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Energy of n identical bosons in a finite volume at $\mathcal{O}(L^{-7})$

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The volume dependence of the ground-state energy of n identical bosons with short-range interactions in a periodic spatial volume with sides of length L is calculated at order L^{-7} in the large-volume expansion. This result will enable a refined determination of the $\pi^+\pi^+\pi^+$ interaction from lattice QCD calculations.

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It is now well established that two-body interactions between hadrons can be studied with lattice QCD as the volume dependence of the energy spectrum of two hadrons is related to their scattering amplitude below inelastic thresholds [1,2]. Recently, this method has been used to determine the $\pi^+\pi^+$ scattering length [3], $a_{\pi^+\pi^+}$, with $\sim 1\%$ precision with a $n_f = 2+1$ fully dynamical mixed-action lattice QCD calculation. In order to extract the many-body interactions from lattice QCD calculations, the energy of multihadron states in a finite volume must be calculated with lattice QCD and combined with the known dependence of this energy on the many-body interactions. The ground-state energy of a system of n identical bosons with short-range interactions in a cubic volume with sides of length L was recently computed at $\mathcal{O}(L^{-6})$ in the large-volume expansion [4]. The underlying motivation for that work, which builds upon the classic works of Refs. [5–8], was to provide a way to determine the three-body interactions between π^+'s from lattice QCD calculations, which first enters at that order [4]. In Ref. [9], this result was used in conjunction with lattice QCD calculations of multi-pion systems to determine the interaction between three π^+'s for the first time. In order to refine the determination of the $\pi^+\pi^+\pi^+$ interaction, here we compute the contribution to the energy shift of n identical bosons at $\mathcal{O}(L^{-7})$ in the large-volume expansion. The energy shift of three identical bosons in a finite volume has been computed recently in Ref. [10], and our $n = 3$ calculation agrees.

The ground-state energy of n identical bosons is calculated using standard Schrödinger perturbation theory, with a Hamiltonian, appropriate to the order we are working in the large-volume expansion, of the form

$$H = \sum_{\mathbf{k}} h^\dagger_{\mathbf{k}} h_{\mathbf{k}}\left(\frac{|\mathbf{k}|^2}{2M} - \frac{|\mathbf{k}|^4}{8M^3}\right) + \frac{1}{(2!)^2}\sum_{\mathbf{Q},\mathbf{k},\mathbf{p}} h^\dagger_{(\mathbf{Q}/2)+\mathbf{k}} h^\dagger_{(\mathbf{Q}/2)-\mathbf{k}} h_{(\mathbf{Q}/2)+\mathbf{p}} h_{(\mathbf{Q}/2)-\mathbf{p}}\left(\frac{4\pi a}{M} + \frac{\pi a}{M}\left(ar - \frac{1}{2M^2}\right)(|\mathbf{k}|^2 + |\mathbf{p}|^2)\right) + \frac{\eta_3(\mu)}{(3!)^2}\sum_{\mathbf{Q},\mathbf{k},\mathbf{p},\mathbf{r},\mathbf{s}} h^\dagger_{(\mathbf{Q}/3)+\mathbf{k}} h^\dagger_{(\mathbf{Q}/3)+\mathbf{p}} h^\dagger_{(\mathbf{Q}/3)-\mathbf{k}-\mathbf{p}} h_{(\mathbf{Q}/3)+\mathbf{r}} h_{(\mathbf{Q}/3)+\mathbf{s}} h_{(\mathbf{Q}/3)-\mathbf{r}-\mathbf{s}}, \tag{1}$$

where the operator $h_{\mathbf{k}}$ annihilates a π^+ with momentum $\mathbf{k}$ with unit amplitude. The divergences that arise at loop level are regulated with dimensional regularization, and therefore the coefficients of the two-body interaction can be readily identified with the parameters describing the scattering amplitude: the scattering length, a, and the effective range, r ($p\cot\delta = -\frac{1}{a} + \frac{1}{2}rp^2 + \ldots$). The terms proportional to M^{-3} in Eq. (1) describe the leading effects of relativity. Only the momentum independent three-body interaction, $\eta_3(\mu)$, is required at $\mathcal{O}(L^{-7})$. Our method of computation is equivalent to the pionless effective field theory (EFT) describing low-energy nucleon-nucleon interactions, EFT($\not\pi$) [11–13] (when modified to describe systems with natural scattering lengths) and the method of pseudopotentials used in our previous work. The divergences that occur in loop diagrams are renormalized order by order in the expansion, preserving the power counting, and hence the explicit dependence of the bare three-body coefficient on the renormalization scale, μ.

