Quark and lepton masses and mixing in $SO(10)$ with a GUT-scale vector matter

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We explore in detail the effective matter fermion mass sum-rules in a class of renormalizable SUSY $SO(10)$ grand unified models where the quark and lepton mass and mixing patterns originate from nondecoupling effects of an extra vector matter multiplet living around the unification scale. If the renormalizable type-II contribution governed by the $SU(2)_L$-triplet in 54_H dominates the seesaw formula, we obtain an interesting correlation between the maximality of the atmospheric neutrino mixing and the proximity of y_s/y_b to $|V_{cb}|$ in the quark sector.

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I. INTRODUCTION

Though being on the market for more than 30 years, the idea of grand unification [1] still receives a lot of attention in various contexts. Not only it is physical—the proton decay, one of its inherent consequences, is experimentally testable—but it has also been very successful in shedding light on some of the deepest mysteries of the standard model (SM), be it the (hyper)charge quantization, the electroweak scale gauge-coupling hierarchy, or the high-scale $b-\tau$ Yukawa coupling convergence.

The advent of precision neutrino physics in the last decade triggered an enormous boost to the field. The unprecedented smallness of the neutrino masses [2], finding a natural explanation in the class of seesaw models [3], indicates a new fundamental scale at around 10^{14} GeV, remarkably close to the gauge-coupling unification scale of simplest grand unifications (GUTs) $M_{\rm GUT} \approx 2 \times 10^{16}$ GeV. Moreover, the high degree of complementarity among the forthcoming large volume neutrino experiments and proton decay searches [4] is promising a further GUT renaissance in the near future.

An important new aspect of the recent developments is the proliferation of detailed studies of the Yukawa sector [5] of various GUTs including reliable data on lepton mixing, with a potential to further constrain the simplest predictive models like e.g. the minimal supersymmetric $SU(5)$ [6] or $SO(10)$ [7]. This, in turn, has nontrivial implications for proton decay [8], absolute neutrino mass scale [9], leptogenesis [10], etc.

In what follows, we shall stick to the class of $SO(10)$-based supersymmetric (SUSY) GUTs. Perhaps the most attractive feature of the $SO(10)$ schemes is their capability of accommodating all the SM matter multiplets within just three 16-dimensional spinor representations, thus providing a simple understanding of the peculiar SM hypercharge pattern. Moreover, the proton decay issue of the minimal SUSY $SU(5)$ can be considerably alleviated in SUSY $SO(10)$, see e.g. [11,12], because $SO(10)$, as a rank 5 group, features higher flexibility in the symmetry breaking pattern [13–15] than $SU(5)$.

Concerning the basic symmetry breaking scenarios, the two most popular variants are distinguished by either making use of 16-dimensional spinors, or the 252-dimensional 5-index antisymmetric tensor (decomposing under parity into $126 \oplus \overline{126}$ self- and anti-self-dual components) in the Higgs sector in order to break the intermediate $SU(2)_R \otimes U(1)_{B-L}$ symmetry down to $U(1)_Y$ of the SM hypercharge. However, as the SM singlets in both $16_H \oplus \overline{16}_H$ and $126_H \oplus \overline{126}_H$ preserve $SU(5)$, extra Higgs multiplets like $45_H \oplus 54_H$ or 210_H are needed (cf. [7,14,15] and references therein) to achieve the full $SO(10) \to$ SM gauge symmetry breakdown. On top of that, the correlations between the effective Yukawa couplings strongly suggest an extra 10_H in the Higgs sector taking part at the final SM $\to SU(3)_c \otimes U(1)_Q$ step.

Though similar in the symmetry breaking strategy, these options differ dramatically at the effective Yukawa sector level. Concerning the $\overline{126}_H \oplus 126_H$ case, the anti-self-dual part $\overline{126}_H$ couples to the spinorial matter bilinear $16_F 16_F$ via renormalizable coupling $16_F 16_F \overline{126}_H$, which (together with the $16_F 16_F 10_H$ vertex) gives rise to simple effective Yukawa and Majorana sector sum-rules featuring a high degree of predictivity in the matter sector [7,11,16,17].

The minimal potentially realistic renormalizable scenario of this kind (MSGUT) [7] (with 10_H, $126_H \oplus \overline{126}_H$, and 210_H in the Higgs sector) became a subject of thorough examination in the past few years [9,17–20]. This was triggered, namely, by the observation [19] of a profound link between the large 23-mixing in the lepton sector and the (GUT-scale) $b-\tau$ Yukawa convergence, if the type-II contribution (coming from a renormalizable coupling $16_F 16_F \overline{126}_H$) governs the seesaw formula. It has also been shown [19,20] that this scenario predicts a relatively large reactor mixing angle (typically $\sin\theta_{13} \approx 0.1$), well

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within the reach of the future neutrino experiments [21]. However, the recent studies [9] revealed a tension between the lower bounds on the absolute neutrino mass scale and the GUT-scale gauge-coupling unification thresholds.

Though these issues can be to some extent relaxed by adding an extra 120-dimensional Higgs multiplet, either as a subleading correction to the minimal model setting [22] or as a full-featured contribution to the relevant Yukawa sum-rules (deferring $\overline{126}_H$ for the neutrino sector purposes [23]), the large Higgs sector generically pulls the Landau pole to the GUT-scale proximity [15], thus questioning the viability of the perturbative approach.

If, on the other hand, $16_H \oplus \overline{16}_H$ is employed [12,24], the complexity of the Higgs sector is reduced and the Landau pole issue can be partially relaxed. With just three matter spinors, the renormalizable operators are incapable of transferring the effects of $SU(2)_R \otimes U(1)_{B-L}$ breaking into the matter sector and thus nonrenormalizable couplings must be invoked. This, however, ruins the Yukawa sector predictivity of such theories, unless extra assumptions (like e.g. family symmetries) reduce the number of free parameters entering the effective mass matrices, see e.g. [5].

A simple way out [25] consists in adding extra matter multiplets, in particular, the 10-dimensional $SO(10)$ vector(s) transmitting the $SU(2)_R \otimes U(1)_{B-L}$ breakdown (driven by $16_H \oplus \overline{16}_H$) to the effective matter sector (spanning over $16_F^i \oplus 10_F$) via renormalizable mixing terms $16_F^i 10_F 16_H$. Though 10_F tends to decouple from the GUT-scale physics upon pushing its $SO(10)$-singlet mass M_{10} far above the GUT scale, the generic proximity of the above-GUT-scale thresholds (being it $M_{\rm Planck}$ or just a higher unification scale) admits for speculation about the lightest such multiplet next to the GUT scale.

With 10_F at hand, the $SU(5)$ breaking (triggered typically by $45_H \oplus 54_H$) can also be transmitted to the matter sector via loops (in non-SUSY context), higher order operators [26], or at renormalizable level via $10_F 10_F 54_H$ or[1] $10_F 10_F 45_H$ couplings. Although there is a number of studies exploiting this mechanism in the literature [25,27], a generic viability analysis of this strategy is still missing.

In this paper, we shall attempt to fill this gap by focusing on the simplest such scenario, a renormalizable model with three matter spinors 16_F^i (for $i = 1, 2, 3$) and one extra vector multiplet 10_F. We will not resort to any extra symmetries or effective operators or make other assumptions to reduce the complexity of the effective Yukawa sum-rules; rather than that we shall scrutinize the *generic* setting analytically (focusing, in particular, on the quark and charged-lepton sector) in order to get as much understanding of the numerical results as possible. The neutrino sector does not admit for such a detailed analysis unless the type-II contribution happens to govern the seesaw formula. In such a case, one of the lepton sector mixing angles is under control and one can derive a new (GUT-scale) relation between the deviation of the lepton sector 23 mixing from maximality and the proximity of y_s/y_b and $|V_{cb}|$ in the quark sector.

The paper is organized as follows: in Sec. II, the relevant SUSY $SO(10)$ framework is defined, with particular attention paid to the generic features of the Yukawa sector. Next, we derive a detailed form of the relevant GUT-scale Yukawa matrices and (after integrating out the heavy degrees of freedom) give the effective 3×3 mass matrices for the SM matter fermions. Section IV is devoted to a thorough analysis of these structures, the relevant parameter counting, and development of tools necessary for analytic understanding of the given numerical results (deferring some of the technicalities into an appendix). Focusing on the heavy sector, we shall examine the correlation between the proximity of y_s/y_b to $|V_{cb}|$ and the large atmospheric lepton mixing.

II. THE MODEL

Following the strategy sketched above, the matter sector of the scenario of our interest consists of three "standard" copies of the $SO(10)$ spinorial matter residing in 16_F^i ($i = 1, 2, 3$) plus one extra $SO(10)$ vector representation 10_F. The $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ decomposition of these multiplets (up to the generation indices) reads:

$$\begin{aligned} 16_F &= (3, 2, +\tfrac{1}{3}) \oplus (1, 2, -1) \oplus (\bar{3}, 1, -\tfrac{4}{3}) \oplus (\bar{3}, 1, +\tfrac{2}{3}) \\ &\quad \oplus (1, 1, 0) \oplus (1, 1, +2) \\ 10_F &= (3, 1, -\tfrac{2}{3}) \oplus (1, 2, +1) \oplus (\bar{3}, 1, +\tfrac{2}{3}) \oplus (1, 2, -1). \end{aligned} \tag{1}$$

As usual, the submultiplets of 16_F will be consecutively referred to as $Q_L, L_L, U_L^c, D_L^c, N_L^c$, and E_L^c, while those in the decomposition of 10_F as Δ_L, Λ_L^c, Δ_L^c, and Λ_L. Therefore, at the SM level, D_L^c can mix with Δ_L^c and L_L with Λ_L giving rise to the physical light down-type quark and charged-lepton mass eigenstates d_L^c and l_L (which then share the features of both 16_F^i and 10_F, in particular, the sensitivity of 10_F to the $SU(2)_R \otimes U(1)_{B-L}$ and $SU(5)$ breakdown).

