Threshold and flavor effects in the renormalization group equations of the MSSM: Dimensionless couplings

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In a theory with broken supersymmetry, gaugino couplings renormalize differently from gauge couplings, as do higgsino couplings from Higgs boson couplings. As a result, we expect the gauge (Higgs boson) couplings and the corresponding gaugino (higgsino) couplings to evolve to different values under renormalization group evolution. We reexamine the renormalization group equations (RGEs) for these couplings in the minimal supersymmetric standard model (MSSM). To include threshold effects, we calculate the β functions using a sequence of (nonsupersymmetric) effective theories with heavy particles decoupled at the scale of their mass. We find that the difference between the SM couplings and their SUSY cousins that is ignored in the literature may be larger than two-loop effects which are included, and further that renormalization group evolution induces a nontrivial flavor structure in gaugino interactions. We present here the coupled set of RGEs for these dimensionless gauge and Yukawa-type couplings. The RGEs for the dimensionful soft-supersymmetry-breaking parameters of the MSSM will be presented in a companion paper.

I. INTRODUCTION

Renormalization group equations (RGEs) have played an important role in extracting phenomenological predictions of theories valid at very high energy scales. One of the best known examples of this is the prediction of the weak mixing angle from the simplest SUSY $SU(5)$ grand unified theories (GUTs): the equality of the gauge couplings at $Q = M_{\text{GUT}}$, which implies $\sin^2\theta_W(M_{\text{GUT}}) = 3/8$, leads to the measured value of $\sin^2\theta_W$ only when these couplings are evolved to the scale $Q \sim 100$ GeV relevant to experiments [1]. Since at least the 1980's, RGEs have played a role in the analysis of the implications of widely different models of new physics, including supersymmetric models, the subject of this paper [2].

Supersymmetry (SUSY) is a novel symmetry that links bosons and fermions, providing a level of synthesis never before attained [3–6]. Since supersymmetry implies a superpartner for every chirality state of the standard model (SM) with the same internal quantum numbers, it implies an approximate doubling of the degrees of freedom. Supersymmetry, even if softly broken, fixes the dimensionless couplings that determine the tree-level interactions of the superpartners with SM particles and with one another (i.e. interactions via mass dimension four), so that in this sense the theory is completely predictive. The interest in SUSY phenomenology was sparked once it was understood that the SUSY nonrenormalization theorem implied the stability of the hierarchy between the weak and GUT or

Planck scales to radiative corrections [7]. Since SUSY is clearly not manifested in the observed spectrum of elementary particles, SUSY must be (explicitly or spontaneously) broken. Moreover, the mass scale of SM superpartners—at least of those that couple sizably to the Higgs sector—must be smaller than $\mathcal{O}(1)$ TeV, in order for SUSY breaking not to destabilize the hard-won hierarchy between the weak and GUT scale just mentioned. In view of our complete ignorance of the mechanism of SUSY breaking, it is expedient to parametrize SUSY breaking by introducing soft-supersymmetry-breaking (SSB) operators with dimension ≤ 3 that respect Lorentz, gauge, and any other symmetries into the effective Lagrangian of the theory. These include gaugino mass parameters, the SSB Higgs mass parameter b, often written as $B\mu$, sfermion mass matrices, and trilinear scalar coupling parameters, usually denoted by A_{ijk} [8].

We emphasize that the proliferation of SSB parameters is a result of our ignorance of how SUSY breaking is communicated to MSSM superpartners. Until this is understood, we are led to resort to a variety of models [9–12]. Typically these models, which are based on differing assumptions about *high scale* physics, lead to different ansätze for the pattern of SSB parameters, in a Lagrangian renormalized at this high scale Q_{high}. This Lagrangian cannot directly be used to perturbatively extract phenomenological predictions at the $E \sim 100$ GeV scale of our experiments, because large logarithms $\log(Q_{\text{high}}/E)$ would invalidate (fixed order) perturbation theory. Instead, RGEs are used to evolve all the Lagrangian parameters to the scale E, and this resulting "low energy" Lagrangian is used to evaluate its implications [2].

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Since supersymmetric models automatically contain many scalars with the same gauge quantum numbers, there are potential new sources of flavor violation [7,13]. Within the parametrization of the SSB terms described above, these are encapsulated in the sfermion mass matrices and in the A parameters (which may be regarded as matrices in the flavor space). Our ultimate goal is to develop a program that will enable a general study of flavor violation, for arbitrary values of SSB parameters at $Q = Q_{high}$. For the present, we confine ourselves to quark flavor violation.[1]

Toward this end, we need to know the RGEs for all dimensionless and dimensionful parameters. These are already known, both in the SM [14] and in the MSSM [15–18] at the two-loop level. Working to two-loop accuracy means that we must include threshold effects in the one-loop RGEs, since these can numerically be as large as the two-loop terms, and perhaps even larger if the sparticle spectrum is very split. It is in this connection that we extend the existing literature [19,20] as described below. As in these studies, we implement SUSY and Higgs particle thresholds as step functions in our evaluation of the β functions. We consider first the evolution of the dimensionless couplings. We assume that for any particle with mass M_i between the weak scale and the scale Q_{high} introduced above, the effective theory used to calculate the β function includes this particle if $Q > M_i$, but not otherwise. Thus, at a high scale Q above the masses of all sparticles and Higgs bosons, the β functions are those of the MSSM. As we come down in scale, the SUSY particles and the heavy Higgs bosons decouple one by one until we are left with just the SM particles, and the β functions for the dimensionless couplings of the SM. Although this sounds quite straightforward in principle, there are several issues that need to be confronted to implement this program.

(i) The step-function decoupling of particles at the scale of their mass makes sense for couplings of fields only in their (at least, approximate) mass basis. For instance, in the MSSM Higgs sector, it is quite clear where the spin-zero mass eigenstates h or H (or A and $H^{\pm}$) decouple, but the decoupling point for the h_u or h_d fields (or for the higgsinos) is ambiguous, because the mass eigenstates are combinations of the h_u- and h_d-type fields (or the corresponding higgsinos). Moreover, since the Higgs bosons and higgsinos acquire mass from different origins, these are not "diagonalized" by a common rotation, so that in order to implement the decoupling we must, in principle, work with Higgs boson/higgsino bases that are

not SUSY transforms of one another. See, however, Sec. III for a simplification.

(ii) Since we decouple the superpartners but not the lighter SM particles, the effective theory below the scale of the heaviest sparticle "knows" about SUSY breaking, and necessarily includes new dimensionless couplings. For instance, while in the SUSY limit, the coupling of the gauge boson to quarks is exactly the same for the corresponding gaugino-quark-squark coupling, below the scale where we begin to decouple heavy particles, these couplings no longer run together. For every gauge coupling g_i, the low energy theory may include additional distinct gaugino-particle-sparticle couplings $\tilde{g}_i^{\Phi}$ (labeled by the scalar Φ) that evolve differently from g_i. For instance, if $m_{\tilde{q}} \ll m_{\tilde{g}}$, below $Q = m_{\tilde{g}}$ the bino coupling $\tilde{g}'^Q$ evolves differently from g'. Moreover, $\tilde{g}'^Q$ and $\tilde{g}'^L$ (as well as other $\tilde{g}'^{\Phi}$ couplings) also do not evolve in the same way. Indeed, as we will see later, these couplings also develop flavor-violating components and so become matrices in the flavor space, which we will denote by $\tilde{\mathbf{g}}_i^{\Phi}$, once again labeled by the scalar Φ.

(iii) The proliferation of these new fermion-fermion-scalar couplings also extends to the Higgs sector. For instance, if the heavier Higgs bosons $H^{\pm}$, A, and H all decouple at the scale $Q \simeq m_H$ that is hierarchically larger than $|\mu|$, and squarks are light, the effective theory in the range $|\mu| < Q < m_H$ includes both higgsino doublets $\tilde{h}_u$ and $\tilde{h}_d$ coupling to various flavors of quarks and squarks, but with quark (and squark) pairs coupling only to the light SM-like Higgs boson h. Thus, we must also have independent higgsino coupling matrices that we denote by $\tilde{\mathbf{f}}_{u,d,e}^{\Phi}$, once again, labeled by the scalar.

We see that even for the dimensionless couplings, there are new complications which (to our knowledge) have not been included in previous studies, but must be taken into account for a proper implementation of threshold effects into the (one-loop) RGEs. Motivated by the fact that our goal is to study flavor violation in sparticle decays in a general way, we derive the RGEs for all the relevant dimensionless couplings of the MSSM, including flavor-violating superpotential interactions, in this paper. The number of RGEs for the dimensionless couplings now expands from those for the three gauge couplings and three Yukawa coupling matrices to those for the three gauge couplings, nineteen scalar-fermion-fermion coupling matrices and four Higgs-higgsino-gaugino couplings. We should mention that, in addition, there are numerous quartic scalar couplings arising from the D-terms or the F-terms in the scalar potential. Although these are given by the "squares" of gauge or superpotential Yukawa couplings in the SUSY limit, they will also renormalize differently below the SUSY thresholds. We do not discuss the

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[1]The analysis can readily be extended to lepton flavor violation. Although lepton flavor violation may be more striking and experimentally easier to access than quark flavor violation, it may depend on yet other unknowns such as the structure of any singlet neutrino mass matrix.

evolution of these couplings which are less important phenomenologically. Fortunately, these couplings do not enter the evolution of the dimensionless couplings that we do consider in this paper. We derive the RGEs for the dimensionful parameters, the scalar masses (matrices in the case of squarks), the trilinear coupling matrices, and the SSB $B\mu$ parameter, which have their own complications, in a companion paper [21].

Since, as we mentioned above, the threshold effects necessarily involve SUSY breaking, we derive our RGEs from the RGEs in a general (i.e. not necessarily supersymmetric) field theory that have been worked out in the literature. We use the seminal papers by Machacek and Vaughn [14] to derive the RGEs for the dimensionless couplings, while we use the work by Luo, Wang, and Xiao [22] to derive those for the dimensionful couplings of the MSSM. These studies both use the two-component spinor formalism. In contrast, most phenomenologists are much more familiar with the four-component formalism. To facilitate the use of the general RGEs, we recast the formulas for the RGEs for dimensionless couplings into four-component notation in Sec. II. In Sec. III we discuss the transition to the "mass basis" necessary for the implementation of the MSSM thresholds. The actual derivation of the RGEs is carried out in Sec. IV. In Sec. V we show numerical examples of the solutions to some of the RGEs and discuss the impact of the new effects we have included in our analysis. We summarize in Sec. VI. The set of RGEs for the dimensionless couplings is listed in the Appendix.

II. FORMALISM

The one-loop β functions for gauge couplings are well known and take the form [23]

$$(4\pi)^2 \beta_g|_{1-\text{loop}} = -g^3 \left[\frac{11}{3} C(G) - \frac{2}{3} \sum_{\text{fermions}} S(R_F) \right.$$
$$\left. - \frac{1}{3} \sum_{\text{scalars}} S(R_S) \right], \quad (1)$$

where $C(G)$ is the quadratic Casimir for the adjoint representation of the associated Lie algebra, and $S(R_F)$ and $S(R_S)$, respectively, are the Dynkin indices for the representations R_F, R_S under which the fermions and (complex) scalars transform. For the Lie algebra of $SU(N)$, $C(G) = N$, while $S(R) = 1/2$ for the fundamental N-dimensional representation, and $S(R) = N$ for the adjoint representa-

tion. For the $U(1)_Y$ gauge coupling g', $C(G) = 0$ while $S(R) = (Y/2)^2$. Heavy scalars and fermions are decoupled simply by excluding them from the sums on the right-hand side of (1).

The derivation of the RGEs for "Yukawa-type" couplings of scalars with fermions is more involved. As discussed in Sec. I our strategy for deriving the RGEs is to first cast the general two-component results in Refs. [14,22] into four-component notation, match the parameters with those of the MSSM Lagrangian density, and write down the RGEs. Toward this end, we begin with the Lagrangian density written in terms of *two-component* spinor fields ψ_p and *real* scalar fields ϕ_a, as

$$\mathcal{L}_{(2)} = i\psi_p^\dagger \sigma^\mu D_\mu \psi_p + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a - \frac{1}{4} F_{\mu\nu A} F_A^{\mu\nu}$$
$$- \frac{1}{2} [(\mathbf{m}_f)_{pq} \psi_p^T \zeta \psi_q + \text{H.c.}] - \frac{1}{2!} \mathbf{m}_{ab}^2 \phi_a \phi_b$$
$$- \left(\frac{1}{2} \mathbf{Y}_{pq}^a \psi_p^T \zeta \psi_q \phi_a + \text{H.c.} \right) - \frac{1}{3!} \mathbf{h}_{abc} \phi_a \phi_b \phi_c$$
$$- \frac{1}{4!} \boldsymbol{\lambda}_{abcd} \phi_a \phi_b \phi_c \phi_d. \quad (2)$$

The antisymmetric matrix $\zeta \equiv i\sigma_2$, which is included to make the spinor bilinears Lorentz invariant, also ensures that the Yukawa coupling matrices $\mathbf{Y}^a$ (labeled by the scalar field that couples to the spinors) are symmetric in the fermion field type indices p, q. The spinor mass matrices $\mathbf{m}_f$ are likewise symmetric, but not necessarily Hermitian. The coefficients $\mathbf{m}_{ab}$, $\mathbf{h}_{abc}$, and $\boldsymbol{\lambda}_{abcd}$ are real and completely symmetric under permutations of all their indices.[2]

We now turn to the corresponding Lagrangian density in the *four-component* notation with Dirac and Majorana spinor fields and *complex* scalar fields, Φ_a, usually used by phenomenologists. The familiar four-component Dirac spinor Ψ_D has unrelated left- and right-chiral components and is, therefore, made up of two independent two-component spinors, ψ_L and ψ_R, according to $\Psi_D \equiv (\psi_L, -\zeta \psi_R^*)$ (where we really mean to write this equation as one for the column matrix Ψ_D). In contrast, the left- and right-chiral components of a Majorana spinor which, by definition, satisfies $\Psi_M = C\bar{\Psi}_M^T$ are related. Thus, for example, a Majorana spinor may be written in terms of a single two-component spinor as $\Psi_M \equiv (\psi_L, -\zeta \psi_L^*)$ or, alternatively, as $(\zeta \psi_R^*, \psi_R)$. The general form of the Lagrangian density that we work with in the context of the MSSM with a conserved R parity then takes the form

