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# Constructing the leptonic unitarity triangle

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Following the analogy of the "$db$" triangle in the quark-mixing case, we have explored the construction of the "$\nu_2.\nu_3$" leptonic unitarity triangle using the Maki-Nakagawa-Sakata matrix obtained by Bjorken *et al.* through generalization of the tribimaximal scenario. In particular, for the $U_{e3}$ range 0.05–0.15, the existence of the leptonic unitarity triangle indicates a fairly good possibility of having nonzero $CP$ violation.

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In the last few years, apart from establishing the hypothesis of neutrino oscillations, impressive advances have been made in understanding the phenomenology of neutrino oscillations through solar neutrino experiments [1], atmospheric neutrino experiments [2], reactor based experiments [3], and accelerator based experiments [4], enabling the determination of the basic form of the Maki-Nakagawa-Sakata (MNS) leptonic mixing matrix [5]. At present, one of the key issues in the context of neutrino oscillation phenomenology is the existence of $CP$ violation in the leptonic sector.

Taking clues from the existence of the unitarity triangle and consequently $CP$ violation in the quark sector [6], several attempts [7–9] have been made to explore such a possibility in the leptonic sector. In particular, Farzan and Smirnov [8] have discussed the desirability of exploring the construction of the leptonic unitarity triangle for finding possible clues to the existence of $CP$ violation in the leptonic sector. Considering the "$e-\mu$" triangle, for $U_{e3}$ values in the range 0.09–0.22, they have examined the detailed implications of different values of the $CP$-violating phase $\delta$ on the possible accuracy required in the measurement of various oscillation probabilities. Very recently, Bjorken *et al.* [9] have constructed a generalization of the tribimaximal scenario and have not only presented a very useful form of the MNS matrix but have also proposed a unitarity triangle, referred to as "$\nu_2.\nu_3$" which could be leptonic analogue of the much talked about "$db$" triangle in the quark sector. This immediately suggests a need for deeper study of the "$\nu_2.\nu_3$" triangle using clues from the "$db$" triangle in the quark sector. In particular, it would be very much desirable, as a complimentary approach to the scenario investigated by Farzan and Smirnov [8], to find the probable values of the $CP$-violating phase $\delta$ suggested by the generalized tribimaximal scenario of Bjorken *et al.* [9].

To this end, taking clues from the "$db$" unitarity triangle in the quark sector, the purpose of the present paper is to explore the possibility of the construction of the leptonic unitarity triangle as well as the existence of $CP$ violation in the leptonic sector. In particular, in the MNS matrix constructed by Bjorken *et al.* [9] we have considered different values of $U_{e3}$ suggested by various theoretical models [10]. Further, we have explored in detail the possibility of finding a nonzero value of $J_l$, the Jarlskog's rephasing invariant parameter in the leptonic sector as well as the related Dirac-like $CP$-violating phase $\delta$.

For ready reference as well as to facilitate a discussion of results, we begin with the neutrino mixing phenomenon expressed in terms of a 3 × 3 neutrino-mixing MNS matrix [5] given by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \tag{1}$$

where $\nu_e$, $\nu_\mu$, $\nu_\tau$ are the flavor eigenstates and $\nu_1$, $\nu_2$, $\nu_3$ are the mass eigenstates. Following Particle Data Group (PDG) representation, involving three angles and the Dirac-like $CP$-violating phase $\delta$ as well as the two Majorana phases $\alpha_1$, $\alpha_2$, the MNS matrix $U$ can be written as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The Majorana phases $\alpha_1$ and $\alpha_2$ do not play any role in neutrino oscillations and henceforth would be dropped from the discussion.

Unitarity implies nine relations, three in terms of normalization conditions, the other six can be defined as

$$\sum_{i=1,2,3} U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta} \qquad (\alpha \neq \beta), \tag{3}$$

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$$\sum_{\alpha=e,\mu,\tau} U_{\alpha i}U^*_{\alpha j} = \delta_{ij} \qquad (i \neq j), \tag{4}$$

where Latin indices run over the mass eigenstates (1, 2, 3) and Greek ones run over the flavor eigenstates $(e, \mu, \tau)$. These six nondiagonal relations can also be expressed through six independent unitarity triangles in the complex plane.

