Age-dependent decay in the landscape

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The picture of the "multiverse" arising in diverse cosmological scenarios involves transitions between metastable vacuum states. It was pointed out by Krauss and Dent that the transition rates decrease at very late times, leading to a dependence of the transition probability between vacua on the age of each vacuum region. I investigate the implications of this non-Markovian, age-dependent decay on the global structure of the spacetime in landscape scenarios. I show that the fractal dimension of the eternally inflating domain is precisely equal to 3, instead of being slightly below 3, which is the case in scenarios with purely Markovian, age-independent decay. I develop a complete description of a non-Markovian landscape in terms of a nonlocal master equation. Using this description I demonstrate by an explicit calculation that, under some technical assumptions about the landscape, the probabilistic predictions of our position in the landscape are essentially unchanged, regardless of the measure used to extract these predictions. I briefly discuss the physical plausibility of realizing non-Markovian vacuum decay in cosmology in view of the possible decoherence of the metastable quantum state.

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I. INTRODUCTION

According to today's accepted cosmological data, the universe is now undergoing accelerated expansion with an approximately constant Hubble rate H_{now}. However, models of string theory suggest that this accelerating state may be merely a metastable vacuum that is destined, after a long time, to decay via quantum tunneling into other states with different values of H. The recently developed paradigm of "string theory landscape" [1] involves a very large number of metastable vacua, corresponding to local minima of an effective potential in field space. The value of the potential at each minimum determines the effective Hubble rate in the corresponding vacuum. A similar scenario combining inflationary evolution and tunneling was proposed earlier in Ref. [2] under the name of "recycling universe." In all these scenarios, the universe becomes a "multiverse," that is, an infinite ensemble of large, causally disconnected spatial regions. Some of these regions contain galaxies and stars, while other regions are undergoing inflation and generating new vast domains of space. Each spatial domain may be in a metastable vacuum state with a sufficiently long decay time, so that reheating can occur and the standard cosmological evolution can proceed before the transition to a different vacuum state.

The theory allows us, in principle, to determine the set of possible vacua but does not predict our position in the landscape with certainty. After many transitions, the position of our observable patch of the universe in the landscape becomes random. Nevertheless, one would like to explain the present value of the cosmological constant and possibly other observables. Therefore one attempts to calculate the *probability* of being in a vacuum of a given kind, for a "typical" observer. It is notoriously difficult to formulate an unambiguous and well-behaved measure on the set of all possible observers such that the typical observers are selected without bias; see e.g. [3,4] for a recent discussion and Refs. [5–8] for reviews of the proposals of observer-based measure.

In this paper I study a different aspect of the measure problem. All currently proposed measures are based on the assumption that the decay of a metastable state proceeds independently of the individual age of that state. In other words, it is assumed that the random process of transitions between different states in the landscape is a Markov chain. Markovian transition probabilities are determined only by the current state and have no memory of previous transitions. (The "memory" effect due to bubble collisions [9] does not modify transition probabilities.) Vacuum decay proceeds through bubble nucleation and is normally described via the nucleation rate per unit 4-volume [10,11],

$$\Gamma^{(4D)} = O(1)H^4 \exp\left[-S_I - \frac{\pi}{H^2}\right], \tag{1}$$

where S_I is the relevant instanton action and H is the Hubble rate of the parent vacuum (I use the Planck units throughout the paper). The transition rates between different metastable vacua can be considered (in principle) known in a given model of the landscape. For a fixed 3-volume V, the probability of nucleating no bubbles after time t is exponentially small, $\propto \exp[-\Gamma^{(4D)}Vt]$.

A statistical description of evolution in the landscape can be obtained [2,12] by considering the fraction $f_\alpha(t)$ of the comoving volume occupied by bubbles of type α at time t. One can approximate the transition to a different vacuum as a series of random nucleation events, each event resulting in an instantaneous conversion of a volume H_α^{-3} of vacuum type α into the same volume of vacuum type β. The rate of this conversion per unit time, denoted $\kappa_{\alpha\to\beta}$,

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can be computed according to Eq. (1) with appropriate normalization factor,

$$\kappa_{\alpha\to\beta} = O(1)H_\alpha \exp\left[-S_{\alpha\to\beta} - \frac{\pi}{H_\alpha^2}\right]. \tag{2}$$

Defining for convenience $\kappa_{\alpha\to\alpha} \equiv 0$, one then writes the master equation describing the evolution of $f_\alpha(t)$,

$$\frac{df_\alpha}{dt} = \sum_\beta (\kappa_{\beta\to\alpha} f_\beta - \kappa_{\alpha\to\beta} f_\alpha). \tag{3}$$

This equation can be solved with initial conditions $f_\alpha(0)$. Some measure prescriptions based on the comoving distribution were proposed in Refs. [13–15].

Another useful distribution is the 3-volume $V_\alpha(t)$ of spatial regions within bubbles of type α at time t. The evolution equation for $V_\alpha(t)$ differs from Eq. (3) by the volume expansion factors,

$$\frac{dV_\alpha}{dt} = 3H_\alpha V_\alpha + \sum_\beta (\kappa_{\beta\to\alpha} f_\beta - \kappa_{\alpha\to\beta} f_\alpha) \tag{4}$$

$$\equiv \sum_\beta M_{\alpha\beta} V_\beta, \tag{5}$$

where the matrix $M_{\alpha\beta}$ is defined by

$$M_{\alpha\beta} \equiv \left(3H_\alpha - \sum_\lambda \kappa_{\alpha\to\lambda}\right)\delta_{\alpha\beta} + \kappa_{\beta\to\alpha}. \tag{6}$$

The volume-weighted master equations are used in volume-based measure prescriptions (e.g. Refs. [12,16]).

All the existing measure prescriptions depend on the properties of the *late-time* behavior of the distributions $f_\alpha(t)$ and $V_\alpha(t)$. The late-time asymptotics of the solutions of Eqs. (3) and (4) are always exponential. For instance, the volume distribution has the late-time asymptotics $V_\alpha \propto c_\alpha e^{\gamma t}$, where $\gamma > 0$ is the dominant eigenvalue of the matrix $M_{\alpha\beta}$. The values of the coefficients c_α are determined by the right eigenvector of $M_{\alpha\beta}$ corresponding to the eigenvalue γ.

Recently, Krauss and Dent [17] called attention to the fact that the decay of metastable states becomes subexponential at very late times. In typical quantum-mechanical metastable systems in d-dimensional space, the probability of not decaying (the "survival probability") initially decreases exponentially as $e^{-\Gamma t}$, where Γ is the decay rate, but eventually starts falling off as t^{-d} after a (very long) crossover time $T \sim 5\Gamma^{-1}\ln(E/\Gamma)$, where E is the energy difference between the metastable state and the final stable state. Effectively, the tunneling rate for all transitions between states goes to zero as $\Gamma(t) \propto t^{-1}$ after a (state-dependent) crossover time. It is important to note that the transition dynamics depends on the "age" of the current state, i.e. on the time elapsed since the last transition. With this modification, the transition process becomes a non-Markov random walk, and Eqs. (3) and (4) no longer apply. In particular, the late-time asymptotics of the bubble distributions $f_\beta(t)$ and $V_\beta(t)$ are no longer purely exponential. For this reason it is interesting to investigate the implications of the non-Markov transitions for the measure calculations, which depend in an essential way on the late-time behavior of $f_\beta(t)$ and $V_\beta(t)$.

