Clock ambiguity and the emergence of physical laws

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The process of identifying a time variable in time-reparameterization invariant theories results in great ambiguities about the actual laws of physics described by a given theory. A theory set up to describe one set of physical laws can equally well be interpreted as describing any other laws of physics by making a different choice of time variable or clock. In this article we demonstrate how this ''clock ambiguity'' arises and then discuss how one might still hope to extract specific predictions about the laws of physics even when the clock ambiguity is present. We argue that a requirement of quasiseparability should play a critical role in such an analysis. As a step in this direction, we compare the Hamiltonian of a local quantum field theory with a completely random Hamiltonian. We find that any random Hamiltonian (constructed in a sufficiently large space) can yield a ''good enough'' approximation to a local field theory. Based on this result we argue that theories that suffer from the clock ambiguity may in the end provide a viable fundamental framework for physics in which locality can be seen as a strongly favored (or predicted) emergent behavior. We also speculate on how other key aspects of known physics such as gauge symmetries and Poincare invariance might be predicted to emerge in this framework.

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I. INTRODUCTION

In attempts to find a physical description of the Universe one has to address many issues forced upon us by consistency with quantum mechanics. A well-known example is an aspect of time that arises in the quantization of gravity. In any theory with time-reparameterization invariance, including Einstein gravity, quantization schemes tend to produce theories in which time is not fundamental, being only recovered after some split of the superspace is performed to identify a time parameter or a ''choice of clock.'' In [1] it was argued that the freedom to choose a clock leads to profound ambiguities in the physics that emerges. In this article we study the implications of taking these ambiguities seriously. Specifically, we consider the fact that the clock ambiguity implies that completely random choices of unitary evolution of the physical systems are on an equal physical footing. A detailed derivation and discussion of the clock ambiguity is presented in Sec. II of this article.

We examine the possibility that the clock ambiguity is a fundamental characteristic of physical laws, which forces us to regard other crucial properties of the physical world such as space, locality, gravity, gauge symmetries, and cosmology as emergent and approximate. In Sec. III we consider how one might best set up the problem so that the emergence of these properties could be studied and understood.

To test the viability of these ideas we compare a random Hamiltonian with that of a local field theory in Sec. IV. Remarkably, we find that in sufficiently large spaces *any* random Hamiltonian appears to give a sufficiently good approximation to a local field theory to account for the viability of local field theory as a description of the observed physical world. Note that our starting point is an arbitrary random *Hamiltonian* (not an arbitrary Hamiltonian *density*). We make no initial assumption about the existence of space, locality, etc. We are claiming that these properties can quite generally be seen as emergent from a random Hamiltonian.

A priori, placing all possible Hamiltonians on an equal footing seems to be hopelessly in conflict with standard approaches to physics. Certainly one possible outcome of this work is to cause the abandonment of at least one of the assumptions that go into stating the clock ambiguity. We discuss this possibility in Sec. II B, but note that this outcome would be significant, in that our assumptions are ones that are widely used in quantum cosmology.

Taking the clock ambiguity at face value, it would seem that extracting the known physical laws from a situation where all possibilities for those laws are initially given equal weight would involve eliminating most of those possibilities for one (probably anthropic) reason or another. Thus we feel our result from Sec. IV is extremely interesting. It shows that good approximations to local field theories can be found very generically in randomly chosen Hamiltonians. We take this as an indication that a framework for fundamental physics with the clock ambiguity rooted firmly in its foundations may not be nearly as problematic as it first seems. We feel our work offers a strong motivation for taking such a framework seriously and making further efforts to explore its ultimate viability.

II. THE CLOCK AMBIGUITY

A. Statement of the clock ambiguity

Here we review the clock ambiguity as discussed in [1]. Our starting point is a standard approach to time in quantum gravity whereby time is defined internally [2]. Any time-reparameterization invariant theory (including general relativity) has the property that the Hamiltonian is zero [3] and the reparameterization can be viewed as a gauge transformation generated by the first-class constraint H. In a covariant approach to quantization this feature is imposed as a constraint equation,

$$H|\psi\rangle = 0, \tag{1}$$

on physical states $|\psi\rangle$ in a "superspace" that includes both matter and metric degrees of freedom. Time evolution is regained by identifying some degree of freedom (or "subsystem") as the "clock" and evaluating correlations between the rest of the Universe and the state of the clock subsystem. For example, several classic papers on quantum cosmology [4–9] use the cosmic scale factor a as the clock degree of freedom.

This approach gives a practical way forward when considering time in quantum gravity, but it also makes intuitive sense. The process of identifying a subsystem of the world as a clock and noting the passage of time in terms of correlations with the clock subsystem gives a good operational picture of how we actually work with time in realistic situations.

In order to describe the clock ambiguity we assume the superspace may be taken to be discrete and finite. Although continuous quantities (fields, the metric, and spacetime itself) are usually used to described physics, observations do not rule out the idea that these continuous quantities are just approximations to a system that is fundamentally discrete and finite. (This fact has been used by others seeking a discrete and finite fundamental description of physics; see, for example, [10–16].) Assuming discreteness and finiteness will make our mathematical manipulations simple.

Formally, if S designates the superspace, then the identification of a clock subsystem involves designating the clock (C) and "rest" (R) subspaces of S so that

$$S = C \otimes R. \tag{2}$$

Let

$$\{|t_i\rangle_C\} \tag{3}$$

be a basis which spans the clock space (eigenstates of the time operator in the language of Isham [17]) and let basis

$$\{|j\rangle_R\} \tag{4}$$

span R. The tensor product of the of the bases spanning C and R spans S, so any state $|\psi\rangle_S$ in superspace can be written

$$|\psi\rangle_S = \sum_{ij} \alpha_{ij} |t_i\rangle_C |j\rangle_R, \tag{5}$$

where α_{ij} are the expansion coefficients. One can the sum up the j's for a fixed time (fixed i) to get the states

$$|\phi_i\rangle_R \equiv \sum_j \alpha_{ij} |j\rangle_R. \tag{6}$$

One can then rewrite Eq. (5) as

$$|\psi\rangle_S = \sum_i |t_i\rangle_C |\phi_i\rangle_R. \tag{7}$$

The state $|\psi(t_i)\rangle_R$ of subsystem R at time t_i is determined by conditioning (projecting) on clock state $|t_i\rangle_C$, giving [18]

$$|\psi(t_i)\rangle_R = |\phi_i\rangle_R. \tag{8}$$

So far we have just summarized a standard approach to time in quantum gravity using a formal discrete notation that will be useful in what follows.

Now we present the argument from [1] that suitably changing the choice of clock subsystem can lead to a description of an arbitrary system experiencing arbitrary time evolution.

To start with, we note that all the information about the state and the time evolution is contained in the α_{ij}'s of Eq. (5). We will show that by choosing different clock subsystems we can get arbitrary α_{ij}'s, which will then correspond to arbitrary states undergoing arbitrary time evolution.

It will be helpful to relabel the tensor product basis for S used in Eq. (5) with a single index. This involves defining some mapping $k(i, j)$ that uniquely assigns an index k to each pair (i, j) so one can write

$$|k(i, j)\rangle_S \equiv |i(k)\rangle_C |j(k)\rangle_R. \tag{9}$$

Then one can write

$$|\psi\rangle_S = \sum_k \alpha_k |k\rangle_S, \tag{10}$$

where

$$\alpha_k \equiv \alpha_{i(k),j(k)}, \tag{11}$$

where the functions $i(k)$ and $j(k)$ simply invert the mapping $k(i, j)$. In this notation, arbitrary α_{ij}'s corresponds to arbitrary α_k's.

But arbitrary α_k's are easy to attain through a change of basis. To see this, suppose one starts with a particular vector given by Eq. (5), or equivalently Eq. (10), and would like to demonstrate an alternative choice of clock describing a specific different state and time evolution. The goal is to construct a new set of subsystems

$$S = C' \otimes R' \tag{12}$$

and the appropriate bases in C' and R' so that

$$|\psi\rangle_S = \sum_{ij} \beta_{ij} |t_i\rangle_{C'} |j\rangle_{R'}, \tag{13}$$

where the β_{ij}'s give the required information about the state and its time evolution (just as α_{ij} did for the original case). The first step is to use the same function $k(i, j)$ discussed above to construct

$$\beta_k \equiv \beta_{i(k),j(k)} \tag{14}$$

and then consider a new vector in the superspace

$$|\psi'\rangle_S \equiv \beta_k |k\rangle_S \tag{15}$$

(note that here the original superspace basis $\{|k\rangle_S\}$ is used). Now consider a unitary [19] transformation $\mathbf{M}$ that transforms $|\psi\rangle_S$ into $|\psi'\rangle_S$:

$$\mathbf{M}\,|\psi\rangle_S = |\psi'\rangle_S \tag{16}$$

(it should be always possible to find at least one such transformation). Operating on both sides of Eq. (16) with $\mathbf{M}^{-1}$ gives

$$|\psi\rangle_S = \mathbf{M}^{-1}|\psi'\rangle_S = \sum_k \beta_k \mathbf{M}^{-1}|k\rangle_S. \tag{17}$$

If one then defines a new basis

$$|k\rangle'_S \equiv \mathbf{M}^{-1}|k\rangle_S, \tag{18}$$

one gets

$$|\psi\rangle_S = \sum_k \beta_k |k\rangle'_S. \tag{19}$$

The desired C' and R' subsystems are constructed using the inverse of the mapping function $k(i, j)$ (the same one used above) to give

$$|i(k), j(k)\rangle'_S \equiv |i\rangle_{C'}|j\rangle_{R'} = |k\rangle'_S \tag{20}$$

leading to

$$|\psi\rangle_S = \sum_{ij} \beta_{ij}|i\rangle_{C'}|j\rangle_{R'} \tag{21}$$

which is the desired result.