For getting viable clues to the construction of the leptonic unitarity triangle, we first consider the case of quarks wherein the Cabibbo-Kobayashi-Maskawa (CKM) matrix [11] is fairly well established and the $CP$-violating phase $\delta$ has also been measured recently [12–15] with a good deal of accuracy. To begin with, we consider the quark-mixing matrix given by PDG 2006 [12] and attempt to reconstruct the $CP$-violating phase $\delta$ using the Jarlskog's rephasing invariant parameter $J$, equal to twice the area of any of the unitarity triangle. In this context, we consider the usual "$db$" triangle, expressed as

$$V_{ud}V_{ub}{}^* + V_{cd}V_{cb}{}^* + V_{td}V_{tb}{}^* = 0, \tag{5}$$

from which one can obtain a histogram of $J$ by considering Gaussian distribution for the CKM matrix elements as well as imposing the constraints $|a| + |c| > |b|$ and $|b| + |c| > |a|$ for the three sides of the triangle $a$, $b$, $c$. From the histogram of $J$, not shown here, one can find

$$J = (3.0 \pm 0.4) \times 10^{-5}. \tag{6}$$

Using the relation between the parameter $J$ and phase $\delta$, e.g.,

$$J = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin\delta, \tag{7}$$

one can obtain the corresponding histogram of $\delta$, shown in Fig. 1, yielding

$$\delta = 55.4° \pm 10.0°. \tag{8}$$

For further details we refer the reader to [16]. Interestingly, we find that the above-mentioned $J$ value has an excellent overlap with that found by PDG through their recent global analysis [12]. Also, this value of $\delta$ is fully compatible with the experimentally determined $\delta$ given by PDG 2006 as well as found by some of the most recent analyses [12–15].

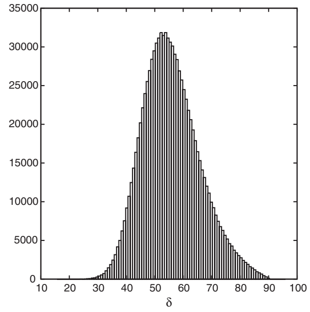

FIG. 1. Histogram of $\delta$ plotted by considering the "$db$" triangle in the case of quarks.

The above discussion immediately provides a clue for exploring the possibility of the existence of $CP$ violation in the leptonic sector, even when the leptonic mixing matrix is approximately known. In this context, we have considered the MNS matrix obtained by Bjorken *et al.* [9], e.g.,

$$U = XY \tag{9}$$

where

$$X = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad Y = \begin{pmatrix} C & 0 & \sqrt{\frac{3}{2}}U_{e3} \\ 0 & 1 & 0 \\ -\sqrt{\frac{3}{2}}U^*_{e3} & 0 & C \end{pmatrix} \tag{10}$$

with

$$C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2}. \tag{11}$$

The matrix $X$ corresponds to the tribimaximal mixing matrix whereas the matrix $Y$ breaks this exact tribimaximal form by small perturbations due to the effects of the element $U_{e3}$. It may be added that the matrix $U$ is unitary by construction.

It needs to be mentioned that the experimental data deviates somewhat from the tribimaximal form; therefore we have modified the matrix $X$ by introducing a modest error of 5% to its "12“ and "23“ elements, governed by solar and atmospheric neutrino data. These elements have been considered independent in the present case; the other elements along with the respective errors have been obtained by using the constraints of unitarity. For $U_{e3}$, it is very well recognized that its value would have deep implications for the neutrino oscillation phenomenology [17–24]. However at present only its upper limit is known; therefore while constructing the matrix $Y$ we have taken its values to be 0.05, 0.10, and 0.15 and attached 10% errors. These values have been considered primarily following a recent detailed analysis by Albright and Chen [10] wherein they have studied the implications of $U_{e3}$ values on various leptonic and grand unified models of neutrino masses and mixings. The errors in the elements of the matrices $X$ and $Y$ have been introduced following Farzan and Smirnov [8] and keeping in mind the accuracy with which these would be measured by the planned neutrino experiments [8,22,25–27]. The matrices corresponding to $U_{e3}$ values $0.05 \pm 0.005$, $0.10 \pm 0.01$, and $0.15 \pm 0.015$ are, respectively, as follows:

$$U = \begin{pmatrix} 0.8150 \pm 0.0204 & 0.5774 \pm 0.0289 & 0.05 \pm 0.005 \\ 0.4508 \pm 0.0646 & 0.5774 \pm 0.0144 & 0.6808 \pm 0.0356 \\ 0.3642 \pm 0.0646 & 0.5774 \pm 0.0144 & 0.7308 \pm 0.0356 \end{pmatrix}, \tag{12}$$

$$U = \begin{pmatrix} 0.8104 \pm 0.0203 & 0.5774 \pm 0.0289 & 0.1 \pm 0.01 \\ 0.4918 \pm 0.0648 & 0.5774 \pm 0.0144 & 0.6518 \pm 0.0363 \\ 0.3186 \pm 0.0648 & 0.5774 \pm 0.0144 & 0.7518 \pm 0.0363 \end{pmatrix}, \tag{13}$$

$$U = \begin{pmatrix} 0.8026 \pm 0.0202 & 0.5774 \pm 0.0289 & 0.15 \pm 0.015 \\ 0.5312 \pm 0.0652 & 0.5774 \pm 0.0144 & 0.6201 \pm 0.0376 \\ 0.2714 \pm 0.0652 & 0.5774 \pm 0.0144 & 0.7701 \pm 0.0376 \end{pmatrix}, \tag{14}$$

wherein we have given the magnitude of the elements, as is usual.

Out of the six triangles defined by Eqs. (3) and (4), Bjorken *et al.* [9] have considered the "$\nu_2.\nu_3$" triangle which is the leptonic analogue of the "*db*" triangle of the quark sector and is expressed as

$$U_{e2}U_{e3}{}^* + U_{\mu 2}U_{\mu 3}{}^* + U_{\tau 2}U_{\tau 3}{}^* = 0. \tag{15}$$

Using the matrices constructed above and following the same procedure as in the quark case, for the "$\nu_2.\nu_3$" triangle we obtain the corresponding respective values of $J_l$ as

$$J_l = 0.009 \pm 0.003, \tag{16}$$

$$J_l = 0.018 \pm 0.006, \tag{17}$$

$$J_l = 0.024 \pm 0.009. \tag{18}$$

Using the relation between $J_l$ and phase $\delta$, Eq. (7), as well as considering the above values of $J_l$ and mixing angles to have Gaussian distributions, one can find the corresponding histograms of $\delta$. Using the histograms, shown in Fig. 2, the $\delta$ values corresponding to $U_{e3}$ values $0.05 \pm 0.005$, $0.10 \pm 0.01$, and $0.15 \pm 0.015$ are, respectively, as follows:

$$\delta \simeq 47^\circ \pm 15^\circ, \tag{19}$$

$$\delta \simeq 45^\circ \pm 15^\circ, \tag{20}$$

$$\delta \simeq 42^\circ \pm 16^\circ. \tag{21}$$

It is interesting to note that the *CP*-violating phase $\delta$ comes out to be around 45° and is not very sensitive to $U_{e3}$ in the range 0.05–0.15. The above calculated values of $\delta$, indicating an almost $2.5\sigma$ deviation from 0°, are in line with the suggestion by several authors [25,26,28] about the expected *CP* violation in the leptonic sector. In particular, the present analysis is broadly in agreement with a similar analysis carried out by Farzan and Smirnov [8] and also with a recent phenomenological analysis carried out by Balaji *et al.* [29]. Further, it is interesting to note that the present analysis carried out purely on phenomenological inputs is very much in agreement with several analyses based on expected outputs from different experimental scenarios [8,22,25,27]. In particular, the analysis of Marciano and Parsa [25] carried out for the BNL-Homestake (2540 km) proposal is in complete agreement with the present analysis in respect to expected error in $\delta$ and the insensitivity of $\delta$ for values of $U_{e3} \gtrsim 0.05$. Therefore, the BNL-Homestake experiment would not only shed light on the existence of *CP* violation in the leptonic sector but would also have implications for the scenario of Bjorken *et al.* [9].