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[2]The factor $\frac{1}{2}$ in front of the Yukawa couplings, which is a reflection of the symmetry of the Yukawa coupling matrices, was included in Ref. [22] but not in Ref. [14].

$$\mathcal{L}_{(4)} = \frac{i}{2}\bar{\Psi}_j\gamma^\mu D_\mu\Psi_j + (D_\mu\Phi_a)^\dagger(D^\mu\Phi_a) - \frac{1}{4}F_{\mu\nu A}F_A^{\mu\nu} - \frac{1}{2}[(\mathbf{m}_X)_{jk}\bar{\Psi}_{Mj}\Psi_{Mk} + i(\mathbf{m}'_X)_{jk}\bar{\Psi}_{Mj}\gamma_5\Psi_{Mk}]$$

$$+ \left[\frac{1}{2!}\mathcal{B}_{ab}\Phi_a\Phi_b + \text{H.c.}\right] - \mathbf{m}^2_{ab}\Phi_a^\dagger\Phi_b - \left[(\mathbf{U}_a^1)_{jk}\bar{\Psi}_{Dj}P_L\Psi_{Dk}\Phi_a + (\mathbf{U}_a^2)_{jk}\bar{\Psi}_{Dj}P_L\Psi_{Dk}\Phi_a^\dagger + (\mathbf{V}_a)_{jk}\bar{\Psi}_{Dj}P_L\Psi_{Mk}\Phi_a\right.$$

$$+ (\mathbf{W}_a)_{jk}\bar{\Psi}_{Mj}P_L\Psi_{Dk}\Phi_a^\dagger + \frac{1}{2}(\mathbf{X}_a^1)_{jk}\bar{\Psi}_{Mj}P_L\Psi_{Mk}\Phi_a + \frac{1}{2}(\mathbf{X}_a^2)_{jk}\bar{\Psi}_{Mj}P_L\Psi_{Mk}\Phi_a^\dagger + \text{H.c.}\bigg]$$

$$+ \left[\frac{1}{2!}\Phi_a^\dagger\mathbf{H}_{abc}\Phi_b\Phi_c + \text{H.c.}\right] - \frac{1}{2!}\frac{1}{2!}\Lambda_{abcd}\Phi_a^\dagger\Phi_b^\dagger\Phi_c\Phi_d - \left[\frac{1}{3!}\Lambda'_{abcd}\Phi_a^\dagger\Phi_b\Phi_c\Phi_d + \text{H.c.}\right]. \tag{3}$$

In (3), Yukawa couplings of Dirac fields to scalars Φ_a are denoted by the matrices $\mathbf{U}_a^1$ and $\mathbf{U}_a^2$, while couplings of scalars to one Majorana and one Dirac field are denoted by the matrices $\mathbf{V}_a$ and $\mathbf{W}_a$. These matrices have no particular symmetry (or hermiticity) properties under interchange of the fermion field indices j and k. These indices label the fermion field type (quark, lepton, gaugino, higgsino) and also carry information of flavor and other quantum numbers (e.g. weak isospin and color). On the other hand, the matrices $\mathbf{X}_a^1$ and $\mathbf{X}_a^2$ that couple scalars to two Majorana fields are symmetric under $j \leftrightarrow k$ because of the symmetry properties [3] of the Majorana spinor bilinears that appear in (3). The scalar mass squared matrix $\mathbf{m}^2$ is Hermitian in the scalar field indices a, b, while the matrix $\mathcal{B}$ is symmetric. The Majorana fermion mass matrices $\mathbf{m}_X$ and $\mathbf{m}'_X$ are both Hermitian and symmetric. The matrix $\mathbf{m}_X$ is the familiar mass matrix, while $\mathbf{m}'_X$ is the coefficient of the CP-violating Majorana fermion bilinear [3], whose effects in the two-component spinor language show up as phases in the entries of the fermion mass matrix $\mathbf{m}_f$ in (2). The trilinear and quartic scalar couplings Λ, Λ', and $\mathbf{H}$ are symmetric under interchanges $a \leftrightarrow b$ and/or $c \leftrightarrow d$ for Λ, under interchanges of b, c, d for Λ', and finally under $b \leftrightarrow c$ for $\mathbf{H}$.

Before proceeding further, we should point out that the Lagrangian density in (3) is not the most general one that we can write. For instance, we have not included mass terms for Dirac fermions. Also, in writing the Yukawa interactions of fermions, we have not included terms with the operators

$$\bar{\Psi}_M P_L \Psi_D \Phi \quad \text{and} \quad \bar{\Psi}_D P_L \Psi_M \Phi^\dagger.$$

Dirac fermion masses appear in the MSSM only upon the spontaneous breakdown of electroweak symmetry, while the omitted Yukawa terms do not appear in the MSSM with R-parity conservation because Dirac fermions then carry a baryon or lepton number which is assumed to be conserved by these renormalizable operators. Likewise, we have also written only the subset of all scalar interactions that we need for our analysis of the MSSM.

Upon substitution of the four-component spinors in terms of their two-component cousins, and decomposing the complex field

$$\Phi_a = \frac{\Phi_{a_R} + i\Phi_{a_I}}{\sqrt{2}}$$

into its real and imaginary pieces, both of which are real fields, we can recast (3) into the general form of (2), involving only two-component spinors and real scalar fields. This then allows us to translate the parameters in (3) into the corresponding parameters in (2). We can now use the general results for the RGEs for the dimensionless [14] and dimensionful [22] parameters of a general quantum field theory to obtain the RGEs for the dimensionless Yukawa coupling matrices as well as for the mass and trilinear coupling parameters that appear in (3). We then use these to obtain the RGEs for the parameters of the MSSM, including the effects of the decoupling of various sparticles.

Since our focus in this paper is on the RGEs for the dimensionless parameters contained in the matrices $\mathbf{U}_a^1$, $\mathbf{U}_a^2$, $\mathbf{V}_a$, $\mathbf{W}_a$, $\mathbf{X}_a^1$, and $\mathbf{X}_a^2$, we begin by recasting the corresponding terms into two-spinor component notation to find

$$\mathcal{L}_{(4)} \ni -\left[\frac{1}{2}(\vec{\psi})^T\zeta\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{0} & (\mathbf{U}_a^{1T} + \mathbf{U}_a^{2T}) & \mathbf{W}_a^T \\ (\mathbf{U}_a^1 + \mathbf{U}_a^2) & \mathbf{0} & \mathbf{V}_a \\ \mathbf{W}_a & \mathbf{V}_a^T & (\mathbf{X}_a^1 + \mathbf{X}_a^2) \end{pmatrix}(\vec{\psi})\Phi_{a_R}\right.$$

$$\left. + \frac{1}{2}(\vec{\psi})^T\zeta\frac{i}{\sqrt{2}}\begin{pmatrix} \mathbf{0} & (\mathbf{U}_a^{1T} - \mathbf{U}_a^{2T}) & -\mathbf{W}_a^T \\ (\mathbf{U}_a^1 - \mathbf{U}_a^2) & \mathbf{0} & \mathbf{V}_a \\ -\mathbf{W}_a & \mathbf{V}_a^T & (\mathbf{X}_a^1 - \mathbf{X}_a^2) \end{pmatrix}(\vec{\psi})\Phi_{a_I}\right] + \text{H.c.} \tag{4}$$

where

$$\vec{\psi} \equiv \begin{pmatrix} \psi_L \\ \psi_R \\ \psi_M \end{pmatrix},$$

with the Dirac spinor $\Psi_D \equiv (\psi_L, -\zeta\psi_R^*)$ and the Majorana spinor $\Psi_M \equiv (\psi_M, -\zeta\psi_M^*)$. Notice that the two-component spinor fields are coupled to *real* scalar fields, Φ_{a_R} and Φ_{a_I}, via symmetric (but not necessarily Hermitian) matrices, as expected. We can now use the one-loop RGE for the Yukawa matrices $\mathbf{Y}$ that appear in (2) [14,22],[3]

$$
\begin{aligned}
(4\pi)^2 \boldsymbol{\beta}_Y^a\big|_{\text{1-loop}} &= \tfrac{1}{2}[\mathbf{Y}_2^T(F)\mathbf{Y}^a + \mathbf{Y}^a\mathbf{Y}_2(F)] + 2\mathbf{Y}^b\mathbf{Y}^{a\dagger}\mathbf{Y}^b \\
&\quad + \mathbf{Y}^b \operatorname{Tr}\{\tfrac{1}{2}(\mathbf{Y}^{b\dagger}\mathbf{Y}^a + \mathbf{Y}^{a\dagger}\mathbf{Y}^b)\} \\
&\quad - 3g^2\{\mathbf{C}_2(F), \mathbf{Y}^a\},
\end{aligned} \tag{5}
$$

to obtain the one-loop β functions for the various dimen-sionless couplings in (3). Here, $\mathbf{Y}_2(F) = \mathbf{Y}^{b\dagger}\mathbf{Y}^b$ and $\mathbf{C}_2(F) = \mathbf{t}^A\mathbf{t}^A$, where $\mathbf{t}^A$ are the group generators in the reducible representation that includes all the fermion fields. Clearly, the matrices $\mathbf{Y}^a$, $\mathbf{Y}_2(F)$, and $\mathbf{C}_2(F)$ all have the same dimensionality, determined by the total number of two-component fermion fields in the system. We also draw the reader's attention to the fact that in the four-component notation the sum over $a, b, \ldots$ runs over the different complex scalar fields, whereas in the two-component notation, we must not only sum over the various field types, but also separately over the real and imaginary parts of the same complex field. In other words, we have two distinct Yukawa matrices $\mathbf{Y}^a$, one when a refers to Φ_{a_R} and the other when a refers to Φ_{a_I}. The trace that appears above is a sum over the *fermion types*.

The β functions for the Yukawa coupling matrices in (3) can then be readily found to be

$$
\begin{aligned}
(4\pi)^2 \boldsymbol{\beta}_{U_a^1}\big|_{\text{1-loop}} &= \tfrac{1}{2}[(\mathbf{U}_b^1\mathbf{U}_b^{1\dagger} + \mathbf{U}_b^2\mathbf{U}_b^{2\dagger} + \mathbf{V}_b\mathbf{V}_b^\dagger)\mathbf{U}_a^1 + \mathbf{U}_a^1(\mathbf{U}_b^{1\dagger}\mathbf{U}_b^1 + \mathbf{U}_b^{2\dagger}\mathbf{U}_b^2 + \mathbf{W}_b^\dagger\mathbf{W}_b)] + 2[\mathbf{U}_b^1\mathbf{U}_a^{2\dagger}\mathbf{U}_b^2 + \mathbf{U}_b^2\mathbf{U}_a^{2\dagger}\mathbf{U}_b^1 \\
&\quad + \mathbf{V}_b\mathbf{X}_a^{2\dagger}\mathbf{W}_b] + \mathbf{U}_b^1 \operatorname{Tr}\{(\mathbf{U}_b^{1\dagger}\mathbf{U}_a^1 + \mathbf{U}_a^{2\dagger}\mathbf{U}_b^2) + \tfrac{1}{2}(\mathbf{X}_b^{1\dagger}\mathbf{X}_a^1 + \mathbf{X}_a^{2\dagger}\mathbf{X}_b^2)\} + \mathbf{U}_b^2 \operatorname{Tr}\{(\mathbf{U}_b^{2\dagger}\mathbf{U}_a^1 + \mathbf{U}_a^{2\dagger}\mathbf{U}_b^1) \\
&\quad + \tfrac{1}{2}(\mathbf{X}_b^{2\dagger}\mathbf{X}_a^1 + \mathbf{X}_a^{2\dagger}\mathbf{X}_b^1)\} - 3g^2[\mathbf{U}_a^1\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{U}_a^1] + \tfrac{1}{2}\mathbf{V}_a(\mathbf{V}_b^\dagger\mathbf{U}_b^1 + \mathbf{X}_b^{2\dagger}\mathbf{W}_b) + 2\mathbf{U}_b^1\mathbf{W}_a^\dagger\mathbf{W}_b \\
&\quad + \mathbf{U}_b^1 \operatorname{Tr}\{\mathbf{W}_a^\dagger\mathbf{W}_b + \mathbf{V}_b^\dagger\mathbf{V}_a\},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
(4\pi)^2 \boldsymbol{\beta}_{U_a^2}\big|_{\text{1-loop}} &= \tfrac{1}{2}[(\mathbf{U}_b^1\mathbf{U}_b^{1\dagger} + \mathbf{U}_b^2\mathbf{U}_b^{2\dagger} + \mathbf{V}_b\mathbf{V}_b^\dagger)\mathbf{U}_a^2 + \mathbf{U}_a^2(\mathbf{U}_b^{1\dagger}\mathbf{U}_b^1 + \mathbf{U}_b^{2\dagger}\mathbf{U}_b^2 + \mathbf{W}_b^\dagger\mathbf{W}_b)] + 2[\mathbf{U}_b^1\mathbf{U}_a^{1\dagger}\mathbf{U}_b^2 + \mathbf{U}_b^2\mathbf{U}_a^{1\dagger}\mathbf{U}_b^1 \\
&\quad + \mathbf{V}_b\mathbf{X}_a^{1\dagger}\mathbf{W}_b] + \mathbf{U}_b^2 \operatorname{Tr}\{(\mathbf{U}_b^{2\dagger}\mathbf{U}_a^2 + \mathbf{U}_a^{1\dagger}\mathbf{U}_b^1) + \tfrac{1}{2}(\mathbf{X}_b^{2\dagger}\mathbf{X}_a^2 + \mathbf{X}_a^{1\dagger}\mathbf{X}_b^1)\} + \mathbf{U}_b^1 \operatorname{Tr}\{(\mathbf{U}_b^{1\dagger}\mathbf{U}_a^2 + \mathbf{U}_a^{1\dagger}\mathbf{U}_b^2) \\
&\quad + \tfrac{1}{2}(\mathbf{X}_b^{1\dagger}\mathbf{X}_a^2 + \mathbf{X}_a^{1\dagger}\mathbf{X}_b^2)\} - 3g^2[\mathbf{U}_a^2\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{U}_a^2] + \tfrac{1}{2}(\mathbf{U}_b^2\mathbf{W}_b^\dagger + \mathbf{V}_b\mathbf{X}_b^{1\dagger})\mathbf{W}_a + 2\mathbf{V}_b\mathbf{V}_a^\dagger\mathbf{U}_b^2 \\
&\quad + \mathbf{U}_b^2 \operatorname{Tr}\{\mathbf{W}_b^\dagger\mathbf{W}_a + \mathbf{V}_a^\dagger\mathbf{V}_b\}.
\end{aligned} \tag{7}
$$

