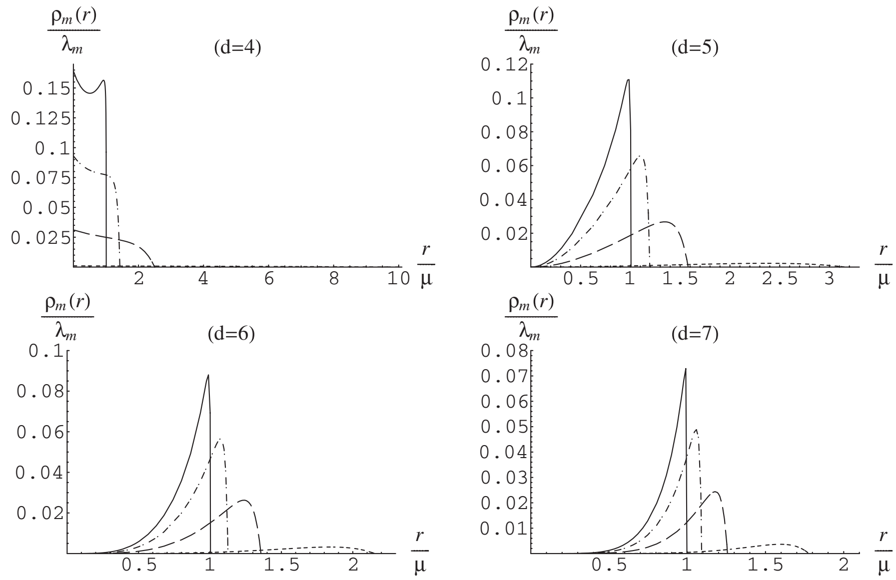

FIG. 5. The normalized relativistic mass density $\rho_{\rm m}(r)/\lambda_{\rm m}$ as a function of r/μ, where $\lambda_{\rm m} = \frac{12}{d-1}\bar{\rho}_{\rm m}$, $\bar{\rho}_{\rm m}$ being a kind of average density (see text), and $\mu \equiv (G_d M/(d-3))^{1/(d-3)}$, for the cases $d = 4, 5, 6, 7$ (as indicated), and with $a = 1$ (upper curve), $a = 0.7$ (dot-dashed curve), $a = 0.4$ (dashed curve), and $a = 0.1$ (lowest curve) for each d. The normalized relativistic mass density $\rho_{\rm m}(r)/\lambda_{\rm m}$ goes to zero at the surface of the star, defining thus the radius r_0 in each plotted case.

plots that the function $1/A(r)$ attains a minimum at $r/\mu = 1 + \varepsilon$, such that $1/A(r) = \varepsilon$, where again, $\mu = (G_d M/(d-3))^{1/(d-3)}$. Also, for such a small but nonzero ε the configuration is regular everywhere with a nonvanishing metric function B. Moreover, in the limit $\varepsilon \to 0$ the interior metric potential $B_{\rm i}$ obeys $B_{\rm i} \to 0$ for all $r/\mu \leq 1$. These three features define a quasi-black hole; see [16,17]. These three features imply, among other things, that (a) there are infinite redshift whole regions rather than surfaces, (b) the object displays naked behavior, i.e., generation of infinite tidal forces in a freely falling frame, (c) outer and inner regions become impenetrable and disjoint, and (d) for external distant observers the spacetime is indistinguishable from that of extremal black holes. The quasi-black hole is on the verge of forming an event horizon, but it never forms one, instead, a quasihorizon appears.

It is of interest to see that in the quasi-black-hole limit the metric is well defined and everywhere regular. We check this for the interior. One defines, from the isotropic radial coordinate R, a new spatial coordinate x by

$$x = \frac{R}{R_0}, \qquad 0 \leq x \leq 1, \tag{86}$$

from which one sees that the surface of the star is now located at $x = 1$. Substituting this transformation into the interior metric functions and choosing a new time coordinate T according to

$$dT = \frac{(d-3)R_0^{d-3}}{G_d M}\,dt, \tag{87}$$

the interior metric is now

$$ds^2 = -\tilde{U}^{-2}dT^2 + \left(\frac{G_d M}{d-3}\right)^{2/(d-3)} \tilde{U}^{2/(d-3)}(dx^2 + x^2 d\Omega_{d-2}^2), \tag{88}$$

where

$$\tilde{U} = 1 + (d-3)\left[\frac{\beta + d - 3}{\alpha(\beta - \alpha)}(1 - x^{\alpha}) + \frac{(\alpha + d - 3)}{\beta(\alpha - \beta)}(1 - x^{\beta})\right]. \tag{89}$$