To summarize, following the analogy of the "*db*" triangle in the quark sector, we have explored the possibility

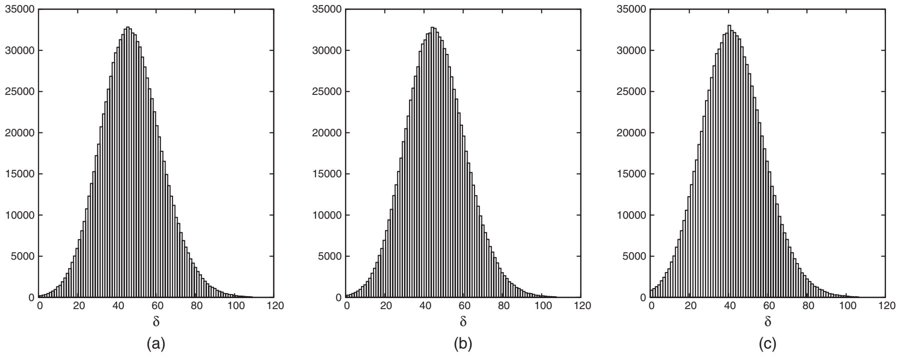

FIG. 2. Histogram of $\delta$ plotted by considering the "$\nu_2.\nu_3$" triangle in the case of neutrinos for (a) $U_{e3} = 0.05 \pm 0.005$, (b) $U_{e3} = 0.1 \pm 0.01$, and (c) $U_{e3} = 0.15 \pm 0.015$.

of the construction of the "$\nu_2.\nu_3$" leptonic unitarity triangle using the MNS matrix obtained by Bjorken *et al.* [9]. In particular, modifying this matrix and considering values of $U_{e3}$ suggested by different theoretical models we have obtained the leptonic mixing matrices. Using these as well as the "$\nu_2.\nu_3$" leptonic unitarity triangle, we have constructed histograms for the Dirac-like $CP$-violating phase $\delta$ in the leptonic sector. Interestingly, from these histograms one can find that for the $U_{e3}$ range 0.05–0.15, there is an almost $2.5\sigma$ likelihood of finding a nonzero Dirac-like $CP$-violating phase $\delta$ with its central value to be around 45°. The present analysis is largely in agreement with the analysis of Marciano and Parsa [25] regarding the expected outcome of the BNL-Homestake (2540 km) proposal.

