Viscous dissipative Chaplygin gas dominated homogenous and isotropic cosmological models

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The generalized Chaplygin gas, which interpolates between a high density relativistic era and a nonrelativistic matter phase, is a popular dark energy candidate. We consider a generalization of the Chaplygin gas model, by assuming the presence of a bulk viscous type dissipative term in the effective thermodynamic pressure of the gas. The dissipative effects are described by using the truncated Israel-Stewart model, with the bulk viscosity coefficient and the relaxation time functions of the energy density only. The corresponding cosmological dynamics of the bulk viscous Chaplygin gas dominated universe is considered in detail for a flat homogeneous isotropic Friedmann-Robertson-Walker geometry. For different values of the model parameters we consider the evolution of the cosmological parameters (scale factor, energy density, Hubble function, deceleration parameter, and luminosity distance, respectively), by using both analytical and numerical methods. In the large time limit the model describes an accelerating universe, with the effective negative pressure induced by the Chaplygin gas and the bulk viscous pressure driving the acceleration. The theoretical predictions of the luminosity distance of our model are compared with the observations of the type Ia supernovae. The model fits well the recent supernova data. From the fitting we determine both the equation of state of the Chaplygin gas, and the parameters characterizing the bulk viscosity. The evolution of the scalar field associated to the viscous Chaplygin fluid is also considered, and the corresponding potential is obtained. Hence the viscous Chaplygin gas model offers an effective dynamical possibility for replacing the cosmological constant, and for explaining the recent acceleration of the universe.

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I. INTRODUCTION

The observations of high redshift supernovae [1] and the Boomerang/Maxima/WMAP data [2], showing that the location of the first acoustic peak in the power spectrum of the microwave background radiation is consistent with the inflationary prediction $\Omega = 1$, have provided compelling evidence for a net equation of state of the cosmic fluid lying in the range $-1 \leq w = p/\rho < -1/3$. To explain these observations, two dark components are invoked: the pressureless cold dark matter (CDM) and the dark energy (DE) with negative pressure. CDM contributes $\Omega_m \sim 0.3$ and is mainly motivated by the theoretical interpretation of the galactic rotation curves and large scale structure formation. DE is assumed to provide $\Omega_{\rm DE} \sim 0.7$ and is responsible for the acceleration of the distant type Ia supernovae. There are a huge number of candidates for DE in the literature (for recent reviews see [3,4]).

One possibility are cosmologies based on a mixture of cold dark matter and quintessence, a slowly varying, spatially inhomogeneous component [5]. An example of implementation of the idea of quintessence is the suggestion that it is the energy associated with a scalar field Q with self-interaction potential $V(Q)$. If the potential energy density is greater than the kinetic one, then the pressure $p = \dot{Q}^2/2 - V(Q)$ associated to the Q-field is negative. Quintessential cosmological models have been intensively investigated in the physical literature [6].

A different line of thought has been followed in [7–9], where the conditions under which the dynamics of a self-interacting Brans-Dicke (BD) field can account for the accelerated expansion of the Universe have been analyzed. Accelerated expanding solutions can be obtained with a quadratic self-coupling of the BD field and a negative coupling constant ω [7].

Dissipative effects, including both bulk and shear viscosity, are supposed to play a very important role in the early evolution of the Universe. A cosmic fluid (pressureless and with pressure) obeying a perfect fluid type equation of state cannot support the acceleration [9]. A solution to this problem, and thus avoiding the necessity of a potential for the BD field, is to assume that some dissipative effects of bulk viscous type take place at the cosmological scale [8]. A combination of a cosmic fluid with bulk dissipative pressure and quintessence matter can drive an accelerated expansion phase of the Universe and also solve

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the coincidence problem (the observational fact that the energy density of cold dark matter and of Q-matter should be comparable today) [10]. The dynamics of a causal bulk viscous cosmological fluid filled flat homogeneous universe in the framework of the BD theory was considered in [11]. The bulk viscous pressure term in the matter energy-momentum tensor leads to a nondecelerating evolution of the Universe.

Neither CDM nor DE have direct laboratory observational or experimental evidence for their existence. Therefore it would be important if a unified dark matter-dark energy scenario could be found, in which these two components are different manifestations of a single fluid [12]. A candidate for such an unification is the so-called generalized Chaplygin gas, which is an exotic fluid with the equation of state $p = -B/\rho^n$, where B and n are two parameters to be determined. It was initially suggested in [13] with $n = 1$, and then generalized in [14] for the case $n \neq 1$. The Chaplygin gas also appears in the stabilization of branes in Schwarzschild-AdS black hole bulks as a critical theory at the horizon [15] and in the stringy analysis of black holes in three dimensions [16]. The Chaplygin equation of state can be derived from Born-Infeld type Lagrangians [14,17]. This simple and elegant model smoothly interpolates between a nonrelativistic matter phase ($p = 0$) and a negative-pressure dark energy dominated phase.

The cosmological implications of the Chaplygin gas model have been intensively investigated in the recent literature [18]. The Chaplygin gas cosmological model has been constrained by using different cosmological observations, like type Ia supernovae [19], the CMB anisotropy measurements [20], gravitational lensing surveys [21], the age measurement of high redshift objects [22], and the X-ray gas mass fraction of clusters [23]. The obtained results are somewhat controversial, with some of them claiming good agreement between the data and the Chaplygin gas model, while the rest ruling it as a feasible candidate for dark matter. In particular, the standard Chaplygin gas model with $n = 1$ is ruled out by the data at a 99% level [23]. The exact solutions of the gravitational field equations in the generalized Randall-Sundrum model for an anisotropic brane with Bianchi type I geometry, with a generalized Chaplygin gas as matter source, were obtained in [24].