Within the MSSM with *R*-parity conservation, the last three terms of each of these equations vanishes because $\mathbf{W}_a$ and $\mathbf{V}_a$ vanish for the a values for which $\mathbf{U}_a^1$ and $\mathbf{U}_a^2$ are nonzero. Continuing,

$$
\begin{aligned}
(4\pi)^2 \boldsymbol{\beta}_{X_a^1}\big|_{\text{1-loop}} &= \tfrac{1}{2}[(\mathbf{W}_b\mathbf{W}_b^\dagger + \mathbf{V}_b^T\mathbf{V}_b^* + \mathbf{X}_b^1\mathbf{X}_b^{1\dagger} + \mathbf{X}_b^2\mathbf{X}_b^{2\dagger})\mathbf{X}_a^1 + \mathbf{X}_a^1(\mathbf{W}_b^*\mathbf{W}_b^T + \mathbf{V}_b^\dagger\mathbf{V}_b + \mathbf{X}_b^{1\dagger}\mathbf{X}_b^1 + \mathbf{X}_b^{2\dagger}\mathbf{X}_b^2)] \\
&\quad + 2[\mathbf{W}_b\mathbf{U}_b^{2\dagger}\mathbf{V}_b + \mathbf{V}_b^T\mathbf{U}_b^{2*}\mathbf{W}_b^T + \mathbf{X}_b^1\mathbf{X}_a^{2\dagger}\mathbf{X}_b^2 + \mathbf{X}_b^2\mathbf{X}_a^{2\dagger}\mathbf{X}_b^1] + \mathbf{X}_b^1 \operatorname{Tr}\{(\mathbf{U}_b^{1\dagger}\mathbf{U}_a^1 + \mathbf{U}_a^{2\dagger}\mathbf{U}_b^2) \\
&\quad + \tfrac{1}{2}(\mathbf{X}_b^{1\dagger}\mathbf{X}_a^1 + \mathbf{X}_a^{2\dagger}\mathbf{X}_b^2)\} + \mathbf{X}_b^2 \operatorname{Tr}\{(\mathbf{U}_b^{2\dagger}\mathbf{U}_a^1 + \mathbf{U}_a^{2\dagger}\mathbf{U}_b^1) + \tfrac{1}{2}(\mathbf{X}_b^{2\dagger}\mathbf{X}_a^1 + \mathbf{X}_a^{2\dagger}\mathbf{X}_b^1)\} \\
&\quad - 3g^2[\mathbf{X}_a^1\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{X}_a^1] + \tfrac{1}{2}[(\mathbf{W}_b\mathbf{U}_b^{2\dagger} + \mathbf{X}_b^1\mathbf{V}_b^\dagger)\mathbf{V}_a + \mathbf{V}_a^T(\mathbf{U}_b^{2*}\mathbf{W}_b^T + \mathbf{V}_b^*\mathbf{X}_b^1)] \\
&\quad + 2[\mathbf{W}_b\mathbf{W}_a^\dagger\mathbf{X}_b^1 + \mathbf{X}_b^1\mathbf{W}_a^*\mathbf{W}_b^T] + \mathbf{X}_b^1 \operatorname{Tr}\{\mathbf{W}_a^\dagger\mathbf{W}_b + \mathbf{V}_b^\dagger\mathbf{V}_a\},
\end{aligned} \tag{8}
$$

---

[3]Equation (5) is slightly modified from that in Ref. [14]. We have written the $Y_2^T(F)$ instead of $Y_2^\dagger(F)$ in the first term and symmetrized the trace with respect to a and b. The second modification also appears in Ref. [22], while the first one preserves the symmetry of the Yukawa coupling matrix.

$$(4\pi)^2\boldsymbol{\beta}_{X_a^2}|_{\text{1-loop}} = \tfrac{1}{2}[(\mathbf{W}_b\mathbf{W}_b^\dagger + \mathbf{V}_b^T\mathbf{V}_b^* + \mathbf{X}_b^1\mathbf{X}_b^{1\dagger} + \mathbf{X}_b^2\mathbf{X}_b^{2\dagger})\mathbf{X}_a^2 + \mathbf{X}_a^2(\mathbf{W}_b^*\mathbf{W}_b^T + \mathbf{V}_b^\dagger\mathbf{V}_b + \mathbf{X}_b^{1\dagger}\mathbf{X}_b^1 + \mathbf{X}_b^{2\dagger}\mathbf{X}_b^2)]$$
$$+ 2[\mathbf{W}_b\mathbf{U}_a^{1\dagger}\mathbf{V}_b + \mathbf{V}_b^T\mathbf{U}_a^{1*}\mathbf{W}_b^T + \mathbf{X}_b^1\mathbf{X}_a^{1\dagger}\mathbf{X}_b^2 + \mathbf{X}_b^2\mathbf{X}_a^{1\dagger}\mathbf{X}_b^1] + \mathbf{X}_b^2\,\text{Tr}\{(\mathbf{U}_b^{2\dagger}\mathbf{U}_a^2 + \mathbf{U}_a^{1\dagger}\mathbf{U}_b^1)$$
$$+ \tfrac{1}{2}(\mathbf{X}_b^{2\dagger}\mathbf{X}_a^2 + \mathbf{X}_a^{1\dagger}\mathbf{X}_b^1)\} + \mathbf{X}_b^1\,\text{Tr}\{(\mathbf{U}_b^{1\dagger}\mathbf{U}_a^2 + \mathbf{U}_a^{1\dagger}\mathbf{U}_b^2) + \tfrac{1}{2}(\mathbf{X}_b^{1\dagger}\mathbf{X}_a^2 + \mathbf{X}_a^{1\dagger}\mathbf{X}_b^2)\}$$
$$- 3g^2[\mathbf{X}_a^2\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{X}_a^2] + \tfrac{1}{2}[(\mathbf{V}_b^T\mathbf{U}_b^{1*} + \mathbf{X}_b^2\mathbf{W}_b^*)\mathbf{W}_a^T + \mathbf{W}_a(\mathbf{U}_b^{1\dagger}\mathbf{V}_b + \mathbf{W}_b^\dagger\mathbf{X}_b^2)]$$
$$+ 2[\mathbf{V}_b^T\mathbf{V}_a^*\mathbf{X}_b^2 + \mathbf{X}_b^2\mathbf{V}_a^\dagger\mathbf{V}_b] + \mathbf{X}_b^2\,\text{Tr}\{\mathbf{W}_b^\dagger\mathbf{W}_a + \mathbf{V}_a^\dagger\mathbf{V}_b\}. \tag{9}$$

The last three terms of each of (8) and (9) vanish in the R-parity conserving MSSM, again because $\mathbf{W}^a$ and $\mathbf{V}^a$ are zero. Finally,

$$(4\pi)^2\boldsymbol{\beta}_{V_a}|_{\text{1-loop}} = \tfrac{1}{2}[(\mathbf{U}_b^1\mathbf{U}_b^{1\dagger} + \mathbf{U}_b^2\mathbf{U}_b^{2\dagger} + \mathbf{V}_b\mathbf{V}_b^\dagger)\mathbf{V}_a + \mathbf{V}_a(\mathbf{W}_b^*\mathbf{W}_b^T + \mathbf{V}_b^\dagger\mathbf{V}_b + \mathbf{X}_b^{1\dagger}\mathbf{X}_b^1 + \mathbf{X}_b^{2\dagger}\mathbf{X}_b^2)]$$
$$+ 2[\mathbf{U}_b^1\mathbf{W}_a^\dagger\mathbf{X}_b^2 + \mathbf{U}_b^2\mathbf{W}_a^\dagger\mathbf{X}_b^1 + \mathbf{V}_b\mathbf{W}_a^*\mathbf{W}_b^T] + \mathbf{V}_b\,\text{Tr}\{\mathbf{W}_a^\dagger\mathbf{W}_b + \mathbf{V}_b^\dagger\mathbf{V}_a\} - 3g^2[\mathbf{V}_a\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{V}_a]$$
$$+ \tfrac{1}{2}[(\mathbf{U}_b^2\mathbf{W}_b^\dagger + \mathbf{V}_b\mathbf{X}_b^{1\dagger})(\mathbf{X}_a^1 + \mathbf{X}_a^2) + (\mathbf{U}_a^1 + \mathbf{U}_a^2)(\mathbf{U}_b^{1\dagger}\mathbf{V}_b + \mathbf{W}_b^\dagger\mathbf{X}_b^2)]$$
$$+ 2[\mathbf{U}_b^2(\mathbf{U}_a^{1\dagger} + \mathbf{U}_a^{2\dagger})\mathbf{V}_b + \mathbf{V}_b(\mathbf{X}_a^{1\dagger} + \mathbf{X}_a^{2\dagger})\mathbf{X}_b^2] + \mathbf{V}_b\,\text{Tr}\{[\mathbf{U}_b^{1\dagger}(\mathbf{U}_a^1 + \mathbf{U}_a^2) + (\mathbf{U}_a^{1\dagger} + \mathbf{U}_a^{2\dagger})\mathbf{U}_b^2]$$
$$+ \tfrac{1}{2}[\mathbf{X}_b^{1\dagger}(\mathbf{X}_a^1 + \mathbf{X}_a^2) + (\mathbf{X}_a^{1\dagger} + \mathbf{X}_a^{2\dagger})\mathbf{X}_b^2]\}, \tag{10}$$

$$(4\pi)^2\boldsymbol{\beta}_{W_a}|_{\text{1-loop}} = \tfrac{1}{2}[(\mathbf{W}_b\mathbf{W}_b^\dagger + \mathbf{V}_b^T\mathbf{V}_b^* + \mathbf{X}_b^1\mathbf{X}_b^{1\dagger} + \mathbf{X}_b^2\mathbf{X}_b^{2\dagger})\mathbf{W}_a + \mathbf{W}_a(\mathbf{U}_b^{1\dagger}\mathbf{U}_b^1 + \mathbf{U}_b^{2\dagger}\mathbf{U}_b^2 + \mathbf{W}_b^\dagger\mathbf{W}_b)]$$
$$+ 2[\mathbf{V}_b^T\mathbf{V}_a^*\mathbf{W}_b + \mathbf{X}_b^1\mathbf{V}_a^\dagger\mathbf{U}_b^2 + \mathbf{X}_b^2\mathbf{V}_a^\dagger\mathbf{U}_b^1] + \mathbf{W}_b\,\text{Tr}\{\mathbf{W}_b^\dagger\mathbf{W}_a + \mathbf{V}_a^\dagger\mathbf{V}_b\} - 3g^2[\mathbf{W}_a\mathbf{C}_2^L(F) + \mathbf{C}_2^R(F)\mathbf{W}_a]$$
$$+ \tfrac{1}{2}[(\mathbf{W}_b\mathbf{U}_b^{2\dagger} + \mathbf{X}_b^1\mathbf{V}_b^\dagger)(\mathbf{U}_a^1 + \mathbf{U}_a^2) + (\mathbf{X}_a^1 + \mathbf{X}_a^2)(\mathbf{V}_b^\dagger\mathbf{U}_b^1 + \mathbf{X}_b^{1\dagger}\mathbf{W}_b)]$$
$$+ 2[\mathbf{W}_b(\mathbf{U}_a^{1\dagger} + \mathbf{U}_a^{2\dagger})\mathbf{U}_b^1 + \mathbf{X}_b^1(\mathbf{X}_a^{1\dagger} + \mathbf{X}_a^{2\dagger})\mathbf{W}_b] + \mathbf{W}_b\,\text{Tr}\{[\mathbf{U}_b^{2\dagger}(\mathbf{U}_a^1 + \mathbf{U}_a^2) + (\mathbf{U}_a^{1\dagger} + \mathbf{U}_a^{2\dagger})\mathbf{U}_b^1]$$
$$+ \tfrac{1}{2}[\mathbf{X}_b^{2\dagger}(\mathbf{X}_a^1 + \mathbf{X}_a^2) + (\mathbf{X}_a^{1\dagger} + \mathbf{X}_a^{2\dagger})\mathbf{X}_b^1]\}. \tag{11}$$

The last three lines of each of (10) and (11) do not contribute in the R-parity conserving MSSM because each of $\mathbf{U}_a^1$, $\mathbf{U}_a^2$, $\mathbf{X}_a^1$, and $\mathbf{X}_a^2$ vanish for the values of a for which $\mathbf{V}_a$ and $\mathbf{W}_a$ do not.

The quantities $\mathbf{C}_2^L(F)$ and $\mathbf{C}_2^R(F)$ that appear in Eqs. (6)–(11) are the quadratic Casimirs $\mathbf{C}_2(F)$ separated into contributions from left-handed, right-handed, and Majorana fermions, so that

$$\mathbf{C}_2(F) = \text{diag}(\mathbf{C}_2^L(F), \mathbf{C}_2^R(F), \mathbf{C}_2^M(F)).$$

The traces that appear in (6)–(11) denote a sum over fermion types, which sometimes, but not necessarily, are traces over fermion flavors when we refer to the MSSM. Note also that, in the context of the MSSM, the index a may itself include a flavor (or even color) index, for instance when $\Phi_a = \tilde{c}_L$.

The reader may legitimately wonder what we have gained by writing the RGE which took the much simpler-looking form (5) in the much more cumbersome-looking and definitely much longer forms given by Eqs. (6)–(11). The reasons for doing so are twofold. First, many authors are more familiar with the MSSM couplings written in four-component notation, so that the longer four-component equations can be more directly used. Second, and perhaps the more important reason, is that *much of the work that would be needed to obtain the MSSM RGEs, even starting with the couplings in two-component notation, has*

already been done when the RGEs are written as in (6)–(11). The many terms that appear here are indeed present in the MSSM RGEs (as we will see), and starting with the compact notation of (5) only means that part of the work has to be repeated each time we use it to get the RGE that we need.