The Higgs sector is taken to be the minimal (concerning dimensionality) leading to a viable symmetry breaking chain (while preserving SUSY down to the electroweak scale), i.e. $10_H \oplus 16_H \oplus \overline{16}_H \oplus 45_H \oplus 54_H$ [15]. We shall further assume an unbroken Z_2 parity distinguishing among the (Z_2-odd) matter multiplets 16_F and 10_F and the Higgs sector fields (that are Z_2-even). The relevant SM decompositions read:

[1] Recall that due to antisymmetry of 45_H the latter option is viable only with more than one 10_F.

$$
\begin{aligned}
10_H &= (3,1,-\tfrac{2}{3}) \oplus \underline{(1,2,+1)} \oplus (\bar{3},1,+\tfrac{2}{3}) \oplus \underline{(1,2,-1)},\\
16_H &= (3,2,+\tfrac{1}{3}) \oplus \underline{(1,2,-1)} \oplus (\bar{3},1,-\tfrac{4}{3}) \oplus (\bar{3},1,+\tfrac{2}{3})\\
&\quad \oplus (1,1,+2) \oplus \underline{(1,1,0)},\\
\overline{16}_H &= (\bar{3},2,-\tfrac{1}{3}) \oplus \underline{(1,2,+1)} \oplus (3,1,+\tfrac{4}{3}) \oplus (3,1,-\tfrac{2}{3})\\
&\quad \oplus (1,1,-2) \oplus \underline{(1,1,0)},\\
54_H &= \underline{(1,1,0)} \oplus (1,3,0) \oplus (1,3,\pm 2) \oplus (\bar{6},1,+\tfrac{4}{3})\\
&\quad \oplus (6,1,-\tfrac{4}{3}) \oplus (8,1,0) \oplus (3,2,+\tfrac{1}{3}) \oplus (3,2,-\tfrac{5}{3})\\
&\quad \oplus (\bar{3},2,-\tfrac{1}{3}) \oplus (\bar{3},2,+\tfrac{5}{3}).
\end{aligned} \tag{2}
$$

The underlined components of the SM singlet type in $16_H \oplus \overline{16}_H$ and 54_H receive GUT-scale vacuum expectation values (VEVs) while the doublets $(1,2,\pm 1)$ enter the light $SU(2)_L$-doublets responsible for the electroweak symmetry breakdown. We shall use the abbreviations: $H_u^{10} \equiv (1,2,+1)_{10}$, $H_d^{10} \equiv (1,2,-1)_{10}$, $H_u^{\overline{16}} \equiv (1,2,+1)_{\overline{16}}$, $H_d^{16} \equiv (1,2,-1)_{16}$, $S^{16} \equiv (1,1,0)_{16}$, $S^{\overline{16}} \equiv (1,1,0)_{\overline{16}}$ and $S^{54} \equiv (1,1,0)_{54}$, and $\langle S^{16}\rangle \equiv V^{16}$, $\langle S^{\overline{16}}\rangle \equiv V^{\overline{16}} = (V^{16})^*$ (from D-flatness), $\langle S^{54}\rangle \equiv V^{54}$, $\langle H_{u,d}^{10}\rangle \equiv v_{u,d}^{10}$, $\langle H_d^{16}\rangle \equiv v_d^{16}$ for the corresponding VEVs.

Though 10_F (unlike 16_F^i) admits an $SO(10)$ singlet mass term $M_{10}10_F10_F$ in the superpotential and thus should decouple in the $M_{10}\to\infty$ limit [28], we shall assume the opposite, i.e. that M_{10} happens to live close to the $SO(10)$ breaking scales V^{54} and V^{16}. In such a case, the effective light matter becomes sensitive to the GUT-symmetry breakdown due to the interactions of its nonvanishing components in the 10_F direction with the $SU(2)_R \otimes U(1)_{B-L}$ breaking VEVs in 16_H (via $10_F16_F16_H$) and the $SU(5)$ breaking VEVs in $45_H \oplus 54_H$ (through $10_F10_F54_H$).

For this to be the case, one must assume that the mixing terms $16_F^i10_F16_H$ are not very suppressed with respect to M_{10} and V^{54} (driving the mass of the heavy part of the matter sector), which, however, is exactly the situation suggested by the gauge-coupling renormalization group running in the SUSY "desert" picture.

A. The superpotential

The most general renormalizable (and Z_2-even) Yukawa superpotential at the $SO(10)$ level reads

$$
W_Y = Y^{ij}16_F^i16_F^j10_H + F^i16_F^i10_F16_H + \lambda 10_F10_F54_H + M_{10}10_F10_F. \tag{3}
$$

Here Y denotes a 3×3 symmetric complex Yukawa matrix, while $\vec{F}$ and λ are the relevant 3-component complex vector and scalar Yukawa couplings, respectively. In components, this gives rise to the following $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ structure (in the LR chirality basis):

$$
\begin{aligned}
W_Y \ni{}& Y^{ij}[Q_L^iU_L^{cj}H_u^{10} + Q_L^iD_L^{cj}H_d^{10} + L_L^iN_L^{cj}H_u^{10} + L_L^iE_L^{cj}H_d^{10}] + F^i[L_L^i\Lambda_L^cS^{16} - D_L^{ci}\Delta_LS^{16} + Q_L^i\Delta_L^cH_d^{16} - E_L^{ci}\Lambda_LH_d^{16}\\
&+ N_L^{ci}\Lambda_L^cH_d^{16}] + \lambda[\Delta_L\Delta_L^c(M_{10} - \tfrac{1}{\sqrt{15}}S^{54}) + \Lambda_L^c\Lambda_L(M_{10} + \tfrac{1}{2}\sqrt{\tfrac{3}{5}}S^{54})] + \text{transp.} + \lambda c_T[\Lambda_L^c\Lambda_L^cT_-^{54} + \Lambda_L\Lambda_LT_+^{54}],
\end{aligned} \tag{4}
$$

where "+transp." stands for the Hermitian-conjugated terms (in the SUSY notation) while $T_\pm^{54} \equiv (1,3,\pm 2)_{54}$ are $SU(2)_L$ triplets that can receive tiny VEVs relevant for the type-II neutrino mass-matrix entry and c_T is the associated Clebsch-Gordon (CG) coefficient.

B. The GUT-scale mass matrices

Once the relevant Higgs fields develop the VEVs specified above, the Yukawa couplings in W_Y give rise to the quark and lepton mass matrices.

Up-type quarks: The up-type quark mass matrix receives a simple 3×3 form as there is no multiplet in 10_F that could mix with the up-type quarks in 16_F^i:

$$
M_u = Yv_u^{10}. \tag{5}
$$

Down-type quarks: Since the right-handed down-quark-type fields D_L^c in 16_F^i mix with Δ_L^c in 10_F, the relevant (GUT-scale) mass matrix is four-dimensional and reads (in the $\{D_L^i, \Delta_L\}$ basis):

$$
M_d = \begin{pmatrix} Yv_d^{10} & -\vec{F}v_d^{16} \\ \vec{F}^TV^{16} & M_{10} - \lambda\tilde{V}^{54} \end{pmatrix}. \tag{6}
$$

The minus signs in the 4th column come from the relevant CG coefficients in (4) with redefinition of $\tilde{V}^{54} \equiv \frac{1}{\sqrt{15}}V^{54}$.

Charged leptons: The situation in the charged-lepton sector is similar to the down-quark case up to the point that it is now the *left-handed* chiral components (L_L in 16_F^i and Λ_L of 10_F) that can mix. The net effect of this difference boils down to the charged-lepton mass-matrix structure very close to the transpose of M_d (in the $\{E_L^i, \Lambda_L^-\}$ basis, where Λ_L^- denotes the charged component of the Λ_L $SU(2)_L$-doublet):

$$
M_l = \begin{pmatrix} Yv_d^{10} & \vec{F}V^{16} \\ -\vec{F}^Tv_d^{16} & M_{10} + \tfrac{3}{2}\lambda\tilde{V}^{54} \end{pmatrix}. \tag{7}
$$

Thus, it is, namely, the difference in the 44-entry CG coefficient that actually makes M_l and M_d feel the $SU(5)$ breakdown. Moreover, this is also the *only* distinction between M_d and M_l^T in the current model and one of our goals will be to see whether such a detail could account for all the difference amongst the charged-lepton and down-quark spectra.

Neutrinos: Since there are in total 8 neutral components in 16_F^i and 10_F (N_L^i, N_L^{ci}, Λ_L^{c0}, and Λ_L^0), the (symmetric) renormalizable neutrino mass matrix (in the $\{N_L^i, N_L^{ci}, \Lambda_L^0, \Lambda_L^{c0}\}$ basis) is more complicated:

$$M_\nu = \begin{pmatrix} 0 & Yv_u^{10} & 0 & \vec{F}V^{16} \\ . & 0 & 0 & \vec{F}v_d^{16} \\ . & . & \lambda c_T w_+ & M_{10} + \frac{3}{2}\lambda\tilde{V}^{54} \\ . & . & . & \lambda c_T w_- \end{pmatrix}. \quad (8)$$

Here we use $w_\pm$ for the VEVs of the electroweak triplets $T_\pm^{54} = (1, 3, \pm 2)_{54}$, which provide the only source of diagonal Majorana masses at the renormalizable level.