The possibility of constraining Chaplygin dark energy models with current integrated Sachs Wolfe effect data was investigated in [25]. In the case of a flat universe the generalized Chaplygin gas models must have an energy density such that $\Omega_c > 0.55$ and an equation of state $w < -0.6$ at 95% confidence level. The extent to which the knowledge of spatial topology may place constraints on the parameters of the generalized Chaplygin gas (GCG) model for unification of dark energy and dark matter was studied in [26]. By using both the Poincaré dodecahedral and binary octahedral spaces as the observable spatial topologies, the current type Ia supernovae (SNe Ia) constraints on the GCG model parameters were examined. An action formulation for the GCG model was developed in [27], and the most general form for the nonrelativistic GCG action consistent with the equation of state has been derived. The thermodynamical properties of dark energy have been investigated in [28]. For dark energy with constant equation of state $w > -1$ and the generalized Chaplygin gas, the entropy is positive and satisfies the entropy bound. Observational constraints on the generalized Chaplygin gas (GCG) model for dark energy from the nine Hubble parameter data points, the 115 SNLS Sne Ia data, and the size of baryonic acoustic oscillation peak at redshift $z = 0.35$ were examined in [29]. At a 95.4% confidence level, a combination of the three data sets gives $0.67 \leq B/\rho_0^{1+n} \leq 0.83$ (where ρ_0 is the present day energy density) and $-0.21 \leq n \leq 0.42$, which is within the allowed parameters ranges of the GCG as a candidate of the unified dark matter and dark energy. However, the standard Chaplygin gas model ($n = 1$) is also ruled out by these data at the 99.7% confidence level. A geometrical explanation for the generalized Chaplygin gas within the context of brane world theories, where matter fields are confined to the brane by means of the action of a confining potential, was considered in [30].

The evolution of the Universe contains a sequence of important dissipative processes, including grand unified theory phase transition, taking place at $t \approx 10^{-34}$ s and a temperature of about $T \approx 10^{27}$ K, when gauge bosons acquire mass, reheating of the Universe at the end of inflation ($t \approx 10^{-32}$ s), when the scalar field decays into particles, decoupling of neutrinos from the cosmic plasma ($t \approx 1$ s, $T \approx 10^{10}$ K), when the temperature falls below the threshold for interactions that keep the neutrinos in thermal contact, nucleosynthesis, decoupling of photons from matter during the recombination era ($t \approx 10$ s, $T \approx 10^3$ K), when electrons combine with protons and no longer scatter the photons, etc. [31].

The first attempts at creating a theory of relativistic dissipative fluids were those of Eckart [32] and Landau and Lifshitz [33]. These theories are now known to be pathological in several respects. Regardless of the choice of the equation of state, all equilibrium states in these theories are unstable and in addition signals may be propagated through the fluid at velocities exceeding the speed of light. These problems arise due to the first-order nature of the theory, that is, it considers only first-order deviations from the equilibrium leading to parabolic differential equations, hence to infinite speeds of propagation for heat flow and viscosity, in contradiction with the principle of causality. Conventional theory is thus applicable only to phenomena which are quasistationary, i.e. slowly varying on space and time scales characterized by mean free path and mean collision time.

A relativistic second-order theory was found by Israel [34] and developed in [35–37] into what is called "transient'' or ''extended'' irreversible thermodynamics. In this model deviations from equilibrium (bulk stress, heat flow, and shear stress) are treated as independent dynamical variables, leading to a total of 14 dynamical fluid variables to be determined. For general reviews on causal thermodynamics and its role in relativity see [31,38]. Causal bulk viscous thermodynamics has been extensively used for describing the dynamics and evolution of the early Universe, or in an astrophysical context [39].

It is the purpose of this paper to consider the effects of a possible existence of a bulk viscosity of the generalized Chaplygin gas on the cosmological dynamics of the Universe. The viscous effects are described by using the truncated Israel-Stewart theory [35]. By using the Laplace transformation and the convolution theorem, the second-order differential equation describing the evolution of the Hubble parameter H is transformed into an integral equation. The field equations are solved by means of an iterative scheme. Then the general solutions of the equations are obtained in a parametric form in the zero, first, second, and mth order approximation, and the relevant cosmological parameters (scale factor, energy density, Hubble parameter, deceleration parameter, etc.) are obtained. The scalar field interpretation of the Chaplygin gas is generalized to take into account the viscosity and dissipative effects.

In order to compare the predictions of the model with the observational data we have fitted the luminosity distance-redshift relation with the latest observational data of the type Ia supernovae. The model fits well these data. From the fitting we determine both the equation of state of the Chaplygin gas, and the parameters characterizing the bulk viscosity. Even by taking into account the effect of the bulk viscosity, the $n = 1$ Chaplygin gas models are ruled out by the observations.

The present paper is organized as follows. The physical model and the basic equations are presented in Sec. II. The evolution equation for the Hubble parameter is studied in Sec. III, and the behavior of the cosmological parameters is obtained. The observational data have been compared with the theoretical predictions of the model in Sec. IV. In Sec. V we discuss and conclude our results. In the present paper we use a system of units so that $8\pi G = c = 1$.

II. GEOMETRY, FIELD EQUATIONS, AND CONSEQUENCES

Perfect fluids in equilibrium generate no entropy and no frictional type heating, since their dynamics is reversible and without dissipation. A perfect fluid model is adequate for the description of many processes in cosmology. However, *real fluids* behave *irreversibly*, and some processes in astrophysics and cosmology cannot be understood except as irreversible processes. An important irreversible effect is bulk viscosity, which typically arises in mixtures, either of different species, as is the case of the radiative fluid, or of the same species, but with different energies, as in a Maxwell-Boltzmann gas. Physically, in cosmology we can think of bulk viscosity as an internal friction due to the different cooling rates in an expanding gas. The dissipation due to bulk viscosity converts kinetic energy of the particles into heat, and thus we expect it to reduce the effective pressure in an expanding fluid [31,38].

For a flat homogeneous Friedmann-Robertson-Walker (FRW) with a line element

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{1}$$

filled with a bulk viscous cosmological fluid the energy-momentum tensor is given by

$$T_i^k = (\rho + p + \Pi)u_i u^k - (p + \Pi)\delta_i^k, \tag{2}$$

where ρ is the energy density, p the thermodynamic pressure, Π the bulk viscous pressure and u_i the four velocity satisfying the condition $u_i u^i = 1$. The effect of the bulk viscosity of the cosmological fluid can be considered by adding to the usual thermodynamic pressure p the bulk viscous pressure Π, and formally substituting the pressure terms in the energy-momentum tensor by $p_{\rm eff} = p + \Pi$. The particle and entropy fluxes are defined according to $N^i = nu^i$ and $S^i = \sigma N^i - (\tau\Pi^2/2\xi T)u^i$, where n is the number density, σ is the specific entropy, $T \geq 0$ is the temperature, ξ is the bulk viscosity coefficient and $\tau \geq 0$ is the relaxation coefficient for transient bulk viscous effect (i.e. the relaxation time). The evolution of the cosmological fluid is subject to the dynamical laws of particle number conservation $N^i_{;i} = 0$ and Gibb's equation $Td\sigma = d(\rho/n) + pd(1/n)$ [31,38]. In the following we shall also suppose that the energy-momentum tensor of the cosmological fluid is conserved, that is $T^k_{i;k} = 0$, where ; denotes the covariant derivative with respect to the metric.