Basically we have used the fact that while a different state evolving under a different Hamiltonian would seem to correspond to a different state $|\psi'\rangle_S$ in superspace, it could just as well be seen as the *same* state in superspace expressed in a different basis (corresponding to a different subdivision of the system into clock and rest subspaces). We have used $|\psi'\rangle_S$ as well as the mapping function $k(i, j)$ to explicitly demonstrate how such a new basis can be constructed for S.

The implication of our result is that given that all possible clocks corresponding to all possible time evolutions can be demonstrated to exist, a physicist trying to interpret $|\psi\rangle_S$ from scratch is equally likely to try any one of these clock subsystems, thus placing all possible types of evolution on an equal footing. Specifically, a single state in superspace can be interpreted as any initial state evolving under any Hamiltonian.

B. Discussion

The result in Sec. II A is radical, but it seems to be an inevitable consequence of standard ideas about quantum gravity. One could take the standard model of particle physics (or one's favorite extension thereof), combine it with gravity, and construct the corresponding superspace. Then someone else could come along using the exact same rules of interpretation you use, but by merely choosing a different clock could come up instead with a world described by the old $O(3)$ model of weak interactions, the minimal supersymmetric standard model, technicolor, or something wildly different from any of these.

It is important to emphasize that the clock ambiguity is *not* equivalent to the statement that it is possible to choose terrible clock subsystems (for example, a firefly) by whose measure the evolution of the Universe appears highly irregular (although the clock ambiguity does incidentally include these cases). The most important implication of the clock ambiguity is that it also includes a multitude of arbitrarily good clocks which describe the Universe evolving under very well defined and "sensible" physical laws. The clock ambiguity tells us that there is nothing about the form of superspace nor the state which we choose in superspace to give a preference of one set of physical laws over another, no matter how hard we may try at the outset to build such a preference into the formalism.

One possible response to our analysis is that one or another of our assumptions is wrong. For example, perhaps there is something truly precious about the continua we use to construct theories of fundamental physics, and our discrete and finite treatment misses some key point. One could also choose to reject the superspace formalism outright as is done, for example, by Banks *et al.* in [20].

One might also object that we should not be allowed freedom to choose a clock subsystem arbitrarily but should stubbornly stick to the one originally designated. That objection seems to run up against commonly held views in quantum cosmology. For example, in "eternal inflation" [21] the system in some regimes is completely dominated by quantum fluctuations. Which combination of some "fundamental" states and operators in superspace ends up representing actual semiclassical observables (including time) will depend strongly on which piece of the wave function one is looking at (or which quantum fluctuations one is following). The idea that one must dig through a more formally constructed space to select observables based on their actual behavior is widely used. See, for example, [22] and also [23,24]. Indeed, in our analysis in the previous section it is not just the clock but all observables that are changed when going from one picture to another.

One might even question the use of the covariant approach—the alternative being the fixing of the original reparameterization symmetry at the classical level by imposing a gauge condition, i.e., a choice of reference time.

While such a choice of an external time is suitable for the study of subsystems, with negligible interaction with the environment (that would naturally set such a reference) it presents no advantage when dealing with the Universe as a whole. In fact, in this case, the absence of external reference and subsequent arbitrariness in the choice of gauge, we believe, is analogous to the arbitrariness in the choice of clock subsystem of Sec. II A. An early discussion of this point of view can be found in [25]. However, these authors chose a framework that was too restrictive to expose the full clock ambiguity.

The original paper on the clock ambiguity [1] gives further discussion of the objections that might be raised about our formalism, and gives responses to these. Also, Isham's review [17] (especially Sec. 6.2) gives a good account of some pros and cons of this formalism (although Isham does not use our assumption of discreteness and finiteness).

Our main position on all of this is that the clock ambiguity is a very important topic. If careful consideration of the clock ambiguity leads to rejection of some of the starting assumptions, we feel that would be a significant outcome since these assumptions are currently widely accepted, especially by those who work with quantum cosmology.

The rest of this paper focuses on another possible outcome, namely, that the formalism used above really does describe fundamental physics. In that case, the clock ambiguity is something we need to face head on. We investigate possible ways forward under that assumption.

III. USEFUL CONDITIONS FOR FINDING A GOOD CLOCK

A. Overview

The clock ambiguity seems to leave us very little to work with. Can a fundamental physical theory that appears to put all possible states evolving under all possible Hamiltonians on an equal footing make any concrete statements about the nature of physical laws? If the clock ambiguity is a real feature of fundamental physics, then the fact that the world is so understandable in terms of specific physical laws must mean that there really must be a way forward.

In this section we consider some possible ways preferences for specific physical laws could emerge in this picture. We continue the approach developed in [1], where fundamental aspects of our experience as observers are identified and considered as selection criteria in choosing our Hamiltonian and state from among all possible ones.

This approach seems to fall under the broad category of ‘‘anthropic reasoning’’(as used, for example, in [26–29]), but as emphasized in [1] and developed further in [30], our approach should be seen as a natural application of the conditional probability analysis that underlies most applications of theoretical physics to actual observations. Probably the most controversial aspect of anthropic reasoning (and rightly so in our view) involves attempts to incorporate general ‘‘physical conditions necessary for life’’ as conditions in conditional probability statements. We do not believe that we (or any other physicists) really know the general physical conditions necessary for life, so we refrain from speculating on those here. Instead, we consider what appear to be general features of our interaction with the rest of the Universe. These features are just as essential to inanimate observers (such as automated data acquisition systems) as they are to us.

If this picture is to succeed, we expect to eventually reach the point where more familiar conditions are applied (such as observations of the electron mass fixing its value in quantum electrodynamics). However the current picture is so far removed from that stage that we only emphasize here more exotic conditions that could offer a glimmer of hope that some sort of preference for specific physical laws could emerge.

B. Time independence of the Hamiltonian

A striking aspect of the clock ambiguity is that it gives no a priori preference for evolution under a time *independent* Hamiltonian. In the notation of Sec. II, the Hamiltonian should generate steps between adjacent discrete times. Specifically,

$$|\psi(t_{i+1})\rangle_R = -i\hbar(t_{i+1} - t_i)\mathbf{H}(\mathbf{t_i})|\psi(t_i)\rangle_R. \qquad (22)$$

But since $|\psi(t_{i+1})\rangle_R$ and $|\psi(t_i)\rangle_R$ are just defined separately by the $\alpha_{i+1,j}$'s and $\alpha_{i,j}$'s, respectively [see Eqs. (6) and (8)], and since the clock ambiguity allows one to consider on an equal footing all possible $\alpha_{i+1,j}$'s, regardless of the values of the α_{ij}'s, there is no a priori reason to assume any particular relationship between $\mathbf{H}(t_i)$ and $\mathbf{H}(t_{i+1})$.

Certainly the constancy of the laws of physics over time appears to be a critical part of our experience as observers. We count on such constancy to learn about our environments (both with our minds, and through genetic evolution) and reap the benefits from the knowledge we gain. So there seems to be hope that this aspect of our existence as observers could be related to (approximate) time independence of $\mathbf{H}$, but at this point do not have a quantitative analysis to offer.

C. Hermiticity of the Hamiltonian

In contrast to the time dependence of the Hamiltonian, the self-adjoint property of $\mathbf{H}$ is realized in a straightforward way. In standard quantum mechanics the Hamiltonian is taken to be Hermitian in order to ensure that time evolution is unitary. Unitary evolution allows wave functions normalized to unit total probability to remain so normalized as time evolves. As emphasized by Isham [17], the formalism we use here makes explicit use of conditional probabilities. For example, you could calculate the probability of measuring a particle at position *x given* that the clock is in state $|t_i\rangle_C$. To do this you would project onto the $|t_i\rangle_C$ state and normalize the answer so that the total probability assigned to all possible outcomes of the measurement (given the time projection) is unity. The overall normalization of the wave function before projection is unimportant when formulating conditional probabilities.