It is clear that this basic texture cannot accommodate the standard seesaw mechanism: the 1-3 rotation, which cancels the large 14 entry (so that all the GUT-scale masses occupy the 3-4 sector), affects only the 11 entry of the 1-2 block (due to the zeros at the 13, 23 and 31, 32 positions). This gives rise to a type-II contribution proportional to $\lambda c_T w_+$ at 11 position, while keeps the other 1-2 block entries intact. We are then left with pseudo-Dirac neutrinos around the electroweak scale, at odds with experiment.

However, the picture changes dramatically beyond the renormalizable level. The SM-singlet zero at the 22 position is not protected by the $SU(2)_L \otimes U(1)_Y$ symmetry and thus receives contributions from the effective operators of the form $\kappa^{ij}16_F^i 16_F^j \overline{16}_H \overline{16}_H/\Lambda_\kappa$ (with Λ_κ denoting the relevant physics scale above M_{GUT}) leading to a naturally suppressed[2] Majorana mass term M_M. Next, the zero at the 23 position can be lifted upon the SM symmetry breakdown (requiring only one VEV insertion for the hypercharge deficit +1) by means of an effective operator with structure $\zeta^i 16_F^i 10_F \overline{16}_H 10_H/\Lambda_\zeta$. Last, a pair of electroweak VEV insertions (coming from the two types of the effective operators above, plus $\sigma 10_F 10_F 10_H 10_H/\Lambda_\sigma$), can give rise to the $Y_W = +2$ entries at the doublet-doublet positions 11, 13, and 33.

With all this at hand, one finds that the only terms that do affect the structure of the seesaw formula are indeed the large Majorana mass $M_M \equiv \kappa(\overline{V^{16}})^2/\Lambda_\kappa$ at the 22 position, the electroweak VEVs at the 23/32 positions, and the tiny diagonal Majorana VEVs $\mu_3 \propto \kappa(\overline{v_u^{16}})^2/\Lambda_\kappa$ and $\mu_\pm \equiv \lambda c_T w_\pm + \mathcal{O}[(v_u^{10})^2/\Lambda_\sigma]$. The neutrino mass matrix beyond the renormalizable level then reads (in the $\{N_L^i, N_L^{ci}, \Lambda_L^0, \Lambda_L^{c0}\}$ basis) at the leading order :

$$M_\nu = \begin{pmatrix} \mu_3 & Yv_u^{10} & . & FV^{16} \\ . & M_M & \zeta\overline{V^{16}}v_u^{10}/\Lambda_\zeta & Fv_d^{16} \\ . & . & \mu_+ & M_{10} + \frac{3}{2}\lambda\tilde{V}^{54} \\ . & . & . & \mu_- \end{pmatrix}. \quad (9)$$

It is obvious that, without extra assumptions, M_ν is not constrained enough to admit for predictions in the neutrino sector (due to the ambiguity in the M_M and μ_3 matrices generated at the effective operator level only). However, as we shall see in the next section, the renormalizable part of the effective type-II contribution to the effective light neutrino mass matrix (coming from the $SU(2)_L$-triplet of 54_H) is calculable up to an overall scale, and if it happens to dominate the seesaw formula, one can obtain an interesting link between the 23-quark sector observables and a large value of the associated lepton sector mixing.

III. EFFECTIVE MASS SUM-RULES

Since the up-quark mass matrix (5) is simple, let us focus on the down-quark and charged-lepton mass matrices given by (6) and (7). First, in order to employ the CG coefficients at the 44 positions of M_d and M_l (to disentangle their effective light spectra), 10_F should not decouple. Therefore, we need M_{10} comparable to V^{54} and V^{16} not far below V^{54}; a detailed discussion can be found in Sec. III B. Thus, the physically viable situation corresponds to $V^{16} \lesssim V^{54} \approx M_{10}$ and from now on we shall always assume this to be the case.

A. Integrating out the heavy degrees of freedom

With one dominant column in M_d and M_l^T, there will always be three light matter states and one superheavy living around the GUT scale. If $V^{16} < V^{54} \approx M_{10}$, the light states are predominantly spanned over the 16_F^i components with a subleading contribution in the 10_F direction, while there is no such clear mapping if $V^{16} \approx V^{54} \approx M_{10}$. Prior to getting to the quantitative analysis of the effective SM spectra and mixings, the heavy degrees of freedom must be integrated out.

Down-type quarks: In the down-quark sector, the right-handed (RH) part of the physical heavy state ($\tilde{\Delta}$) can be readily identified:

$$\begin{aligned} \tilde{\Delta}_L^c &= \frac{1}{M_\Delta}[F^i V^{16} D_L^{ci} + (M_{10} - \lambda\tilde{V}^{54})\Delta_L^c] \\ &\equiv C_d^i D_L^{ci} + D_d \Delta_L^c, \end{aligned} \quad (10)$$

where $M_\Delta = \sqrt{F^\dagger F (V^{16})^2 + (M_{10} - \lambda\tilde{V}^{54})^2}$ is a real normalization factor and

$$\vec{C}_d = \vec{F}V^{16}/M_\Delta, \qquad D_d = (M_{10} - \lambda\tilde{V}^{54})/M_\Delta \quad (11)$$

denote the relevant weight coefficients in the $16_F^i \oplus 10_F$ space. The RH components of the three light states (d_L^{ci})

[2] With respect to the typical GUT scale M_M generated at renormalizable level in models with $\overline{126}_H$ in the Higgs sector.

live in the orthogonal subspace defined by the relevant 4×4 unitary transformation

$$\begin{pmatrix} d_L^{c1} \\ d_L^{c2} \\ d_L^{c3} \\ \tilde{\Delta}_L^c \end{pmatrix} = \begin{pmatrix} A_d^{11} & A_d^{12} & A_d^{13} & B_d^1 \\ A_d^{21} & A_d^{22} & A_d^{23} & B_d^2 \\ A_d^{31} & A_d^{32} & A_d^{33} & B_d^3 \\ C_d^1 & C_d^2 & C_d^3 & D_d \end{pmatrix} \begin{pmatrix} D_L^{c1} \\ D_L^{c2} \\ D_L^{c3} \\ \Delta_L^c \end{pmatrix}. \tag{12}$$

The A_d^{ij} and B_d^i coefficients are constrained only from unitarity, and there is a lot of ambiguity in this sector. Introducing a compact notation

$$U_d \equiv \begin{pmatrix} A_d & \vec{B}_d \\ \vec{C}_d^T & D_d \end{pmatrix} \tag{13}$$

the defining basis down-quark fields can be recast in terms of the physical ones as follows:

$$\begin{pmatrix} \vec{D}_L^c \\ \Delta_L^c \end{pmatrix} = \begin{pmatrix} A_d^\dagger & \vec{C}_d^* \\ \vec{B}_d^\dagger & D_d^* \end{pmatrix} \begin{pmatrix} \vec{d}_L^c \\ \tilde{\Delta}_L^c \end{pmatrix}. \tag{14}$$

The relevant piece of the Yukawa Lagrangian then reads

$$\mathcal{L}_d \ni Y^{ij} D_L^i [(A_d^\dagger)^{jk} d_L^{ck} + C_d^{j*} \tilde{\Delta}_L^c] v_d^{10} - F^i D_L^i [B_d^{*k} d_L^{ck} + D_d^* \tilde{\Delta}_L^c] v_d^{16} + M_\Delta \Delta_L \tilde{\Delta}_L^c, \tag{15}$$

and thus the down-quark mass matrix (in the $\{D_L^i, \Delta_L\}$, $\{d_L^{ck}, \tilde{\Delta}_L^c\}$ bases) becomes block-diagonal with zero at the $\Delta_L d_L^c$ position. Since the subsequent left-handed (LH) rotation is suppressed by $\mathcal{O}(v/M_{\rm GUT})$, the left-handed physical components d_L^i, $\tilde{\Delta}_L$ can be (at leading order) identified with the defining ones $d_L^i \equiv D_L^i$, $\tilde{\Delta}_L \equiv \Delta_L$.

With all this at hand, the effective mass matrix for the down quarks d^i obeys:

$$M_d^{ik} = Y^{ij}(A_d^\dagger)^{jk} v_d^{10} - F^i B_d^{*k} v_d^{16} + \mathcal{O}\left(\frac{v^2}{M_{\rm GUT}}\right), \tag{16}$$

while the heavy state $\tilde{\Delta}$ has a mass M_Δ. Recall that the $A_d^\dagger$ matrix and the $\vec{B}_d^*$ vector are just (Hermitian conjugates of) the upper left and upper right blocks of the unitarity transformation (13), which can be partially determined from the lower left $\vec{C}_d$ and lower right D_d components of U_d from the unitarity conditions $A_d^\dagger \vec{B}_d = -\vec{C}_d^* D_d$ and $|\vec{C}_d|^2 + |D_d|^2 = 1$.

Charged leptons: The situation in the charged-lepton sector is analogous to the down quarks with the relevant parameters equipped by a subscript l instead of d. Taking into account the similarity of M_d and M_l^T one obtains:

$$M_l^{ik} = (A_l^*)^{ij} Y^{jk} v_d^{10} - B_l^{*i} F^k v_d^{16} + \mathcal{O}\left(\frac{v^2}{M_{\rm GUT}}\right), \tag{17}$$

where, as before, A_l and $\vec{B}_l$ complement the relevant $\vec{C}_l$ and D_l defined as

$$\vec{C}_l = \vec{F} V^{16}/M_\Lambda, \qquad D_l = (M_{10} + \tfrac{3}{2}\lambda \tilde{V}^{54})/M_\Lambda \tag{18}$$

with $M_\Lambda = \sqrt{F^\dagger F (V^{16})^2 + (M_{10} + \frac{3}{2}\lambda \tilde{V}^{54})^2}$ denoting the mass of the GUT-scale state Λ.