The gravitational field equations together with the continuity equation $T^k_{i;k} = 0$ imply

$$3H^2 = \rho, \tag{3}$$

$$2\dot{H} + 3H^2 = -p - \Pi, \tag{4}$$

$$\dot{\rho} + 3(\rho + p)H = -3H\Pi, \tag{5}$$

where $H = \dot{a}/a$ is the Hubble parameter.

For the evolution of the bulk viscous pressure Π we adopt the truncated evolution equation [31,38], obtained in the simplest way (linear in Π) to satisfy the H-theorem (i.e. for the entropy production to be nonnegative, $S^i_{;i} = \Pi^2/\xi T \geq 0$ [35]. The evolution equation for Π is given in the framework of the truncated Israel-Stewart theory by [31]

$$\tau\dot{\Pi} + \Pi = -3\xi H, \tag{6}$$

where ξ the bulk viscosity coefficient and τ the relaxation time. The truncated equation is a good approximation of the full causal transport equations if the condition

$|\Pi d(a^3\tau/\xi T)/dt| << a^3 H/T$ holds [31,38]. In order to close the system of Eqs. (4) and (6) we have to give the equation of state for p and specify τ and ξ.

We assume that the isotropic pressure p of the cosmological fluid obeys a modified Chapylin gas equation of state [40],

$$p = \gamma\rho - \frac{B}{\rho^n}, \tag{7}$$

where $0 \le \gamma \le 1$ and $0 \le n \le 1$. B is a positive constant.

When $\gamma = 1/3$ and the comoving volume of the Universe is small ($\rho \to \infty$), this equation of state corresponds to a radiation dominated era. When the density is small, $\rho \to 0$, the equation of state corresponds to a cosmological fluid with negative pressure (the dark energy). Generally the modified Chaplygin equation of state corresponds to a mixture of ordinary matter and dark energy. For $\rho = (B/\gamma)^{1/(n+1)}$ the matter content is pure dust with $p = 0$. The speed of sound $v_s = (\partial p/\partial \rho)^{1/2}$ in the Chaplygin gas is given by

$$v_s^2 = \gamma(1+n) - \frac{np}{\rho}. \tag{8}$$

For the bulk viscosity coefficient and for the relaxation time of the viscous Chaplygin gas we assume the following phenomenological laws

$$\xi = \alpha\rho^s, \qquad \tau = \xi\rho^{-1} = \alpha\rho^{s-1}, \tag{9}$$

where $0 \le \gamma \le 1$, $\alpha \ge 0$, and $s \ge 0$ are constants [41]. Equations (9) are standard in cosmological models, whereas the equation for τ is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light.

The truncated Israel-Stewart theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, that is the nonequilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure $|\Pi| << p = \gamma\rho - B/\rho^n$. If this condition is violated then one is effectively assuming that the linear theory holds also in the nonlinear regime far from equilibrium. However, for a fluid description of the matter, the condition ought to be satisfied.

To see if a cosmological model accelerates or not it is convenient to introduce the deceleration parameter

$$q = \frac{dH^{-1}}{dt} - 1 = \frac{\rho + 3p + 3\Pi}{2\rho} = \frac{1}{2} + \frac{3\gamma(n+1)}{2n}\left[1 - \frac{v_s^2}{(n+1)\gamma}\right] + \frac{3\Pi}{2\rho}. \tag{10}$$

The positive sign of the deceleration parameter corresponds to standard decelerating models whereas the negative sign indicates accelerated expansion.

By using the assumptions given by Eqs. (9) for the bulk viscosity coefficient and the relaxation time, the evolution equation for the Hubble parameter H for the viscous dissipative Chaplygin gas dominated flat homogeneous cosmological models is obtained from the field equations as

$$\ddot{H} + \left[3(\gamma+1)H + \frac{nB}{3^n}H^{-2n-1} + \frac{3^{1-s}}{\alpha}H^{2-2s}\right]\dot{H} - \frac{3^{1-n-s}}{2\alpha}BH^{2-2s-2n} + \frac{3^{2-s}(\gamma+1)}{2\alpha}H^{4-2s} - \frac{9}{2}H^3 = 0. \tag{11}$$

III. ITERATIVE SOLUTIONS OF THE EVOLUTION EQUATION

In order to obtain a simpler form of Eq. (11) we introduce the dimensionless functions θ and h by means of the definitions

$$H = (3^s\alpha)^{1/(1-2s)}h, \qquad \theta = \frac{3}{\sqrt{2}}(3^s\alpha)^{1/(1-2s)}t, \qquad s \neq \frac{1}{2}, \tag{12}$$

and we denote $\lambda_0 = (B/3^{n+1})(3^s\alpha)^{-2(1+n)/(1-2s)}$, $s \neq 1/2$.

In these variables Eq. (11) takes the form

$$\frac{d^2h}{d\theta^2} + \sqrt{2}[(\gamma+1)h + n\lambda_0 h^{-1-2n} + h^{2(1-s)}]\frac{dh}{d\theta} - \lambda_0 h^{2(1-s-n)} + (\gamma+1)h^{2(2-s)} - h^3 = 0, \qquad s \neq \frac{1}{2}. \tag{13}$$

Introducing the new variables $h = \sqrt{y}$ and $\eta = \int \sqrt{y}d\theta$, respectively, Eq. (13) becomes

$$\frac{d^2y}{d\eta^2} + \sqrt{2}(\gamma+1)\frac{dy}{d\eta} - 2y + \sqrt{2}(n\lambda_0 y^{-n-1} + y^{(1/2)-s})\frac{dy}{d\eta} - 2\lambda_0 y^{(1/2)-s-n} + 2(\gamma+1)y^{(3/2)-s} = 0, \qquad s \neq \frac{1}{2}. \tag{14}$$

Therefore for a fixed equation of state and known values of s and n the evolution of the viscous Chaplygin gas cosmological models is determined by a single numerical parameter λ_0.