Before closing this section, we remark that the RGEs for the dimensionless gauge couplings and those for the dimensionless "Yukawa-like" couplings form a closed set, uncoupled to the RGEs for the dimensionful parameters. Dimensionless quartic couplings of scalars also do not enter these RGEs. Since we have made no assumption about supersymmetry, this is true even with threshold effects included.[4] In the next section, we turn to a discussion of how threshold effects are to be included in the MSSM RGEs. Of course, since we want to work to two-loop accuracy, it suffices to include the threshold effects only in the one-loop terms, which is the reason for our focus on the one-loop RGEs of a general quantum field theory. We mention here that the authors of Refs. [14,22], which is our starting point, use the $\overline{\text{MS}}$ renormalization scheme rather than the $\overline{\text{DR}}$ scheme better suited for the analysis of radiative corrections in supersymmetric theories. We can, however, use their general expressions (only

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[4]Of course, in this case, information about the SUSY spectrum enters via the location of the thresholds.

for the one-loop part) to obtain our RGEs, since there is no scheme dependence at this level. We can then augment these with the two-loop terms (without threshold corrections) of the MSSM RGEs [15] to obtain two-loop accuracy in the $\overline{\text{DR}}$ scheme.

III. PARTICLE DECOUPLING

We are now equipped to derive the RGEs for the dimensionless couplings in any quantum field theory with an arbitrary set of Dirac and Majorana spinor fields together with spin-zero fields (with nontrivial gauge quantum numbers) whose Lagrangian can be written in the form (3). This is, as we have already noted, not the most general form for the Lagrangian density since some field operators, such as mass terms for Dirac fermions or dimension four operators of the form $\bar{\Psi}_M P_L \Psi_D \Phi$ and $\bar{\Psi}_D P_L \Psi_M \Phi^\dagger$, that do not occur in the MSSM with R-parity conservation imposed have been omitted. Since (3) makes no reference to supersymmetry, we can continue to use it even when SUSY is broken via the inclusion of SSB terms. The procedure for obtaining the one-loop RGEs is now straightforward. We simply write down the Lagrangian density for the MSSM, including the SSB terms consistent with Lorentz and MSSM gauge symmetries and the assumed conservation of R parity, compare the dimensionless couplings with those in (3) to construct the matrices $\mathbf{U}_a^1$, $\mathbf{U}_a^2$, ..., $\mathbf{W}_a$, and then use (6)–(11) to get the required RGEs. This program has two complications that we must take care of in its implementation. Although we have referred to these in Sec. I, it does not hurt to repeat them here.

(i) For Q values above all SUSY thresholds, the effective theory is supersymmetric, and couplings of SM particles are related by SUSY to corresponding sparticle couplings, so that the particle and sparticle couplings evolve the same way. This is, however, not necessarily the case below the SUSY thresholds. Then, while the β function for, say, the hypercharge gauge coupling still depends only on the hypercharge gauge coupling (because of Ward identities, its one-loop evolution is governed by just the "vacuum polarization corrections"), the β functions for the corresponding "Yukawa-like" couplings $\tilde{\mathbf{g}}^{i\bar{f}}$ of the bino to the fermion-sfermion pair will, in general, depend on gauge as well as *all* "Yukawa" couplings, including those from superpotential interactions. There is no principle that precludes this once supersymmetry is broken. However, above all SUSY thresholds, this β function must reduce to that for the gauge coupling, so that all dependence on superpotential Yukawa couplings must cancel. This provides a nontrivial check on our formulas. Thus our system must be extended to include the couplings $\tilde{\mathbf{g}}_i^\Phi$ as well as the couplings $\tilde{\mathbf{f}}_i^\Phi$, the additional "Yukawa-like" couplings of gauginos and higgsinos, respec-

tively, which as we have noted become matrices in the flavor space.

(ii) To implement SUSY and Higgs particle thresholds, we need to have an idea of the spectrum as well as the couplings in the corresponding mass basis. Of course, the exact spectrum is only obtained after extracting the weak scale values of all SSB parameters, and then diagonalizing the various mass matrices. Since the positions of the thresholds only enter logarithmically in the solutions to the RGEs for the various gauge and "Yukawa-like" couplings, it suffices to include these in a reasonable approximation. Obviously, this entails rewriting the interactions in terms of the approximate mass basis for the fields. A discussion of how we do so forms the subject of the remainder of this section.

The mass basis and MSSM thresholds

In addition to the sparticles of the SM, the MSSM includes squarks and sleptons, gluinos, charginos, and neutralinos, and the additional spin-zero particles A, H, and $H^\pm$ in its Higgs sector. At a scale Q larger than the masses of all these particles, the RGEs are given by those of the MSSM, and as we reduce Q, these morph into those of effective theories that interpolate between the MSSM and the SM, and ultimately at the scale $Q \sim m_t$, reduce to those of the SM, assuming that all sparticles and additional Higgs bosons are significantly heavier than m_t. This will serve as another check of our formulas.

Assuming for the moment that $\tilde{f}_L$ and $\tilde{f}_R$ are also approximate sfermion mass eigenstates, i.e. that mixing among the sfermions is modest, decoupling the sfermions is straightforward. Simply exclude any sfermion field (and all its couplings) in the effective theory at a scale Q below its mass, and evaluate the RGEs using this truncated theory. The same is true for gluinos which, being color octets, cannot mix with any other fermion since $SU(3)_C$ is unbroken. The decoupling of charginos, neutralinos, and the additional Higgs bosons is more complicated because these are frequently strongly mixed.

Our procedure for decoupling these fields is guided by the observation that threshold effects that we are trying to evaluate are significant only if there are several well-separated mass scales M_i in the spectrum. If this is not the case, then we can decouple all the additional particles at a common scale M_{SUSY}, since any error would only be of $\mathcal{O}(\frac{1}{16\pi^2} \ln\frac{M_{\text{SUSY}}}{M_i})$, and so would become negligible as $M_i \to M_{\text{SUSY}}$.

In the Higgs boson sector, threshold effects are, therefore, important only if the mass scale of the additional bosons $\gg m_h$, in which case $m_A \simeq m_H \simeq m_{H^\pm} \gg m_h < m_t$. We can then decouple all the additional spin-zero bosons at a common scale that we take to be m_H, below which h plays the role of the SM Higgs boson. To imple-

ment the decoupling, we must rewrite the Lagrangian couplings in terms of the (approximate) mass eigenstates contained in the $SU(2)$ doublets[5]

$$\begin{pmatrix} G^+ \\ \mathsf{h} \end{pmatrix} = s \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix} + c \begin{pmatrix} h_d^{-*} \\ h_d^{0*} \end{pmatrix}, \qquad (12)$$

$$\begin{pmatrix} H^+ \\ \mathcal{H} \end{pmatrix} = c \begin{pmatrix} h_u^+ \\ h_u^0 \end{pmatrix} - s \begin{pmatrix} h_d^{-*} \\ h_d^{0*} \end{pmatrix}, \qquad (13)$$

where the electrically neutral, *complex* fields h and $\mathcal{H}$ are given by

$$\mathsf{h} = \frac{h + iG^0}{\sqrt{2}}, \qquad \mathcal{H} = \frac{-H + iA}{\sqrt{2}},$$

and $s = \sin\beta$ and $c = \cos\beta$. Above $Q = m_H$, the theory includes two Higgs doublets which we may take to be h_u and h_d, or *any two orthogonal* combinations of these. However, at $Q = m_H$, it is specifically the doublet (13) that decouples, and the effective theory includes just one spin-zero doublet which couples to quarks and leptons via Yukawa coupling matrices $\boldsymbol{\lambda}_u = \mathbf{f}_u \sin\beta$, $\boldsymbol{\lambda}_d = \mathbf{f}_d \cos\beta$, and $\boldsymbol{\lambda}_e = \mathbf{f}_e \cos\beta$, with these relations imposed at $Q = m_H$. For $Q < m_H$, we will be interested in the evolution of the $\boldsymbol{\lambda}_i$'s, which are the couplings of the doublet (12) that remains in the theory. We stress again that supersymmetry is broken, and the decoupling of the second doublet makes no statement about the decoupling of the corresponding higgsinos. The effective theory below $Q = m_H$ may or may not contain higgsinos, depending on, as we will see next, the value of μ.

In the chargino-neutralino sector, threshold corrections are small if $|\mu|$ and the SSB gaugino masses all have a magnitude $\sim M_Z$, since their effects will then persist down to the weak scale. Threshold effects are significant only when at least one of these is hierarchically larger than the weak scale, in which case gaugino-higgsino mixing effects are small, and the gaugino-higgsino mass eigenstates can be approximated by the bino, the wino-triplet, and the two Majorana higgsino doublet fields,[6]

$$\tilde{h}_{1,2} = \frac{\psi_{h_d} \mp \psi_{h_u}}{\sqrt{2}}. \qquad (14)$$

To implement decoupling, we must rewrite interactions of gauginos and higgsinos in the basis consisting of bino, winos, and these two higgsino doublets. Above the scale of the higgsinos, we could have worked with any two orthogonal linear combinations of the higgsino doublets, whereas below $Q = |\mu|$, both higgsinos decouple from the effective theory. We note here that we have to perform different rotations in the spin-zero and spin-$\frac{1}{2}$ Higgs sectors to go to the mass basis. This appears to be different[7] from what has been done in the literature [19,20]. There is one additional complication that enters when we rewrite the Lagrangian density in the (approximate) "mass basis." Some of the fermion fields have negative eigenvalues, and we are led to make field redefinitions [3] $\Psi_k \rightarrow (i\gamma_5)^{\theta_k}\Psi_k$, where $\theta_k = 0(1)$ if the corresponding mass eigenvalue is positive (negative), on Majorana spinors to get their masses in canonical form. While this introduces additional γ_5's and phases in the interactions, we have checked that the RGEs are independent of these, as may be expected since these RGEs do not depend on the masses.

Finally, let us return to mixing effects in the sfermion sector. We have ignored these in our analysis. For the squarks of the first two generations, flavor-physics constraints typically restrict the size of intergeneration mixing effects (in the basis where the corresponding quark Yukawa couplings are diagonal). There is, however, no principle that dictates intrageneration mixing to be small, even though this is usually assumed to be the case for first and second generation fermions in many models, while third generation sfermions are expected to mix via their Yukawa couplings.[8] This said, barring large, accidental cancellations, we would expect that the physical sfermion masses are approximately of the same order, in which case threshold effects from intrageneration mixing are not important, or the diagonal SSB terms are themselves hierarchical so that the unmixed sfermions $\tilde{f}_L$ and $\tilde{f}_R$ are approximate mass eigenstates, and can be decoupled at their physical masses.

IV. APPLICATION TO THE MSSM

A. Interactions

We now use the RGEs for the general quantum field theory that we obtained in Sec. II to derive the RGEs for the dimensionless couplings of the R-parity conserving MSSM. In the notation of Ref. [3], the superpotential for the MSSM is given by

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[5]We use the notation of Ref. [3] where the left-chiral up-type Higgs superfield $\hat{H}_u$ transforms as a **2** of $SU(2)$, while the left-chiral down-type Higgs superfield transforms as a **2***. The doublets that make up the linear combinations in (12) and (13) both transform as a **2** of $SU(2)$ and have a positive weak hypercharge.

[6]ψ_{h_u} and ψ_{h_d} are the fermionic components of the $SU(2)$ doublet left-chiral superfields $\hat{H}_u$ and $\hat{H}_d$. Following Ref. [3], we denote the charged and neutral components of ψ_{h_u} by $(\psi_{h_u^+}, \psi_{h_u^0})$ and of ψ_{h_d} by $(\psi_{h_d^-}, \psi_{h_d^0})$. $\psi_{h_u^+}$ is a Majorana spinor whose left-chiral component destroys positively charged higgsinos, while its right-chiral component destroys negatively charged higgsinos, and similarly for $\psi_{h_d^-}$.

[7]We mention though that, because we choose to decouple both higgsinos at the common scale $|\mu|$, this rotation of the higgsino fields is irrelevant for practical purposes.

[8]Substantial intragenerational mixing may occur via the trilinear sfermion-fermion-Higgs boson SSB parameter, which is, after all, independent of the corresponding superpotential Yukawa coupling.

$$\hat{f} = \tilde{\mu}\hat{H}_u^a\hat{H}_{da} + (\mathbf{f}_u)_{ij}\epsilon_{ab}\hat{Q}_i^a\hat{H}_u^b\hat{U}_j^c + (\mathbf{f}_d)_{ij}\hat{Q}_i^a\hat{H}_{da}\hat{D}_j^c + (\mathbf{f}_e)_{ij}\hat{L}_i^a\hat{H}_{da}\hat{E}_j^c, \tag{15}$$

while the SSB terms may be written as

$$
\begin{aligned}
\mathcal{L}_{\text{SSB}} = &-\{\tilde{u}_{Lk}^\dagger(\mathbf{m}_Q^2)_{kl}\tilde{u}_{Ll} + \tilde{d}_{Lk}^\dagger(\mathbf{m}_Q^2)_{kl}\tilde{d}_{Ll} + \tilde{u}_{Rk}^\dagger(\mathbf{m}_U^2)_{kl}\tilde{u}_{Rl} + \tilde{d}_{Rk}^\dagger(\mathbf{m}_D^2)_{kl}\tilde{d}_{Rl} + \tilde{\nu}_{Lk}^\dagger(\mathbf{m}_L^2)_{kl}\tilde{\nu}_{Ll} + \tilde{e}_{Lk}^\dagger(\mathbf{m}_L^2)_{kl}\tilde{e}_{Ll} \\
&+ \tilde{e}_{Rk}^\dagger(\mathbf{m}_E^2)_{kl}\tilde{e}_{Rl} + m_{H_u}^2 h_u^{0\dagger}h_u^0 + m_{H_u}^2 h_u^{+\dagger}h_u^+ + m_{H_d}^2 h_d^{0\dagger}h_d^0 + m_{H_d}^2 h_d^{-\dagger}h_d^-\} + \{\tilde{u}_{Rk}^\dagger(\mathbf{a}_u^T)_{kl}\tilde{u}_{Ll}h_u^0 - \tilde{u}_{Rk}^\dagger(\mathbf{a}_u^T)_{kl}\tilde{d}_{Ll}h_u^+ \\
&+ \tilde{d}_{Rk}^\dagger(\mathbf{a}_d^T)_{kl}\tilde{d}_{Ll}h_d^0 + \tilde{d}_{Rk}^\dagger(\mathbf{a}_d^T)_{kl}\tilde{u}_{Ll}h_d^- + \tilde{e}_{Rk}^\dagger(\mathbf{a}_e^T)_{kl}\tilde{\nu}_{Ll}h_d^- + \tilde{e}_{Rk}^\dagger(\mathbf{a}_e^T)_{kl}\tilde{e}_{Ll}h_d^0 + \text{H.c.}\} \\
&- \frac{1}{2}[M_1\bar{\lambda}_0\lambda_0 + M_2\bar{\lambda}_p\lambda_p + M_3\bar{\tilde{g}}_A\tilde{g}_A] - \frac{i}{2}[M_1'\bar{\lambda}_0\gamma_5\lambda_0 + M_2'\bar{\lambda}_p\gamma_5\lambda_p + M_3'\bar{\tilde{g}}_A\gamma_5\tilde{g}_A] + \{b(h_u^+h_d^- + h_u^0h_d^0) + \text{H.c.}\},
\end{aligned}
\tag{16}
$$

where the matrix indices $i, j, \cdots$ label the (s)particle flavors. The sfermion mass squared matrices $\mathbf{m}_Q^2, \mathbf{m}_U^2, \ldots$ are Hermitian, whereas the Yukawa coupling matrices $\mathbf{f}_{u,d,e}$ that appear in the superpotential, as well as the trilinear SSB scalar coupling matrices $\mathbf{a}_{u,d,e}$, have no hermiticity or symmetry property. Finally, the gaugino mass parameters M_a and M_a' are real, while the SSB Higgs mass parameter may, in general, be complex. Since the RGEs for the dimensionless couplings do not depend on any dimensionful parameters, we do not really need $\mathcal{L}_{\text{SSB}}$ for their derivation. The SSB parameters only enter the analysis in that they determine the positions of the various thresholds.