The seesaw for neutrinos: After some tedium deferred to the appendix, the standard seesaw formalism yields

$$M_\nu \doteq M_\nu^{\rm II} - D_\nu M_M^{-1} D_\nu^T \tag{19}$$

with the effective type-II and Dirac mass matrices obeying

$$\begin{aligned} M_\nu^{\rm II} &\equiv (\vec{B}_l^* \otimes \vec{B}_l^*)\mu_+ + A_l^* \mu_3 A_l^\dagger, \\ D_\nu &\equiv A_l^* Y v_u^{10} + (\vec{\zeta} \otimes \vec{B}_l^*) v_u^{10} V^{\overline{16}}/\Lambda_\zeta. \end{aligned} \tag{20}$$

The matrix A_l and the vector $\vec{B}_l$ are the same parameters that enter the charged-lepton sector analysis above, cf. formula (17) and the comments in the appendix.

As it was already mentioned, we shall assume that the first term in (20), i.e. the renormalizable part of the type-II contribution associated to the $SU(2)_L$-triplet in 54_H, dominates over the nonrenormalizable μ_3-piece as well as the type-I contributions in the formula (19). Notice, however, that in such a case $M_\nu^{\rm II}$ has only one nonzero eigenvalue and thus the nonrenormalizable type-II and/or type-I corrections should account, at some level, for the second nonzero neutrino mass and thus can never be entirely neglected.

Let us finish this section with a brief recapitulation of the four sum-rules we have obtained so far:

$$M_u = Y v_u^{10}, \tag{21}$$

$$M_d = Y A_d^\dagger v_d^{10} - \vec{F} \otimes \vec{B}_d^* v_d^{16}, \tag{22}$$

$$M_l = A_l^* Y v_d^{10} - \vec{B}_l^* \otimes \vec{F} v_d^{16}, \tag{23}$$

$$M_\nu \propto \vec{B}_l^* \otimes \vec{B}_l^* + \ldots \tag{24}$$

The proportionality sign $\propto$ in (24) reflects the fact that the overall scale of the (triplet driven) type-II dominated neutrino mass matrix is unknown and the $\otimes$ symbol in $(\vec{x} \otimes \vec{y})_{ij} \equiv x_i y_j$ represents the outer products of the vectors $\vec{F}$ and $\vec{B}_{d,l}$ in Eqs. (16) and (17). The formulae (21)–(24) shall be the subject of a detailed analysis in the remainder of this work.

B. Physical understanding and decoupling

One can check the consistency of formulae (21)–(24) by exploring the various limiting cases where different intermediate symmetries should be restored and the corresponding effective mass sum-rules revealed.

$V^{16}, V^{54} \ll M_{10}$: This setting corresponds to decoupling of 10_F, so $\vec{C}_{d,l} \to 0$, $D_{d,l} \to 1$ and (from unitarity of $U_{d,l}$, cf. Eq. (13)) $\vec{B}_{d,l} \to 0$ so the $A_{d,l}$ matrices become unitary. The light spectra are sensitive only to the electroweak $SU(2)_L \otimes U(1)_Y$ breakdown, but there is no means to transfer therein the information about the $SU(5)$ or Pati-Salam breaking at the renormalizable level. This, as expected, leads to degenerate spectra of $M_d = M_l^T \propto M_u$ *à la* $SO(10)$ with a single "Yukawa-active" Higgs multiplet 10_H.

$V^{54} \ll V^{16}$: This scenario features an intermediate $SU(5)$ symmetry—though the CG coefficients in D_d and D_l remain "visible" for $M_{10} \lesssim V^{54}$, one still has $M_\Delta \approx M_\Lambda$, $\vec{C}_d \approx \vec{C}_l \neq 0$, and $D_d \approx D_l \neq 1$. This gives $\vec{B}_d \approx \vec{B}_l$ and thus the subsequent $SU(5)$ breaking affects the down quarks and charged leptons in the same manner, and we get $M_d = M_l^T$ along the $SU(5)$ lines. However, the spectra of the up- and down-type quarks are disentangled. Apart from the potential problem with the proton decay there is also no handle on the Cabibbo-Kobayashi-Maskawa (CKM) mixing in this case.

$V^{16} \ll V^{54}$, $M_{10} \lesssim V^{54}$: In this regime 10_F does feel the $SU(5)$ breaking in 54_H, but due to the weakness of its interaction with the matter spinors (suppressed by V^{16}/M_{10} or V^{16}/V^{54}), it cannot transmit the information to the light sector (because $\vec{C}_{d,l} \to 0$, $D_{d,l} \to 1$) and again $\vec{B}_{d,l} \to 0$. Moreover, $A_{d,l}$ become unitary, leading to the same shape of the effective matter spectrum as in the decoupling case V^{16}, $V^{54} \ll M_{10}$.

As already mentioned above, a potentially realistic scenario could arise only if $V^{16} \lesssim V^{54}$ and M_{10} low enough not to screen the CG coefficients in $D_{d,l}$. In such a case, one gets a good reason for the smallness of the CKM mixing (because $V_{\rm CKM} \to \mathbb{1}$ for $V^{16} \ll V^{54}$) while the 1st and 2nd generation Yukawa degeneracy can be lifted. The proximity of the third generation Yukawas could be reconciled with the third generation mass hierarchies for large $\tan\beta$ values. Recall that, indeed, $V^{16} \lesssim V^{54}$ is also the hierarchy suggested by the SUSY gauge-coupling unification.

IV. ANALYSIS AND DISCUSSION

Let us now inspect in detail the sum-rules (21)–(24). Apart from the 3×3 Yukawa matrix Y common to M_u, M_d and M_l, the latter two contain extra factors $A_{d,l}$ and $\vec{B}_{d,l}$ arising upon integrating out the heavy sector. How much do we actually know about the $A_{d,l}$ matrices and the $\vec{B}_{d,l}$ vectors given the couplings Y, $\vec{F}$, λ and the high-scale parameters M_{10}, V^{54}, and V^{16}?

A. General prerequisites

All the information we have about these quantities comes from the unitarity of the $U_{d,l}$ matrices (12):

$$A_{d,l}^\dagger \vec{B}_{d,l} = -\vec{C}_{d,l}^* D_{d,l}, \qquad |\vec{C}_{d,l}|^2 + |D_{d,l}|^2 = 1. \qquad (25)$$

Several comments are worth making at this point. First, due to reality of $M_{\Delta,\Lambda}$, the phases in $\vec{C}_{d,l}$ are aligned, while those of D_d and D_l can differ. Second, the physical observables (i.e. spectra and mixings) coming from (21)–(24) should be blind to any (unphysical) change of basis in the light sector.

To check this explicitly, recall that a general 4×4 unitary matrix U (parametrized by 6 angles and 10 phases) can be written as a product of a unitary 3×3 matrix U_3 (which depends on 3 angles $\xi_{1,2,3}$ corresponding to rotations in the 2-3, 3-1, and 1-2 planes, respectively, and 6 phases $\rho_{1,\ldots,6}$), acting on the first three indices only, and a "unitary remnant" U_4 accounting for the remaining 3 mixings α_1, α_2, α_3, corresponding to rotations in the 1-4, 2-4, and 3-4 planes, plus the remaining 4 phases $\psi_{1,\ldots,4}$). We get

$$U = \begin{pmatrix} U_3(\xi_{1,2,3}; \rho_{1,\ldots,6}) & 0 \\ 0 & 1 \end{pmatrix} U_4(\alpha_{1,2,3}; \psi_{1,\ldots,4}), \qquad (26)$$

where

$$U_4(\alpha_{1,2,3}; \psi_{1,\ldots,4}) = e^{i\psi_4} R_{14}(\alpha_1, \psi_1) R_{24}(\alpha_2, \psi_2) R_{34}(\alpha_3, \psi_3). \qquad (27)$$

The $R_{i4}(\alpha_i, \psi_i)$ matrices in Eq. (27), given by

$$R_{i4}(\alpha_i, \psi_i) = \begin{pmatrix} 1 & . & . & . \\ . & \cos\alpha_i & . & -\sin\alpha_i e^{i\psi_i} \\ . & . & 1 & . \\ . & \sin\alpha_i e^{-i\psi_i} & . & \cos\alpha_i \end{pmatrix}, \qquad (28)$$

represent the elementary unitary transformations in the i-4 planes. The point here is that if we employ the parametrization (26) for $U_{d,l}$, the lower two sub-blocks of such unitary matrices are simple functions of $\alpha_{1,2,3}$ and $\psi_{1,\ldots,4}$ (thus leading to a convenient parametrization of $\vec{C}_{d,l}$ and $D_{d,l}$). Indeed, performing the multiplications in (26) and (27) one obtains

$$\begin{aligned} \vec{C}^T &= e^{i\psi_4}(s_1 e^{-i\psi_1}, c_1 s_2 e^{-i\psi_2}, c_1 c_2 s_3 e^{-i\psi_3}), \\ D &= e^{i\psi_4} c_1 c_2 c_3, \end{aligned} \qquad (29)$$

where the standard shorthand notation $s_i \equiv \sin\alpha_i$, $c_i \equiv \cos\alpha_i$ has been used and all the flavor indices d, l distinguishing among the down-quark and charged-lepton sector quantities were dropped for simplicity.

The main benefit from (29) is the independence of $\vec{C}_{d,l}$ and $D_{d,l}$ on the parameters driving the "unphysical" $U_3^{d,l}$ rotations. In fact, U_3 enters only the formulae for the A-matrices obeying $A = U_3 V$ (with the family indices suppressed), where V denotes the 3×3 upper left block of the U_4 matrix:

$$V \equiv e^{i\psi_4} \begin{pmatrix} c_1 & -e^{i(\psi_1-\psi_2)} s_1 s_2 & -e^{i(\psi_1-\psi_3)} c_2 s_1 s_3 \\ 0 & c_2 & -e^{i(\psi_2-\psi_3)} s_2 s_3 \\ 0 & 0 & c_3 \end{pmatrix}. \qquad (30)$$

Note that V actually measures the nonunitarity of A, since V becomes unitary if and only if $\vec{C} \to 0$, $|D| \to 1$, i.e. in the decoupling limit $M_{10} \to \infty$. Note that it also depends only on the reduced set of parameters $\alpha_{1,2,3}$ and $\psi_{1,\ldots,4}$.