Because of the complicated nonlinear character of the evolution Eq. (14), it is very difficult to obtain exact solutions of this equation in the framework of the truncated Israel-Stewart theory. The cosmological model presented above could be robust if the cosmological solutions of Eq. (14), depicting the causal bulk viscous FRW spacetime, could be studied for an arbitrary range of values of s, n, and γ in the hope of leading to the possibility of correct physical description of a well-determined period in the evolution of our universe. By using the Laplace transform and convolution theorem, the differential Eq. (14) is equivalent with the following integral equation:

$$y(\eta) = \int_0^\eta F(\eta - x)[\sqrt{2}(n\lambda_0 y^{-n-1} + y^{(1/2)-s})\frac{dy}{d\eta} - 2\lambda_0 y^{(1/2)-s-n} + 2(\gamma+1)y^{(3/2)-s}]dx + y_0(\eta), \tag{15}$$

where

$$F(\eta - x) = \frac{1}{2\delta}[e^{-(\delta+((\gamma+1)/\sqrt{2}))(\eta-x)} - e^{(\delta-((\gamma+1)/\sqrt{2}))(\eta-x)}], \tag{16}$$

$$y_0(\eta) = e^{-((\gamma+1)/\sqrt{2})\eta}(Me^{-\delta\eta} + Ne^{\delta\eta}), \tag{17}$$

$$M = \frac{n_+ y(0) - y^*(0)}{2\delta}, \qquad N = \frac{y^*(0) - n_- y(0)}{2\delta}, \tag{18}$$

and we denoted $\delta = \sqrt{\gamma^2 + 2\gamma + 5}/\sqrt{2}$, $n_\pm = -(\gamma+1)/\sqrt{2} \pm \delta$, and $y^*(0) = (dy/d\eta)_{\eta=0}$, respectively.

The solution of the integral Eq. (15) can be easily obtained by using the method of successive approximations or method of iteration to obtain a solution to any desired accuracy. Taking as an initial approximation the solution of the linear part of Eq. (14), the general solution of the integral Eq. (15) can be expressed in the first and mth order approximation, $m \in N$, as follows:

$$y_1(\eta) = \int_0^\eta F(\eta - x)[\sqrt{2}(n\lambda_0 y_0^{-n-1}(x) + y_0^{(1/2)-s}(x))y_0'(x) - 2\lambda_0 y_0^{(1/2)-s-n}(x) + 2(\gamma+1)y_0^{(3/2)-s}(x)]dx + y_0(\eta), \tag{19}$$

$$\cdots$$

$$\cdots$$

$$y_m(\eta) = \int_0^\eta F(\eta - x)[\sqrt{2}(n\lambda_0 y_{m-1}^{-n-1}(x) + y_{m-1}^{(1/2)-s}(x))y_{m-1}'(x) - 2\lambda_0 y_{m-1}^{(1/2)-s-n}(x) + 2(\gamma+1)y_{m-1}^{(3/2)-s}(x)]dx + y_{m-1}(\eta), \tag{20}$$

$$y(\eta) = \lim_{m\to\infty} y_m(\eta). \tag{21}$$

We can express the iterative solutions of the gravitational field equations for a bulk viscous fluid filled FRW universe in the framework of the truncated Israel-Stewart theory for $s \neq 1/2$ in the following parametric form (in the following equations we write σ for the variable of integration in order to distinguish it from the independent variable):

$$\theta - \theta_0 = \int_{\eta_0}^{\eta} \frac{1}{\sqrt{y(\sigma)}} d\sigma, \qquad a = a_0 e^{(\sqrt{2}/3)\eta}, \tag{22}$$

$$\rho = \rho_0 y(\eta), \qquad p = \rho_0\left[\gamma y(\eta) - \frac{\lambda_0}{y^n(\eta)}\right], \tag{23}$$

$$\xi = \sqrt{\frac{\rho_0}{3}} y^s(\eta), \qquad \tau = \sqrt{\frac{\rho_0}{27}} y^{s-1}(\eta), \tag{24}$$

$$q = -\frac{3}{2\sqrt{2}}\frac{1}{y(\eta)}\frac{dy}{d\eta} - 1, \tag{25}$$

$$\Pi = \rho_0\left[-(\gamma+1)y(\eta) + \frac{\lambda_0}{y^n(\eta)} - \frac{1}{\sqrt{2}}\frac{dy}{d\eta}\right], \tag{26}$$

where a_0, and t_0 are arbitrary constants of integration, and we denoted $\rho_0 = 3(3^s\alpha)^{2/(1-2s)}$.

In order to solve the evolution equation iteratively we need to chose the initial conditions for the cosmological model. The initial value of the function $y(\eta)$ can be obtained by fixing the initial value of the Hubble function or, equivalently, of the density, by using the equation $y(0) = 3H^2(0)/\rho_0 = \rho(0)/\rho_0$. The initial value of $dy/d\eta|_{\eta=0}$ can be obtained by fixing the initial value of the deceleration parameter, so that $dy/d\eta|_{\eta=0} = -(2\sqrt{2/3})[q(0)+1]$. In this way the mathematical initial conditions are fixed by the physical characteristics of the Universe.

The behavior of the cosmological parameters of the bulk viscous Chaplygin gas filled homogeneous and isotropic dust universe, with $\gamma = 0$, and, consequently, $p = -B/\rho^n$, are represented for some fixed values of n and s and for different values of λ_0 in Figs. 1–5. In Fig. 1 the evolution of the scale factor a is represented as a function of the dimensionless time θ.

The time variation of the density of the matter is plotted against the time in Fig. 2. In the expanding universe the density is a monotonically decreasing function of the cosmic time.