In this paper I study the evolution of the landscape assuming that the late-time asymptotic of the survival probability becomes subexponential at a state-dependent crossover time. The main results of this first study are as follows. I show that the fractal dimension of the inflating domain is exactly equal to 3, while it is always slightly below 3 in Markovian models. Then I develop an explicit non-Markovian description of the transition dynamics in terms of a master equation that is nonlocal in time. Using that equation, I derive the late-time asymptotics of the volume distributions $V_\alpha(t)$ using the proper time coordinate t. The results show explicitly, within a controlled approximation, that the volume ratios $V_\alpha(t)/V_\beta(t)$ approach a constant at late times and are approximately the same as those computed within the Markovian situation, except for the volume in bubbles of type 0 having the largest Hubble rate $H_0 = \max_\alpha H_\alpha$. The bubbles of type 0 now entirely dominate the volume of the universe at a fixed time t, whereas their volume fraction was large but finite in Markovian scenarios. These results (obtained using the proper time gauge) applied to landscapes where a single vacuum type has the largest Hubble rate of all available vacuum types. I also show that the comoving volume distributions remain essentially unchanged in the non-Markovian regime. This suggests that the results obtained in any measure prescription (whether volume-based or worldline-based) do not need any modification in view of the modified late-time decay. I conclude with a brief discussion of the viability of the non-Markovian assumption in the cosmological context.

II. NON-MARKOVIAN SIERPIŃSKI CARPET

I begin by examining the global structure of the spacetime undergoing non-Markovian vacuum decay. A particular version of the random Sierpiński carpet, or "inflation in a box," was considered in Ref. [18] as a drastically simplified toy model mimicking the global geometry of such a spacetime. In this model, time elapses in discrete steps, and the space is reduced to a two-dimensional square domain $0 < x, y < 1$, where x, y are the *comoving* coordinates. The entire initial Hubble-size domain is assumed to be initially inflating. To imitate inflation during one time step, one subdivides the initial inflating square into $N \times N$ equal subsquares of size $N^{-1} \times N^{-1}$; at the next step, each subsquare will again have the Hubble proper size. Then one randomly marks some of the smaller squares as "thermalized," assuming that each Hubble-size inflating square continues inflation with a probability q (where $0 < q < 1$) and thermalizes with probability $1 - q$, independently of all other squares. The selection of thermalized squares concludes the simulation for one time step. At the next time step, the same procedure of subdivision and random thermalization is applied to each Hubble-sized inflating square, while the "thermalized" squares do not evolve any further (see Fig. 1). This process is continued indefinitely and generates a fractal set of measure zero consisting of points that never enter any thermalized squares (called the "eternal points" in Ref. [19] where rigorous definitions are given). This set represents the eternally inflating subdomain of the spacetime. Under the condition $N^2 q > 1$, the fractal dimension of the eternally inflating domain is $\gamma = 2 + \ln q / \ln N > 0$, and future-eternal inflation occurs with a nonzero probability [18].

As formulated, the model is Markovian since the thermalization probability at each step is independent of the age of the inflating square. The probability of remaining in the inflationary regime (the "survival probability") after t time steps is $q^t = e^{-\alpha t}$, where $\alpha \equiv \ln\frac{1}{q}$. Let us now modify this toy model by assuming that the survival probability is given by a function $S(t)$ that interpolates between the initially exponential falloff $S(t) = e^{-\alpha t}$ for $t \ll T$ and the power-law asymptotic $S(t) \approx S_0 t^{-p}$ for $t \gg T$, where p is a fixed constant and T is the crossover time. We would like to compute the fractal dimension of the set of eternal points in this non-Markovian model.

Let us denote by $X(t)$ the probability of the presence of at least one eternal point within an inflating square at time t. The quantity $X(t)$ can be computed explicitly, but it is sufficient for the present purposes to obtain the asymptotic value of $X(t)$ at $t \to \infty$. Since the thermalization probability per step goes to zero at late times, the value of $X(t)$ approaches 1 as $t \to \infty$. More precisely, $X(t)$ is the nonzero solution of the equation

$$1 - X(t) = p(t) + (1 - p(t))(1 - X)^{N^2}. \qquad (7)$$

An approximate solution of this equation for $p(t) \ll 1$ is $1 - X \approx p(t)$. Since $p(t) \to 0$ as $t \to \infty$, we have $X(t) \to 1$. Hence the average number of inflating squares containing at least one eternal point at a late time t is $\approx S(t) N^{2t}$, while the linear size of each square is N^{-1}. So the fractal dimension of the eternally inflating set is

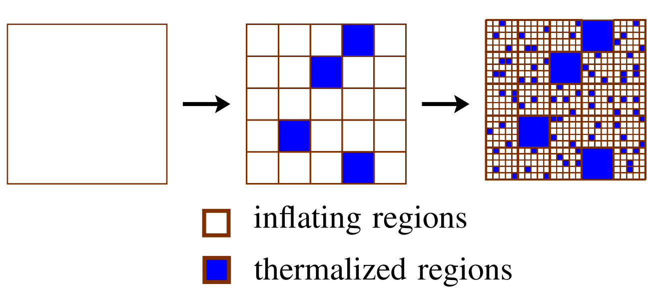

FIG. 1 (color online). First steps in the construction of a random Sierpiński carpet with $N = 5$ and $q = 5/6$.

$$\gamma = 2 - \lim_{t\to\infty} \frac{\ln S(t)}{\ln N^{-t}} = 2 + \lim_{t\to\infty} \frac{\ln(S_0 t^{-p})}{t \ln N} = 2. \qquad (8)$$

It can be shown that the eternally inflating domain consists of an infinite merged cluster when it is formed as a random Sierpiński carpet with fractal dimension 2. It is important to note that the eternal set still has measure zero because every comoving point will reach thermalization with probability 1.

By analogy, one can investigate the eternally inflating domain in a three-dimensional space and conclude that its fractal dimension is 3. A quick argument leading to this conclusion consists of estimating the growth of the 3-volume of the inflating domain as $V(t) \propto e^{3Ht} t^{-p}$ in the regime of power-law decay at late times. A domain growing as $V(t) \propto e^{\gamma Ht}$ is interpreted as a lacunary fractal with dimension γ [18,20], regardless of subexponential corrections. Therefore, the fractal dimension of the inflating domain is always equal to 3 in the non-Markovian case. This is only a small correction to the results obtained in typical scenarios of eternal inflation where the fractal dimension is very slightly below 3 (see, for instance, Refs. [20,21]). Therefore, the global geometry of the spacetime is not significantly modified in these scenarios even if the late-time decay is subexponential.