One way to put this is that given that only conditional probability questions will be posed, all possible time evolutions will be treated in a way that makes them effectively unitary. Any nonunitary aspect of the α_{ij}'s will drop out of the final analysis.

D. Quasiseparability and locality

1. Overview

Our experience in the Universe is characterized by the fact that we are minuscule subsystems of the Universe that are able to survive and even thrive with respect to our interactions with the rest of the Universe. We are able to keep our interactions with the rest of the Universe from destroying us for a period of time (with luck, several decades). Furthermore, the state of the rest of the Universe can be accounted for in a highly simplified manner that allows us to model the rest of the Universe with our tiny little brains in a way that usually is sufficient for our survival: even though a bus has many more microscopic physical degrees of freedom than our brains, on average we manage to model the behavior of those degrees of freedom sufficiently well to avoid being hurt by the bus, and even to utilize it for transportation. Planets, stars, and galaxies (with vastly more internal degrees of freedom than a bus) are even easier to handle.

This behavior is extremely different from that which these various subsystems (galaxies, stars, planets, buses, and us) would experience under evolution given by an arbitrary Hamiltonian. In the most general case one would not expect the interactions between these subsystems to be particularly weak or predictable. In fact, there would be no reason to expect the interactions to be at all subdominant to these subsystems' self-interactions. The generic result would seem to be subsystems that are rapidly torn apart by their interactions with the rest of the Universe in a manner that prevents them from keeping any identity as subsystems.

In this section we note that the observed peaceful coexistence of subsystems reflects the quasiseparable nature of the Hamiltonian that actually governs our world. We take the point of view that the quasiseparability is sufficiently important (and sufficiently nongeneric) that it should be taken as a key condition to impose as we search among all arbitrary Hamiltonians for ones that might be relevant to the physical world we experience.

We then note that the quasiseparability we actually experience is closely related to the *locality* that is manifested by the fundamental physical laws we observe. We then speculate on the degree to which imposing the quasiseparable requirement on arbitrary Hamiltonians could strongly favor local physics, perhaps even sufficiently strongly to favor Hamiltonians approximating local quantum field theories with local gauge symmetries and gravity.

2. Locality

In our experience, the key to the quasiseparable nature of our world is the locality of physics. As long as we occupy different locations from the buses, planets, stars, and galaxies, we have a reasonable shot at not being destroyed by them. Formally, this locality comes about because Hamiltonians that describe known physics take the form

$$H = \int \mathcal{H}(x) d^3x. \tag{23}$$

This is certainly very far from the most general case. A general Hamiltonian would allow arbitrary interactions between matter at any two points. Even within the local formalism, there are two long-range forces: gravity and electromagnetism. The overall neutrality of the Universe cuts back greatly on the impact of electromagnetism, and the overall (homogeneous and isotropic) state of the Universe limits the impact of the long-range gravitational forces between objects. Also, the time scale for gravity to have its full impact (such as earth's orbit decaying and plunging us into the sun) is long compared to time scales that interest us.

The critical role locality plays in realizing the quasiseparability that is so important to us leads us to speculate that locality could turn out to be a "generic" way for quasiseparability to emerge as one sifts through arbitrary Hamiltonians. It could be that when quasiseparability is sought that optimizes the evolution of small successful observers it tends to naturally lend itself to interpretation in a "local" language. Since locality is a crucial piece of the construction of quantum field theory, perhaps one could even use arguments such as these "derive" quantum field theory as a foundation of our understanding of matter.

A key part of locality is the definition of space and of distances between points in that space. A more general realization of these features will come about if one allows distances in the space to be defined in terms of an arbitrary metric g_{ij}. When one sifts through random systems and selects out ones that exhibit locality, presumably many more examples will turn up with complicated metrics than with simple ones. Such a tendency toward nontrivial spatial metrics might lay the groundwork for Einstein gravity to emerge in this picture.

3. The speed of light

Another key aspect of known physics is the bounding of all speeds by the finite speed of light. In the picture we describe here, if we enforce locality then propagation speeds should be finite (but not necessarily equal) in all directions. At each point, and in each direction, there will be a maximum propagation speed experienced by certain degrees of freedom. Perhaps it will turn out to be natural to define all other propagation speeds relative to this maximum speed.

The more quantitative analysis of Sec. IV suggests an interesting perspective on the emergence of full Poincare invariance, which we discuss briefly in Sec. IV E.

E. Spin statistics and gauge symmetry

If Poincare invariance and locality do indeed emerge in this picture, presumably this will lead to the emergence of field operators in various representations of the Lorentz group. Since the spin statistics relation is understood to be a consequence of locality [31,32], the critical role of locality in our picture should enforce the usual spin statistics relation.

Since one can argue that gauge symmetries are necessary for the consistency of massless spin one fields (see, for example, [33] and Chap. 8 of [34]), the random appearance of some massless spin one fields might be all it takes for gauge symmetries to emerge in this picture [35]. One possible outcome is that the probabilities for the emergence of particles in different representations of the Lorentz group lead to preferences of particular gauge symmetries over others.

F. The arrow of time and the state of the Universe

The statistical foundations of the thermodynamic arrow of time make it natural to associate the arrow of time with special (low entropy) initial conditions of the Universe. This point was first made in a modern cosmological context by Penrose [36]. Starting with Boltzmann [37], some physicists have been interested in a cosmological picture with an eternal equilibrium state that occasionally fluctuates so as to produce a region of (temporarily) low entropy. The regions of increasing entropy associated with these fluctuations then become candidates to describe our world [1,15,30,38,39]. In general this picture is believed to suffer from the ‘‘Boltzmann brain’’ problem [37,39,40], whereby small fluctuations containing only one observer for a brief moment dominate the predictions. One of us has argued in [30,39] that the Boltzmann brain problem could be resolved by a period of cosmic inflation, and that the resolution of the Boltzmann brain problem is in fact one of the key attractive features offered by inflation.

We find it interesting to compare the picture developed in this paper with Boltzmann's ‘‘fluctuation from equilibrium’’ picture. Formally, one might think that since we consider a finite system there should be quasiperiodic recurrences of the sort that were considered in [15,39]. However, just because a subsystem C has ‘‘good clock’’ behavior for a sufficiently long period to describe our observations does not mean it would be a good clock over a *complete* set of clock states $|t_i\rangle_C$ that span C. It is quite possible that most realistic depictions of our Universe in this formalism would involve the breakdown of the ‘‘good clock’’ behavior at some point outside of the observed domain [17,20,41–43]. That would make it hard to define Boltzmann's fluctuating equilibrium state over ‘‘eternity’’ (or in other words, a complete recurrence time). This is an intriguing point, especially since in the Boltzmann picture it seems a bit of a waste to have time well defined over the extremely long equilibrium period when it is of no real use to us without a thermodynamic arrow.

Still, it is quite possible that something similar to the arguments in [39] will apply in formalism described in this paper. In that case the need for a thermodynamic arrow of time will not only play a key role as a condition for searching for realistic clocks, it will also play a critical part in biasing the initial conditions of the observed Universe toward those that were subject to an early period of inflation.

G. Dimensionality of space, classicality, and other considerations

There are a number of other factors that could have an important impact on the selection of a good clock. For example, the term ‘‘classicality’’ is often applied to the various combinations of phenomena (including the dominance of quantum path integrals by saddle points and the stability of a measurement apparatus after a quantum measurement). More generally, one needs spacetime itself to behave in a classical manner to accurately describe the world we see around us. Many of the phenomena associated with classicality have already been mentioned in this section (for example, the emergence of space, locality, and the thermodynamic arrow of time). It is possible that requiring additional aspects of physics that lead to classical behavior produces additional constraints in the sort of analysis envisioned here.

Also, in a picture where space is emergent, one naturally wonders if there are any preferences for one number of space dimensions over another. Several ideas along these lines have been been put forward over the years (see, for example, [44]). We will return briefly to this issue in Sec. IV E, but so far we do not yet feel we have a compelling argument that a particular number of space dimensions would be favored.

IV. SEARCHING FOR A FIELD THEORY IN A RANDOM HAMILTONIAN

A. Overview

In Sec. III we considered possible ways forward to extract meaningful physics out of quantum gravity, despite the clock ambiguity described in Sec. II. There is clearly far to go if that approach is to really bear fruit. In this section we take a ‘‘reverse engineered’’ approach and ask to what extent the known laws of physics might match on to a random Hamiltonian.