The behavior of the bulk viscous pressure Π of the Chaplygin gas is shown in Fig. 3. The negative bulk

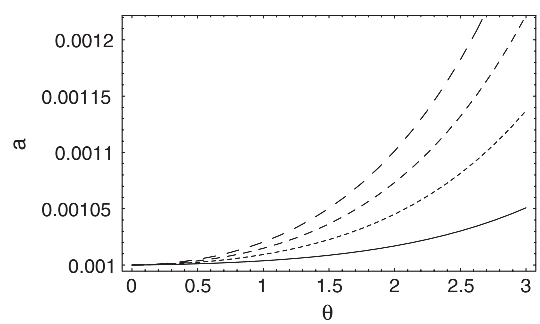

FIG. 1. The scale factor a of the bulk viscous Chaplygin gas filled homogeneous and isotropic dust universe ($\gamma = 0$) as a function of the dimensionless time $\theta = \alpha^{1/(1-2s)}t$, for $n = 0.1$, $s = 1/4$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

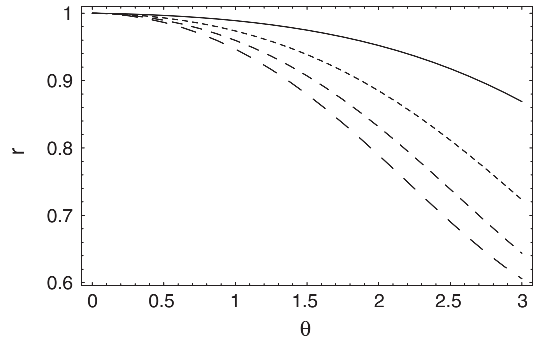

FIG. 2. The dimensionless density function $r = \rho/\alpha^{2/(1-2s)}$ of the bulk viscous Chaplygin gas filled homogeneous and isotropic dust universe ($\gamma = 0$) as a function of the dimensionless time $\theta = \alpha^{1/(1-2s)}t$, for $n = 0.1$, $s = 1/4$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

viscous pressure gives a significant contribution to the total negative pressure of the Chaplygin gas.

The time variation of the bulk viscosity coefficient of the Chaplygin gas is represented in Fig. 4. Similarly to the energy density, the bulk viscosity coefficient is a monotonically decreasing function of time.

The time variation of the deceleration parameter q is represented in Fig. 5. In the limit of the large times $q < 0$, showing that the viscous Chaplygin gas filled universe experiences an accelerated cosmological dynamics. For large values of θ, after experiencing a superaccelerated phase with $q < -1$, the viscous Chaplygin gas filled universe ends in a de Sitter regime, with $q = -1$.

When the bulk viscosity coefficient ξ is proportional to the square root of the density, $\xi \sim \rho^{1/2}$, that is, for $s = 1/2$, the transformations introduced in Eqs. (12) cannot be applied. In this case a set of dimensionless variable is given by

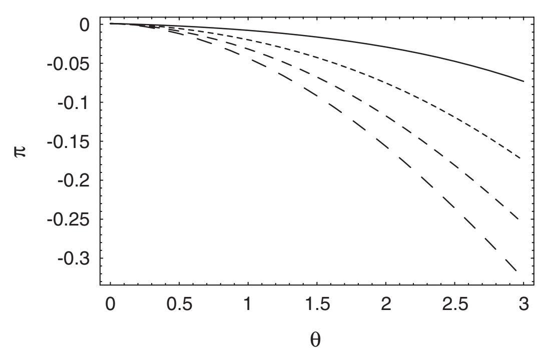

FIG. 3. The dimensionless bulk viscous pressure $\pi = \Pi/\alpha^{2/(1-2s)}$ of the Chaplygin gas for a homogeneous and isotropic dust universe ($\gamma = 0$) as a function of the dimensionless time $\theta = \alpha^{1/(1-2s)}t$, for $n = 0.1$, $s = 1/4$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

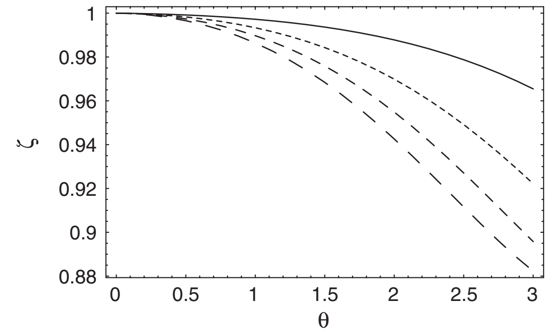

FIG. 4. The dimensionless bulk viscosity coefficient $\zeta = \xi/\alpha$ of the Chaplygin gas for a homogeneous and isotropic dust universe ($\gamma = 0$) as a function of the dimensionless time $\theta = \alpha^{1/(1-2s)}t$, for $n = 0.1$, $s = 1/4$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

$$H = \left(\frac{3^{n+1}}{B}\right)^{-(1/(2(1+n)))} h, \qquad \theta = \frac{3}{\sqrt{2}}\left(\frac{3^{n+1}}{B}\right)^{-(1/(2(1+n)))} t, \tag{27}$$

while the dynamics of the Universe is determined by the parameter $\chi = 1/\sqrt{3}\alpha$. In these variables and for $s = 1/2$ the evolution Eq. (11) takes the form

$$\frac{d^2h}{d\theta^2} + \sqrt{2}[(\gamma + 1 + \chi)h + nh^{-1-2n}]\frac{dh}{d\theta} - \chi h^{1-2n} + [(\gamma + 1)\chi - 1]h^3 = 0, \qquad s = \frac{1}{2}. \tag{28}$$

The transformations $h = \sqrt{y}$ and $\eta = \int \sqrt{y} d\theta$ reduces Eq. (28) to

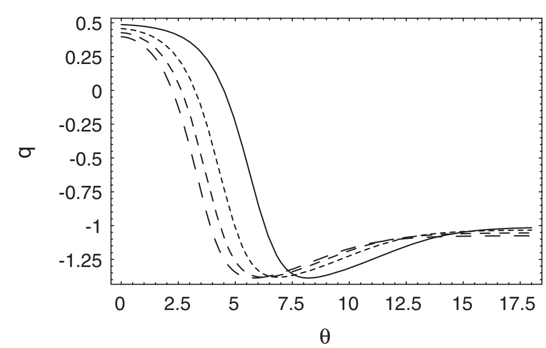

FIG. 5. The deceleration parameter q of the bulk viscous Chaplygin gas filled homogeneous and isotropic dust universe ($\gamma = 0$) as a function of the dimensionless time $\theta = \alpha^{1/(1-2s)}t$, for $n = 0.1$, $s = 1/4$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