In order to be able to compare our results (above all thresholds) with the standard results already in the litera-ture [15–18], we need the following conversion between the notation used here and that in Ref. [15]: $\mathbf{f} \equiv \mathbf{Y}^T$, $\mathbf{a} \equiv -\mathbf{h}^T$, $b \equiv -B$. Moreover, the *complex* gaugino mass parameters (M_a) in Ref. [15] are equivalent to $M_a - iM_a'$ in (16).

It is clear that, in order to obtain the RGEs, we need to extract the various matrices $\mathbf{U}_\Phi^{1,2}$, $\mathbf{X}_\Phi^{1,2}$, $\mathbf{V}_\Phi$, and $\mathbf{W}_\Phi$, whose elements are the couplings between the various scalars, Φ, and fermions of the MSSM. Thus fermion-fermion-scalar interactions play an important role in our derivation. The interactions of quarks and squarks with Higgs bosons and higgsinos may be obtained from the superpotential, and are given by

$$
\begin{aligned}
\mathcal{L} \ni &-[\bar{u}_j(\mathbf{f}_u)_{ji}^Th_u^0P_Lu_i - \bar{u}_j(\mathbf{f}_u)_{ji}^Th_u^+P_Ld_i + \bar{d}_j(\mathbf{f}_d)_{ji}^Th_d^-P_Lu_i + \bar{d}_j(\mathbf{f}_d)_{ji}^Th_d^0P_Ld_i + \bar{e}_j(\mathbf{f}_e)_{ji}^Th_d^-P_L\nu_i + \bar{e}_j(\mathbf{f}_e)_{ji}^Th_d^0P_Le_i + \text{H.c.}] \\
&- [\bar{\Psi}_{h_u^0}\tilde{u}_{Rj}^\dagger(\tilde{\mathbf{f}}_u^{u_R})_{ji}^TP_Lu_i + \bar{u}_j(\tilde{\mathbf{f}}_u^Q)_{ji}^T\tilde{u}_{Li}P_L\Psi_{h_u^0} - \bar{\Psi}_{h_u^+}\tilde{u}_{Rj}^\dagger(\tilde{\mathbf{f}}_u^{u_R})_{ji}^TP_Ld_i - \bar{u}_j(\tilde{\mathbf{f}}_u^Q)_{ji}^T\tilde{d}_{Li}P_L\Psi_{h_u^+} + \bar{\Psi}_{h_d^-}\tilde{d}_{Rj}^\dagger(\tilde{\mathbf{f}}_d^{d_R})_{ji}^TP_Lu_i \\
&+ \bar{d}_j(\tilde{\mathbf{f}}_d^Q)_{ji}^T\tilde{u}_{Li}P_L\Psi_{h_d^-} + \bar{\Psi}_{h_d^0}\tilde{d}_{Rj}^\dagger(\tilde{\mathbf{f}}_d^{d_R})_{ji}^TP_Ld_i + \bar{d}_j(\tilde{\mathbf{f}}_d^Q)_{ji}^T\tilde{d}_{Li}P_L\Psi_{h_d^0} + \bar{\Psi}_{h_d^-}\tilde{e}_{Rj}^\dagger(\tilde{\mathbf{f}}_e^{e_R})_{ji}^TP_L\nu_i + \bar{e}_j(\tilde{\mathbf{f}}_e^L)_{ji}^T\tilde{\nu}_{Li}P_L\Psi_{h_d^-} \\
&+ \bar{\Psi}_{h_d^0}\tilde{e}_{Rj}^\dagger(\tilde{\mathbf{f}}_e^{e_R})_{ji}^TP_Le_i + \bar{e}_j(\tilde{\mathbf{f}}_e^L)_{ji}^T\tilde{e}_{Li}P_L\Psi_{h_d^0} + \text{H.c.}].
\end{aligned}
\tag{17}
$$

We have written these interactions with a left-chiral projector sandwiched between the four-component spinors to facilitate comparison with Eq. (3). Notice that the couplings of matter fermions to Higgs bosons are denoted the same way as the superpotential Yukawa matrices. In contrast, we have denoted the couplings of the fermion-sfermion system by a matrix with a tilde on it, labeled by the scalar that enters the corresponding interaction. Above the scale of all particles and Higgs boson thresholds, the effective theory is the MSSM and these $\tilde{\mathbf{f}}^\Phi$ couplings are the same as the corresponding superpotential couplings $\mathbf{f}$, but may be different below this scale. For instance, if $|\mu| < m_H$, the low energy theory just below $Q = m_H$ contains two higgsinos, but just the one scalar doublet in (12). The effective theory below $Q = m_H$ then contains only the couplings $\boldsymbol{\lambda}_{u,d,e}$ along with the couplings $\tilde{\mathbf{f}}_{u,d,e}^\Phi$, but no $\mathbf{f}_{u,d,e}$. As we will see, the evolution of $\tilde{\mathbf{f}}_{u,d,e}^\Phi$ differs from the "would-have-been" evolution of $\mathbf{f}_{u,d,e}$ for this range of Q.

The remaining fermion-fermion-scalar interactions are those with the gauginos, and are given by

$$
\begin{aligned}
\mathcal{L} \ni &-\frac{1}{\sqrt{2}}\left\{(\tilde{u}_{Lj}^\dagger, \tilde{d}_{Lj}^\dagger)\mathbf{G}_QP_L\begin{pmatrix}u_i\\d_i\end{pmatrix} + (\tilde{\nu}_{Lj}^\dagger, \tilde{e}_{Lj}^\dagger)\mathbf{G}_LP_L\begin{pmatrix}\nu_i\\e_i\end{pmatrix} + \bar{u}_j(\tilde{\mathbf{g}}'^{u_R})_{ji}\left(-\frac{4}{3}\right)\tilde{u}_{Ri}P_L\lambda_0 + \bar{d}_j(\tilde{\mathbf{g}}'^{d_R})_{ji}\left(\frac{2}{3}\right)\tilde{d}_{Ri}P_L\lambda_0\right. \\
&+ \left.\bar{e}_j(\tilde{\mathbf{g}}'^{e_R})_{ji}(2)\tilde{e}_{Ri}P_L\lambda_0 + \text{H.c.}\right\} - \sqrt{2}\left\{(\tilde{\mathbf{g}}_s^{\tilde{q}_L})_{ji}(-i)^{\theta_{\tilde{g}}}\tilde{q}_{Lj}^\dagger\bar{\tilde{g}}_A\frac{\lambda_A}{2}P_Lq_i - (\tilde{\mathbf{g}}_s^{\tilde{q}_R})_{ji}(-i)^{\theta_{\tilde{g}}}\bar{q}_j\frac{\lambda_A}{2}P_L\tilde{g}_A\tilde{q}_{Ri} + \text{H.c.}\right\} \\
&-\frac{1}{\sqrt{2}}\left\{(h_u^{+\dagger}, h_u^{0\dagger})\mathbf{G}_{h_u}P_L\begin{pmatrix}\Psi_{h_u^+}\\\Psi_{h_u^0}\end{pmatrix} + (h_d^{-\dagger}, h_d^{0\dagger})\mathbf{G}_{h_d}P_L\begin{pmatrix}\Psi_{h_d^-}\\\Psi_{h_d^0}\end{pmatrix} + \text{H.c.}\right\},
\end{aligned}
\tag{18}
$$

where we have defined the following matrices:

$$\mathbf{G}_Q = \begin{pmatrix} (\tilde{\mathbf{g}}^Q)_{ji}\bar{\lambda}_3 + \frac{1}{3}(\tilde{\mathbf{g}}'^Q)_{ji}\bar{\lambda}_0 & (\tilde{\mathbf{g}}^Q)_{ji}(\bar{\lambda}_1 - i\bar{\lambda}_2) \\ (\tilde{\mathbf{g}}^Q)_{ji}(\bar{\lambda}_1 + i\bar{\lambda}_2) & -(\tilde{\mathbf{g}}^Q)_{ji}\bar{\lambda}_3 + \frac{1}{3}(\tilde{\mathbf{g}}'^Q)_{ji}\bar{\lambda}_0 \end{pmatrix},$$

(19)

$$\mathbf{G}_L = \begin{pmatrix} (\tilde{\mathbf{g}}^L)_{ji}\bar{\lambda}_3 - (\tilde{\mathbf{g}}'^L)_{ji}\bar{\lambda}_0 & (\tilde{\mathbf{g}}^L)_{ji}(\bar{\lambda}_1 - i\bar{\lambda}_2) \\ (\tilde{\mathbf{g}}^L)_{ji}(\bar{\lambda}_1 + i\bar{\lambda}_2) & -(\tilde{\mathbf{g}}^L)_{ji}\bar{\lambda}_3 - (\tilde{\mathbf{g}}'^L)_{ji}\bar{\lambda}_0 \end{pmatrix},$$

(20)

$$\mathbf{G}_{h_u} = \begin{pmatrix} \tilde{g}^{h_u}\bar{\lambda}_3 + \tilde{g}'^{h_u}\bar{\lambda}_0 & \tilde{g}^{h_u}(\bar{\lambda}_1 - i\bar{\lambda}_2) \\ \tilde{g}^{h_u}(\bar{\lambda}_1 + i\bar{\lambda}_2) & -\tilde{g}^{h_u}\bar{\lambda}_3 + \tilde{g}'^{h_u}\bar{\lambda}_0 \end{pmatrix},$$

(21)

$$\mathbf{G}_{h_d} = \begin{pmatrix} -\tilde{g}^{h_d}\bar{\lambda}_3 - \tilde{g}'^{h_d}\bar{\lambda}_0 & \tilde{g}^{h_d}(-\bar{\lambda}_1 - i\bar{\lambda}_2) \\ \tilde{g}^{h_d}(-\bar{\lambda}_1 + i\bar{\lambda}_2) & \tilde{g}^{h_d}\bar{\lambda}_3 - \tilde{g}'^{h_d}\bar{\lambda}_0 \end{pmatrix}.$$

(22)

Here, λ_0 and λ_p denote the (Majorana) hypercharge and $SU(2)$ gauginos. Just as the higgsino couplings in (17) may differ from the corresponding Higgs boson couplings, below the various thresholds the effective theory is no longer supersymmetric, and the couplings of gauginos may deviate from the corresponding gauge boson couplings. Moreover, unlike gauge couplings, the renormalization of these fermion-fermion-scalar interactions depends also on the couplings $\mathbf{f}$ and $\tilde{\mathbf{f}}^\Phi$, so that the $\tilde{\mathbf{g}}^\Phi$'s can acquire a nontrivial flavor structure, as allowed for in (18) above. Above the scale of the highest thresholds, however, the RGEs for these $\tilde{\mathbf{g}}^\Phi$-type couplings must coincide with those for the gauge couplings (in the matrix sense).