The critical point of comparison is the eigenvalue spectrum: We draw a Hamiltonian at random by choosing a random clock. If its eigenvalue spectrum matches one corresponding to that of the standard model of particle physics, then we are ''done'' in the sense that there is nothing in principle stopping us from carefully identifying the requisite field operators, observables, etc., that describe the theory in the usual way in terms of the eigenstates of H. We do not know how easy this would be in practice, but we do not believe it would run into any issues of principle.

Having stated our approach, we discuss (in the next section) some general results from the theory of random Hamiltonians. Then, in Sec. IV C we consider the eigenvalue spectrum of a free field theory (as a first step toward the eigenvalue spectrum of a full interacting theory). In Sec. IV D we attempt a comparison between the field theory spectrum and that of a random Hamiltonian. Although at first glance comparison seems futile, we suggest an intriguing way forward which appears to hold considerable promise.

B. Properties of random Hamiltonians

There is an extensive literature on random Hamiltonians (see [45,46] and references therein). The basic idea is to select each matrix element of the Hamiltonian from some distribution and look at the ensemble that emerges. It turns out that a wide variety of such random Hamiltonians end up obeying the ''Wigner semicircle rule''

$$\frac{dN}{dE} = \begin{cases} \frac{2N_H}{\pi E_M}\sqrt{1-\left(\frac{E}{E_M}\right)^2} & |E| < E_M, \\ 0 & \text{otherwise}. \end{cases} \tag{24}$$

(We derive this in Appendix A for the Gaussian case.) Here E_M is the maximum eigenvalue and N_H is the size of the random Hamiltonian. One can wonder if there might be subtleties in the process of generating random Hamiltonians through the ''choice of clock'' process that do not generate eigenvalue spectra of exactly this form. Thus we consider a slightly generalized form

$$\frac{dN}{dE} = \begin{cases} a\frac{N_H}{E_m}\left(1-\left(\frac{E}{E_M}\right)^{\beta}\right)^{\gamma} & |E| < E_M, \\ 0 & \text{otherwise}. \end{cases} \tag{25}$$

So we can see how possible variations on the standard form might affect our final results. An illustrative example from this class of functions is depicted in Fig. 1.

C. Density of states of a free field theory

Ideally, at this point we would write down a formula for dN/dE for the standard model of particle physics and compare it with Eq. (25). Since we do not know this function, we seek some initial insights by considering dN/dE for a free field theory. We are not aware of prior calculations of this quantity either, but in 1 + 1 dimensions

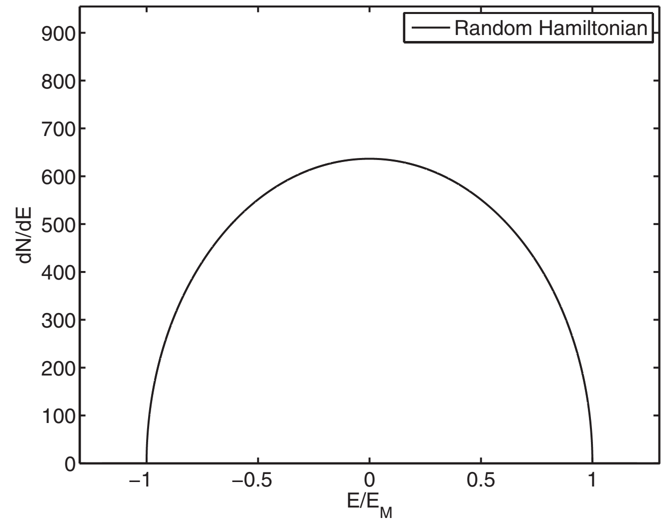

FIG. 1. A plot of the density of states for a random Hamiltonian as given by the Wigner semicircle rule. [We plot Eq. (24) with $E_M = 1$ and $N_H = 1000$.]

we show in Appendix B that

$$\frac{dN}{dE} \sim \frac{1}{4\sqrt{3}E}\exp\left\{\pi\sqrt{\frac{2E}{3\Delta k}}\right\}, \qquad E \gg \Delta k, \tag{26}$$

for a free boson. The quantity Δk reflects the fact that we have regulated the field theory by putting it in a box of size $L = 2\pi/\Delta k$. A similar expression is also found for free fermions. Thus, we consider in this case the following generalization,

$$\frac{dN}{dE} = \frac{B}{E}\exp\left\{b\left(\frac{E}{\Delta k}\right)^{\alpha}\right\}, \tag{27}$$

for large E, which for $b = 1/2$ contains, as special cases, the 1 + 1 expressions of Appendix B and the higher dimensional generalization proposed by Verlinde in [47,48]. When both Fermi and Bose fields are combined, the density of states is dominated by the Bose fields which is why Eq. (27) reflects the Bose form. An illustrative example of this type of function is shown in Fig. 2.

D. Analysis

The forms of Eqs. (25) and (27) (also depicted in Figs. 1 and 2) are dramatically different. At first look this suggests that finding a field theory by randomly choosing a clock (and thus generating a random Hamiltonian) is a very unrewarding endeavor. At best, only Hamiltonians on highly exponentially suppressed tails of the random distribution might give the needed eigenvalue spectrum. It is perhaps not surprising that the ''conditions for a good clock'' discussed in Sec. III would seek out atypical cases. Still, the striking difference between these two functional forms seems to indicate how extremely selective these

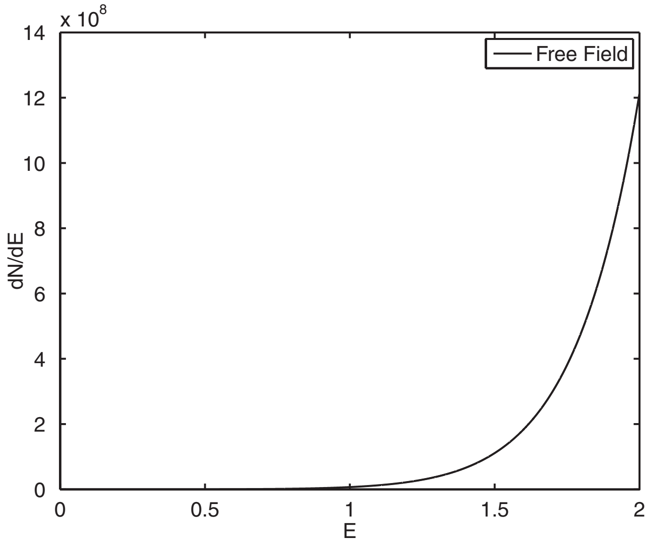

FIG. 2. A plot of the field theory density of states given by Eq. (27) (using $B = b = 1$, $\alpha = 1/2$, and $\Delta k = 0.001$).

conditions would have to be in order to allow this whole approach to succeed.

However, there might be a much easier way forward from here. The key is to consider the fact that we only actually explore dN/dE of the Universe in some relatively narrow range of energies ΔE around a mean energy E_0 (the "energy of the Universe"). Given this, we can ask if Eqs. (25) and (27) can look similar in that particular range of energies.

We start with Eq. (25), the generalized form of the Wigner semicircle result

$$\frac{dN}{dE} = \begin{cases} a\frac{N}{E_m}\left(1-\left(\frac{E}{E_M}\right)^{\beta}\right)^{\gamma} & |E| < E_M, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

and the generalized form of the free field theory result [Eq. (27)]

$$\frac{dN_F}{dE} = \frac{B}{E}\exp\left\{b\left(\frac{E}{\Delta k}\right)^{\alpha}\right\}. \tag{29}$$

We attempt to equate these two equations order by order in a Taylor expansion around E_0.