III. EVOLUTION IN A NON-MARKOVIAN LANDSCAPE

The next issue is whether the results of applying the various landscape measure proposals are modified when non-Markovian decay is assumed. In this section I derive the suitably modified versions of Eqs. (3) and (4) and obtain their late-time asymptotics. Since all the different measure proposals require computing the late-time behavior of these same evolution equations, the results of the present calculation will be equally relevant to every measure proposal.

To describe the evolution of spacetime in a landscape scenario with non-Markovian transitions, one needs to specify the transition rate $\Gamma_{\alpha\to\beta}(t)$ between vacua α and β as a function of the age t of the parent vacuum α. The precise form of $\Gamma_{\alpha\to\beta}(t)$ will be model-dependent except for the properties $\Gamma_{\alpha\to\beta}(t) \approx \kappa_{\alpha\to\beta} = \text{const}$ for $t < T_{\alpha\to\beta}$, where $T_{\alpha\to\beta}$ is the crossover time, and $\Gamma_{\alpha\to\beta}(t) \to 0$ for $t \gg T_{\alpha\to\beta}$. For simplicity I will assume below that the crossover time $T_{\alpha\to\beta} \equiv T$ is independent of α and β. Without this technical assumption, the analysis will be more complicated without yielding significantly different results. If transitions $\alpha \to \beta$ have different crossover times $T_{\alpha\to\beta}$, the results of the present analysis will be approximately applicable at sufficiently late times t such that $t \gg T \equiv \max_{\alpha,\beta} T_{\alpha\to\beta}$.

Since transition probabilities depend on the age, it is not sufficient to consider the probability distributions $f_\beta(t)$ and $V_\beta(t)$ mentioned above. One needs to introduce more detailed distributions that include information about the times of the previous transitions.

A. Volume distributions

I first consider the volume distribution. Assume for convenience that there is a single initial bubble of type α_0 formed at time $t = 0$ with unit volume, and that we are interested in describing only the evolution of the interior of the initial bubble and any bubbles nucleated in it. (The case of several initial bubbles is a straightforward extension.) Let $V_\alpha(t_0, t)dt_0$ denote the volume at time t of bubbles of type α that were formed at an earlier time between t_0 and $t_0 + dt_0$. By definition, we set $V_\alpha(t_0, t) = 0$ for $t_0 > t$. The volume remaining from the initial bubble could be included in $V_{\alpha_0}(t_0, t)$ as a contribution of the form $\delta(t_0)V^{(0)}_{\alpha_0}(t)$, but it is technically more convenient to *exclude* the initial bubble from $V_\alpha(t_0, t)$ and to account for its volume $V^{(0)}_{\alpha_0}(t)$ separately. The quantity $V^{(0)}_{\alpha_0}(t)$ represents the proper volume that remains from the initial bubble and has not decayed by time t. The volume $V^{(0)}_{\alpha_0}(t)$ of the initial bubble grows with the rate $3H_{\alpha_0}$ and decreases due to nucleation of other bubbles:

$$\frac{dV^{(0)}_{\alpha_0}(t)}{dt} = 3H_{\alpha_0}V^{(0)}_{\alpha_0}(t) - \sum_\beta \Gamma_{\alpha_0\to\beta}(t)V^{(0)}_{\alpha_0}(t). \quad (9)$$

Integrating Eq. (9) with the initial condition $V^{(0)}_{\alpha_0}(0) = 1$, we find

$$V^{(0)}_{\alpha_0}(t) = \exp[3H_{\alpha_0}t]S_{\alpha_0}(t), \quad (10)$$

$$S_\alpha(t) \equiv \exp\left[-\int_0^t \sum_\beta \Gamma_{\alpha\to\beta}(t')dt'\right]. \quad (11)$$

The auxiliary function $S_\alpha(t)$ is the survival probability of a bubble of type α and age t.

The evolution equation for $V_\alpha(t_0, t)$ accounts for the growth of volume at rate $3H_\alpha$, age-dependent decay into bubbles of different kinds, and age-dependent nucleation of zero-age bubbles of kind α from other bubbles (including the original bubble):

$$\begin{aligned}\frac{\partial V_\alpha(t_0, t)}{\partial t} &= 3H_\alpha V_\alpha(t_0, t) - \sum_\beta \Gamma_{\alpha\to\beta}(t - t_0)V_\alpha(t_0, t) \\ &+ \delta(t - t_0)\int_0^t d\tilde{t}_0 \sum_\beta \Gamma_{\beta\to\alpha}(t - \tilde{t}_0)V_\beta(\tilde{t}_0, t) \\ &+ \delta(t - t_0)\Gamma_{\alpha_0\to\alpha}(t)V^{(0)}_{\alpha_0}(t). \end{aligned} \quad (12)$$

The factors $\delta(t - t_0)$ account for the fact that bubbles nucleated at time t have zero age at that time and therefore contribute to the distribution $V_\alpha(t_0, t)$ only at $t_0 = t$.

Noting that $V_\alpha(t_0, t)$ with $t \neq t_0$ is decoupled from other $V_\beta(t_0, t)$, we have (for $t > t_0$)

$$V_\alpha(t_0, t) = V_\alpha(t_0, t_0)\exp[3H_\alpha(t - t_0)]S_\alpha(t - t_0). \quad (13)$$

It remains to determine the function $V_\alpha(t_0, t_0) \equiv U_\alpha(t_0)$. Integrating Eq. (12) in t over an infinitesimal interval around $t = t_0$ and using Eq. (13) and the condition $V_\alpha(t_0, t) = 0$ for $t < t_0$, we obtain a closed system of integral equations for $U_\alpha(t)$,

$$U_\alpha(t) = \sum_\beta \int_0^t d\tilde{t}_0 U_\beta(\tilde{t}_0)e^{3H_\beta(t-\tilde{t}_0)}S_\beta(t - \tilde{t}_0)\Gamma_{\beta\to\alpha}(t - \tilde{t}_0) \quad (14)$$

$$+ \Gamma_{\alpha_0\to\alpha}(t)e^{3H_{\alpha_0}t}S_{\alpha_0}(t). \quad (15)$$

It remains to determine the asymptotic behavior of the functions $U_\alpha(t)$.