B. Hiding unphysical parameters $\xi_{1,\ldots,3}$ and $\rho_{1,\ldots,6}$

The parametrization introduced in Sec. IV A allows for recasting the sum-rules (22) and (23) for M_d and M_l as

$$\begin{aligned} M_d A_d &= Y A_d^\dagger A_d v_d^{10} - \vec{F} \otimes (A_d^\dagger \vec{B}_d)^* v_d^{16}, \\ M_l^T A_l &= Y A_l^\dagger A_l v_d^{10} - \vec{F} \otimes (A_l^\dagger \vec{B}_l)^* v_d^{16}, \end{aligned} \tag{31}$$

where the $U_3^{d,l}$ matrices cancel in $A_{d,l}^\dagger A_{d,l} = V_{d,l}^\dagger V_{d,l}$ (with $V_{d,l}$ of the generic form (30)) and due to unitarity (25), the right-hand side (RHS) of relations (31) becomes $\xi_{1,2,3}$- and $\rho_{1,\ldots,6}$-independent. The only trace of the $U_3^{d,l}$ rotations remains in the $A_{d,l}$ on the left-hand side (LHS) of (31), but this can be dealt with by multiplying (31) with $V_{d,l}^{-1}$:

$$\begin{aligned} M_d U_3^d &= Y V_d^\dagger v_d^{10} + (\vec{F} \otimes \vec{C}_d) D_d^* V_d^{-1} v_d^{16}, \\ M_l^T U_3^l &= Y V_l^\dagger v_d^{10} + (\vec{F} \otimes \vec{C}_l) D_l^* V_l^{-1} v_d^{16}. \end{aligned} \tag{32}$$

The unitary transformations $U_3^{d,l}$ pending on the LHS of (32) can then be eliminated upon looking at quantities like LHS. LHS† or by a suitable redefinition of d_R or l_L, which of course does not affect $V_{\rm CKM}$, but can be relevant for the lepton mixing.

Concerning the type-II dominated neutrino sector, the relevant mass matrix in the basis we used for the charged-lepton sum-rule (32) reads

$$M_\nu \propto V_l^{-1T} A_l^T \vec{B}_l^* \otimes (V_l^{-1T} A_l^T \vec{B}_l^*)^T \tag{33}$$

which (using the unitarity conditions (25)) can be rewritten in the form (dropping the D_l^{*2} factor due to the overall scale ambiguity):

$$M_\nu \propto V_l^{-1T} (\vec{C}_l \otimes \vec{C}_l) V_l^{-1}. \tag{34}$$

As mentioned before, such a mass matrix has only 1 nonzero eigenvalue and must be clearly subject to subleading corrections coming from the type-I sector in order to lift at least one of the two remaining neutrino masses. This means that the only piece of information one can derive from (34) is the mixing angle between the heaviest third and the lighter second generation θ_{23}^l. Therefore, we shall not consider neutrinos in the 3×3 analysis in Sec. IV E.

C. Physical parameter counting

Apart from the 18 parameters ($\xi_{1,\ldots,3}^{d,l}$ and $\rho_{1,\ldots,6}^{d,l}$) hidden in the $U_3^{d,l}$ matrices in (32), we can eliminate 6 other quantities by exploiting the close connection between the M_d and M_l^T matrices (which are identical up to one CG coefficient, cf. formulae (6) and (7)).

Since the ψ_4^d and ψ_4^l phases entering $\vec{C}_{d,l}$, $D_{d,l}$, and $V_{d,l}$ given by the generic formulae (29) and (30) act only as global rephasing on the RHS of (32), they can be absorbed into the definition of $U_3^{d,l}$. Denoting $\vec{C}'_{d,l} \equiv e^{-i\psi_4^{d,l}} \vec{C}_{d,l}$, $D'_{d,l} \equiv e^{-i\psi_4^{d,l}} D_{d,l}$, $V'_{d,l} \equiv e^{-i\psi_4^{d,l}} V_{d,l}$, and $U_3'^{d,l} \equiv e^{-i\psi_4^{d,l}} U_3^{d,l}$, one can rewrite Eq. (32) as

$$\begin{aligned} M_d U_3'^d &= Y V_d'^\dagger v_d^{10} + (\vec{F} \otimes \vec{C}'_d) D'_d V_d'^{-1} v_d^{16}, \\ M_l^T U_3'^l &= Y V_l'^\dagger v_d^{10} + (\vec{F} \otimes \vec{C}'_l) D'_l V_l'^{-1} v_d^{16}, \end{aligned} \tag{35}$$

where the reality of $D'_{d,l}$ has been used to drop the star.

Next, using (11) and (18), one can connect the down-quark and the charged-lepton sector $\vec{C}'_{d,l}$ vectors by means of a single real and positive parameter $s \equiv M_\Delta / M_\Lambda$:

$$\vec{C}'_l = e^{-i\phi} s \vec{C}'_d, \tag{36}$$

where $\phi \equiv \psi_4^l - \psi_4^d$ is the only physical remnant of the $\psi_4^{d,l}$ phases. Notice that formula (36) together with (29) admits for trading all $\psi_{1,2,3}^l$ and $\alpha_{1,2,3}^l$ for $\psi_{1,2,3}^d$ and $\alpha_{1,2,3}^d$ with s and ϕ only. Furthermore, one can exploit the proportionality (11) to express also $\vec{F}$ in terms of this reduced set of parameters: $\vec{F} = e^{i\psi_4^d} M_\Delta / V^{16} \vec{C}'_d \equiv x \vec{C}'_d$. Subsequently, using $Y = M_u / v_u$ and defining $v_d^{10} \cot\beta / v_d \equiv r e^{i\psi_r}$, $x v_d^{16} e^{-i\psi_r} \equiv q e^{i\psi_q}$, and $U_3'^{d,l} e^{-i\psi_r} \equiv \tilde{U}_{d,l}$ one obtains

$$\begin{aligned} M_d \tilde{U}_d &= M_u V_d'^\dagger r + q e^{i\psi_q} (\vec{C}'_d \otimes \vec{C}'_d) D'_d V_d'^{-1}, \\ M_l^T \tilde{U}_l &= M_u V_l'^\dagger r + q e^{i\psi_q} (\vec{C}'_d \otimes \vec{C}'_l) D'_l V_l'^{-1}. \end{aligned} \tag{37}$$

The last trick consists in observing that the complicated structure of the $V'_{d,l}$ matrices (30) can be further simplified by means of (suppressing the flavor indices)[3]:

$$\begin{aligned} (\vec{C}' \otimes \vec{C}') V'^{-1} &= (\vec{C}' \otimes \vec{C}') N(\vec{C}'), \\ V'^\dagger &= [1 - P(\vec{C}')] N(\vec{C}'), \end{aligned} \tag{38}$$

where N is a real and diagonal matrix function defined for a generic complex vector $\vec{z}$ by $N(\vec{z}) \equiv \mathrm{diag}^{-1}(n_1, n_2, n_3)$ with

$$\begin{aligned} n_1 &= \sqrt{1 - |z_1|^2}, \\ n_2 &= \sqrt{1 - |z_1|^2} \sqrt{1 - |z_1|^2 - |z_2|^2}, \\ n_3 &= \sqrt{1 - |z_1|^2 - |z_2|^2} \sqrt{1 - |\vec{z}|^2}, \end{aligned} \tag{39}$$

while $P(\vec{z})$ obeys

$$P(\vec{z}) \equiv \begin{pmatrix} |z_1|^2 & 0 & 0 \\ z_1 z_2^* & |z_1|^2 + |z_2|^2 & 0 \\ z_1 z_3^* & z_2 z_3^* & |\vec{z}|^2 \end{pmatrix}. \tag{40}$$

Notice that unlike V', it is trivial to invert a diagonal matrix to get N from (39) and neither $N(\vec{z})$ nor $P(\vec{z})$ depends on the

[3] The point is that $\vec{C}' \otimes \vec{C}'$ is a projector to its only nonzero eigenvector which leads to a reduction of complexity.

global phase of $\vec{z}$. With this at hand, one can rewrite (34) and (37) into the final form

$$M_d\tilde{U}_d = \{M_u[1-P(\vec{f})]r + q(\vec{f}\otimes\vec{f})\sqrt{1-|\vec{f}|^2}\}N(\vec{f}),$$
$$M_l^T\tilde{U}_l = \{M_u[1-P(s\vec{f})]r + qse^{-i\phi}(\vec{f}\otimes\vec{f}) \times\sqrt{1-|s\vec{f}|^2}\}N(s\vec{f}), \quad (41)$$

$$M_\nu \propto N(s\vec{f})(\vec{f}\otimes\vec{f})N(s\vec{f}), \quad (42)$$

where $\vec{f}\equiv e^{i\psi_q/2}\vec{C}'_d$ has been used to absorb the remaining unphysical parameter ψ_q.