First we note that we are not trying to consider gravity at this point. As discussed in Sec. III D 2 gravity could potentially emerge through deeper insights into the emergence of locality (and thus a metric). Without gravity, we can assume in this very simpleminded comparison that the overall zero point of E does not have any physical meaning (it just causes an unobservable overall phase shift in the time evolution). Thus we allow a zero point shift E_S when comparing the Wigner and free field theory expressions. Specifically we relate E_R, the energy in the Wigner random Hamiltonian expression [Eq. (25)], to E_F, the energy in the free field theory expression [Eq. (27)], according to

$$E_R = E_F - E_S. \tag{30}$$

We keep Eq. (29) unchanged and absorb the shift into Eq. (25) to give

$$\frac{dN}{dE} = \begin{cases} a\frac{N}{E_m}\left(1-\left(\frac{E-E_S}{E_M}\right)^{\beta}\right)^{\gamma} & |E| < E_M, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

We then Taylor expand each of the expressions for the density of states around the central energy E_0. Expanding the generalized Wigner formula [Eq. (31)] gives

$$\frac{dN_R}{dE} = a\frac{N_H}{E_M}\left(1-\left(\frac{E_0-E_S}{E_M}\right)^{\beta}\right)^{\gamma}\left\{1-\beta\gamma Q\frac{\Delta E}{E_0} + \frac{1}{2}\gamma Q[(\gamma-1)Q-(\beta-1)]\left(\frac{\Delta E}{E_0}\right)^2+\ldots\right\}, \tag{32}$$

where $1/Q \equiv ((E_0 - E_S)/E_M)^{\beta} - 1$. In turn, the field theory formula [Eq. (27)] gives

$$\frac{dN_F}{dE} = B\frac{1}{E_0}\exp\left\{b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right\}\left[1+\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\frac{\Delta E}{E_0} + \frac{1}{2}\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\left[\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha}+(\alpha-1)\right]\left(\frac{\Delta E}{E_0}\right)^2+\ldots\right]. \tag{33}$$

Demanding equality at 0th order and solving the resulting expression for N_H leads to

$$N_H = \frac{B}{a}\frac{E_M}{E_0}\left[1-\left(\frac{E_0-E_S}{E_M}\right)^{\beta}\right]^{-\gamma}\exp\left[b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right]. \tag{34}$$

This expression is completely dominated by the exponential (even though we will soon argue that the quantity in square brackets is extremely small). Thus 0th order equality of the two densities of states just sets the size of the space of the random Hamiltonian to be some specific exponentially large number. It seems reasonable to regard Eq. (34) as a fundamental relation in our formalism. We note that since data only give an upper bound on the field theory regulator Δk, Eq. (34) should really be seen as giving a lower bound on N_H.

Requiring equality between the free field and the generalized Winger expression at first order (as well as at zeroth order) leads to

$$-\beta\gamma\frac{E_0}{E_0-E_S}\frac{\left(\frac{E_0-E_S}{E_m}\right)^{\beta}}{\{1-[\frac{E_0-E_S}{E_M}]^{\beta}\}} = \alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha}. \tag{35}$$

The right-hand side is expected to be an exponentially large quantity (the ratio of the energy of the Universe field theory regulator Δk). To achieve equality for Eq. (35) requires the quantity in square brackets to be exponentially close to unity. This leads to

$$E_S = E_0 - E_M(1-\varepsilon). \tag{36}$$

Here ε [equal to the quantity in curly brackets in Eq. (35)] must be exponentially small [determined implicitly from Eq. (35)] and must be positive (by the nature of the Wigner formula). Also, one must have

$$E_S > E_0, \tag{37}$$

in order to get the overall sign right. It seems reasonable to also take Eq. (35) as a fundamental relation for our scheme, giving the value of the shift energy E_S.

Since we are considering a finite system, there will be a finite gap Δ_G between energy eigenvalues which can be estimated by

$$\Delta_G = \left(\frac{dN_R}{dE}\bigg|_{E_0}\right)^{-1} = \left(\frac{dN_F}{dE}\bigg|_{E_0}\right)^{-1} = \frac{E_0}{B}\exp\left\{-b\left(c\frac{E_0}{\Delta k}\right)^{\alpha}\right\}. \tag{38}$$

Comparing this gap with the field theory regulator gives

$$\frac{\Delta_G}{\Delta k} = \frac{E_0}{\Delta k}\frac{1}{B}\exp\left\{-b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right\}. \tag{39}$$

This suggests that as long as we set Δk small enough to respect the phenomenological successes of continuum field theory, any effects due to the finiteness of the random Hamiltonian are exponentially suppressed and are unlikely to be unobservable. Similar arguments suggest that random fluctuations in density of states due to specific realizations of the random Hamiltonian will be highly subdominant, although we have not exhaustively investigated this question.

It is also irresistible to note that Eq. (36) includes E_0, the energy of the Universe, in the energy offset. In the absence of a specific notion of how gravity emerges in this picture it is really too early to speculate, but we cannot help but wonder if this offset might end up offering an interesting insight into the cosmological constant.

Equality of the two densities of states at second order gives

$$\frac{1}{\gamma}\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha} = \alpha - 1 + \beta, \tag{40}$$

where we have also used the conditions of equality at zeroth and first order [Eqs. (34) and (35)]. The left-hand side of Eq. (40) is generally an extremely large number (the ratio of the energy of the Universe to the k-space regulator of the field theory). The right-hand side is definitely of order unity. Clearly one cannot expect to impose exact equality at second order. Specifically, the second order difference between the two densities of states is given by

$$\begin{aligned}\left(\frac{dN_F}{dE} - \frac{dN_R}{dE}\right)_2 &= \frac{1}{2}\frac{B}{E_0}\exp\left\{b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right\}\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha} \\ &\quad\times\left(\alpha - 1 - \frac{1}{\gamma}\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha} + \beta\right)\left(\frac{\Delta E}{E_0}\right)^2 \\ &\approx \frac{dN}{dE}\bigg|_{E_0}\left(\alpha b\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right)^2\left(\frac{\Delta E}{E_0}\right)^2 \\ &\approx \frac{dN}{dE}\bigg|_{E_0}\left(\left(\frac{E_0}{\Delta k}\right)^{\alpha}\right)^2\left(\frac{\Delta E}{E_0}\right)^2,\end{aligned} \tag{41}$$

where we have dropped subdominant terms as well as factors of order unity to reach the final line. This leads to possible fractional corrections to the density of states given by

$$\frac{\Delta\frac{dN}{dE}}{\frac{dN}{dE}|_{E_0}} \approx \left(\left(\frac{E_0}{\Delta k}\right)^{\alpha}\frac{\Delta E}{E_0}\right)^2 + \mathcal{O}((\Delta E)^3), \tag{42}$$

and we label the second order piece as

$$\Delta_2 \equiv \left(\left(\frac{E_0}{\Delta k}\right)^{\alpha}\frac{\Delta E}{E_0}\right)^2. \tag{43}$$

Now we consider the overall size of Δ_2. We take E_0 to be the energy of the observed Universe, namely,

$$\begin{aligned}E_0 &= \rho_c R_H^3 \approx (H_0^2 m_P^2)H_0^{-3} = m_P^2 H_0^{-1} \\ &\approx \frac{10^{19}\ \mathrm{GeV}}{10^{-42}\ \mathrm{GeV}} m_P = 10^{61} m_P = 10^{80}\ \mathrm{GeV},\end{aligned} \tag{44}$$

where R_H is the Hubble length, H_0 is the Hubble constant today, and $m_P \approx 10^{19}$ GeV is the Planck mass. We have chosen the zero point of the energy as the point with zero particle excitations, so E_0 in Eq. (44) is the correct value to use in the field theory density of states. Taking care of the contribution of the dark energy to this estimate could lead to $O(1)$ correction factors that do not concern us here.

The quantity Δk gives the scale of discreteness for the field theory. The fact that so far we have no evidence for discreteness suggests some pretty low upper bounds on Δk. We consider one value of Δk given by the current bound on the photon mass $\Delta k = m_\gamma \approx 10^{-25}$ GeV [49,50]. We also consider $\Delta k = H_0 \approx 10^{-42}$ GeV, the wave number of a wave the size of the observed Universe.

The quantity ΔE should give the range of energy eigenvalues over which we expect field theory to a give good representation of physics. That is, the range over which we hope the two densities of states will coincide to a good approximation. Physically, a lower bound on ΔE is set by the shortest time δt over which we successfully model observed phenomena using field theory. The two are related by

$$\Delta E \geq \frac{\hbar}{\delta t} \tag{45}$$

We can look to ultrahigh energy cosmic rays ($\delta t^{-1} \approx 10^{11}$ GeV) or the highest energy elementary particle accelerators ($\delta t^{-1} \approx 10^3$ GeV) to set values of δt [51]

Table I gives the values of Δ_2 which correspond to a selection of different values for the quantities in Eq. (43) mentioned in the above discussion. The upshot is that as long as $\alpha = 1/2$ (the value given in our formula for 1 + 1 and also in Verlinde's suggested generalization to higher dimensions) the second order fractional correction to the density of states is very small ($\Delta_2 \approx 10^{-8}$ or even $\Delta_2 \approx 10^{-24.5}$). This small deviation between the free field theory and Wigner formula might account for interactions seen within the context of field theory or perhaps yet-to-be-observed deviations from field theoretical behavior in the real physical world. Also, the field theory density of states assumes a Minkowski space, so deviations of the real spacetime from the Minkowski form will show up as discrepancies in this analysis. (We do not yet have a calculation of any of these phenomena.) Also, we note that the small values of Δ_2 encourage us to believe that contributions from the higher order terms in ΔE will be even smaller for the values of ΔE that are of interest.

We have thus shown that it is possible to find "field-theory-like behavior" in the density of states of any random Hamiltonian. The key is to only look for equivalence between the field theory and the random Hamiltonian over a *finite range* of energy eigenvalues ΔE. Specifically, we have shown that for sufficiently large spaces [N_H set by Eq. (34)] the offset energy (E_S) can be suitably chosen [using Eq. (35)] so that the field theory and random densities of states are identical to zeroth and first orders and only differ by a small amount [given by Δ_2 from Eq. (43)] at second order.