1. Markovian regime

I first consider times t before the crossover time scale, $0 < t < T$. At these times, the behavior of the system is (approximately) Markovian, and one expects to recover the standard equations (4). In Eq. (15) we may approximately set

$$\Gamma_{\beta\to\alpha}(t - \tilde{t}_0) \approx \kappa_{\beta\to\alpha} = \text{const}, \quad (16)$$

$$S_\beta(t - \tilde{t}_0) \approx \exp[-(t - \tilde{t}_0)\Gamma_\beta], \quad (17)$$

where we denoted by $\Gamma_\beta \equiv \sum_\alpha \kappa_{\beta\to\alpha}$ the total decay rate of the vacuum type β in the Markovian regime. Then Eq. (15) is rewritten as

$$\begin{aligned}U_\alpha(t) \approx& \sum_\beta \kappa_{\beta\to\alpha}\int_0^t d\tilde{t}_0 U_\beta(\tilde{t}_0)e^{(3H_\beta-\Gamma_\beta)(t-\tilde{t}_0)} \\ &+ \Gamma_{\alpha_0\to\alpha}e^{(3H_{\alpha_0}-\Gamma_{\alpha_0})t}.\end{aligned} \quad (18)$$

Although this system of equations appears to be nonlocal in time, it can be reduced explicitly to a Markovian system. We pass to new variables

$$V_\alpha(t) \equiv \int_0^t d\tilde{t}_0 U_\alpha(\tilde{t}_0)e^{(3H_\alpha-\Gamma_\alpha)(t-\tilde{t}_0)} + \delta_{\alpha\alpha_0}e^{(3H_{\alpha_0}-\Gamma_{\alpha_0})t}. \quad (19)$$

The quantity $V_\alpha(t)$ represents the total volume inside bubbles of type α at time t integrated over the bubble ages and also including the volume of the initial bubble. The variables $U_\alpha(t)$ are expressed through $V_\alpha(t)$ as

$$\begin{aligned}U_\alpha(t) &= e^{(3H_\alpha-\Gamma_\alpha)t}\partial_t[e^{-(3H_\alpha-\Gamma_\alpha)t}V_\alpha(t)] \\ &= \dot{V}_\alpha - (3H_\alpha - \Gamma_\alpha)V_\alpha.\end{aligned} \quad (20)$$

Hence the volumes $V_\alpha(t)$ satisfy the differential equation

$$\dot{V}_\alpha = (3H_\alpha - \Gamma_\alpha)V_\alpha + \sum_\beta \kappa_{\beta\to\alpha}V_\beta = \sum_\beta M_{\alpha\beta}V_\beta \quad (21)$$

with the initial condition $V_\alpha(0) = \delta_{\alpha\alpha_0}$, which is equivalent to Eqs. (4); the matrix $M_{\alpha\beta}$ is defined by Eq. (6). The late-time asymptotic of solutions is exponential,

$$V_\alpha(t) = c_\alpha e^{\gamma t}, \tag{22}$$

where γ is the largest eigenvalue of the matrix $M_{\alpha\beta}$. It is important to note that $\gamma > 3H_\beta - \Gamma_\beta$ for all β.

The eigenvalue γ and the corresponding eigenvectors of $M_{\alpha\beta}$ can be estimated explicitly under some technical assumptions. To be specific, let us denote by H_0 and H_1 the first and the second largest values among all the H_α, and let us assume that the nucleation rates are small,

$$\kappa_{\alpha\to\beta} \ll H_0 - H_1 \quad \text{for all } \alpha, \beta. \tag{23}$$

Since the nucleation rates are typically exponentially small, one can disregard terms of higher order in $\kappa_{\alpha\to\beta}$. Then the matrix $M_{\alpha\beta}$ can be represented as a diagonal matrix $\delta_{\alpha\beta}(3H_\alpha - \Gamma_\alpha)$ with a small perturbation of order $\kappa_{\alpha\to\beta}$, and the dominant eigenvalue is found by standard perturbation theory as

$$\gamma = 3H_0 - \Gamma_0 + \sum_{\alpha\neq 0} \frac{\kappa_{\alpha\to 0}\kappa_{0\to\alpha}}{3H_0 - 3H_\alpha} + O(\kappa^3_{\alpha\to\beta}). \tag{24}$$

The second-order term in γ will play a role below.

The coefficients c_α in Eq. (22) are proportional to the components of the (right) dominant eigenvector $r_{\alpha 0}$ of $M_{\alpha\beta}$, so that $c_\alpha/c_\beta = r_{\alpha 0}/r_{\beta 0}$. The ratios of components of the eigenvector $r_{\alpha 0}$ can be found approximately as

$$\frac{r_{\alpha 0}}{r_{00}} = \frac{\kappa_{0\to\alpha}}{3H_0 - 3H_\alpha} + O(\kappa^2_{\alpha\to\beta}), \qquad \alpha \neq 0. \tag{25}$$

It is useful to compute also the absolute normalization of the coefficients c_α, which will yield an explicit late-time asymptotic $V_\alpha(t) = c_\alpha e^{\gamma t}$ as a function of the initial conditions $V_\alpha(0) = \delta_{\alpha\alpha_0}$. The time-dependent solution $V_\alpha(t)$ can be decomposed as

$$V_\alpha(t) = \sum_n v_n r_{\alpha n} e^{\gamma_n t}, \tag{26}$$

where $\gamma_0, \gamma_1, \ldots$ and $r_{\alpha 0}, r_{\alpha 1}, \ldots$ are the eigenvalues and the corresponding (right) eigenvectors of $M_{\alpha\beta}$. The late-time behavior of $V_\alpha(t)$ is dominated by $e^{\gamma_0 t}$, where $\gamma_0 \equiv \gamma$ is the largest eigenvalue.

The coefficients v_n are found by decomposing the initial condition vector $V_\alpha(0)$ in the basis $\{r_{\alpha n}\}$,

$$V_\alpha(0) = \sum_n v_n r_{\alpha n}. \tag{27}$$

The coefficients v_n are computed as the products of the *left* eigenvectors $l_{\alpha n}$, $n = 0, 1, \ldots$ of $M_{\alpha\beta}$ with the initial condition vector $V_\alpha(0) \equiv \delta_{\alpha\alpha_0}$,

$$v_n = \sum_\alpha l_{\alpha n} V_\alpha(0) = l_{\alpha_0 n}, \tag{28}$$

where we assumed that the dual bases $\{l_{\alpha n}\}$ and $\{r_{\alpha n}\}$ are normalized,

$$\sum_\alpha l_{\alpha m} r_{\alpha n} = \delta_{mn}. \tag{29}$$

We are interested only in the coefficients $V_{\alpha 0}$ corresponding to the dominant eigenvalue $\gamma \equiv \gamma_0$, so $v_0 = l_{\alpha_0 0}$. The vector $l_{\alpha 0}$ is determined perturbatively under the assumption (23) through the ratios

$$\frac{l_{\alpha 0}}{l_{00}} = \frac{\kappa_{\alpha\to 0}}{3H_0 - 3H_\alpha} + O(\kappa^2_{\alpha\to\beta}), \qquad \alpha \neq 0. \tag{30}$$

Hence, a suitable normalization of the eigenvectors is

$$r_{00} = 1, \qquad r_{\alpha 0} = \frac{\kappa_{0\to\alpha}}{3H_0 - 3H_\alpha} + O(\kappa^2_{\alpha\to\beta}), \qquad \alpha \neq 0; \tag{31}$$

$$l_{00} = 1, \qquad l_{\alpha 0} = \frac{\kappa_{\alpha\to 0}}{3H_0 - 3H_\alpha} + O(\kappa^2_{\alpha\to\beta}), \qquad \alpha \neq 0. \tag{32}$$

Now we may compute explicitly

$$c_\alpha = v_0 r_{\alpha 0} = r_{\alpha 0} l_{\alpha_0 0}. \tag{33}$$

The full solution $U_\alpha(t)$ can be written as

$$U_\alpha(t) = \sum_n l_{\alpha_0 n} r_{\alpha n}(\gamma_n - 3H_\alpha + \Gamma_\alpha) e^{\gamma_n t}, \qquad t < T. \tag{34}$$

The late-time (but still Markovian) behavior of $U_\alpha(t)$ is

$$U_\alpha(t) \approx r_{\alpha 0} l_{\alpha_0 0}(\gamma - 3H_\alpha + \Gamma_\alpha) e^{\gamma t}. \tag{35}$$

Although the absolute values of the coefficients c_α depend on the initial conditions, the ratios c_α/c_β do not. This is the standard property of Markovian models: the late-time asymptotics do not depend on the initial conditions.