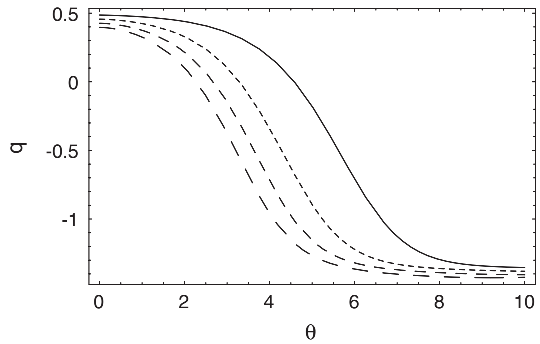

FIG. 6. The deceleration parameter q of the homogeneous and isotropic dust universe ($\gamma = 0$) filled with a viscous Chaplygin gas, with bulk viscosity proportional to the square root of the energy density ($s = 1/2$) as a function of the dimensionless time $\theta = t/\sqrt{\alpha}$, for $n = 0.1$ and different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

$$\frac{d^2y}{d\eta^2} + \sqrt{2}\left[\gamma + 1 + \chi + ny^{-1-n}\right]\frac{dy}{d\eta} - 2\chi y^{-n} + 2[(\gamma + 1)\chi - 1]y = 0. \quad (29)$$

The general behavior of the viscous Chaplygin gas models with $s = 1/2$ is qualitatively similar to the case $s \neq 1/2$. Therefore we present only the time evolution of the deceleration parameter q, which is shown in Fig. 6.

In the limit of large times the viscous Chaplygin universe with $s = 1/2$ ends in a superaccelerated state, with $q \approx -1.25$.

IV. COMPARISON WITH OBSERVATIONAL DATA

From an observational point of view, fundamental tests of cosmological models can be performed from the study of the propagation in a curved space-time of the light emitted by a source in a distant galaxy (like, for example, a supernova) and detected on Earth on a telescope mirror. The luminosity of the source is defined as $L = dE_{\rm em}/dt_{\rm em}$, that is, the luminosity is the total energy emitted by the source in unit time; the suffix em refers to emission. A telescope detects a photon flux $F = dE_{\rm rec}/dt_{\rm rec}/A_M$, where the suffix rec refers to reception. The flux is the energy detected on the telescope mirror surface A_M (assumed to be perpendicular to the incident light beam) per unit time interval [42].

An important observational parameter, the redshift z is defined as $1 + z = a_0/a$, where a_0 is the present day value of the scale factor, which is usually conventionally taken as 1, $a_0 = 1$. From the definition of z we obtain $da/dt = -[a_0/(1 + z)^2]dz/dt$. From the definition of the Hubble function we have $H = (1/a)(da/dt) = -[1/(1 + z)] \times (dz/dt)$, which gives $dz/dt = -(1 + z)H$. Because of the cosmological expansion the elementary area changes as a^2 and the frequency ω of the light is redshifted during the cosmic evolution so that $\omega \propto 1/a$. Therefore $F/L = (1/A_{\rm tot})(a/a_0)^2$, where $A_{\rm tot}$ represents the proper area of a sphere centered in the light source and containing at the time of reception the reception point on its surface. The luminosity distance is defined as [42]

$$d_L(z) = \sqrt{\frac{L}{4\pi F}} = a_0 r_{\rm em}(1 + z), \quad (30)$$

where $r_{\rm em}$ is the comoving radius. The comoving coordinate $r_{\rm em}$ can be written in terms of another radial comoving coordinate $\chi_{\rm em}$, so that

$$\chi_{\rm em} = \chi(a_{\rm em}) = \int_{a_{\rm em}}^{a_0} \frac{da}{a^2H(a)} = \frac{1}{a_0}\int_0^z \frac{dz'}{H(z')}. \quad (31)$$

In a flat ($k = 0$) FRW geometry we have $d_L(z) = a_0(1 + z)\chi_{\rm em}$. The luminosity distance-redshift relation is given by [42]

$$d_L(z) = (1 + z)\int_0^z \frac{dz'}{H(z')}. \quad (32)$$

In the case of a bulk viscous Chaplygin gas filled universe the luminosity distance $d_L(z)$ can be obtained by simultaneously solving the following system of differential equations, with z as an independent variable

$$-2(1 + z)H\frac{dH}{dz} + 3H^2 = -\gamma\rho + \frac{B}{\rho^n} - \Pi, \quad (33)$$

$$-(1 + z)\frac{d\rho}{dz} + 3\left[(1 + \gamma)\rho - \frac{B}{\rho^n}\right] = -3\Pi, \quad (34)$$

$$-\alpha(1 + z)\rho^{s-1}H\frac{d\Pi}{dz} + \Pi = -3\alpha\rho^sH, \quad (35)$$

and

$$\frac{dd_L(z)}{dz} - \frac{1}{1 + z}d_L(z) = \frac{1 + z}{H(z)}, \quad (36)$$

respectively. In order to simplify this system we introduce a set of dimensionless variables, defined as

$$H(z) = H_0h(z), \qquad \rho(z) = 3H_0^2r(z),$$
$$\Pi(z) = 3H_0^2\pi(z), \qquad d_L(z) = \frac{D_L(z)}{H_0}. \quad (37)$$

We denote $\lambda = B/3^nH_0^{2n+2}$, and choose α so that $3^{s-1}\alpha H_0^{2s-1} = 1$. Substitution into Eqs. (33)–(36) transform these equations into the form

$$-2(1 + z)h\frac{dh}{dz} + 3h^2 = -3\gamma r + \frac{\lambda}{r^n} - 3\pi, \quad (38)$$

$$-(1 + z)\frac{dr}{dz} + 3\left[(1 + \gamma)r - \frac{\lambda}{3r^n}\right] = -3\pi, \quad (39)$$

$$-(1+z)r^{s-1}h\frac{d\pi}{dz}+\pi=-3r^{s}h, \tag{40}$$

and

$$\frac{dD_L(z)}{dz}-\frac{1}{1+z}D_L(z)=\frac{1+z}{h(z)}, \tag{41}$$

respectively. The initial conditions for the system of Eqs. (38)–(41) are $h(0)=1$, $r(0)=1$, $\pi(0)=\pi_0$, and $D_L(0)=0$, respectively. In the equation of state one can take $\gamma=0$. The physical luminosity distance can be written as $d_L(z;\lambda;n;s)=H_0^{-1}D_L(z;\lambda;n;s)$. Once the dimensionless function $D_L(z;\lambda;n;s)$ is known from the numerical integration of the system, the fitting with the observational data will fix the numerical values of the parameters λ, n, s.