Figs. 3 and 4 give an illustration of how the two densities of states can be made to coincide over a finite range of energies, despite their radically different overall shapes depicted in Figs. 1 and 2. For this illustrative example we have chosen $a = 2/\pi$, $E_M = 10$, $B = 1$, $b = 1$, $\alpha = 1/2$, and $\Delta k = 0.001$. For these parameters solving Eq. (34) gives $N_H = 2.15 \times 10^{21}$ and solving Eq. (35) implicitly gives $E_S = 11.9533$, which we use for these plots.

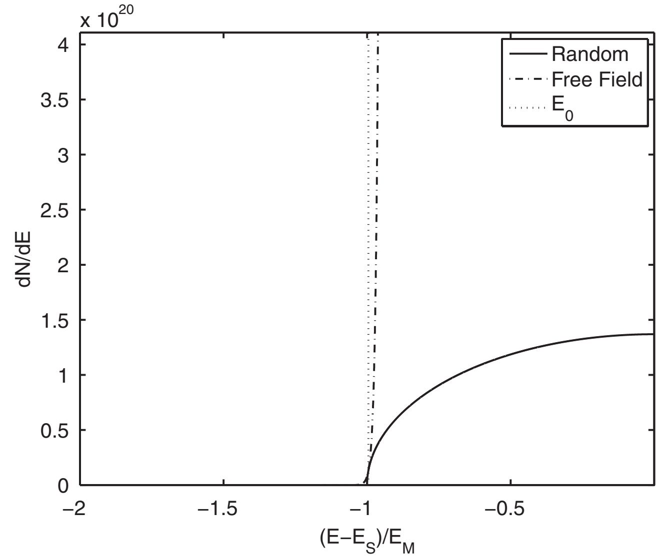

FIG. 3. This figure plots curves for dN_R/dE and dN_F/dE from Eqs. (27) and (31), respectively. Eqs. (34) and (35) have been imposed to cause the zeroth and first order terms in Taylor expansions to be equal at E_0 (marked by the vertical line). The point of coincidence is chosen to be close to the edge of the circle, as discussed around Eq. (36), but for easier viewing the value of ε (which measures the proximity to the circle edge) in this plot is much larger than the exponential small values discussed in the text.

TABLE I. Value of Δ_2 for different choices of the exponent α in Eq. (29), field theory regulator Δk, and observable energy range ΔE. As long as $\alpha = 1/2$, Δ_2 takes on small values, suggesting the random Hamiltonian is giving a good approximation to the field theory and also validating the Taylor expansion.

α	Δk	ΔE	Δ_2
1/2	$m_\gamma = 10^{-25}$ GeV	1 TeV	$10^{-24.5}$
1/2	$m_\gamma = 10^{-25}$ GeV	10^{11} GeV	$10^{-16.5}$
1/2	$H_0 = 10^{-42}$ GeV	1 TeV	10^{-16}
1/2	$H_0 = 10^{-42}$ GeV	10^{11} GeV	10^{-8}
1	$m_\gamma = 10^{-25}$ GeV	1 TeV	10^{28}
1	$m_\gamma = 10^{-25}$ GeV	10^{11} GeV	10^{36}
1	$H_0 = 10^{-42}$ GeV	1 TeV	10^{45}
1	$H_0 = 10^{-42}$ GeV	10^{11} GeV	10^{53}
2	$m_\gamma = 10^{-25}$ GeV	1 TeV	10^{133}
2	$m_\gamma = 10^{-25}$ GeV	10^{11} GeV	10^{141}
2	$H_0 = 10^{-42}$ GeV	1 TeV	10^{167}
2	$H_0 = 10^{-42}$ GeV	10^{11} GeV	10^{175}

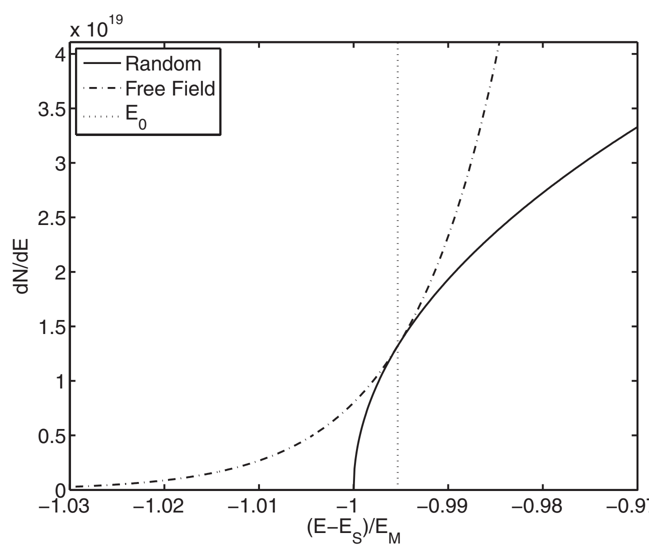

FIG. 4. The same plot as Fig. 3 but zoomed in to show more detail where the curves coincide.

E. Discussion

Our results suggest the following picture: A physicist analyzes a state in superspace by choosing a clock subsystem at random, resulting in the particular story about the time evolution of the rest of the space which is associated with that clock. The physicist then poses the question ‘‘how well can this time evolution accommodate tiny observer subsystems such as ourselves?" The idea here is that the clock choices that maximize the viability of observer subsystems give laws of physics more likely to be the ones we observe (assuming we are ‘‘randomly selected’’ observer subsystems found in our superspace).

By the arguments of Sec. III B clocks which give Hamiltonians which are (at least approximately) time independent should be favored. Since separability is very helpful to observer subsystems, the laws of physics which maximize separability, namely, local field theory, are the ones that are most favored by this selection process. The results from the previous subsection tell us that any random Hamiltonian can be interpreted as a local field to a very good approximation. Thus the search for separability appears to *predict* the emergence of local field theory.

One then must go about analyzing the implications of the deviations from a free field theory represented by Δ_2 and terms higher order in ΔE. We speculate that the goal of optimizing a local interpretation (and thus good separability) will be very significant in interpreting these deviations from free field theory. We expect it to lead to allocations of these deviations among a variety of corrections to the simple ‘‘free field in Minkowski space’’ starting point. These corrections include deviations from Minkowski space (nontrivial metrics), the allocation of degrees of freedom among different particles with the spin statistics relation imposed (as usual) by the need for locality, as well as interactions between the different particles. As argued in Sec. III B, this analysis also might be expected to reveal emergent gauge symmetries. We also note that while the approach of Sec. IV D was to set the free field and random densities of states precisely equal at zeroth and first order, optimizing the overall interpretation of the random matrix as an interacting field theory with gravity might involve slightly relaxing strict equality at those orders.

A very interesting issue is the degree to which the process of assigning local interpretations to the Δ_2, etc., results in any predictive power. Perhaps this process could eventually be understood to give some very powerful predictions about particle content, symmetries, etc., based on the same sort of statistical arguments that operate to give arbitrary random Hamiltonians a unique spectrum given by the Wigner semicircle [52].

One intriguing direction is to consider the role of Poincare invariance in this picture. In our analysis, Poincare invariance has an impact on the form of the density of states of a field theory due to its role in defining the dispersion relation of free particles. It is possible that local theories that have dispersion relations inconsistent with Poincare invariance will not exhibit the good behavior under the Taylor expansion noted in this paper. For example, they may manifest the exponentially large second order ‘‘corrections’’ seen in some cases discussed above (see, for example, the $\alpha = 1$ and $\alpha = 2$ cases in Table I). It is conceivable that such considerations could lead to a sharp preference for Poincare invariant physics, a possibility we are currently actively investigating.

When setting up this analysis, we introduced extra parameters to produce generalized forms of the density of states for free field theory and a random Hamiltonian (these parameters are a, γ, β, B, b, and α). These parameters were introduced to evaluate the robustness of our analysis. At the broad level of our current discussion which is focused on the degree to which random Hamiltonians can approximate a local field theory, the only one of these parameters that seems to matter is α. The other parameters show up in the equations, but changing their values does not change our main points.

We expect the number of space dimensions to enter through the parameter b in the field theory density of states [Eq. (29)]. The lack of sensitivity to b in our current discussion means we have yet to uncover any factors that would prefer one number of space dimensions over another. As discussed in Sec. III G, other considerations might lead to such a preference.