2. Non-Markovian regime

Having computed the early-time behavior of $U_\alpha(t)$, I now consider the asymptotics of $U_\alpha(t)$ at late times t for which the survival probabilities $S_\alpha(t)$ are subexponential. Since the decay rate is the logarithmic derivative of the survival probability, it follows that $\Gamma_{\alpha\to\beta}(t) \propto t^{-1}$ at those times. To simplify calculations, I assume that

$$S_\alpha(t)\Gamma_{\alpha\to\beta} = R_\alpha(t)\kappa_{\alpha\to\beta} \quad \text{for all } \beta, \tag{36}$$

where the function $R_\alpha(t)$ describes the transition from the Markovian to the non-Markovian regime as

$$R_\alpha(t) = \begin{cases} \exp[-\Gamma_\alpha t], & t < T; \\ \exp[-\Gamma_\alpha T](\frac{T}{t})^{p_\alpha}, & t > T, \end{cases} \tag{37}$$

where T is the crossover time and $p_\alpha > 0$ are constants of order 1. (For the cited examples of subexponential decay with $S_\alpha(t) \propto t^{-3}$ one will have to set $p_\alpha = 4$.) The assumption of a common time profile $R_\alpha(t)$ and a common crossover time T, independent of the vacuum type α and of the decay channel $\alpha \to \beta$, may be insufficiently precise in some scenarios. Here I employ this technical assumption as a first step towards a more complete calculation.

Let us first determine the ansatz for the asymptotics of $U_\alpha(t)$ by examining Eq. (15). Since $U_\alpha(t)$ receives contributions from all the subexponentially decaying states $\beta \neq \alpha$ according to Eq. (15), the late-time asymptotics of $U_\alpha(t)$ must grow at least as fast as the fastest-growing function among $e^{3H_\alpha t}R_\alpha(t)$ for all α. Hence, the exponential part of the asymptotic is $U_\alpha(t) \propto e^{\tilde{\gamma}t}$, where $\tilde{\gamma}$ is not less than $3H_0$ and H_0 is the largest available value among H_α. However, the function $U_\alpha(t)$ cannot grow *faster* than $e^{3H_0 t}$, i.e. as $e^{\tilde{\gamma}t}$ with $\tilde{\gamma} > 3H_0$, because in that case the integral in line (14) is dominated by $\tilde{t}_0 \approx t$ (recently nucleated bubbles) where the survival probabilities $S_\beta(t - \tilde{t}_0)$ are Markovian. So the Markovian calculation leading to Eq. (18) still holds and yields the contradictory result $\tilde{\gamma} = \gamma \approx 3H_0 - \Gamma_0 < 3H_0$. Hence, $\tilde{\gamma} = 3H_0$. We need to allow for the possibility that $U_\alpha(t)$ contains also a subexponential asymptotic, $U_\alpha(t) \propto e^{3H_0 t}Q(t)$, where $Q(t)$ is a subexponential function decaying not faster than $R_\alpha(t)$ at $t \gg T$. (Below I will show that $Q(t) \propto R_0(t)$, but at this point the behavior of $Q(t)$ is not yet determined.) Thus, the late-time asymptotics of $U_\alpha(t)$ are of the form

$$U_\alpha(t) \approx q_\alpha e^{3H_0 t}Q(t), \qquad t > T, \tag{38}$$

while the Markovian behavior was determined above in Eq. (34). The task at hand is to determine the coefficients q_α and the function $Q(t)$ for the non-Markovian asymptotics (38).

Let us define the auxiliary quantities

$$W_\alpha(t) \equiv \int_0^t dt_0 U_\alpha(t_0)e^{3H_\alpha(t-t_0)}R_\alpha(t-t_0) + \delta_{\alpha\alpha_0}e^{3H_\alpha t}R_\alpha(t), \tag{39}$$

so that Eq. (15) becomes

$$U_\alpha(t) = \sum_\beta \kappa_{\beta\to\alpha}W_\beta(t). \tag{40}$$

We will first determine the asymptotics of the quantities $W_\alpha(t)$ for $t \gg T$.

The definition of $W_\alpha(t)$ involves an integral over t_0 that needs to be estimated. It is convenient to estimate it separately for $\alpha \neq 0$ and $\alpha = 0$. For $\alpha \neq 0$, the function $U_\alpha(t_0)$ grows as $e^{\gamma t_0}$ until $t_0 = T$; subsequently $U_\alpha(t_0)$ grows even faster, as $e^{3H_0 t}$. This function is multiplied by a decay factor $e^{-3H_\alpha t_0}R_\alpha(t - t_0)$ that never compensates the growth of $U_\alpha(t_0)$ if $\alpha \neq 0$ because $\gamma - 3H_\alpha \gg \Gamma_\alpha$ for $\alpha \neq 0$. Therefore, the integral over t_0 is dominated by the contribution near the upper limit $t_0 \approx t$ where $R_\alpha(t - t_0)$ is Markovian while $U_\alpha(t) \propto e^{3H_0 t}$. One obtains the asymptotic estimate

$$W_\alpha(t) \approx \frac{q_\alpha Q(t)e^{3H_0 t}}{3H_0 - 3H_\alpha + \Gamma_\alpha}, \qquad \alpha \neq 0, \tag{41}$$

where the term $\propto e^{3H_\alpha t}R_\alpha(t)$ can be disregarded since it is exponentially smaller at late times.

Estimating the quantity $W_0(t)$ requires somewhat more work. One needs to split the integral in the definition of $W_\alpha(t)$ into three subintervals $[0, T]$, $[T, t - T]$, and $[t - T, t]$ where different factors in the integrand have either Markovian or non-Markovian behavior. These three integrals are estimated as follows. The first integral,

$$\int_0^T dt_0 U_0(t_0)e^{3H_0(t-t_0)}R_0(t-t_0), \tag{42}$$

is dominated by the contribution of $t_0 \approx 0$ because $U_0(t_0)$ in the Markovian regime grows as $e^{\gamma t_0}$, while $\gamma < 3H_0$. Using Eq. (34), we find

$$\int_0^T dt_0 U_0(t_0)e^{3H_0(t-t_0)}R_0(t-t_0) \approx e^{3H_0 t}R(t)\sum_n v_n r_{0n}\frac{\gamma_n - 3H_0 + \Gamma_0}{3H_0 - \gamma_n}. \tag{43}$$