Parameter counting: apart from the up-quark masses in M_u, there are in total 3 angles $\alpha_{1,2,3}\equiv\alpha^d_{1,2,3}$ and 3 phases $\gamma_{1,2,3}\equiv 2\psi^d_{1,2,3}-\psi_q$ in the generic complex vector $\vec{f}$ entering (41) and (42):

$$\vec{f}^{\,T}\equiv(\mathrm{s}_1e^{-i(\gamma_1/2)}, \mathrm{c}_1\mathrm{s}_2e^{-i(\gamma_2/2)}, \mathrm{c}_1\mathrm{c}_2\mathrm{s}_3e^{-i(\gamma_3/2)}). \quad (43)$$

On top of that, there are 4 other real parameters q, r, s, and ϕ on the RHS of (41) and (42), so altogether we are left with 10 free parameters to match the 6 eigenvalues of M_d and M_l and 4 CKM mixing parameters in the quark sector. Remarkably enough, there is a simple vocabulary that can be used to get the charged-lepton sum-rule (41) out of the down-quark one:

$$\vec{f}\to e^{-i\phi/2}s\vec{f}, \qquad q\to q/s. \quad (44)$$

We shall exploit this feature in the physical analysis in the next section.

D. Extracting the physical information

Given M_u, one can first exploit the sum-rule (41) to fit the three down-quark masses and all the CKM parameters. For any set of values of $\alpha_{1,2,3}$, $\gamma_{1,2,3}$, q, and r, the RHS of Eq. (41) (to be denoted by R) is fully specified. In the basis in which M_u is diagonal, one can decompose $M_d\tilde{U}_d$ on the LHS of (41) as

$$M_d\tilde{U}_d = V^0_{\mathrm{CKM}}\mathcal{D}_dW = R, \quad (45)$$

where $\mathcal{D}_d$ is a diagonal form of M_d and V^0_{CKM} is a "raw" form of the CKM matrix $V^0_{\mathrm{CKM}} = P_LV_{\mathrm{CKM}}P_R$ ($P_L\equiv\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ and $P_R\equiv\mathrm{diag}(e^{i\phi_4}, e^{i\phi_5}, 1)$ denote the phase factors necessary to bring V_{CKM} into the standard form [29]), while W represents a generic unitary right-handed rotation. One can first get rid of W and P_R by focusing on the combination $R.R^\dagger$:

$$P_LV_{\mathrm{CKM}}|\mathcal{D}_d|^2V^\dagger_{\mathrm{CKM}}P_L^\dagger = R.R^\dagger. \quad (46)$$

Second, the diagonal entries, the principal minors, and the full determinant are insensitive to P_L and, remarkably enough, some of these combinations can be further simplified. Denoting $d_{ij}\equiv(V_{\mathrm{CKM}}|\mathcal{D}_d|^2V^\dagger_{\mathrm{CKM}})_{ij}$, the equality of the diagonal elements in (46) yields

$$d_{ii} = r^2m_u^{i2}(1-g_i) + g_iq^2\Sigma g_j + 2rq\cos\gamma_i m_u^i g_i\sqrt{1-\Sigma g_j} \quad (47)$$

(no summation over i), where $g_i\equiv|f_i|^2$ (for $i=1,\ldots,3$) are real numbers $\in\langle 0,1)$ and m_u^i correspond to the relevant up-type quark masses. On the other hand, the three main minors $\Delta_{i<j}\equiv d_{ii}d_{jj}-d_{ij}d_{ji}=d_{ii}d_{jj}-|d_{ij}|^2$ obey

$$\Delta_{i<j} = -m_u^{i2}m_u^{j2}r^4 + r^2(d_{ii}m_u^{j2}+d_{jj}m_u^{i2}) - r^2q^2g_ig_j[m_u^{i2}+m_u^{j2}-2m_u^im_u^j\cos(\gamma_i-\gamma_j)]. \quad (48)$$

It is crucial that d_{ii} and $\Delta_{i<j}$ depend only on the physical quark sector data so the 6 relatively simple constraints (47) and (48) can be (at least in principle) used to solve for 6 out of the 8 unknown quark sector parameters g_i, γ_i, r, and q, given m_u^i, m_d^i, and V_{CKM}.

For example, from (48) one readily gets three independent combinations g_ig_j, $i\neq j$

$$g_ig_j = b_{ij}/q^2, \quad (49)$$

where the b_{ij} coefficients defined as

$$b_{ij}\equiv\frac{|d_{ij}|^2-(d_{ii}-r^2m_u^{i2})(d_{jj}-r^2m_u^{j2})}{r^2[m_u^{i2}+m_u^{j2}-2m_u^im_u^j\cos(\gamma_i-\gamma_j)]}\equiv\frac{b_{ij}^{(n)}}{b_{ij}^{(d)}} \quad (50)$$

depend on r^2 and γ_i only. From (49), the individual g_i's are then given by

$$g_i = \frac{1}{|q|}\sqrt{\frac{b_{ij}b_{ik}}{b_{jk}}}, \qquad (i\neq j\neq k\neq i). \quad (51)$$

These quantities may be used in (47) to recast the γ_i phases as functions of r and q which become the only pending quark sector parameters. Recall also that consistency of Eqs. (49) and (51) requires r and q such that $g_i\in\langle 0,1)$ and $g_ig_j\in\langle 0,\frac{1}{4})$ for $i\neq j$.

Concerning the charged-lepton sector, there is no analogue of decomposition (45) because the neutrino mass matrix remains unconstrained. All we can write is $M_l^T\tilde{U}_l = V_l^L\mathcal{D}_lV_l^{R\dagger}$, where $V_l^{L,R}$ are unitary diagonalization matrices without immediate physical significance. Nevertheless, one can look at the spectrum of M_l by means of the three basic invariants—the trace, the sum of the main minors, and the determinant of $M_l.M_l^\dagger$. Moreover, the right-hand sides of the trace and sum-of-the-minors formulae can be obtained from (the sum of) the right-hand sides of Eqs. (47) and (48) upon replacing

$$g_i\to s^2g_i, \qquad q\to q/s, \quad \text{and} \quad \gamma_i\to\gamma_i+\phi, \quad (52)$$

which is just the vocabulary (44) rewritten for g_i and γ_i.

E. Numerical analysis

Before getting to the full-featured three-generation fit, let us inspect in brief the basic features of the 2×2 case focusing on the second and third generation of quarks and leptons. The reason is that in such a case a further constraint on one lepton mixing angle can be derived from the type-II dominated seesaw formula (20). One can then expect the higher order corrections coming from the effective operators (or further vector multiplets above the GUT scale) to account for the structure of the light sector. However, the effects of 10_F must be compatible with the second and third generation spectra already at this level, should the current approach be viable at all.

1. 2×2 *heavy charged sector analysis*

Forgetting for a while about the first row and column in the matrix relations (41) (which is technically achieved by $\alpha_1 \to 0$ yielding also $g_1 = 0$, with γ_1 left unconstrained), the formulae (47) and (48) are affected accordingly. Denoting $d_{ij}^{(2)} \equiv (V_{\rm CKM}^{(2)}|\mathcal{D}_d^{(2)}|^2 V_{\rm CKM}^{(2)\dagger})_{ij}$ where

$$\mathcal{D}_d^{(2)} \equiv {\rm diag}(m_s, m_b), \qquad V_{\rm CKM}^{(2)} \equiv \begin{pmatrix} \cos\theta_{23}^q & \sin\theta_{23}^q \\ -\sin\theta_{23}^q & \cos\theta_{23}^q \end{pmatrix}$$

are the 23 blocks of $\mathcal{D}_d$ and $V_{\rm CKM}$, one arrives at

$$d_{22}^{(2)} = r^2 m_c^2(1-g_2) + g_2 q^2(g_2+g_3) + 2rq\cos\gamma_2 m_c g_2\sqrt{1-g_2-g_3}, \tag{53}$$

$$d_{33}^{(2)} = r^2 m_t^2(1-g_3) + g_3 q^2(g_2+g_3) + 2rq\cos\gamma_3 m_t g_3\sqrt{1-g_2-g_3}, \tag{54}$$

$$\det[d^{(2)}] = -m_c^2 m_t^2 r^4 + r^2(d_{22}^{(2)} m_t^2 + d_{33}^{(2)} m_c^2) - r^2 q^2 g_2 g_3[m_c^2 + m_t^2 - 2m_c m_t\cos(\gamma_2-\gamma_3)].$$

The corresponding lepton sector relations can be derived from $\Delta_{2<3}$ in (48) and the sum[4] of Eqs. (53) and (54) using the vocabulary (52):

$$m_\mu^2 + m_\tau^2 = (g_2+g_3)^2 s^2 q^2 + r^2[m_c^2(1-s^2 g_2) + m_t^2(1-s^2 g_3)] + 2rsq\sqrt{1-s^2(g_2+g_3)} \times [g_2 m_c\cos(\gamma_2+\phi) + g_3 m_t\cos(\gamma_3+\phi)], \tag{55}$$

$$m_\mu^2 m_\tau^2 = \{r^2 m_t^2 m_c^2[1-(g_2+g_3)s^2] + s^2 q^2[m_t^2(g_2)^2 + m_c^2(g_3)^2] + 2m_c m_t sq[g_2 g_3 sq\cos 2(\gamma_2-\gamma_3) + r\sqrt{1-s^2(g_2+g_3)} \times (g_2 m_t\cos(\gamma_2-\phi) + g_3 m_c\cos(\gamma_3-\phi))]\}r^2. \tag{56}$$

Formulae (53)–(56) allow for a full reconstruction of the 5 relevant measurables (apart from m_t and m_c that we count amongst inputs), namely m_s, m_b, m_μ, m_τ and the 23 CKM mixing angle θ_{23}^q (recall that there is no CP phase in the 2×2 quark sector) in terms of 5 real parameters (q, r, s, α_2, and α_3) and 3 phases (γ_2, γ_3, ϕ).

Thus, it is natural to constrain the fit furthermore by sticking to the CP-conserving (i.e. real) case, which corresponds to 0 or π of the $\gamma_{1,2}$ and ϕ phases. This, however, makes the fit nontrivial because $\alpha_{1,2} \in \langle 0, 2\pi)$ live in a compact domain, s must be an $\mathcal{O}(1)$ number (to avoid extra fine-tuning in $M_{\Delta,\Lambda}$), and q should (for consistency reasons[5]) be within a few GeV range. Thus, the only free parameter in the real 2×2 case is r.