V. SUMMARY AND CONCLUSIONS

We are used to doing physics by stating the physical laws which we believe may be true, and then calculating predictions based on those laws in order to test them against observations of the physical world. The clock ambiguity appears to completely undermine this approach to physics. In time-reparameterization invariant theories such as Einstein gravity, the process of identifying a time variable creates the clock ambiguity. Even if one carefully sets up the system to reflect a particular set of physical laws, the exact same state in superspace can be viewed from the point of view of a different time variable or clock which causes the system to exhibit completely different laws of physics.

A reasonable response to this observation is to reject one or more of the assumptions that go into demonstrating the clock ambiguity. To this end we have carefully identified these assumptions in this paper, and we have argued that rejecting any of these assumptions would be an interesting development since the assumptions we used are widely accepted among physicists.

Most of this article has focused on the possibility that the clock ambiguity is a central feature of fundamental physics, a feature that we are going to have to learn to live with. We first considered the type of analysis that might allow some concrete predictions about the physical world to emerge, despite the profound ambiguities introduced by the choice of clock. Specifically, we envisioned an approach where the fact that we are tiny subsystems of the entire Universe which are able to survive and thrive plays a key role in selecting the type of physical laws we observe. We argued that laws of physics that allow subsystems to do well will preferentially be those that are observed and analyzed by such subsystems.

We identified a number of features of physical laws that would promote the success of small subsystems, and gave special attention to quasiseparability of the Hamiltonian. This is the feature that allows small subsystems to interact much more strongly among themselves than with their environment and thus keep their identities. Quasiseparability also allows subsystems to successfully model their environment based on a simple set of collective coordinates without knowing much about every single degree of freedom. We noted that, in our experience, it is the locality of the laws of physics that leads to the quasiseparability on which we so heavily depend, and we argued that local physics (as expressed by local field theories) seems to be the optimal way of achieving maximal amounts of quasiseparability.

In order to probe this line of thinking in a more quantitative manner, we investigated the extent to which a local quantum field theory could be approximated by a completely random Hamiltonian. Remarkably, we discovered that if the random Hamiltonian is constructed in a large enough space it can *always* approximate a free local field theory to a sufficient degree. Here ‘‘sufficient’’ means part of the spectrum of eigenvalues (or density of states) can coincide with that of a free field theory over a range of eigenvalues. The range of eigenvalues, centered on E_0, the energy of the observed Universe, need only be sufficient to reproduce all observed phenomena that we believe are explained by local field theories. This is very different from matching the full eigenvalue spectrum of a field theory. For example, the field theory ground state on which so much of the formal construction of field theory is based is not part of the spectrum at all. Nonetheless, such an approximation to a true field theory may be sufficient to account for the success of field theoretic models of the physical world.

Initially our picture seemed to be one in which one would reject an enormous fraction of all possible clock subsystems based on the inappropriateness of the corresponding Hamiltonian evolution. We thought that surely conditions such as quasiseparability must be very far from universal. Our result that good approximations to field theory can always be found in sufficiently large random Hamiltonians changes this story considerably. It now appears that any random Hamiltonian can be optimized for quasiseparability by constructing a local field theoretic interpretation. In this sense local field theories might actually be seen as a prediction.

Many questions still remain. How well is the possible time dependence of the Hamiltonian constrained in our picture? Is there some further optimization process that can lead to concrete predictions about gauge symmetries, Poincare invariance, general relativity, etc.? We have speculated along these lines, but so far we do not have concrete results. Still, we find it intriguing that local field theories are much easier to come by than we initially expected. We feel this result offers hope that a framework for fundamental physics which suffers from the clock ambiguity may in the end prove viable.

One of us has argued elsewhere [39] that statistical arguments offer a much more powerful approach to cosmological initial conditions than the more traditional approach of making an *ad hoc* statement of preference. It is possible that eventually the statistical approach to laws of physics described in this paper could achieve that sort of standing. While the outcome is still far from clear, we feel the results in this paper motivate a further investigation of this possibility.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE WIGNER SEMICIRCLE DISTRIBUTION

We consider an ensemble of random Hermitian $N_H \times N_H$ matrices representing the possible Hamiltonians that describe the evolution of states arising in different choices of clock. As we argue in Sec. IV, when comparing a random Hamiltonian with the laws of physics as we know them the key point of comparison is the eigenvalue spectrum.

In the limit of large matrices, the eigenvalue spectrum (or density of states) approaches a unique form. As an illustration, Fig. 5 shows a histogram of the eigenvalues of a 1000×1000 Hermitian matrix where the real and imaginary parts of each matrix element were drawn from a normal distribution of width σ_E. As N_H becomes larger, the fluctuations settle down and the eigenvalue spectrum (or density of states dN/dE) approaches the form given by the ‘‘Wigner semicircle rule’’ [Eq. (24)].

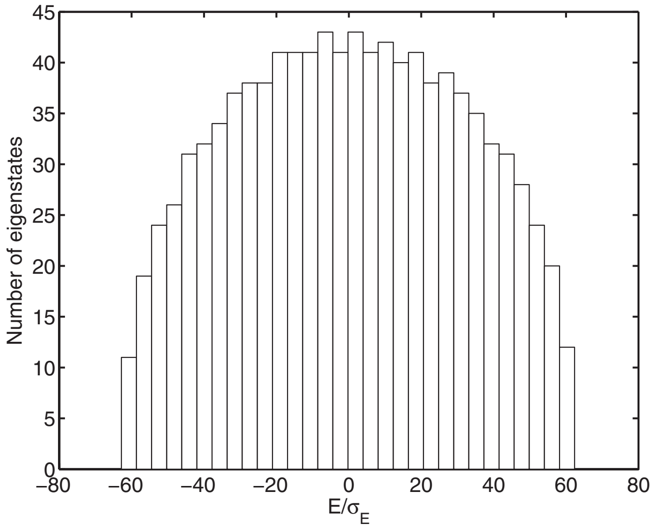

FIG. 5. A histogram of the eigenvalues of a 1000×1000 Hermitian matrix where the real and imaginary parts of each matrix element are drawn from a normal distribution of zero mean and width σ_E.

A derivation of this result (see, for example, [45]) starts by considering a random Gaussian distribution of the matrix elements of H with flat measure $d^{N_H^2}H$ over the N_H^2 real variables $\mathrm{Re}H_{ij}$, $\mathrm{Im}H_{ij}$ to form the partition function,

$$Z = \int d^{N_H^2} H \mathrm{e}^{-(1/2\sigma^2)\,\mathrm{tr}H^2}. \tag{A1}$$

The partition function is then conveniently rewritten in terms of independent variables (the N_H eigenvalues E_i of H and the elements of the matrix U that brings H to diagonal form Λ) by inserting $1 = \int dU \delta(UHU^\dagger - \Lambda)\Delta^2(E)$, where $\Delta(E)$ is the Fadeev-Popov determinant of Vandermonde form $\prod_{i<j}(E_j - E_i)$. Integrating over H first and then factoring out the U integration,

$$Z = \int \prod_i dE_i \Delta^2(E) \mathrm{e}^{-(1/2\sigma^2)\sum_i E_i^2}, \tag{A2}$$

which has a stationary point at $E_i = 2\sigma^2 \sum_j' (E_i - E_j)^{-1}$. In the large N_H limit, a continuous distribution of eigenvalues can be taken instead,

$$E_i = \sqrt{N_H} E(i/N_H), \tag{A3}$$

spread over the interval $(-E_M, E_M)$ with variance $\sigma^2 = E_M^2/N$ and density dN/dE such that

$$\frac{1}{N_H} \int_{-E_M}^{E_M} dE \frac{dN}{dE} = 1. \tag{A4}$$

The stationary condition is now

$$\frac{1}{2} E = \int_{-E_M}^{E_M} dE' \frac{dN}{dE'} \frac{E_M^2}{2N_H} \frac{1}{E - E'}, \tag{A5}$$

solved by the Wigner semicircle distribution:

$$\frac{dN}{dE} = \begin{cases} \frac{2N_H}{\pi E_M^2} \sqrt{E_M^2 - E^2} & |E| < E_M, \\ 0 & \text{otherwise.} \end{cases} \tag{A6}$$

APPENDIX B: DENSITY OF STATES FOR A FREE FIELD THEORY

Here we discuss an example in which the density of states can be computed analytically: a massless free field theory in 1 + 1 dimensions with coordinates (σ, t). In this case the degeneracy of states at a given energy level can be obtained by studying the appropriate generating function.

1. Bosons

Let us first consider a boson confined to an interval of the spacial dimension, $\sigma \in [0, \pi]$. The resulting free theory has a discreet spectrum, with mode decomposition of the form,

$$\phi = \sum_{n \in Z}' \frac{1}{\sqrt{n}} a_n \mathrm{e}^{-int} \sin n\sigma, \tag{B1}$$

and states with arbitrary occupation number N_n for each mode, $|N_1, N_2, \cdots\rangle$, with energy $E = \Delta k \sum_{n=1} n N_n$ (the energy eigenvalue spacing Δk equals one in the present case).