The sum in the last line can be estimated without actually computing all the eigenvectors r_{0n} by noting that $3H_0 - \gamma_n \gg \Gamma_0$ for $n \neq 0$, and thus the factor

$$\frac{\gamma_n - 3H_0 + \Gamma_0}{3H_0 - \gamma_n} \approx -1 + O(\kappa_{\alpha\to\beta}), \qquad n \neq 0. \tag{44}$$

For $n = 0$ this factor is negligible,

$$\frac{\gamma - 3H_0 + \Gamma_0}{\Gamma_0} = O(H_0^{-1}\kappa_{\alpha\to\beta}), \tag{45}$$

where we used Eq. (24). By splitting off the $n = 0$ term from the sum in Eq. (43), one now obtains

$$\sum_n v_n r_{0n}\frac{\gamma_n - 3H_0 + \Gamma_0}{3H_0 - \gamma_n} \approx -\sum_{n\neq 0} v_n r_{0n}. \tag{46}$$

The last sum can be evaluated using Eq. (27),

$$V_0(0) = \sum_n v_n r_{0n} = v_0 r_{00} + \sum_{n\neq 0} v_n r_{0n} = \delta_{0\alpha_0}, \tag{47}$$

and we find

$$\sum_n v_n r_{0n}\frac{\gamma_n - 3H_0 + \Gamma_0}{3H_0 - \gamma_n} \approx v_0 - \delta_{0\alpha_0}. \tag{48}$$

Hence, the expression (43) is estimated as

$$e^{3H_0 t}R_0(t)(v_0 - \delta_{0\alpha_0}). \tag{49}$$

The integral over the second interval,

$$\int_T^{t-T} dt_0 U_0(t_0)e^{3H_0(t-t_0)}R_0(t-t_0), \tag{50}$$

involves both $U_0(t_0)$ and $R_0(t - t_0)$ in the non-Markovian regime. We find

$$\int_T^{t-T} dt_0 U_0(t_0) e^{3H_0(t-t_0)} R_0(t-t_0) \approx q_0 e^{3H_0 t} \int_T^{t-T} dt_0 Q(t_0) R_0(t-t_0). \quad (51)$$

Since both $R_0(t)$ and $Q(t)$ are decaying functions, we may estimate the integral in Eq. (51) as the sum of the contributions from intervals of order T at the two ends $t_0 = T$ and $t_0 = t - T$,

$$q_0 e^{3H_0 t}[R_0(t)Q(T)O(T) + Q(t)R_0(T)O(T)]. \quad (52)$$

This precision is sufficient since these terms will not play a significant role in the final result.

The integral over the third interval involves the Markovian $R_0(t - t_0)$ and is dominated by $t_0 \approx t$,

$$\int_{t-T}^{t} dt_0 U_0(t_0) e^{3H_0(t-t_0)} R_0(t-t_0) \approx \frac{q_0}{\Gamma_0} Q(t) e^{3H_0 t}, \quad (53)$$

where we disregarded $e^{-\Gamma_0 T} \ll 1$. (Note that $\Gamma_0 T \gg 1$.)

Putting together the contributions of the three intervals as well as the last term in Eq. (39), we obtain

$$W_0(t) \approx e^{3H_0 t} R_0(t) v_0 + q_0 e^{3H_0 t} \Gamma_0^{-1} Q(t) + q_0 e^{3H_0 t} R_0(t) Q(T) O(T), \quad (54)$$

where we disregarded

$$Q(t) R_0(T) O(T) \ll q_0 e^{3H_0 t} \Gamma_0^{-1} Q(t) \quad (55)$$

because

$$R_0(T) O(\Gamma_0 T) = e^{-\Gamma_0 T} O(\Gamma_0 T) \ll 1. \quad (56)$$

Finally, we substitute the ansatz (38) and the estimates (41) and (54) into Eqs. (40) for $U_\alpha(t)$. In the limit $t \gg T$, we may divide through by the factor $e^{3H_0 t} Q(t)$ and obtain a system of equations for q_α and $Q(t)$,

$$q_0 = \sum_\beta \kappa_{\beta \to 0} \frac{q_\beta}{3H_0 - 3H_\beta + \Gamma_\beta}, \quad (57)$$

$$q_\alpha = \sum_\beta \kappa_{\beta \to \alpha} \frac{q_\beta}{3H_0 - 3H_\beta + \Gamma_\beta} + \kappa_{0\to\alpha}[v_0 + q_0 Q(T) O(T)] \lim_{t\to\infty} \frac{R_0(t)}{Q(t)}, \qquad \alpha \neq 0. \quad (58)$$

This is an inhomogeneous linear system for $\{q_\alpha\}$.

Let us consider the possible values of $\lim_{t\to\infty} R_0(t)/Q(t)$ that show whether $Q(t)$ is asymptotically dominant over $R_0(t)$. Since $Q(t)$ in any case does not decay faster than $R_0(t)$, there are only two possibilities: either the limit is zero or it is nonzero. I will now show that this limit must be nonzero.

If $\lim_{t\to\infty} R_0(t)/Q(t) = 0$, we rewrite Eqs. (57) and (58) as

$$q_\alpha = \sum_\beta \kappa_{\beta \to \alpha} \frac{q_\beta}{3H_0 - 3H_\beta + \Gamma_\beta}, \qquad \alpha = 0, 1, \ldots \quad (59)$$

Passing to auxiliary variables

$$s_\alpha \equiv \frac{q_\alpha}{3H_0 - 3H_\alpha + \Gamma_\alpha}, \quad (60)$$

we find

$$3H_0 s_\alpha = \sum_\beta M_{\alpha\beta} s_\beta. \quad (61)$$

Since the largest eigenvalue of $M_{\alpha\beta}$ is $\gamma < 3H_0$, it follows that $3H_0$ is not an eigenvalue of $M_{\alpha\beta}$. Hence, the only solution of the homogeneous system (59) is $q_\alpha = 0$. This contradicts the assumption that $q_\alpha e^{3H_0 t} Q(t)$ is the leading asymptotic of $U_\alpha(t)$. Therefore, $Q(t)$ decays exactly as $R_0(t)$ at late times.

Since Eq. (58) depends only on the ratio $Q(T)/Q(t)$, the normalization of the $Q(t)$ could then be adjusted such that $\lim_{t\to\infty} R_0(t)/Q(t) = 1$. The value $Q(T)$ is of order $e^{-\Gamma_0 T}$ due to the continuity requirement

$$U_\alpha(T) \approx q_\alpha e^{3H_0 T} Q(T) \approx c_\alpha e^{3\gamma T}. \quad (62)$$

Therefore, the term $q_0 Q(T) O(T)$ in Eq. (58) is exponentially small and can be neglected. We note, however, that its magnitude depends on the initial conditions through the coefficient $c_\alpha \sim O(\Gamma_0)$, which introduces, strictly speaking, an exponentially small dependence on initial conditions, of order $O(\Gamma_0 T) e^{-\Gamma_0 T}$.