This metric is regular throughout the interior region and also at the surface of the star. Moreover, at $x = 1$ one has $\tilde{U} = 1$. This means that even in the quasi-black-hole limit the surface of the star is timelike for internal observers. On the other hand, one can verify that, being the exterior metric the extremal Reissner-Nordström metric, the quasi-black-hole limit gives a null surface for external observers. There is thus a mismatch, implying in this case that the interior and exterior regions are disjoint, as was fully analyzed in [16,17].

(iii) Quasi-Newtonian limit: Newtonian Bonnor stars discussed previously:—It is expected that in the weak field approximation a relativistic Bonnor star reduces to a Newtonian Bonnor star. Here, we show that indeed the relativistic star studied in this section, i.e., Sec. III B 2, reduces to the Newtonian star studied in Sec. II B 2.

In the relativistic theory two coordinate systems are involved in the solutions, the isotropic and the Schwarzschild spherical coordinates. Initially we show that to first order approximation in the weak field limit the two coordinate systems are identical. In order to deal with the issue we take the special case considered in paragraph (b)(i) of Sec. III B 2. The weak field limit inside the spherical star corresponds to small values of the parameter $a = G_d M/((d-3)r_0^{d-3})$. Hence, considering the approximation of Eq. (85) up to the first order in a it follows that

$$\frac{R^{d-3}}{R_0^{d-3}} = \frac{r^{d-3}}{r_0^{d-3}}\left\{1 - \frac{G_d}{d-3}\frac{M}{r_0^{d-3}}\left[1 - 2\left(\frac{r^{d-3}}{r_0^{d-3}}\right)^2 + \left(\frac{r^{d-3}}{r_0^{d-3}}\right)^3\right]\right\}. \tag{90}$$

At the lowest order approximation it results in

$$\frac{R^{d-3}}{R_0^{d-3}} = \frac{r^{d-3}}{r_0^{d-3}}, \tag{91}$$

as expected. Therefore, when comparing the first order approximation of the relativistic solution to the Newtonian solution one may work with the isotropic coordinates, identifying the radial coordinate R with the Newtonian radial coordinate r.

The next step is obtaining the potentials and densities in the weak field approximation and comparing them to the Newtonian case. In such a limit one has the relation $U = 1 - V$, where V is the Newtonian potential. Now using the relation (91) and Eq. (72) one can write U_{i} up to first order in M/r_0^{d-3},

$$U_{\mathrm{i}} = 1 + \frac{G_d}{d-3}\frac{M}{r_0^{d-3}}\left(1 + \frac{(d-3)(\beta+d-3)}{\alpha(\beta-\alpha)} \times \left[1 - \left(\frac{r}{r_0}\right)^{\alpha}\right] - \frac{(d-3)(\alpha+d-3)}{\beta(\beta-\alpha)}\left[1 - \left(\frac{r}{r_0}\right)^{\beta}\right]\right). \tag{92}$$

From this equation, and from the exterior solution U_{e}, one then finds the potential in Newtonian approximation, $V = 1 - U$, as

$$V = \begin{cases} V_{\mathrm{i}} = -\frac{G_d}{d-3}\frac{M}{r_0^{d-3}}\left(1 + \frac{(d-3)(\beta+d-3)}{\alpha(\beta-\alpha)}\left[1 - \left(\frac{r}{r_0}\right)^{\alpha}\right] - \frac{(d-3)(\alpha+d-3)}{\beta(\beta-\alpha)}\left[1 - \left(\frac{r}{r_0}\right)^{\beta}\right]\right) & r \le r_0 \\ V_{\mathrm{e}} = -\frac{G_d}{d-3}\frac{M}{r^{d-3}} & r > r_0. \end{cases} \tag{93}$$