Remarkably enough, even such a constrained setting admits good fits of all the relevant experimental data. A pair of illustrative solutions is given in Table I. One can see that r plays the role of the m_t/m_b hierarchy "compensator", while all the other parameters fall into their proper domains specified above. The relative smallness of $g_{2,3}$ indicates that we are indeed in the $V^{16} \lesssim V^{54}$ regime, as suggested by the qualitative arguments in Sec. III B.

2. Large lepton mixing in the CP-conserving 2×2 *case*

In the 2×2 case, one can extend the current analysis to the neutrino sector because the 2×2 version of formula (42) can be a good leading order neutrino mass-matrix contribution (for the hierarchical case). Sticking again to the CP conserving setting, one can rewrite the core of formula (42) (taking for simplicity $\gamma_2 = \gamma_3$, that accounts only for an irrelevant overall sign) in the form:

$$\vec{f}\otimes\vec{f} = \begin{pmatrix} g_2 & \sqrt{g_2 g_3} \\ \cdot & g_3 \end{pmatrix} \tag{57}$$

and

$$N(s\vec{f}) = {\rm diag}[\sqrt{1-s^2 g_2}, \sqrt{1-s^2(g_2+g_3)}]^{-1},$$

which in the regime suggested by the charged-lepton fit ($s^2 g_i \ll 1$) leads to an approximate formula[6]

[4]Recall the individual diagonal entries of M_l are unknown due to the RH-rotation ambiguity in $M_l^T\tilde{U}_l = V_l^L\mathcal{D}_l V_l^{R\dagger}$ while the trace remains fixed.

[5]While the third family hierarchy should be compensated by a suitable choice of r, q governs the second family scales and thus (in order to have $g_i q$ in (56) around m_c for $g_i \approx 10^{-1}$) q should be within a few GeV range.

[6]The charged-lepton contribution to the lepton mixing is negligible because of the hierarchical nature of the LHS of Eq. (41).

TABLE I. A pair of illustrative examples of the 2×2 fits (i.e. focusing on the second and third generation) in the real setting (i.e. all phases set to 0 or π). A sample set of GUT-scale inputs was taken from [30] for $\tan\beta \approx 10$.

parameter	Sample solution 1 value	deviation	Sample solution 2 value	deviation
	Input		Input	
γ_2	π	—	π	—
γ_3	π	—	π	—
ϕ	π	—	π	—
m_c [GeV]	0.209	c.value	0.209	c.value
m_t [GeV]	90	c.value	70	$\sim 1\sigma$
	Free parameters		Free parameters	
r	0.0143	—	0.0184	—
s	2.3825	—	2.2968	—
g_2	0.0187	—	0.0199	—
g_3	0.0415	—	0.0414	—
q [GeV]	1.6296	—	1.5787	—
	Output		Output	
m_s [GeV]	0.0299	c.value	0.0299	c.value
m_b [GeV]	1.200	$\sim 1\sigma$	1.200	$\sim 1\sigma$
m_μ [GeV]	0.0756	c.value	0.0756	c.value
m_τ [GeV]	1.292	c.value	1.292	c.value
$\sin\theta_{23}^q$	0.036	c.value	0.036	c.value

$$\tan 2\theta_{23}^l \approx 2\sqrt{g_2 g_3}/|g_2 - g_3|, \tag{58}$$

so the proximity of g_2 and g_3 leads to a large 2-3 lepton sector mixing. Numerically, for $g_2 \approx 0.0187$ and $g_2 \approx 0.0415$ (solution 1 in Table I) one gets $\sin^2 2\theta_{23}^l \approx 0.85$, which is remarkably close to the observed nearly maximal atmospheric mixing [2].

It is interesting that the $g_2 \sim g_3$ case is not accidental. Notice first that (neglecting the small $P(\vec{f}) \ll 1$ and the almost unity matrices $N(\vec{f})$ and $\tilde{U}_d$), the down-quark mass sum-rule in (41) reads at leading order:

$$M_d \approx \begin{pmatrix} m_c & 0 \\ 0 & m_t \end{pmatrix} r + q\begin{pmatrix} g_2 & \sqrt{g_2 g_3} \\ \sqrt{g_2 g_3} & g_3 \end{pmatrix}. \tag{59}$$

The hierarchies in Table I suggest that the second term in (59) dominates over the first one in all but the 22 entry. Thus, we have approximately

$$M_d \approx \begin{pmatrix} q g_2 & q\sqrt{g_2 g_3} \\ q\sqrt{g_2 g_3} & r m_t \end{pmatrix} \Rightarrow m_s \approx q g_2, \qquad m_b \approx r m_t. \tag{60}$$

Since M_u is diagonal, the 23 quark sector mixing angle θ_{23}^q comes entirely from M_d, i.e. $V_{cb} \approx \sin\theta_{23}^q \approx q\sqrt{g_2 g_3}/m_b$. Solving for g_2 and g_3 and substituting into (58) one finally obtains

$$\tan 2\theta_{23}^l \approx 2x/|1 - x^2| \quad \text{where } x \equiv \frac{y_b}{y_s}\sin\theta_{23}^q, \tag{61}$$

in agreement with the numerical example given above.

Therefore, we have obtained an interesting correlation between $x \sim 1$ (i.e. the proximity of $|V_{cb}|$ to y_s/y_b) in the quark sector and the large atmospheric mixing θ_{23}^l for the type-II dominated neutrino mass matrix, quite along the lines of the connection of the θ_{23}^l-maximality and the $b - \tau$ Yukawa convergence in the context of SUSY $SO(10)$ models with $\overline{126}_H$ in the Higgs sector [19].

3. *Full-featured* 3×3 *charged sector analysis*

Though the full three-generation case is much more sensitive to all sorts of higher order corrections it can still be instructive to look at the fit of the charged sector formulae (47) and (48). Perhaps the simplest approach would be to perturb the 2×2 fits studied in the previous section by relaxing the conditions imposed on α_1, $\gamma_{2,3}$, and ϕ and admitting slight changes in the other parameters as well, in order to fit the first generation quantities m_u, m_d, θ_{12}^q, θ_{13}^q, and $\delta_{\rm CKM}$.

It is very interesting that this simple strategy fails due to the tension emerging already at the level of the pure quark sector fit, which (being even underconstrained) should be essentially trivial. In particular, we found that the quark sector data are reproduced only for the price of pushing the r-parameter very far from its natural domain $r \approx m_b/m_t \approx 10^{-2}$ suggested by the 2×2 fit, cf. Table I. Instead, all the numerical quark sector fits we found give $r \approx m_s/m_c \approx 10^{-1}$ and a g_3 value very close to 1 indicating a high degree of fine-tuning in formula (53). An interested reader can find a sample set of relevant data in Table II.

This, however, has dramatic consequences for the charged-lepton sector. With $r \sim 10^{-1}$ incapable of compensating the m_t/m_b hierarchy without an extra aid from q, there is not enough freedom left to account for the m_μ/m_s hierarchy and (for a fixed m_μ) m_s turns out to be generically too large.

One can use the formula (49) to understand this peculiar instability analytically. Notice first that the denominators $b_{ij}^{(d)}$ of all the b_{ij} coefficients in (49) are always positive, so in order to have $g_i g_j \geq 0$ for all $i \neq j$, all the numerators $b_{ij}^{(n)}$ must be positive as well.

Suppose first that the CKM mixing can be neglected, i.e. $d_{ij} = (V_{\rm CKM}|\mathcal{D}_d|^2 V_{\rm CKM}^\dagger)_{ij} = m_d^{i2}\delta_{ij}$, (no summation over i). The numerators $b_{ij}^{(n)}$ in such a case read:

$$b_{ij}^{(n)}(r) = -p_i(r)p_j(r), \tag{62}$$

where $p_i(r) = m_d^{i2} - r^2 m_u^{i2}$ are simple polynomials in r. However, given (62), there is no[7] r that would lead to

[7] At least one product of a pair of any three real numbers is always non-negative.

TABLE II. An example exhibiting the generic features of all the 3×3 quark sector fits we found. No χ^2 is given because the number of free parameters exceeds the number of observables in the quark sector and the fit is in principle (though not technically) trivial. Notice, namely, the value of r being 1 order of magnitude higher than in the 2×2 case (cf. Table I) as well as the singular behavior of g_3. As before, the sample set of inputs was taken from [30] for $\tan\beta \approx 10$.

Physical inputs	
m_u [MeV]	0.720
m_c [GeV]	0.209
m_t [GeV]	90
Free parameters	
γ_1	2.122
γ_2	1.725
γ_3	3.795
r	0.1527
g_1	0.0002
g_2	0.0048
g_3	0.9945
q [GeV]	0.5436
Physical outputs	
m_d [MeV]	1.562
m_s [GeV]	0.030
m_b [GeV]	1.090
$\sin\theta^q_{12}$	0.2229
$\sin\theta^q_{13}$	0.0365
$\sin\theta^q_{23}$	0.0032
$\delta_{\rm CKM}$	60°

$g_i g_j \geq 0$ for all $i \neq j$ (except $p_i(r) = 0$ for at least one i which, however, corresponds to $g_i = 0$, cf. (51), taking us back to the 2×2 case). Graphically, the three numerators $b^{(n)}_{ij}(r)$ correspond to three functions sharing roots on the real axis, as depicted by the dashed curves in Fig. 1. This means that the 3×3 quark spectra (regardless of their particular shape) cannot be accommodated in the simplest renormalizable model *unless* $V_{\rm CKM} \neq \mathbb{1}$.