Finally, we rewrite Eqs. (57) and (58) through the variables s_α as

$$3H_0 s_\alpha - \sum_\beta M_{\alpha\beta} s_\beta = v_0 \kappa_{0\to\alpha}. \quad (63)$$

This is an inhomogeneous system of equations with a nondegenerate matrix, and so the solution is unique. It follows that all s_α are of order $v_0 \kappa_{\alpha\to\beta}$, so an approximate expression for the solution is readily found as

$$s_0 \approx \frac{v_0}{\Gamma_0} \sum_\beta \frac{\kappa_{\beta\to 0} \kappa_{0\to\beta}}{3H_0 - 3H_\beta}, \quad (64)$$

$$s_\alpha \approx \frac{v_0 \kappa_{0\to\alpha}}{3H_0 - 3H_\alpha}, \qquad \alpha \neq 0. \quad (65)$$

The corresponding values of q_α (neglecting higher orders of $\kappa_{\alpha\to\beta}$) are

$$q_0 \approx v_0 \sum_\beta \frac{\kappa_{\beta\to 0} \kappa_{0\to\beta}}{3H_0 - 3H_\beta}, \quad (66)$$

$$q_\alpha \approx v_0 \kappa_{0\to\alpha}, \qquad \alpha \neq 0. \quad (67)$$

We note that the solution depends on the initial bubble through v_0 only in the overall normalization; the ratios q_α/q_β are independent of v_0.

Having determined the auxiliary quantities $U_\alpha(t)$, we can now compute the non-Markovian volume distribution $V_\alpha(t)$ as

$$V_\alpha(t) = \int_0^t dt_0 U_\alpha(t_0) e^{3H_\alpha(t-t_0)} S_\alpha(t-t_0) + \delta_{\alpha\alpha_0} e^{3H_{\alpha_0}t} S_{\alpha_0}(t). \quad (68)$$

For $\alpha \neq 0$, the integral in Eq. (68) is dominated by $t_0 \approx t$, which yields a term $\propto e^{3H_0 t}$, so the second term in Eq. (68) is negligible. Hence, by setting $Q(t) \approx R_0(t)$ and $q_\alpha = v_0 \kappa_{0\to\alpha}$ one obtains the estimate

$$V_\alpha(t) \approx \frac{v_0 \kappa_{0\to\alpha}}{3H_0 - 3H_\alpha + \Gamma_\alpha} R_0(t) e^{3H_0 t}, \qquad \alpha \neq 0. \quad (69)$$

We note that the ratios of volumes $V_\alpha(t)/V_\beta(t)$ are independent of the initial condition parameter v_0 and of time, indicating a "stationarity" of the solutions $V_\alpha(t)$ with $\alpha \neq 0$. Moreover, these ratios are equal to the ratios obtained in the Markovian regime,

$$\lim_{t\to\infty} \frac{V_\alpha(t)}{V_\beta(t)} \approx \frac{c_\alpha}{c_\beta}, \qquad \alpha, \beta \neq 0. \quad (70)$$

The imprecision in the above equality is exponentially small, of order $O(\Gamma_0 T)e^{-\Gamma_0 T}$, as noted before. [To simplify calculations, we also carried an imprecision of order $\kappa_{\alpha\to\beta}/(H_0 - H_1)$ in the expressions for c_α, but Eq. (70) also carries that imprecision. This limitation is due to the approximations adopted in the present paper.]

It remains to compute $V_0(t)$. For $\alpha = 0$, the estimation of the integral in Eq. (68) proceeds similarly to the argument leading to Eq. (54), except that $R_0(t)$ is replaced by $S_0(t)$ which decays slower. The result is

$$V_0(t) \approx e^{3H_0 t}[v_0 S_0(t) + q_0 \Gamma_0^{-1} Q(t)]. \quad (71)$$

Since at large t

$$Q(t) \approx R_0(t) = S_0(t) \frac{\Gamma_{0\to\alpha}(t)}{\kappa_{0\to\alpha}} \ll S_0(t), \quad (72)$$

the dominant asymptotic for $V_0(t)$ for $t \gg T$ is

$$V_0(t) \approx e^{3H_0 t} v_0 S_0(t). \quad (73)$$

B. Discussion

We will now interpret the results of the calculation in the previous section. Since $Q(t) \ll S_0(t)$ at late times, the volume $V_0(t)$ within bubbles of type 0 grows asymptotically faster than all other $V_\alpha(t)$ for $\alpha \neq 0$,

$$\lim_{t\to\infty} \frac{V_0(t)}{V_\alpha(t)} \propto \lim_{t\to\infty} \frac{S_0(t)}{R_0(t)} = \infty, \qquad \alpha \neq 0. \quad (74)$$

This indicates that the 3-volume at time t is entirely dominated by the bubbles of type 0, which we have labeled as those having the largest Hubble rate $H_0 = \max_\alpha H_\alpha$. Moreover, since the integral in Eq. (68) for $\alpha = 0$ is dominated by $t_0 \approx 0$, it follows that almost all of the volume in bubbles of type 0 at time t is in the *very old* regions of type 0. These regions of type 0 either belong to the original bubble (if $\alpha_0 = 0$), or were nucleated early on (if $\alpha_0 \neq 0$) and, by chance, have remained without decay for almost all of the time t. This dominance does not depend on the initial conditions and is due to the fact that nonexponential decay makes the nucleation of other types of bubbles less likely in very old regions. The absolute dominance of bubbles of type 0 will set in after time T. This is different from the Markovian situation[1] where bubbles of type 0 dominate with a *finite* (but very large) ratio,

$$\lim_{t\to\infty} \frac{V_0^{\mathrm{Markov}}(t)}{V_\alpha^{\mathrm{Markov}}(t)} = \frac{c_0}{c_\alpha} \approx \frac{3H_0 - 3H_\alpha}{\kappa_{0\to\alpha}} \gg 1. \quad (75)$$

Thus the qualitative picture of the distribution of volume in space has changed due to the non-Markovian decay, but the change is not drastic. This conclusion is similar in spirit to that obtained in Sec. II, where the fractal dimension of the eternally inflating domain was modified from $3 - \varepsilon$, where $\varepsilon \ll 1$, to exactly 3.

On the other hand, the 3-volumes $V_\alpha(t)$ within other types of bubbles $\alpha \neq 0$ grow proportionally to each other, and the ratios V_α/V_β are almost the same (up to exponentially small corrections) as those obtained in a Markovian calculation. Therefore, any measure prescription that depends on the asymptotic ratios of volumes, V_α/V_β, will give unchanged predictions as long as one asks about the volumes of bubbles of subdominant types ($\alpha \neq 0$). Since the bubbles of type 0 (presumably, with a Planck-scale H_0) are not especially interesting observationally, one can conclude that a possible non-Markovian decay has no effect on predictions obtained via any measure prescriptions based on volume ratios.

The considerations in the present paper are limited to proper time gauge and to landscape scenarios satisfying the assumptions (23). The methods developed here are applicable to landscapes of any type, and future work will show whether the conclusions hold in more general cases.