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[1] R. Davis, Prog. Part. Nucl. Phys. **32**, 13 (1994); B. T. Cleveland *et al.*, Astrophys. J. **496**, 505 (1998); J. N. Abdurashitov *et al.* (SAGE Collaboration), Phys. Rev. C **60**, 055801 (1999); M. Altmann *et al.* (GNO Collaboration), Phys. Lett. B **616**, 174 (2005); W. Hampel *et al.* (GALLEX Collaboration), Phys. Lett. B **447**, 127 (1999); S. Fukuda *et al.* (Super-Kamiokande Collaboration), Phys. Lett. B **539**, 179 (2002); Q. R. Ahmad *et al.* (SNO Collaboration), Phys. Rev. Lett. **89**, 011301 (2002); S. N. Ahmad *et al.*, Phys. Rev. Lett. **92**, 181301 (2004).
[2] Y. Fukuda *et al.* (SuperKamiokande Collaboration), Phys. Rev. Lett. **81**, 1562 (1998); A. Surdo (MACRO Collaboration), arXiv:hep-ex/9905028; M. Sanchez (Soudan 2 Collaboration), Phys. Rev. D **68**, 113004 (2003).
[3] K. Eguchi *et al.* (KamLAND Collaboration), Phys. Rev. Lett. **90**, 021802 (2003); **94**, 081801 (2005).
[4] M. H. Ahn *et al.* (K2K Collaboration), Phys. Rev. Lett. **90**, 041801 (2003).
[5] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
[6] L.-L. Chau and W.-Y. Kueng, Phys. Rev. Lett. **53**, 1802 (1984); J. D. Bjorken, Phys. Rev. D **39**, 1396 (1989); C. Jarlskog and R. Stora, Phys. Lett. B **208**, 268 (1988); J. L. Rosner, A. I. Sanda, and M. P. Schmidt, *Proceedings of the Workshop on High Sensitivity Beauty Physics, Batavia, IL, 1987*; C. Hamzaoui, J.L. Rosner, and A.I. Sanda, *Proceedings of the Workshop on High Sensitivity Beauty Physics, Batavia, IL, 1987*; G. Branco and L. Lavoura, Phys. Lett. B **208**, 123 (1988).
[7] J. A. Aguilar-Saavedra and G. C. Branco, Phys. Rev. D **62**, 096009 (2000); J. Sato, Nucl. Instrum. Methods Phys. Res., Sect. A **472**, 434 (2001); H. Fritzsch and Z. Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000); Z. Xing and H. Zhang, Phys. Lett. B **618**, 131 (2005); H. Zhang and Z. Xing, Eur. Phys. J. C **41**, 143 (2005).
[8] Y. Farzan and A. Yu. Smirnov, Phys. Rev. D **65**, 113001 (2002).
[9] J. D. Bjorken, P. F. Harrison, and W. G. Scott, Phys. Rev. D **74**, 073012 (2006).
[10] C. H. Albright and M. C. Chen, Phys. Rev. D **74**, 113006 (2006).
[11] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[12] W.-M. Yao *et al.*, J. Phys. G **33**, 1 (2006), updated results available at http://pdg.lbl.gov.
[13] J. Charles *et al.* (CKMfitter Group), Eur. Phys. J. C **41**, 1 (2005), updated results available at http://ckmfitter.in2p3.fr.
[14] M. Bona *et al.* (UTfit Collaboration), J. High Energy Phys. 10 (2006) 081, updated results available at http://www.utfit.org.
[15] E. Barberio *et al.* (Heavy Flavor Averaging Group (HFAG)), arXiv:hep-ex/0603003, updated results available at http://www.slac.stanford.edu/xorg/hfag.
[16] M. Randhawa and M. Gupta, Phys. Lett. B **516**, 446 (2001).
[17] H. Minakata and A. Yu. Smirnov, Phys. Rev. D **70**, 073009 (2004).
[18] A. S. Joshipura and A. Yu. Smirnov, Nucl. Phys. **B750**, 28 (2006).
[19] S. Goswami and A. Yu. Smirnov, Phys. Rev. D **72**, 053011 (2005).
[20] H. Fritzsch and Z. Z. Xing, Phys. Lett. B **634**, 514 (2006).
[21] B. Kayser, arXiv:hep-ph/0605005.
[22] R. N. Mohapatra *et al.*, Rep. Prog. Phys. **70**, 1757 (2007).
[23] M. Lindner, A. Merle, and W. Rodejohann, Phys. Rev. D **73**, 053005 (2006).
[24] H. Minakata and S. Uchinami, New J. Phys. **8**, 143 (2006).
[25] W. Marciano and Z. Parsa, arXiv:hep-ph/0610258.
[26] H. Minakata, arXiv:hep-ph/0604088.
[27] H. Sugiyama, arXiv:hep-ph/0411209.
[28] C. Giunti and M. Tanimoto, Phys. Rev. D **66**, 113006 (2002).
[29] K. R. S. Balaji, G. Couture, and C. Hamzaoui, Phys. Rev. D **74**, 033013 (2006).