However, once the CKM mixing is turned on, the $p_i(r)$ polynomials change to $p'_i(r) = d_{ii} - r^2 m^{i2}_u$ and an extra positive term shows up in the relevant analogue of formula (62), cf. also Eq. (49):

$$b^{(n)\prime}_{ij}(r) = |d_{ij}|^2 - p'_i(r) p'_j(r), \tag{63}$$

with the net effect of slightly distorting and lifting the three dashed curves corresponding to the $V_{\rm CKM} = \mathbb{1}$ case. As a consequence, a small "physical" window for r can open around the point where two of the original $b^{(n)}_{ij}(r)$ functions shared a root (i.e. around the roots of $p_i(r)$'s) while the third was positive, see again Fig. 1. Unfortunately, the value of r where this happens corresponds to $r \approx m_s/m_c \approx 0.15$ and not $r \approx m_b/m_t \approx 0.015$ that would

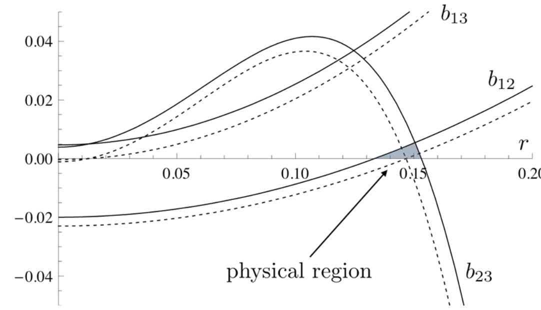

FIG. 1 (color online). The qualitative r-behavior of numerators of the b_{ij} coefficients in the $V_{\rm CKM} = \mathbb{1}$ limit (dashed curve) and in the physical $V_{\rm CKM} \neq \mathbb{1}$ setting (plain curve). Apart from the two common roots of the dashed curves at $r \approx 0.015$ and $r \approx 0.15$ there is a third one outside the displayed region at $r \approx 2.2$. The physical requirement $b_{ij} > 0$ (for all $i \neq j$) can be satisfied only if $V_{\rm CKM} \neq \mathbb{1}$ and $r \approx 0.15$, which leads to an intrinsic instability in the quark sector fits, cf. formula (63) and discussion around. For optical reasons, b_{12} was magnified by a multiplicative factor of 10^7 while b_{13} by 10^2.

be compatible with the "natural" solution we have obtained in the 2×2 case.

Therefore, a single extra 10_F in the matter sector, and, in particular, its nondecoupling effects (that have been shown to account for all the quark and lepton masses and mixings in the two-generation case), cannot provide the only source of physics contributing to the first generation observables.

V. CONCLUSIONS

In this paper, we have scrutinized the effective Yukawa sector emerging in a class of renormalizable SUSY $SO(10)$ GUT models with $16_H \oplus \overline{16}_H$ Higgs fields driving the $SU(2)_R \otimes U(1)_{B-L}$ breakdown.

An extra $SO(10)$-vector matter multiplet 10_F with an accidentally small singlet mass term (around the GUT scale) would not decouple from the GUT-scale physics and, under certain conditions, can provide a nonvanishing component of the light matter states (spanning in traditional case only on the three $SO(10)$ spinors 16^i_F) through the mixing term $16^i_F 10_F 16_H$ in the superpotential. The sensitivity of 10_F to $SU(5)$ and $SU(2)_R \otimes U(1)_{B-L}$ breaking (through the $10_F 10_F 54_H$ and $16^i_F 10_F 16_H$ interactions) lifts the typical high degree of degeneracy in the effective low-energy Yukawa couplings, giving rise to a characteristic pattern of nondecoupling effects in the effective mass matrices. This, however, could render the Yukawa sector of the model potentially realistic.

In order to deal with the complicated structure of the emerging effective matter sector mass sum-rules, a thorough analysis of the would-be ambiguities emerging upon integrating out the heavy parts of the matter spectra has been provided and the relevant parameter counting was given. This admits for a detailed numerical analysis of the quark sector in the full three-generation case. If the renormalizable part of the type-II contribution associated to the $SU(2)_L$ triplet in 54_H governs the seesaw formula, the neutrino mass matrix becomes partly calculable. Focusing on the second and third generation, the 23 mixing (for hierarchical neutrino spectrum) can be estimated. In such a case, we found a striking (GUT-scale) correlation between the proximity of $|V_{cb}|$ and y_s/y_b and a large 23 mixing angle in the lepton sector: $\tan 2\theta^l_{23} \approx 2x/|1-x^2|$ where $x \equiv (y_b/y_s)|V_{cb}|$.

Concerning the charged sector Yukawa sum-rules, any successful fit of the quark spectra in the simplest renormalizable scenario (with a single nondecoupling vector matter multiplet) requires a nontrivial CKM mixing $V_{\rm CKM} \neq \mathbb{1}$ and we provide a detailed analytical understanding of this peculiarity. However, with the charged-lepton sector spectrum taken into account, a generic tension in the m_μ/m_s hierarchy is revealed, calling for extra sources of corrections affecting the first generation observables, be it e.g. contributions from higher order operators or additional vector matter multiplets.

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APPENDIX A: THE SEESAW

Including all the effective contributions sketched in Sec. II B, the full 8×8 neutrino mass matrix (in the $\{N_L, N_L^c, \Lambda_L^0, \Lambda_L^{c0}\}$ basis) receives the following order of magnitude form (forgetting about the CG coefficients):

$$M_\nu = \begin{pmatrix} \kappa(v_u^{\overline{16}})^2/\Lambda_\kappa & Yv_u^{10} & \vec{\zeta} v_u^{\overline{16}} v_u^{10}/\Lambda_\zeta & \vec{F}V^{16} \\ . & M_M & \vec{\zeta}V^{\overline{16}}v_u^{10}/\Lambda_\zeta & \vec{F}v_d^{16} \\ . & . & \mu_+ & M_{\Lambda\Lambda^c} \\ . & . & . & \mu_- \end{pmatrix} \tag{A1}$$

with $M_{\Lambda\Lambda^c} \equiv M_{10} + \frac{3}{2}\lambda\tilde{V}^{54}$ and $\mu_\pm \equiv \lambda c_T w_\pm + \sigma(v_u^{10})^2/\Lambda_\sigma$. The first row GUT scale entries at the 14 position can be cancelled by a suitable rotation in the N_L-Λ_L^0 plane, identical to the charged sector U_l transformation obtained from (13) upon replacing $d \to l$ (recall the $SU(2)_L$-doublet nature of L_L and Λ_L). Using the decompositions $N_L = A_l^\dagger n_L + \vec{B}_l^* \tilde{\Lambda}_L^0$ and $\Lambda_L^0 = \vec{B}_l^* . n_L + D_l^* \tilde{\Lambda}_L^0$, which is just the lepton sector analogue of formula (14), one obtains (in the $\{n_L, N_L^c, \tilde{\Lambda}_L^0, \Lambda_L^{c0}\}$ basis):

$$M_\nu = \begin{pmatrix} M_\nu^{\rm II} & D_\nu & \vec{B}_l^* D_l^* \mu_+ & 0 \\ . & M_M & \vec{C}^{*T} Y v_u^{10} + \vec{\zeta} V^{\overline{16}} D_l^* v_u^{10}/\Lambda_\zeta & \vec{F} v_d^{16} \\ . & . & D_l^* D_l^* \mu_+ & M_\Lambda \\ . & . & . & \mu_- \end{pmatrix}, \tag{A2}$$

where M_Λ is the lepton sector heavy state mass given by Eq. (18), $M_\nu^{\rm II} \equiv \vec{B}_l^* \otimes \vec{B}_l^* \mu_+ + A_l^* \mu_3 A_l^\dagger$ and $D_\nu \equiv A_l^* Y v_u^{10} + (\vec{\zeta} \otimes \vec{B}_l^*) v_u^{10} V^{\overline{16}}/\Lambda_\zeta$. The seesaw formula yields

$$M_\nu = M_\nu^{\rm II} - (D_\nu, \vec{B}_l^* D_l^* \mu^+, 0) M_{234}^{-1} (D_\nu, \vec{B}_l^* D_l^* \mu^+, 0)^T, \tag{A3}$$

where M_{234} is the $\{N_L^{ci}, \tilde{\Lambda}_L^0, \Lambda_L^{c0}\}$ sector 5×5 (i.e. lower right) submatrix of (A2). Denoting

$$m \equiv (\vec{C}^{*T} Y v_u^{10} + \vec{\zeta} D_l^* v_u^{10} V^{\overline{16}}/\Lambda_\zeta, F v_d^{16}) \tag{A4}$$

and

$$\tilde{M}_\Lambda \equiv \begin{pmatrix} 0 & M_\Lambda \\ M_\Lambda & 0 \end{pmatrix},$$

which stand for the leading contributions in the upper right (m) 3×2 and lower right $(\tilde{M}_\Lambda)$ 2×2 blocks of M_{234}, respectively, one can use $|m| \ll M_\Lambda, M_M$ to analytically invert M_{234}:

$$M_{234}^{-1} \approx \begin{pmatrix} M_2 & -M_M^{-1} m \tilde{M}_\Lambda^{-1} \\ -\tilde{M}_\Lambda^{-1} m^T M_M^{-1} & M_{34} \end{pmatrix}, \tag{A5}$$

where $M_2 \equiv M_M^{-1} + M_M^{-1} m \tilde{M}_\Lambda^{-1} m^T M_M^{-1}$ and $M_{34} \equiv \tilde{M}_\Lambda^{-1} + \tilde{M}_\Lambda^{-1} m^T M_M^{-1} m \tilde{M}_\Lambda^{-1}$, see [31] for details. The seesaw formula (A3) then yields (19) up to higher order terms.

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