B. Gauge coupling RGEs

The one-loop RGEs for gauge couplings (including threshold effects) are well known. We will, however, write these for completeness. As explained in Sec. I, we will decouple particles using the "step approximation" [19,20]. Toward this end, for any particle $\mathcal{P}$, we define

$$\theta_\mathcal{P} = 1 \quad \text{if } Q > M_\mathcal{P},$$
$$0 \quad \text{if } Q < M_\mathcal{P}.$$

Throughout this paper, we will assume that $Q > m_t$ so that the $SU(2)$ gauge symmetry is unbroken in the effective theory, and that there are three matter generations. Applying (1) for the MSSM particle content, and using

$$N_{\tilde{f}} = \sum_{i=1}^3 \theta_{\tilde{f}_i},$$

leads to the familiar gauge coupling RGEs,

$$(4\pi)^2\beta_{g_1}|_{1\text{-loop}} = g_1^3\left[4 + \frac{1}{30}N_{\tilde{Q}} + \frac{4}{15}N_{\tilde{u}_R} + \frac{1}{15}N_{\tilde{d}_R}\right.$$
$$+ \frac{1}{10}N_{\tilde{L}} + \frac{1}{5}N_{\tilde{e}_R} + \frac{1}{10}(\theta_h + \theta_H)$$
$$\left. + \frac{1}{5}(\theta_{\tilde{h}_1} + \theta_{\tilde{h}_2})\right],$$

(23)

$$(4\pi)^2\beta_g|_{1\text{-loop}} = g^3\left[-\frac{22}{3} + 4 + \frac{1}{2}N_{\tilde{Q}} + \frac{1}{6}N_{\tilde{L}}\right.$$
$$\left. + \frac{1}{6}(\theta_h + \theta_H) + \frac{1}{3}(\theta_{\tilde{h}_1} + \theta_{\tilde{h}_2}) + \frac{4}{3}\theta_{\tilde{W}}\right],$$

(24)

$$(4\pi)^2\beta_{g_s}|_{1\text{-loop}} = g_s^3\left[4 + \frac{1}{3}N_{\tilde{Q}} + \frac{1}{6}N_{\tilde{u}_R} + \frac{1}{6}N_{\tilde{d}_R}\right.$$
$$\left. + 2\theta_{\tilde{g}} - 11\right].$$

(25)

Here, g_1 is the scaled hypercharge gauge coupling that unifies with the $SU(2)$ and $SU(3)$ couplings when the MSSM is embedded in a GUT: $g'^2 = \frac{3}{5}g_1^2$. We mention that, although we have shown two distinct higgsino thresholds above, since we are working in the approximation that both higgsinos have the mass $|\mu|$, we will, from now on, write

$$\theta_{\tilde{h}} = \theta_{\tilde{h}_1} = \theta_{\tilde{h}_2}.$$

C. RGEs for Yukawa and Yukawa-type couplings

Turning now to the RGEs for fermion-fermion-scalar couplings, we see that there is a large number of such "Yukawa-type" couplings in the Lagrangian. These are as follows: the usual Yukawa couplings to Higgs bosons, $\mathbf{f}_u$, $\mathbf{f}_d$, and $\mathbf{f}_e$; the couplings of higgsinos to the various fermion-sfermion pairs, $\tilde{\mathbf{f}}_u^Q$, $\tilde{\mathbf{f}}_d^Q$, $\tilde{\mathbf{f}}_e^L$, $\tilde{\mathbf{f}}_u^{u_R}$, $\tilde{\mathbf{f}}_d^{d_R}$, $\tilde{\mathbf{f}}_e^{e_R}$; hypercharge gaugino couplings, $\tilde{\mathbf{g}}'^Q$, $\tilde{\mathbf{g}}'^L$, $\tilde{\mathbf{g}}'^{u_R}$, $\tilde{\mathbf{g}}'^{d_R}$, $\tilde{\mathbf{g}}'^{e_R}$, $\tilde{g}'^{h_u}$, $\tilde{g}'^{h_d}$; the $SU(2)$ gaugino couplings, $\tilde{\mathbf{g}}^Q$, $\tilde{\mathbf{g}}^L$, $\tilde{g}^{h_u}$, $\tilde{g}^{h_d}$; and finally, the gluino couplings, $\tilde{\mathbf{g}}_s^Q$, $\tilde{\mathbf{g}}_s^{u_R}$, $\tilde{\mathbf{g}}_s^{d_R}$. While the RGEs for the gauge couplings still form a closed set, those for the usual matter fermion Yukawa couplings do not. It is only the RGEs for *all* these fermion-fermion-scalar couplings that, together with those for the gauge couplings, form a closed system. These new couplings must coincide with the corresponding gauge or usual Yukawa coupling in the SUSY limit, i.e. above the scale of the highest threshold. This serves as the boundary condition for the RGEs for these new couplings.

To implement decoupling of particles as described in Sec. III, we must construct the matrices $\mathbf{U}_\Phi^1$, $\mathbf{U}_\Phi^2$, $\mathbf{V}_\Phi$, $\mathbf{W}_\Phi$, $\mathbf{X}_\Phi^1$, and $\mathbf{X}_\Phi^2$, for all Φ, in the (approximate) mass basis for the various fermion fields. We choose this basis to com-

prise the Dirac fermions $\{u_i, d_i, \nu_i, e_i\}$ together with the Majorana fermions $\{\tilde{h}_1^0, \tilde{h}_2^0, \tilde{h}_1^\pm, \tilde{h}_2^\pm, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \tilde{g}_A\}$. Here, $\tilde{h}_{1,2}^0$ and $\tilde{h}_{1,2}^\pm$ are the neutral and charged components of the Majorana spinor $\tilde{h}_{1,2}$ introduced in (14).[9] The subscript i (which runs from 1 to 3) for the Dirac fermions is a flavor index which can be suppressed if the Yukawa terms are written as matrices in flavor space. This means that, in the MSSM, $\mathbf{U}_\Phi^1$ and $\mathbf{U}_\Phi^2$ will be (4×4) blocks of (3×3) matrices when Φ is one of the Higgs bosons in (12) or (13). Similarly, since flavor is carried by the sfermion scalar index, $\Phi = \tilde{f}_i$, $\mathbf{V}_{\tilde{f}_i}$ will be a (4×9) matrix where the number of rows can be further expanded to show each of the three matter fermion flavors. Likewise, $\mathbf{W}_{\tilde{f}_i}$ is a (9×4) matrix where now the number of columns can be similarly expanded to exhibit these flavors. Thus, when fully written out, $\mathbf{V}_{\tilde{f}_i}$ is a (12×9) matrix, while $\mathbf{W}_{\tilde{f}_i}$ is a (9×12) matrix; see (29)–(32). Finally, $\mathbf{X}_\Phi^1$ and $\mathbf{X}_\Phi^2$ will both be (9×9) matrices.

Within the MSSM with R-parity conservation, the matrices $\mathbf{U}_\Phi^{1,2}$ and $\mathbf{X}_\Phi^{1,2}$ are nonzero only for $\Phi = \{h, \mathcal{H}, G^+, H^+\}$. They can be readily worked out by comparing (3) with the MSSM Yukawa interactions in (17). It would take up too much space to display each of these matrices explicitly, so we will only present one example for each type of matrix. If $\Phi = h$, we have

$$\mathbf{U}_h^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & c\mathbf{f}_d^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & c\mathbf{f}_e^T \end{pmatrix}, \qquad (26)$$

where we use a bold-type $\mathbf{0}$ to indicate a (3×3) matrix in flavor space whose entries are all zero. The corresponding $\mathbf{U}_h^1$ matrix has its only nonvanishing entry in the $(1,1)$ block. The matrix $\mathbf{X}_h^2$ takes the form

$$\mathbf{X}_h^2 = \frac{1}{2} \begin{pmatrix} \mathbf{0}_{(4\times4)} & \boldsymbol{\chi}_h^2 \\ \boldsymbol{\chi}_h^{2T} & \mathbf{0}_{(5\times5)} \end{pmatrix}, \qquad (27)$$

where

$$\boldsymbol{\chi}_h^2 = \begin{pmatrix} -s\tilde{g}'^{h_u} & 0 & 0 & s\tilde{g}^{h_u} & 0 \\ s\tilde{g}'^{h_u} & 0 & 0 & -s\tilde{g}^{h_u} & 0 \\ 0 & -s\tilde{g}^{h_u} & -is\tilde{g}^{h_u} & 0 & 0 \\ 0 & s\tilde{g}^{h_u} & is\tilde{g}^{h_u} & 0 & 0 \end{pmatrix}. \qquad (28)$$

The eight $\mathbf{X}_\Phi^1$ and $\mathbf{X}_\Phi^2$ matrices can be obtained from (18) in conjunction with (21) and (22). The reader will need to

write these as symmetric matrices using the symmetry properties of Majorana spinor bilinears. Note that all the entries in the last row and last column of these matrices are zero because the Higgs boson has no coupling to gluinos.[10] It is also clear from the MSSM Lagrangian that $\mathbf{X}_\Phi^1$ contains only the $\tilde{g}'^{h_d}$ and $\tilde{g}^{h_d}$ couplings, while $\mathbf{X}_\Phi^2$ contains only $\tilde{g}'^{h_u}$ and $\tilde{g}^{h_u}$.

The $\mathbf{V}_\Phi$ and $\mathbf{W}_\Phi$ matrices, on the other hand, are nonzero only for $\Phi = \{\tilde{u}_L, \tilde{d}_L, \tilde{e}_L, \tilde{\nu}_L, \tilde{u}_R, \tilde{d}_R, \tilde{e}_R\}$. From the interactions in (17) and (18), we can see that for left-handed sfermions, $\mathbf{V}_{\tilde{f}}$ contains just $\tilde{\mathbf{f}}^\Phi$-type couplings while $\mathbf{W}_{\tilde{f}}$ contains only $\tilde{\mathbf{g}}^\Phi$-type couplings. The opposite is the case for the right-handed sfermions. In the case where $\tilde{f} = \tilde{u}_{Li}$, we have

$$\mathbf{V}_{\tilde{u}_{Li}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{V}_{\tilde{u}_{Li}} & \mathbf{0}_{(6\times5)} \\ \mathbf{0}_{(6\times4)} & \mathbf{0}_{(6\times5)} \end{pmatrix}, \qquad (29)$$

with

$$\mathcal{V}_{\tilde{u}_{Li}} = \begin{pmatrix} -(\tilde{\mathbf{f}}_u^Q)_{\bullet i}^T & (\tilde{\mathbf{f}}_u^Q)_{\bullet i}^T & \mathbf{0}_{(3\times1)} & \mathbf{0}_{(3\times1)} \\ \mathbf{0}_{(3\times1)} & \mathbf{0}_{(3\times1)} & (\tilde{\mathbf{f}}_d^Q)_{\bullet i}^T & (\tilde{\mathbf{f}}_d^Q)_{\bullet i}^T \end{pmatrix}. \qquad (30)$$

As alluded to above, since the scalar labels on $\mathbf{V}_{\tilde{f}}$ and $\mathbf{W}_{\tilde{f}}$ carry *sfermion* flavor, we need to be careful when writing these if we choose not to expand out the *matter fermion* flavor. Here we denote this suppressed quark flavor index by the $\bullet$, so that $(\tilde{\mathbf{f}}_u^Q)_{\bullet i}^T$ is really a (3×1) column matrix (since the index i is fixed by the squark flavor). Similarly

$$\mathbf{W}_{\tilde{u}_{Li}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{0}_{(4\times6)} & \mathbf{0}_{(4\times6)} \\ \mathcal{W}_{\tilde{u}_{Li}} & \mathbf{0}_{(5\times6)} \end{pmatrix}, \qquad (31)$$

with

$$\mathcal{W}_{\tilde{u}_{Li}} = \begin{pmatrix} \frac{1}{3}(\tilde{\mathbf{g}}'^Q)_{i\bullet} & \mathbf{0}_{(1\times3)} \\ \mathbf{0}_{(1\times3)} & (\tilde{\mathbf{g}}^Q)_{i\bullet} \\ \mathbf{0}_{(1\times3)} & -i(\tilde{\mathbf{g}}^Q)_{i\bullet} \\ (\tilde{\mathbf{g}}^Q)_{i\bullet} & \mathbf{0}_{(1\times3)} \\ 2(\frac{\lambda_A}{2})(\tilde{\mathbf{g}}_s^Q)_{i\bullet} & \mathbf{0}_{(1\times3)} \end{pmatrix}. \qquad (32)$$

Notice that the positions of the suppressed fermion index and the (fixed) sfermion flavor index, i, are swapped between the matrices (30) and (32). Once again, we have suppressed the color index on the quark (and squark) field. Also note that the couplings to the gluino enter only via the $\mathbf{W}_{\tilde{f}}$ matrix for $\tilde{q}_L$, and only via the $\mathbf{V}_{\tilde{f}}$ matrix for $\tilde{q}_R$, for all flavors of squarks.

We are now ready to proceed with the derivation of the RGEs. Before we begin our real derivation, we perform a pedagogical exercise to facilitate comparison with earlier literature and also provide checks on our procedure. Since earlier studies did not make the distinction between couplings of quarks to gauge and Higgs bosons (or g's and $\mathbf{f}$'s)

---

[9]As already mentioned, since we are working with a common threshold for higgsinos, we could have stayed in the original basis. We will see below that performing this rotation (by an arbitrary angle) gives us another check on our procedure. We also draw attention to the fact that we have combined the higgsinos of the **2** and **2*** representations, so that although $\tilde{h}_{1,2}^\pm$ are Majorana spinors, their chiral components do not annihilate a definite sign of the charge.

---

[10]We have suppressed the color index which would otherwise expand the last row/column.

and the corresponding couplings ($\tilde{\mathbf{g}}$'s and $\tilde{\mathbf{f}}$'s) of the quark-squark system to the gauginos and higgsinos, we first derive the RGE for ($s\mathbf{f}_u$) without separating out $\tilde{\mathbf{f}}$'s and $\tilde{\mathbf{g}}$'s from $\mathbf{f}$'s and g's, respectively. It is also instructive to illustrate the additional complexity that results from keeping distinct thresholds for A, H, and $H^\pm$, and an arbitrary rotation angle for the higgsinos instead of the 45° rotation used in (14); i.e.