[1]The 3-volume is not a gauge-invariant quantity, and statements about dominance of 3-volume at fixed time depend sensitively on the choice of the time variable [18]. In particular, in Markovian models the 3-volume is *not* dominated by fastest-expanding bubbles if one chooses the e-folding time $\tau \equiv \ln a$ as the time coordinate. A similar gauge dependence is expected in the non-Markovian case. The present calculation focuses on the effects of non-Markovian decay, which are arguably more pronounced in the proper time gauge.

C. Comoving distributions

I now turn to considering the comoving distribution. One can define the distribution $f_\alpha(t_0, t)dt_0$ as the fraction of comoving volume at time t in bubbles of type α that were formed at an earlier time between t_0 and $t_0 + dt_0$. As before, we set $f_\alpha(t_0, t) = 0$ for $t_0 > t$; the volume remaining from the initial bubble is not included in $f_\alpha(t_0, t)$ but accounted for separately as the function $f^{(0)}_{\alpha_0}(t)$. The formalism and the equations for the distribution $f_\alpha(t_0, t)$ are quite similar to those developed above for the volume distribution $V_\alpha(t_0, t)$ except for the absence of the volume growth factors H_α.

Instead of writing out the equations and the solutions for $f_\alpha(t_0, t)$, a simple consideration suffices to show that non-Markovian effects are irrelevant for the distributions of comoving volume. The comoving volume fractions $f_\alpha(t)$, defined regardless of age, exponentially quickly become constant because the total comoving volume is conserved, and the dominant eigenvalue of the relevant Markovian matrix is equal to zero. The nucleation of bubbles will be always dominated by new bubbles rather than by "aged" comoving volume, simply because the comoving fraction of the aged volume quickly goes to zero. In fact, the aged comoving volume has a smaller nucleation rate, $\Gamma_{\alpha\to\beta}(t) \to 0$ for $t \to \infty$, and therefore plays an even less significant role in nucleation of new bubbles as in Markovian models. This is in contrast to the situation with the volume-weighted distributions, where the aged volume is rewarded by an exponentially large extra growth factor $e^{3H_0 t}$ compared with the new volume that grows slower, as $e^{(3H_0-\Gamma_0)t}$. Therefore, the non-Markovian decay law will introduce only a vanishingly small correction to the predictions obtained through comoving-volume measure prescriptions.

IV. IS AGE-DEPENDENT DECAY COSMOLOGICALLY RELEVANT?

A subexponential asymptotic at late times is a generic feature of quantum-mechanical systems. This feature can be understood heuristically as follows [17]. Decay is due to the spreading of the wave function away from the initial metastable state. However, the wave packet keeps spreading even after tunneling out of the initial domain. If the evolution proceeds without any wave function collapse due to measurements, the tail of the outgoing wave packet will reach back to the initial state. Since the spreading is a power-law process (the root mean square uncertainty in position grows proportionally to time), there will be a power-law tail of the wave packet that overlaps with the initial domain. Hence, the probability of remaining in the initial state has a power-law late-time asymptotic. These considerations apply to tunneling processes in field theory as well because tunneling occurs essentially along a one-dimensional path in field space, corresponding to the instanton solution.

On the technical level, a necessary condition for the existence of the subexponential asymptotic is that the Hamiltonian of the system must be bounded (either from below or from above). An elementary consideration is as follows. The probability of remaining in the metastable state $|\psi\rangle$ is

$$P(t) = |\langle\psi|e^{i\hat{H}t}|\psi\rangle|^2, \tag{76}$$

where $\hat{H}$ is the total Hamiltonian of the system. Let us assume that the spectrum of $\hat{H}$ is bounded from below, say by $E = E_0$. Using the spectral decomposition,

$$\hat{H} = \int_{E_0}^{\infty} E\hat{P}_E dE, \tag{77}$$

where $\hat{P}_E$ is an orthogonal projector onto the subspace of energy E, we find

$$\langle\psi|e^{i\hat{H}t}|\psi\rangle = \int_{E_0}^{\infty} e^{iEt}\langle\psi|\hat{P}_E|\psi\rangle dE \equiv \int_{-\infty}^{\infty} e^{iEt}\rho(E)dE, \tag{78}$$

where, by definition, the function $\rho(E)$ identically vanishes for $E < E_0$. Because of the nonanalyticity of $\rho(E)$ at $E = E_0$, the Fourier transform of $\rho(E)$ necessarily has a power-law asymptotic $\propto t^{-d}$ at $t \to \infty$, where the power d is determined by the order of the (upper) nonzero derivative of $\rho(E)$ at $E = E_0$.

I conclude with some general comments regarding the plausibility of the age-dependent decay in cosmological landscape scenarios. The subexponential asymptotic was obtained by a quantum-mechanical consideration without regard for gravitational effects. However, gravitation plays a central role in vacuum decay [10]. Since the assumption of a bounded Hamiltonian is important, while the Hamiltonian for general relativity is unbounded, it is not immediately clear that the subexponential late-time decay will be manifest also when the effects of gravity is taken into account.

Another relevant consideration is the influence of measurements and decoherence on the vacuum decay. The power-law asymptotic of the survival probability holds only if the evolution of the wave function of the metastable system is unitary and proceeds according to the Schrdinger equation. The power-law decay can occur only if no wave function collapse takes place during that evolution. Therefore, a direct observation of the power-law decay is possible only if the metastable system as well as any decay products are perfectly isolated and do not have any possibility of interacting with any environment at least until times $t \sim T$. It is clear that such a perfect and long-lasting isolation is impossible in practice. Any realistic metastable system and its decay products will interact with an environment long before the crossover time T. After an interaction, the wave function will effectively collapse back to the initial metastable state, and the effects of the slow spreading of the wave packet will be removed.

However, one needs to be careful when applying quantum-mechanical considerations in the cosmological context. Since the potential observers of vacuum decay are inside the decaying field configuration, it is unclear whether they are able to effect a collapse of the wave function of the entire Hubble patch around them. Several points of view are possible. One could assume that a "measurement" of the field in the false vacuum state already occurs if sufficiently many gravitationally interacting macroscopic bodies are present. In that case, the wave function of the decaying field is continuously collapsing back to the false vacuum configuration, and so it would appear that all vacuum decay is *entirely* inhibited due to the quantum Zeno effect (QZE), whereby a metastable system does not collapse when continuously measured. This conclusion appears implausible. On the other hand, it is hard to implement a measurement of the field values on cosmological superhorizon scales by any causal system. Hence, one could assume that "measurements" are absent until a tunneling event is completed and a causally autonomous Hubble-size bubble of true vacuum is formed. Then one finds that the late-time decay asymptotic is indeed relevant to describing the landscape dynamics. Alternatively, one can suppose that measurements due to gravitationally induced decoherence are effectively "performed" only on super-Hubble time and distance scales, as is the case in the decoherence of primordial quantum fluctuations in an inflationary universe [22–25]. In this case, the QZE sets in only if the Hubble time is smaller than the time scale of onset of the exponential decay law. In principle, the QZE time scale can be estimated in a particular model of vacuum decay.

Presently, I merely summarized possible viewpoints on the relevance of decoherence, the quantum Zeno effect, and subexponential decay to cosmological evolution of false vacuum. More work is needed to clarify this fundamental issue.

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