The resulting expression is to be compared to the gravitational potential of the Newtonian star as given in Eq. (19). The two expressions become identical if one identifies the gravitational constant G_d, the radial coordinate r, and the mass of the star M in both equations. We have already shown that, in the weak field approximation, it results $R = r + O(M/r_0^{d-3})$, and also $U(R) = U(r) = 1 + O(M/r_0^{d-3})$. Therefore, substituting such results into Eq. (73) we find the first order approximation for the relativistic mass density,

$$\rho_{\mathrm{m}} = \begin{cases} \frac{(\alpha+d-3)(\beta+d-3)}{(d-3)(\beta-\alpha)}\frac{M}{S_{d-2}r_0^{d-1}}\left[\left(\frac{r}{r_0}\right)^{\alpha-2} - \left(\frac{r}{r_0}\right)^{\beta-2}\right] & r \le r_0 \\ 0 & r > r_0. \end{cases} \tag{94}$$

In order for this result to be identical to Eq. (20) the mass M and the coordinate r must be the same in both equations. It is then straightforward to show that the weak field limits of other relativistic quantities such as the metric functions $B(r)$ and $A(r)$, the electric charge density, and electric potential all agree with their Newtonian counterparts, as expected.

Notice that units have been normalized in such a way that the gravitational coupling constant in Einstein equations equals the Newtonian gravitational coupling constant in Poisson equation (see Appendix A). Furthermore, the mass densities carry identical units and normalizations due to the similarity between Poisson equation for Newtonian gravity, Eq. (4), and the corresponding equation coming from Majumdar-Papapetrou relativistic system, Eq. (37).

IV. CONCLUSIONS

We have studied d-dimensional Bonnor star solutions, spherical distributions of extremal charged dust joined to extremal charged vacua, both in Newtonian gravity and general relativity. We have found that the relativistic solutions present many interesting properties such as forming an extreme d-dimensional quasi-black hole, when the mass to radius ratio reaches a critical value. We have also found that the Newtonian solutions are limiting cases of the relativistic ones. In this connection it is interesting to note that the Bonnor star solutions in Majumdar-Papapetrou Newtonian gravity, when contrasted to those Bonnor solutions in Majumdar-Papapetrou general relativity, display clearly the departing of the high density structures that may arise in the strong field regime of each theory, mild singularities in one theory, quasi-black holes in the other. Moreover, whereas there are no solutions for Newtonian stars supported by degenerate pressure in higher dimensions, and so no general relativistic solutions either, higher-dimensional Bonnor stars, supported by electric repulsion do indeed have solutions. This means that the existence of stars in higher dimensions depends on the number of dimensions itself, and on the underlying field content of those stars, as expected.

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APPENDIX A: NEWTON'S GRAVITATIONAL CONSTANT G_d IN d SPACETIME DIMENSIONS

Within Newtonian gravity, the Poisson equation for the gravitational field is given by

$$\nabla^2 V = k\rho_{\rm m}, \tag{A1}$$

where k is a constant, related to Newton's gravitational constant G_d in d spacetime dimensions, to be determined. Integrating over the space volume $\mathcal{V}$ and using the Gauss theorem, one obtains

$$\int_{\mathcal{V}} \nabla^2 V d^{d-1}x = \oint_{S_{d-2}} \nabla_i V n^i dS_{d-2} = k\int_{\mathcal{V}} \rho_{\rm m} d^{d-1}x = kM, \tag{A2}$$

where S_{d-2} is the boundary surface surrounding the volume $\mathcal{V}$, and n^i is the unit normal to the surface S_{d-2}. Considering now spherical symmetry, i.e.,

$$\nabla_i V n^i = -g_r, \tag{A3}$$

where g_r is defined to be the radial component of the gravitational field, one finds