$$\Psi_{\tilde{h}_1^0} = (i\gamma_5)^{\theta_1}(c'\Psi_{h_d^0} - s'\Psi_{h_u^0}),$$
$$\Psi_{\tilde{h}_2^0} = (i\gamma_5)^{\theta_2}(s'\Psi_{h_d^0} + c'\Psi_{h_u^0}), \qquad (33)$$

and

$$\Psi_{\tilde{h}_1^\pm} = (i\gamma_5)^{\theta_1}(c'\Psi_{h_d^-} - s'\Psi_{h_u^+}),$$
$$\Psi_{\tilde{h}_2^\pm} = (i\gamma_5)^{\theta_2}(s'\Psi_{h_d^-} + c'\Psi_{h_u^+}), \qquad (34)$$

where $s' = \sin\beta'$ (set equal to β in Ref. [19]) so that this rotation is independent of the one for the scalars. The factors $(i\gamma_5)^{\theta_{1,2}}$ allow for the possibility that either higgsino could have had a negative mass eigenvalue, as noted below (14). As we saw earlier, the RGE is not dependent on where or whether we include the $(i\gamma_5)^\theta$ factors. We stress again that we do not mean for the equation that immediately follows to be the correct RGE, but write it only to facilitate comparison, and to illustrate some issues. Ignoring the separation of tilde terms on the couplings in the Lagrangian, we find

$$
\begin{aligned}
(4\pi)^2 \frac{d(s\mathbf{f}_u)_{ij}}{dt} =& \frac{3s}{2}\left[\frac{s^2}{3}(\theta_h + \theta_{G^0} + \theta_{G^+}) + \frac{c^2}{3}(\theta_H + \theta_A + \theta_{H^+})\right](\mathbf{f}_u\mathbf{f}_u^\dagger\mathbf{f}_u)_{ij} \\
&+ \frac{s}{2}\left[\{s'^2(\theta_{\tilde{h}_1^0} + \theta_{\tilde{h}_1^\pm}) + c'^2(\theta_{\tilde{h}_2^0} + \theta_{\tilde{h}_2^\pm})\}\theta_{\tilde{Q}_k}(\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik}(\mathbf{f}_u)_{kj} + \{s'^2\theta_{\tilde{h}_1^0} + c'^2\theta_{\tilde{h}_2^0}\}\theta_{\tilde{u}_k}(\mathbf{f}_u)_{ik}(\mathbf{f}_u^\dagger\mathbf{f}_u)_{kj}\right] \\
&+ s[s^2(\theta_h - \theta_{G^0}) + c^2(\theta_H - \theta_A)](\mathbf{f}_u\mathbf{f}_u^\dagger\mathbf{f}_u)_{ij} + \frac{s}{2}[c^2\theta_{G^+} + s^2\theta_{H^+} - 4c^2(\theta_{G^+} - \theta_{H^+})](\mathbf{f}_d\mathbf{f}_d^\dagger\mathbf{f}_u)_{ij} \\
&+ \frac{s}{2}[\{c'^2\theta_{\tilde{h}_1^\pm} + s'^2\theta_{\tilde{h}_2^\pm}\}\theta_{\tilde{d}_k}(\mathbf{f}_d)_{ik}(\mathbf{f}_d^\dagger\mathbf{f}_u)_{kj}] + s(\mathbf{f}_u)_{ij}[3(s^2\theta_h + c^2\theta_H)\,\mathrm{Tr}\{\mathbf{f}_u^\dagger\mathbf{f}_u\} + c^2(\theta_h - \theta_H) \\
&\times \mathrm{Tr}\{3\mathbf{f}_d^\dagger\mathbf{f}_d + \mathbf{f}_e^\dagger\mathbf{f}_e\}] - (\mathbf{f}_u)_{ij}\left[\frac{3}{5}g_1^2\left\{s\frac{17}{12} - s\left(\frac{1}{36}\theta_{\tilde{Q}_j} + \frac{4}{9}\theta_{\tilde{u}_i}\right)\theta_{\tilde{B}}\right.\right. \\
&- \left(\frac{s}{2}[(s's + c'c)^2\theta_{\tilde{h}_1^0} + (s'c - c's)^2\theta_{\tilde{h}_2^0}]\theta_h + \frac{c}{2}(s'c - c's)(s's + c'c)(\theta_{\tilde{h}_1^0} - \theta_{\tilde{h}_2^0})\theta_H\right. \\
&+ \left.\left.[s'(s's + c'c)\theta_{\tilde{h}_1^0} - c'(s'c - c's)\theta_{\tilde{h}_2^0}]\left\{\frac{1}{3}\theta_{\tilde{Q}_j} - \frac{4}{3}\theta_{\tilde{u}_i}\right\}\right)\theta_{\tilde{B}}\right\} \\
&+ g_2^2\left\{s\frac{9}{4} - s\frac{3}{4}\theta_{\tilde{Q}_j}\theta_{\tilde{W}} - \left(\frac{s}{2}[(s's + c'c)^2\theta_{\tilde{h}_1^0} + (s'c - c's)^2\theta_{\tilde{h}_2^0} + 2(s'^2s^2 + c'^2c^2)\theta_{\tilde{h}_1^\pm}\right.\right. \\
&+ 2(s'^2c^2 + c'^2s^2)\theta_{\tilde{h}_2^\pm}]\theta_h + \frac{c}{2}[(s'c - c's)(s's + c'c)(\theta_{\tilde{h}_1^0} - \theta_{\tilde{h}_2^0}) + 2sc(s'^2 - c'^2)(\theta_{\tilde{h}_1^\pm} - \theta_{\tilde{h}_2^\pm})]\theta_H \\
&- \left.\left.\left.[s'(s's + c'c)\theta_{\tilde{h}_1^0} - c'(s'c - c's)\theta_{\tilde{h}_2^0} + 2s(s'^2\theta_{\tilde{h}_1^\pm} + c'^2\theta_{\tilde{h}_2^\pm})]\theta_{\tilde{Q}_j}\right)\theta_{\tilde{W}}\right\} + g_3^2\left\{8s - s\frac{4}{3}(\theta_{\tilde{Q}_j} + \theta_{\tilde{u}_i})\theta_{\tilde{g}}\right\}\right].
\end{aligned}
$$
$$(35)$$

A sum over repeated flavor indices $k, l, \dots$ is implied, including over those with three repeated indices, one of which is a $\theta_{\tilde{f}_k}$, e.g. $\theta_{\tilde{Q}_k}(\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik}(\mathbf{f}_u)_{kj}$ on line 2. Several comments are worth noting.

  (i) In writing the RGE as in (35), where we have ignored the differences between the usual gauge/Yukawa couplings and their tilde counterparts, we have retained distinct thresholds for each of the Higgs bosons as well as for the higgsinos. This is partly to facilitate comparison with Ref. [19], and partly to indicate the origins of the various terms.
  (ii) We see that if we take the MSSM limit, where all the θ's are set equal to unity, we recover the usual, well-known MSSM RGE [15–18]. This is evident if the rotation angle that defines the doublets in (12) and

(13) is treated as a scale-independent parameter. In this case, the right-hand side of (35) is proportional to $\sin\beta$, which cancels out leaving the MSSM RGE for $\mathbf{f}_u$.[11] Notice also that, in the MSSM limit, the dependence on the higgsino rotation angle, β', also disappears, as it must.

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[11] If instead we consider scale-dependent rotations, then the rotated fields themselves have an explicit scale dependence. The RGEs in Refs. [14,22] that were our starting point assume, of course, no such scale dependence of fields. However, we know that, above all thresholds, the rotation (which is merely a field redefinition) cannot change the RGEs. In other words, there must be additional compensating terms in the RGEs which effectively allow us to take the $\sin\beta$ out of the derivative on the left-hand side, so the RGE once again reduces to the MSSM RGE.

(iii) If, on the other hand, we set all the θ's other than θ_h, θ_{G^0}, and θ_{G^+} to zero, the theory includes just the SM fermions, gauge bosons, and the single Higgs doublet (12) that couples to SM quarks and leptons via the coupling matrices $\boldsymbol{\lambda}_u = \mathbf{f}_u \sin\beta$, $\boldsymbol{\lambda}_d = \mathbf{f}_d \cos\beta$, and $\boldsymbol{\lambda}_e = \mathbf{f}_e \cos\beta$. It is then easy to see that we recover the relevant RGE for the Yukawa couplings in the SM [24]. Indeed, all dependence on the $\mathbf{f}$'s in (35) disappears in favor of dependence on the SM $\boldsymbol{\lambda}$'s.

(iv) Since $SU(2)_L$ remains a symmetry of the theory at low energy, we should expect that the couplings related to one another by $SU(2)_L$ should have the same RGE. We have checked that this is indeed the case, *but only if we set a common value for all the heavy Higgs boson thresholds, and an independent common value for the higgsino thresholds.* This should not be surprising since any splitting between the extra Higgs bosons, or between the higgsinos, is an $SU(2)_L$ breaking effect (and would entail introducing even more couplings into the low energy theory). We do not regard as reliable those terms in (35) that come from $SU(2)$ breaking effects because we are then using an inconsistent approximation.

(v) Below the mass scale of the heavy Higgs bosons we may expect that, since there is just one Higgs boson doublet in the theory, only the $\boldsymbol{\lambda}_{u,d,e}$ couplings enter. We can easily see that this is not the case if higgsinos are lighter than m_H. There will still be two higgsino doublets in the low energy theory, and these couple via the matrices $\tilde{\mathbf{f}}_{u,d,e}^{\Phi}$ (notice the tilde on the $\mathbf{f}$).

(vi) We have compared (35) using a common threshold for the higgsinos with the corresponding RGE in Ref. [19]. We find agreement for all but the terms involving $SU(2)_L$ and $U(1)_Y$ gauge couplings. We find

$$(4\pi)^2 \frac{d(s\mathbf{f}_u)_{ij}}{dt} \ni -s(\mathbf{f}_u)_{ij}\left[\frac{3}{5}g_1^2\left\{\frac{17}{12} - \left(\frac{1}{36}\theta_{\tilde{Q}_j} + \frac{4}{9}\theta_{\tilde{u}_i}\right)\theta_{\tilde{B}} - \left(\frac{1}{2}\theta_h + \frac{1}{3}\theta_{\tilde{Q}_j} - \frac{4}{3}\theta_{\tilde{u}_i}\right)\theta_{\tilde{h}_1^0}\theta_{\tilde{B}}\right\} \right.$$
$$\left. + g_2^2\left\{\frac{9}{4} - \frac{3}{4}\theta_{\tilde{Q}_j}\theta_{\tilde{W}} - \left(\frac{3}{2}\theta_h - 3\theta_{\tilde{Q}_j}\right)\theta_{\tilde{h}_1^0}\theta_{\tilde{W}}\right\}\right],$$

to be contrasted with [19]

$$(4\pi)^2 \frac{d(s\mathbf{f}_u)_{ij}}{dt} \ni -s(\mathbf{f}_u)_{ij}\left[\frac{3}{5}g_1^2\left\{\frac{17}{12} + \frac{3}{4}\theta_{\tilde{h}_1^0} - \left(\frac{1}{36}\theta_{\tilde{Q}_j} + \frac{4}{9}\theta_{\tilde{u}_i} + \frac{1}{4}\theta_{\tilde{h}_1^0}\right)\theta_{\tilde{B}}\right\} + g_2^2\left\{\frac{9}{4} + \frac{9}{4}\theta_{\tilde{h}_1^0} - \frac{3}{4}(\theta_{\tilde{Q}_j} + \theta_{\tilde{h}_1^0})\theta_{\tilde{W}}\right\}\right].$$

There is a similar point of disagreement in the RGEs for the down-type Yukawa RGEs.[12]

(vii) We found it difficult to perform a corresponding comparison with Ref. [20], where there the RGEs appear to be written without doing any rotation of the Higgs fields, so that we could not abstract the relationship between the Higgs bosons h, A, H, and $H^{\pm}$, and the thresholds that appear in their RGE. For the same reason, we could not see how to reduce this RGE to the SM RGE below the scale of all sparticles and heavy Higgs bosons. The MSSM limit is, however, correctly obtained.

We reiterate that (35) is not the correct RGE to use. If we derive this RGE keeping the distinction between g_i and $\tilde{\mathbf{g}}_i^{\Phi}$ and between $\mathbf{f}_{u,d,e}$ and $\tilde{\mathbf{f}}_{u,d,e}^{\Phi}$, and independently set common thresholds for the heavy Higgs scalars (θ_H) and the higgsinos ($\theta_{\tilde{h}}$), along with a common threshold at m_h for the "light Higgs doublet" (θ_h), we find that the RGE for the coupling of the up-type quarks to Higgs bosons becomes

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[12]The SM as well as the MSSM limits come out right. Since an even number of SUSY particles couple at any vertex, we do not, however, understand how $\theta_{\tilde{h}_i}$ could enter without being multiplied by a second θ for another SUSY particle.

$$(4\pi)^2 \frac{d(s\mathbf{f}_u)_{ij}}{dt} = \frac{s}{2}\{3[s^2\theta_h + c^2\theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik} + [c^2\theta_h + s^2\theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik} + 4c^2[-\theta_h + \theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik}\}(\mathbf{f}_u)_{kj}$$

$$+ s(\mathbf{f}_u)_{ik}\left[\theta_{\tilde{h}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_u^Q)_{lj} + \frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}^{\prime u_R})_{kl}^*(\tilde{\mathbf{g}}^{\prime u_R})_{lj}^T + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^*(\tilde{\mathbf{g}}_s^{u_R})_{lj}^T\right]$$

$$+ \frac{s}{4}\left[2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^\dagger + 2\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^\dagger + 3\theta_{\tilde{W}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^Q)_{ik}^T(\tilde{\mathbf{g}}^Q)_{kl}^* + \frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^{\prime Q})_{ik}^T(\tilde{\mathbf{g}}^{\prime Q})_{kl}^*\right.$$

$$\left. + \frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)_{ik}^T(\tilde{\mathbf{g}}_s^Q)_{kl}^*\right](\mathbf{f}_u)_{lj} + s\theta_{\tilde{h}}\theta_{\tilde{Q}_k}\left[-3\theta_{\tilde{W}}(\tilde{g}^{h_u})^*(\tilde{\mathbf{g}}^Q)_{ik}^T + \frac{1}{3}\theta_{\tilde{B}}(\tilde{g}^{\prime h_u})^*(\tilde{\mathbf{g}}^{\prime Q})_{ik}^T\right](\tilde{\mathbf{f}}_u^Q)_{kj}$$

$$- \frac{4}{3}s\theta_{\tilde{B}}\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{g}^{\prime h_u})^*(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{g}}^{u_R})_{kj}^T + s(\mathbf{f}_u)_{ij}[(s^2\theta_h + c^2\theta_H)\operatorname{Tr}\{3\mathbf{f}_u^\dagger\mathbf{f}_u\} + c^2(\theta_h - \theta_H)\operatorname{Tr}\{3\mathbf{f}_d^\dagger\mathbf{f}_d + \mathbf{f}_e^\dagger\mathbf{f}_e\}]$$

$$+ \frac{s}{2}\theta_{\tilde{h}}(\mathbf{f}_u)_{ij}\{3\theta_{\tilde{W}}[|\tilde{g}^{h_u}|^2(s^2\theta_h + c^2\theta_H) + |\tilde{g}^{h_d}|^2(c^2\theta_h - c^2\theta_H)]$$

$$+ \theta_{\tilde{B}}[|\tilde{g}^{\prime h_u}|^2(s^2\theta_h + c^2\theta_H) + |\tilde{g}^{\prime h_d}|^2(c^2\theta_h - c^2\theta_H)]\} - s(\mathbf{f}_u)_{ij}\left[\frac{17}{12}g^{\prime 2} + \frac{9}{4}g_2^2 + 8g_3^2\right]. \tag{36}$$

Here, as with (35) and in all the RGEs that follow, including those in the Appendix, repeated flavor indices $k, l, \ldots$ are summed over, including indices repeated thrice, once in a θ and twice in couplings.