The calculation of the energy-shift of n identical bosons at $\mathcal{O}(L^{-7})$ due to the interactions defined in Eq. (1) is straightforward but tedious. We will not delve into the details, referring the reader to our previous work [4] and that of Ref. [10], and simply state the result. The energy shift of the ground state is

$$E_0(n,L) = \frac{4\pi a}{ML^3}\binom{n}{2}\Big\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2[\mathcal{I}^2 + (2n-5)\mathcal{J}] - \left(\frac{a}{\pi L}\right)^3[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + (5n^2-41n+63)\mathcal{K}]$$
$$+ \left(\frac{a}{\pi L}\right)^4[\mathcal{I}^4 - 6\mathcal{I}^2\mathcal{J} + (4+n-n^2)\mathcal{J}^2 + 4(27-15n+n^2)\mathcal{I}\mathcal{K} + (14n^3-227n^2+919n-1043)\mathcal{L}]\Big\}$$
$$+ \binom{n}{2}\frac{8\pi^2 a^3 r}{ML^6}\left[1 + \left(\frac{a}{\pi L}\right)3(n-3)\mathcal{I}\right] + \binom{n}{3}\frac{1}{L^6}\left[\eta_3(\mu) + \frac{64\pi a^4}{M}(3\sqrt{3}-4\pi)\log(\mu L) - \frac{96a^4}{\pi^2 M}\mathcal{S}\right]$$
$$\times\left[1 - 6\left(\frac{a}{\pi L}\right)\mathcal{I}\right] + \binom{n}{3}\left[\frac{192a^5}{M\pi^3L^7}(\mathcal{T}_0 + \mathcal{T}_1 n) + \frac{6\pi a^3}{M^3L^7}(n+3)\mathcal{I}\right] + \mathcal{O}(L^{-8}). \quad (2)$$

where the geometric constants that enter are[1]

$$\mathcal{I} = -8.9136329, \qquad \mathcal{T}_0 = -4116.2338,$$
$$\mathcal{J} = 16.532316, \qquad \mathcal{T}_1 = 450.6392,$$
$$\mathcal{K} = 8.4019240, \qquad \mathcal{S}_{\rm MS} = -185.12506, \quad (3)$$
$$\mathcal{L} = 6.9458079,$$

and $\binom{n}{k} = n!/(n-k)!/k!$. The last term in the last bracket of Eq. (2) is the leading relativistic contribution to the energy shift. Deviations from the energy shift of n-bosons computed with nonrelativistic quantum mechanics arise only for three or more particles as the two-particle energy shift has the same form when computed in nonrelativistic quantum mechanics and in quantum field theory [1,2]. In Eq. (3), $\mathcal{S}_{\rm MS}$ is the value of the scheme-dependent quantity $\mathcal{S}$ in the minimal subtraction (MS) scheme that we have employed to renormalize the theory (a change in scheme results in a change in $\mathcal{S}$ and a compensating change in $\eta_3(\mu)$).[2] The $\mathcal{T}_i$ are renormalization scheme independent. Our result at $n = 2$ agrees with large-volume expansion of Ref. [1,2], and at $n = 3$ agrees with the previous computation by Shina Tan [10].

The renormalization-scale independent, but volume dependent, quantity

$$\bar{\eta}_3^L = \eta_3(\mu) + \frac{64\pi a^4}{m}(3\sqrt{3}-4\pi)\log(\mu L) - \frac{96a^4}{\pi^2 m}\mathcal{S} \quad (4)$$

was determined in recent lattice QCD calculations [9]. It was found to be nonvanishing in systems of three, four and five π^+'s at pion masses of $m_\pi \sim 290$ and 350 MeV in a $\sim(2.5\ {\rm fm})^3$ volume, when extracted at $\mathcal{O}(L^{-6})$ in the large-volume expansion. Its size was found to be consistent with expectations based upon naive dimensional analysis, $\bar{\eta}_3^L \sim 1/(m_\pi f_\pi^4)$. Our result will allow for further refinement of such extractions.

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[1]The constants $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ were defined previously in Ref. [4], while the constant $\mathcal{L}$ is defined to be the integer-triplet sum

$$\mathcal{L} = \sum_{\mathbf{n}\neq\mathbf{0}}\frac{1}{|\mathbf{n}|^8},$$

and is equal to $\mathcal{L} = \alpha_4$ in the notation of Ref. [10]. The constants $\mathcal{T}_{0,1}$ arise from combinations of up to three-loop diagrams, and involve three-, six-, and nine-dimensional sums over integers, and can be written in terms of constants defined in Ref. [10] plus one additional sum, S_1,

$$\mathcal{T}_0 + \mathcal{T}_1 n = \tfrac{1}{4}\alpha_{1AA1} - \mathcal{I}\alpha_{1A1} + \tfrac{1}{2}(2n-9)\alpha_{2A1} + \tfrac{3}{4}(n-4)\alpha_{1B1} - \tfrac{1}{4}(7n-29)\mathcal{L} + 2(n-3)S_1,$$

where

$$S_1 = \sum_{\mathbf{n},\mathbf{j}\neq\mathbf{0}}\frac{1}{|\mathbf{n}|^2|\mathbf{j}|^4[|\mathbf{n}|^2 + |\mathbf{n}+\mathbf{j}|^2]} = 92.42215.$$

[2]In the notation of Ref. [4], $\mathcal{S}_{\rm MS} = 2\mathcal{Q} + \mathcal{R}$. The numerical value in Eq. (3) corrects a minor error in $\mathcal{Q}$ in Ref. [4].

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