$$\oint_{S_{d-2}} \nabla_i V n^i dS_{d-2} = -g_r S_{d-2} r^{d-2}. \tag{A4}$$

Then (A2) and (A4) yield

$$g_r = -\frac{k}{S_{d-2}}\frac{M}{r^{d-2}}. \tag{A5}$$

The choice in [59] for k is given by

$$k = G_d S_{d-2}. \tag{A6}$$

This is an interesting choice because it gives

$$g_r = -\frac{G_d M}{r^{d-2}}, \tag{A7}$$

i.e., a straight generalization of Newtons force law to d spacetime dimensions, although it puts Einstein's equation into a slightly awkward form,

$$G_{\mu\nu} = \frac{d-2}{d-3} S_{d-2} G_d T_{\mu\nu}, \tag{A8}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor. The choice in [68] for k is given by

$$k = 8\pi G_d \frac{d-3}{d-2}. \tag{A9}$$

This is also an interesting choice because although it gives

$$g_r = -\frac{8\pi G_d}{S_{d-2}}\frac{d-3}{d-2}\frac{M}{r^{d-2}}, \tag{A10}$$

Einstein's equation is written as

$$G_{\mu\nu} = 8\pi G_d T_{\mu\nu}, \tag{A11}$$

i.e., a straight generalization of Einstein's equation to d spacetime dimensions. Both definitions of k give the correct definition in four dimensions for $G_{d=4} = G_4 \equiv G$. In this paper we have opted for the definition (A6), which yields (A7) and (A8).

APPENDIX B: MASS DEFINITIONS

1. Mass functions in isotropic coordinates

Throughout the paper we used the mass function $m(R)$ defined in Eq. (42). In the literature, another mass function, $M(R)$, is sometimes used. The connection between the two definitions is given below. Using Eq. (42) one gets

$$U(R) = 1 - G_d \int^R \frac{m(R)}{R^{d-2}} dR, \tag{B1}$$

where an integration constant has been made equal to unity. Equation (B1) is consistent with the usual form of the potential U outside the mass and charge distributions, i.e., $R > R_0$. In fact, if we take $m(R) = M = \text{const}$, Eq. (B1) yields $U(R) = 1 + G_d M/((d-3)R^{d-3})$, where M is total mass of the source. The other mass function of a charged dust distribution $M(R)$ can then be defined in analogy with the result for vacuum. This is done by taking $U(R)$ inside the dust in the same form as outside,

$$U(R) = 1 + \frac{G_d}{d-3}\frac{M(R)}{R^{d-3}}. \tag{B2}$$

Hence, it follows the relation

$$M(R) = -(d-3)R^{d-3} \int^R \frac{m(R)}{R^{d-2}} dR. \tag{B3}$$

And so, one sees that the two masses $m(R)$ and $M(R)$ are in general different from each other. The two definitions agree just in the region outside the dust fluid, in which case $m(R) = M(R) = M$ is the total mass of the source.

2. Mass functions in Schwarzschild coordinates

The mass definition in Schwarzschild coordinates used in the paper is given by Eq. (56). Besides such a definition, there is a different route to define another mass function $M(r)$. Usually, in the literature the mass within a certain sphere of radius r, $M(r)$, is defined through a relation of the form

$$A = \frac{1}{1 - [G_d/(d-3)][M(r)/r^{d-3}]}. \tag{B4}$$

Interestingly, this mass coincides with $\mathcal{M}(r)$ as defined in Eq. (56). This can be shown as follows. From the last equation it follows

$$M(r) = \frac{d-3}{G_d} r^{d-3}\left(1 - \frac{1}{A}\right). \tag{B5}$$

Moreover, using the expression for dB/dr in terms of A obtained from (54) one gets

$$(d-3)r^{d-3}\left(1 - \frac{1}{A}\right) = \frac{r^{d-2}}{AB}\frac{dB}{dr}. \tag{B6}$$

Therefore, comparing Eqs. (B5) and (B6) one obtains

$$\frac{r^{d-2}}{AB}\frac{dB}{dr} = G_d M(r). \tag{B7}$$

Substituting this result into Eq. (53) and integrating one has

$$M(r) = S_{d-2}\int_0^r \rho_m(r)A(r)r^{d-2}dr, \tag{B8}$$

which is exactly $\mathcal{M}(r)$ as defined in Eq. (56). So, one has the identity $M(r) = \mathcal{M}(r)$.

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