The generating function for this system is given by

$$Z_B = \mathrm{tr}\, \mathrm{e}^{\beta E} = \sum_{E=1}^{\infty} d_E z^E, \tag{B2}$$

where we have defined $z = \mathrm{e}^\beta$, d_E is the degeneracy of states of energy E (the quantity we are interested in), and the trace is taken over the state space.

On the one hand, an exact expression is available for Z_B,

$$Z_B = \prod_{n=1} \sum_{N_n=0} (z^n)^{N_n} = \prod_{n=1} \frac{1}{1 - z^n} = z^{-(1/24)} \eta(\tau), \tag{B3}$$

where η stands for the Dedekind function and $\tau = \beta/2\pi i$. On the other, d_E is easily expressible, from Eq. (B2), as a contour integral,

$$d_E = \frac{1}{2\pi i} \oint \frac{Z_B(z)}{z^{E+1}} dz. \tag{B4}$$

By noting that the integrand in Eq. (B4) is sharply peaked around $z = 1$ the integral can be estimated by a saddle point approximation. The value of $Z(z \to 1)$ can be deduced from the Hardy-Ramanujan formula that exploits the modular property of the Dedekind function ($\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau)$),

$$\frac{1}{Z_B(z)} = \sqrt{\frac{-2\pi}{\log z}} z^{-(1/24)} q^{(1/12)} \frac{1}{Z_B(q^2)}, \qquad q = \mathrm{e}^{(2\pi^2/\log z)}, \tag{B5}$$

and realizing that $z \to 1$ corresponds to $q \to 0$ and that $Z_B(0) = 1$. Therefore, the large E asymptotic behavior of d_E is found by considering the following approximation to Eq. (B4),

$$d_E \sim \frac{1}{2\pi i} \oint \frac{1}{z^{E+1}} \sqrt{-\log z}\, \mathrm{e}^{-(\pi^2/6 \log z)} dz. \tag{B6}$$

The integrand has a stationary point at $\log z = -\pi/\sqrt{6(E+1)}$ ($z \sim 1$ for large E) that readily gives the asymptotic value of d_E:

$$d_E \equiv \frac{dN}{dE} \sim \frac{1}{4\sqrt{3}E} \mathrm{e}^{\sqrt{(2E/3\Delta k)}\pi}, \qquad E \gg \Delta k, \tag{B7}$$

where we have reinserted, to facilitate the generalization of this formula, the spacing between energy eigenvalues Δk (equal to 1 in this case due to the interval of σ chosen).

In the case of a compactified spacial dimension, i.e., imposing periodicity at the boundaries, the left- and right-moving modes become independent, and the generating function is, therefore, a product of the two factors ($Z = |Z_B|^2$).

2. Fermions

Let us consider now a free fermion ψ in 1 + 1 dimensions with right(left)-moving components $\psi_{-(+)}$,

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \tag{B8}$$

and choose, for example, periodic boundary conditions $\psi_+ = \psi_-$ at $\sigma = 0$ and π. The mode expansions are

$$\psi_\pm = \sum_{n \in Z}{}' b_n \mathrm{e}^{-in(t \pm \sigma)}, \tag{B9}$$

and the space of states is labeled by the occupation numbers of each mode as in the bosonic case $|N_1, N_2, \cdots\rangle$ (with energy $E = \sum_{n=1} n N_n$) except that each N can only be either 0 or 1. The generating function in this case is

$$Z_F = \mathrm{tr}\, \mathrm{e}^{\beta E} = \sum_{E=1} d_E z^E, \tag{B10}$$

for which there is also an exact expression,

$$Z_F = \prod_{n=1} (z^0 + z^n) = \frac{\sqrt{2}}{z^{(1/24)}} \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}, \tag{B11}$$

where θ_i, $i = 1 \cdots 4$ are the Jacobi theta functions. Considering the modular property $(\theta_2(-1/\tau) = \sqrt{-i\tau}\theta_4(\tau))$ we find

$$Z_F(z) = \frac{\sqrt{2}}{z^{(1/24)}} q^{-(1/24)} \prod_{r=1/2} (1 - q^{2r}), \qquad q = \mathrm{e}^{(2\pi^2/\log z)}. \tag{B12}$$

Using the same method as in the bosonic case (focusing in the $z \to 1$, $q \to 0$ limit) we obtain an asymptotic expression suitable to obtaining the large E behavior,

$$Z_F(z \sim 1) \sim \sqrt{2} \mathrm{e}^{-(\pi^2/12 \log z)}. \tag{B13}$$

Thus,

$$d_E \sim \frac{1}{2\pi i} \oint \frac{\sqrt{2}}{z^{E+1}} \mathrm{e}^{-(\pi^2/12 \log z)} dz. \tag{B14}$$

The stationary point at $\log z = -\pi/2\sqrt{3(E+1)}$ yields in this case

$$d_E \equiv \frac{dN}{dE} \sim \frac{1}{2(3\Delta k)^{(1/4)} E^{(3/4)}} \mathrm{e}^{\sqrt{(E/3\Delta k)}\pi}, \qquad E \gg \Delta k, \tag{B15}$$

where we have reinserted the eigenvalue spacing Δk in the final step as in the bosonic case.

As mentioned in the previous subsection, a compactified spacial direction implies the right- and left-moving modes are independent, and therefore, the generating function becomes the product of the two corresponding factors.

We note that the exponent in the fermion density of states [Eq. (B15)] is a factor of $\sqrt{2}$ smaller than for the Bose case [Eq. (B7)]. For the huge exponents that concern us in this article this makes the fermion density of states highly subdominant versus the Bose case at the same energy.

3. Bosonization

Notice that for both fermions and bosons the density of states grows exponentially with the square root of the energy. This leads us to the following question: is the dominant contribution to the density of states coming from states with many modes singly excited (given that it is the only possibility for fermions)?

The similarity in the behaviors of the density of states for bosons and fermions can be traced to the close relation between bosons and fermions in the particular case of 1 + 1 dimensions that leads to the concept of bosonization.

The rule that relates the left-moving part of a boson field $\phi(z)$ and the left-moving component of a fermion $\psi(z)$ is

$$\psi = \mathrm{e}^{i\phi}, \tag{B16}$$

where we have switched to complex-plane variables $(\sigma, t) \to (z, \bar{z})$ defining $z = \rho \mathrm{e}^{i\sigma}$ ($i \log\rho = t$) for later convenience. Now we take into account the mode expansions,

$$\phi(z) = \sum a_n z^{-n}, \qquad \psi(z) = \sum b_r z^{-r-(1/2)}, \tag{B17}$$

to express the creation operators in the following way:

$$a_n = \frac{1}{2\pi i}\oint \frac{\phi}{z^{-n+1}},$$
$$b_r = \frac{1}{2\pi i}\oint \frac{\psi}{z^{-r+(1/2)}} = \frac{1}{2\pi i}\oint \frac{e^{i\phi}}{z^{-r+(1/2)}}. \quad \text{(B18)}$$

And through Eq. (B16) we obtain

$$b_r = \frac{1}{2\pi i}\oint \frac{e^{i\phi}}{z^{-r+(1/2)}} = \frac{1}{2\pi i}\sum_m \oint \frac{(i\sum_p a_p z^p)^m}{m! z^{-r+(1/2)}}. \quad \text{(B19)}$$

The implications of Eq. (B19) can be illustrated by the following example: take a singly excited fermionic state of energy $E = 10$,

$$b_{-10-(1/2)}|0\rangle = \frac{1}{10!}\oint \frac{(a_{-1}z)^{10}}{z^{11}} + \frac{1}{5!}\oint \frac{(a_{-2}z^2)^5}{z^{11}} + \frac{1}{2!}\oint \frac{(a_{-5}z^5)^2}{z^{11}} + \oint \frac{(a_{-10}z^{10})}{z^{11}}|0\rangle$$
$$= \left(\frac{1}{10!}(a_{-1})^{10} + \frac{1}{5!}(a_{-2})^5 + \frac{1}{2}(a_{-5})^2 + a_{-10}\right)|0\rangle = \frac{1}{\sqrt{10!}}|n_1 = 10\rangle + \frac{1}{\sqrt{5!}}|n_2 = 5\rangle + |n_{10} = 1\rangle; \quad \text{(B20)}$$

it is equivalent to a linear combination of multiply excited bosonic states of levels given by the divisors of E. In the last line of Eq. (B20) we used normalized states $|n_i = m\rangle \equiv \frac{1}{\sqrt{m!}}(a_i^\dagger)^m|0\rangle$.

In general, a highly energetic singly excited fermionic state $b_r|0\rangle$ is equivalent to a bosonic state of singly excited bosons contaminated by small components of high multiplicity.

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