The function $d_L(z)$ can be measured for distant type Ia supernovae. The luminosity is evaluated by photometry, while the redshift is evaluated from the spectroscopic analysis of the host galaxy. Each cosmological model has its own prediction for the function $d_L(z)$. Therefore the measured $d_L(z)$ data are powerful tests of the cosmological models, and the luminosity distance can be used to fit the free parameters of the model by using the observational results.

Riess *et al.* [43] recently published a new set of 182 gold supernovae, including new Hubble Space Telescope observations, and recalibrations of the previous measurements.

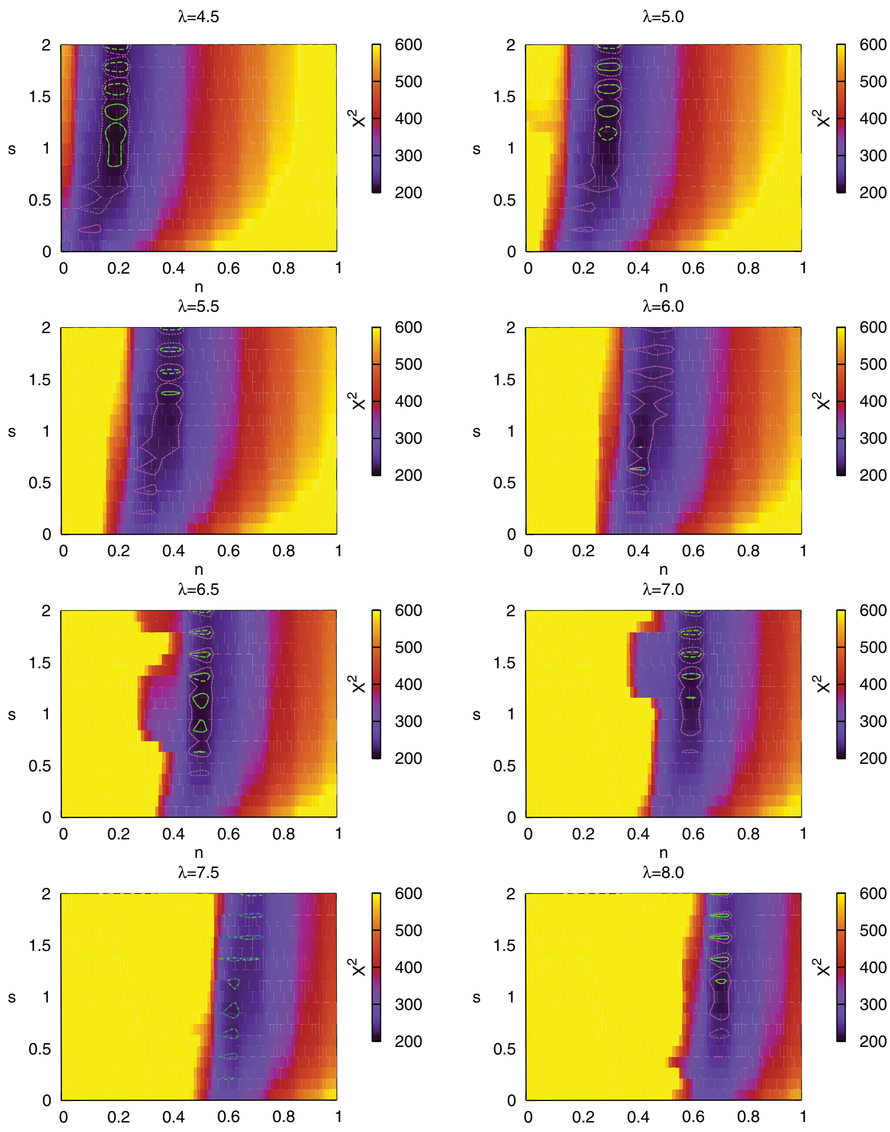

FIG. 7 (color). The fit of the luminosity distance—redshift relation of the bulk viscous Chaplygin gas model to the Gold 2006 [43] supernova data. Different panels show models for different λ values, the inner (magenta) contours indicate the $1-\sigma$ confidence regions, the outer (turquoise) contours border the $2-\sigma$ confidence regions. For all plotted values of λ an accurate value of n with a good fit can be established, while the fit remains acceptable for a wide range of the parameter s.

TABLE I. The preferred n values and the variation with s for the probed λ values.

λ	n	s
4.0	0.1	0.8 ($s > 0.2$)
4.5	0.2	0.8 ($s > 0.3$)
5.0	0.25	0.8 ($s > 0.5$)
5.5	0.35	$s > 1$
6	0.4	1.7 ($s > 0.8$)
6.5	0.5	$s > 0.7$
7	0.6	1 ($s > 0.7$)
7.5	0.65	poor fit for any s
8	0.7	$s > 1$

We applied the same tests to the Gold 2006 data set [43], as described in [42], by taking $\gamma = 0$. Here n and s were adjusted, and fixed values of $\lambda = 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0$ were taken into account. For any probed values of λ, there exists a typical value of n, where a wide range of s offers a fit to the supernovae data, as shown in Fig. 7. As all panels in Fig. 7 show, the acceptable fits occur in several disjunct regions, which are slightly separated by s. The best-fit n value was searched in the marginal projections of n ($0 < s < 2$), instead of finding a global minimum. The varying s caused only slight modifications of the fit, which is characterized by the marginal projection of s *within* the $2-\sigma$ confidence region.

The preferred n values and the variation with s for the probed λ values are represented in Table I. The first column represents the values of λ we have used for comparison. The best fits for n are represented in the second column, while the best-fit values for s as well as the allowed range of the parameter is represented in the third column. In the cases $\lambda = 5.5$ and $\lambda = 8.0$, respectively, a value of $s > 1$ is required, and the fit quality increases with increasing s.

These results show that the model behaves with some complexity in the prediction of the luminosity distance, as the parameters do not behave monotonically (with the exception of n). However, the fitting ellipsoids are well defined (see Fig. 7), and in general the predictions of the model fit well the supernova data.

V. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered the dynamics of a bulk viscous Chaplygin gas filled flat homogeneous and isotropic universe. We have derived and formulated the evolution equations of the system, we have considered their behavior by using both analytical and numerical techniques, and we have compared the predictions of our model with the supernova data. The most attractive feature of the Chaplygin gas is that it could explain the main observational properties of the Universe without appealing to an effective cosmological constant. Generally, the obtained analytical and numerical solutions of the gravitational field equations describes an accelerating universe, with the effective negative pressure induced by the Chaplygin gas and the bulk viscous pressure driving the acceleration.

From the equation of state of the Chaplygin gas with $\gamma = 0$ it follows that for the critical values p_c and ρ_c of the pressure and density the parameter $w_c = p_c/\rho_c$ is given by $w_c = -B/\rho_c^{n+1} - \Pi_c/\rho_c$. Evaluating this relation at the present time when $\rho_c = \rho_{c0}$ gives $B = -w_{c0}\rho_{c0}^{n+1} - \Pi(\rho_{c0})\rho_{c0}^{n}$. The Chaplygin gas behaves like a cosmological constant for $w_{c0} = -1$, which gives the relation between the constant B and the present day value of the bulk viscous pressure as

$$B = \rho_{c0}^{n+1}\left[1 - \frac{\Pi(\rho_{c0})}{\rho_{c0}}\right] = \left(\frac{3H_0^2}{8\pi G}\right)^{n+1}\left[1 - \frac{\Pi(\rho_{c0})}{\rho_{c0}}\right], \tag{42}$$

where $H_0 = 3.24 \times 10^{-18}\ h\mathrm{s}^{-1}$, $0.5 \leq h \leq 1$ is the Hubble constant [3,4]. Since $\Pi(\rho_{c0}) < 0$, the presence of the bulk viscous effects can significantly increase the value of B.

By comparing the model with $\gamma = 0$ to the Gold 2006 supernova data, it turns out that a good agreement with these observations can be established for a wide range of the power $s \in (0.2, 2)$ which occurs in the phenomenological laws (9), which characterize the bulk viscosity coefficient. The other viscosity parameter α can be obtained from the equation $3^{s-1}\alpha H_0^{2s-1} = 1$, and by choosing a value for the Hubble parameter. For $h = 0.7$ ($H_0 = 2.268 \times 10^{-18}\ \mathrm{s}^{-1}$) we obtain $\alpha = (6.2385 \times 10^{-11}\ \mathrm{s}^{-0.6}, 2.8573 \times 10^{52}\ \mathrm{s}^3)$ for the above-established range of the parameter s. As for the equation of state of the Chaplygin gas, by taking into account the definition of λ, $\lambda = B/3^n H_0^{2n+2}$, and for the same value of the Hubble parameter, the confrontation with supernova data selects the pairs (n, B) represented in Fig. 8.

Scalar fields are supposed to play a fundamental role in the evolution of the early universe. The Chaplygin gas

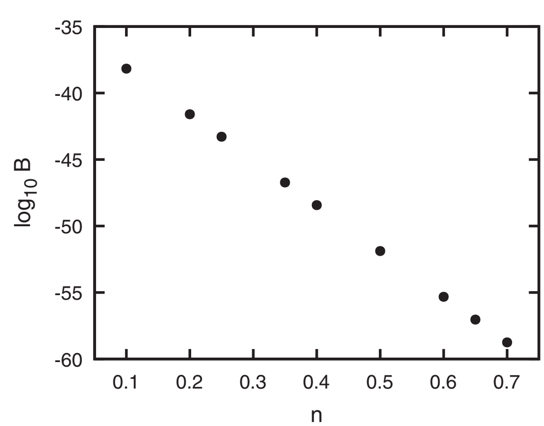

FIG. 8. The parameter values B corresponding to the best-fit values (λ, n) represented in a logarithmic scale as a function of n.

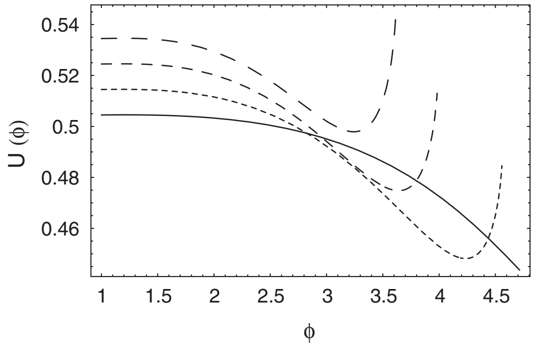

FIG. 9. The potential $U(\phi)$ of the viscous Chaplygin gas associated scalar field as a function of the scalar field ϕ for a dust universe ($\gamma = 0$), $n = 0.1$, $s = 1/4$, and for different values of λ_0: $\lambda_0 = 0.01$ (solid curve), $\lambda_0 = 0.03$ (dotted curve), $\lambda_0 = 0.05$ (dashed curve), and $\lambda_0 = 0.07$ (long dashed curve).

model can be also described from a field theoretical point of view by introducing a scalar field ϕ and a self-interacting potential $U(\phi)$, with the Lagrangian [13,14,19,40,44]

$$L_\phi = \frac{1}{2}\dot{\phi}^2 - U(\phi). \tag{43}$$

The energy density and the pressure associated to the scalar field ϕ associated to the bulk viscous Chaplygin gas are given by

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + U(\phi) = \rho, \tag{44}$$

and

$$p_\phi = \frac{\dot{\phi}^2}{2} - U(\phi) = \gamma\rho - \frac{B}{\rho^n} + \Pi, \tag{45}$$

respectively.

The scalar field and the potential can be obtained from the equations

$$\phi(t) - \phi_0 = \int_{t_0}^{t} \sqrt{(1+\gamma)\rho - \frac{B}{\rho^n} + \Pi}\, dt, \tag{46}$$

and

$$U(t) = \frac{1}{2}\left[(1-\gamma)\rho + \frac{B}{\rho^n} - \Pi\right], \tag{47}$$

respectively, where ϕ_0 is an arbitrary constant of integration.

The dependence of the potential $U(\phi)$ on the scalar field ϕ is represented in Fig. 9.

In conclusion, we have found that the viscous Chaplygin gas model offers a real possibility for replacing the effective cosmological constant and to explain the recent acceleration of the universe.

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