It is easy to see the reduction to the MSSM, and as before, to the SM RGE for the Yukawa coupling matrix $\boldsymbol{\lambda}_u$. Indeed, for $Q < m_H$, this RGE becomes the RGE for the coupling, $\boldsymbol{\lambda}_u = \sin\beta\mathbf{f}_u$, of up-type quarks to the light Higgs doublet in (12), even if higgsinos, gauginos, or matter sfermions remain in the low energy theory. The factors of $\sin\beta$ and $\cos\beta$ in the first pair of curly braces can clearly be absorbed to turn all the $\mathbf{f}$'s into the corresponding $\boldsymbol{\lambda}$'s. This is not, however, true in the next set of terms in square brackets, where there is just one factor of $\sin\beta$ that combines with the $(\mathbf{f}_u)_{ik}$ outside the square bracket to yield $(\boldsymbol{\lambda}_u)_{ik}$. Notice, however, that the remaining couplings on this line have tildes on them, and do not correspond to the quark-quark-gauge/Higgs boson couplings. Indeed, it is interesting to see that we get just the right powers of $\sin\beta$ and $\cos\beta$ on the right-hand side for all the MSSM couplings of matter fermions to Higgs bosons to reduce to those of the SM when all θ's other than θ_h vanish. Notice also that the $\tilde{\mathbf{g}}$ couplings have a nontrivial matrix structure. Since it is these couplings (and not the corresponding gauge boson couplings) that directly enter the decays of squarks and sleptons, it behooves us to keep careful track of these in a study of flavor physics of sparticles.

While it is straightforward to derive the MSSM RGEs starting from the general forms in (6)–(11), we reiterate that the reader should keep clear the distinction between the use of $i, j, \ldots$ as fermion field type indices in the general form from their use as flavor indices in the MSSM RGEs such as (36). Thus, in contrast to the trace in (36), the trace in (6) refers to a sum over the fermion field types, not just flavor. This means that not all trace terms in (6)–(11) lead to a trace in the RGEs of the MSSM, and further, sometimes the trace in the MSSM RGEs may originate in terms that are not traces in the general RGEs. We illustrate this with some examples. For instance, in the derivation of (36), starting from (6):

(i) Since the matrix indices j, k on $(\mathbf{U}_a^1)_{jk}$ and $(\mathbf{U}_a^2)_{jk}$ denote Dirac fermion type and flavor, there will be a different matrix index for each quark and lepton flavor. As a result, the term $\mathbf{U}_b^1 \operatorname{Tr}\{\mathbf{U}_b^{1\dagger}\mathbf{U}_a^1 + \mathbf{U}_a^{2\dagger}\mathbf{U}_b^2\}$ in (6) leads to the trace over flavors found in (36).

(ii) The matrix indices in $(\mathbf{X}_a^1)_{jk}$ and $(\mathbf{X}_a^2)_{jk}$ refer to Majorana fermions, and so do not carry any flavor. Tracing over these in (6) can, therefore, never lead to a trace in the MSSM RGEs. This trace term results in the term that immediately follows the trace term in (36).

(iii) The situation is also different for the $(\mathbf{V}_a)_{jk}$ and $(\mathbf{W}_a)_{jk}$ in (6). For $\mathbf{V}_a$, flavor is located in the a and j indices, and for $\mathbf{W}_a$, in the a and k indices. When there is a sum over squarks or sleptons (i.e. a sum over the scalar index), we include a $\theta_{\tilde{q}_i}$, where i is a flavor index, to keep only the active sfermions at that scale. This means that we can keep usual matrix multiplication with the proviso that the contribution from each squark flavor is associated with a different θ. For example, the sum over left-handed squarks in $2\mathbf{V}_b\mathbf{X}_a^{2\dagger}\mathbf{W}_b$ leads to $-3s\theta_{\tilde{h}}\theta_{\tilde{Q}_k}\theta_{\tilde{W}}(\tilde{g}^{h_u})^*(\tilde{\mathbf{g}}^Q)_{ik}^T(\tilde{\mathbf{f}}_u^Q)_{kj}$.

As a different example, consider the RGE for $(\tilde{\mathbf{g}}^{\prime u_R})$ derived using (10):

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}'^{u_R})_{ij}}{dt} = [s^2\theta_h + c^2\theta_H](\mathbf{f}_u^T\mathbf{f}_u^*)_{ik}(\tilde{\mathbf{g}}'^{u_R})_{kj} + \left[\frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}(\tilde{\mathbf{g}}'^{u_R})_{kl}^{\dagger} + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{ik}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^{\dagger} + \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)_{ik}^T(\tilde{\mathbf{f}}_u^Q)_{kl}^*\right]$$

$$\times (\tilde{\mathbf{g}}'^{u_R})_{lj} + \frac{1}{2}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})_{ij}\left[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)_{kl}^{\dagger}(\tilde{\mathbf{g}}'^Q)_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)_{kl}^{\dagger}(\tilde{\mathbf{g}}'^L)_{lk} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}'^{u_R})_{lk}\right.$$

$$+ \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}'^{d_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}'^{d_R})_{lk} + 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}'^{e_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}'^{e_R})_{lk}\right] + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{h}}(\tilde{\mathbf{g}}'^{u_R})_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}'^{h_u}|^2$$

$$+ [c^2\theta_h + s^2\theta_H]|\tilde{g}'^{h_d}|^2\} - 3\theta_{\tilde{B}}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H]\tilde{g}'^{h_u}(\mathbf{f}_u)_{ik}^T(\tilde{\mathbf{f}}_u^{u_R})_{kj}^* - \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)_{ik}^T(\tilde{\mathbf{g}}'^Q)_{kl}(\tilde{\mathbf{f}}_u^{u_R})_{lj}^*$$

$$+ 2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^T(\tilde{\mathbf{f}}_u^{u_R})_{lj}^* + \theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}\left[\frac{8}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}'^{u_R})_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}_s^{u_R})_{lj}\right]$$

$$- (\tilde{\mathbf{g}}'^{u_R})_{ij}\left[\frac{4}{3}g'^2 + 4g_3^2\right]. \tag{37}$$

Here, the terms involving a trace over flavors may not be immediately evident. One such trace term, found on line 2, is

$$\frac{1}{2}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})_{ij}[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)_{kl}^{\dagger}(\tilde{\mathbf{g}}'^Q)_{lk}],$$

in which the trace is not explicitly written since we need to keep information about the position of the squark thresholds. We point out the following:

(i) The terms involving the trace over flavors in (37) do not originate from the trace term in (10). The trace in (37) can only originate when the external quark and squark flavor indices on the left-hand side occur in the *same* $\mathbf{V}_a$ or $\mathbf{W}_a$ matrix on the right-hand side. We see upon inspection that this is the case only for the second term on the first line of (10), which necessarily leads to the trace over flavors in (37). An analogous discussion applies to the derivation of the RGE for, say, $\tilde{\mathbf{g}}'^Q$, starting from Eq. (11). Note also that, when we are checking reduction to the MSSM, gaugino coupling matrices such as $(\tilde{\mathbf{g}}'^{u_R})$ reduce to $g' \times \mathbb{1}$, so that we obtain an extra factor of 3 when we take a trace over the flavors.

(ii) When inserting thetas, we must, of course, only put thetas which correspond to summed internal indices. To obtain, for example, the term $-\theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)_{ik}^T(\tilde{\mathbf{g}}'^Q)_{kl}(\tilde{\mathbf{f}}_u^{u_R})_{lj}^*$, which comes from $2\mathbf{V}_b\mathbf{W}_a^*\mathbf{W}_b^T$ in (10), where we have picked out the left-handed squarks in the sum over b, we insert a $\theta_{\tilde{Q}_k}$, with k summed over squark flavors. Similarly, we insert a $\theta_{\tilde{h}}$ which comes from picking out the higgsino terms in the sum over Majorana fermions.[13] Finally, even though we have $\tilde{\mathbf{f}}_u^{u_R}$ in our example term, there is no $\theta_{\tilde{u}}$ because the $\tilde{u}_R$ squark is the same as the one which appears on the left-hand side, and hence is necessarily an active squark.

(iii) In the one-loop RGEs for $\tilde{\mathbf{g}}_s^{u_R}$, $\tilde{\mathbf{g}}_s^{d_R}$, and $\tilde{\mathbf{g}}_s^Q$ there are no contributions from $\mathbf{X}^1$ or $\mathbf{X}^2$, which is to be expected since, as mentioned before, the Higgs boson does not couple to the gluino.

(iv) Special care must be taken when considering interactions with gluinos to correctly evaluate the sum over colors. For example, the RGE for $\tilde{\mathbf{g}}_s^{u_R}$ has some terms which may initially seem to lead to a trace which includes $\tilde{\mathbf{g}}'^{u_R}$. However, the term in question is

$$\sim \delta_{dc}\left(\frac{\lambda_A}{2}\right)_{cd}(\tilde{\mathbf{g}}'^{u_R})_{kl}^{\dagger}(\tilde{\mathbf{g}}_s^{u_R})_{lk},$$

where l, k are flavor indices and c, d are color indices. This term is clearly zero, since $\mathrm{Tr}\{\frac{\lambda_A}{2}\} = 0$.

The RGEs for all the dimensionless couplings can be similarly obtained. We list these in the Appendix.

V. SOLUTIONS TO THE RGES AND FLAVOR VIOLATION IN THE MSSM

Since our ultimate goal is to examine flavor violation in sparticle interactions, and in sparticle decays in particular, we are naturally led to examine the flavor structure of various couplings, renormalized at the scale of the sparticle mass, i.e. at scales Q typically between ~ 100 GeV and a few TeV.

The couplings of neutral gauge bosons remain flavor diagonal under renormalization group evolution even at higher loops, since any intergeneration couplings would violate current conservation. This does not mean that their evolution is independent of Yukawa couplings: Yukawa couplings can, and do, enter the evolution of gauge couplings via two-loop contributions to the RGEs, but only as a "flavor-blind trace." As a result of this, and the fact that the charged current weak interaction only couples to the left-handed fermions and their superpartners, intergeneration couplings in the charged current weak interaction are completely determined by the Kobayashi-Maskawa [25] (KM) matrix.

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[13] We also sum over quark flavors, l, but there is no corresponding theta since we are always working at scales above m_t. Therefore, all quarks are active over the whole range of validity of our RGEs.

We thus turn to an examination of the flavor structure of fermion-fermion-scalar interactions. These include the familiar Yukawa couplings of quarks with Higgs bosons, as well as the couplings of the quark-squark system with gluinos, charginos, and neutralinos. Although we will confine ourselves to the quark/squark sector, our considerations readily extend to the lepton sector. In this case, however, the MSSM would need to be extended to include effects from the flavor structure of the singlet neutrino/ sneutrino mass matrix and interactions.

It may appear simplest to work in a basis where the Yukawa coupling matrices in (17) are diagonal at one chosen scale, so that quarks are mass eigenstates, rather than in the "current quark basis" which we have used to write the superpotential (15) and the interactions (16)–(18) as well as the RGEs. The two bases are related by the transformations

$$\mathbf{V}_L(u)u_L^M = u_L, \qquad \mathbf{V}_L(d)d_L^M = d_L;$$
$$\mathbf{V}_R(u)u_R^M = u_R, \qquad \mathbf{V}_R(d)d_R^M = d_R, \qquad (38)$$

where the unitary matrices $\mathbf{V}_L(u)$ and $\mathbf{V}_R(u)$ [$\mathbf{V}_L(d)$ and $\mathbf{V}_R(d)$] diagonalize the Yukawa coupling matrix $\mathbf{f}_u$ [$\mathbf{f}_d$] via

$$\mathbf{V}_L^T(u)\mathbf{f}_u\mathbf{V}_R^*(u) = \mathbf{f}_u^{\mathrm{diag}}, \qquad \mathbf{V}_L^T(d)\mathbf{f}_d\mathbf{V}_R^*(d) = \mathbf{f}_d^{\mathrm{diag}}. \quad (39)$$

Of course, the matrices $\mathbf{f}_u^{\mathrm{diag}}$ and $\mathbf{f}_d^{\mathrm{diag}}$ are diagonal only at one scale, which we take to be $Q = m_t$. The KM matrix that enters the interactions of $W^\pm$ bosons with quarks is then given by

$$\mathbf{K} = \mathbf{V}_L^\dagger(u)\mathbf{V}_L(d).$$

The problem with choosing to work in the quark mass basis is that using different transformations for the left-handed up and down quarks breaks the $SU(2)_L$ symmetry. As a result, interactions of, say, the charged and neutral Higgs bosons with quarks are no longer simply related by an $SU(2)_L$ transformation, but involve an additional matrix (which in the case of the MSSM is the KM matrix).[14]

It is convenient to work instead in the basis where just one of the up- or down-type (but not both) Yukawa coupling matrices is diagonal at m_t. The interactions and the RGEs that we have written in the last section continue to hold in this special "current quark" basis, provided we define the squark fields as the superpartners of these quark fields. In the following, we will choose the basis so that the up-quark Yukawa couplings are diagonal at m_t, i.e. the up-type quarks are in their mass basis at this scale. *This does not imply that the up-type squarks are also in the mass eigenstate basis.* This would be the case in the exact SUSY limit, but the SSB squark mass matrices and the trilinear

SSB scalar couplings may independently violate flavor in the squark sector. In writing our RGEs with independent thresholds for the squarks, we have, however, assumed that mixing between up-type squarks is small in this basis so that $\tilde{u}_{Li}$ and $\tilde{u}_{Ri}$ are approximately also mass eigenstates, and likewise for down-type squarks.

With this preamble, let us turn to an examination of the flavor-violating dimensionless couplings of the MSSM. Let us begin with the Yukawa couplings of quarks to neutral Higgs bosons. We will assume that we have diagonalized the SM Yukawa coupling matrices at the scale $Q = m_t$, and examine the solutions for the various elements of the up-quark Yukawa coupling matrix for larger values of Q. We thus need to solve the system of RGEs listed in the Appendix. Notice that, though the RGEs for the dimensionful SSB parameters are decoupled from the RGEs for dimensionless couplings, the weak scale values of the dimensionful parameters nonetheless enter our analysis in that they determine the various thresholds. This dependence is, fortunately, only logarithmic, and it suffices to approximate the location of the thresholds as discussed in Sec. III. To solve the RGEs we rotate the Yukawa coupling matrices to the current basis using (39), with matrices $\mathbf{V}_{L,R}(q)$ chosen to reproduce the KM matrix. Next, we evolve these along with the values of gauge couplings to the scale $Q = M_{\mathrm{GUT}}$ using the RGEs with all the θ's set to unity, i.e. the MSSM RGEs. We then evolve back down to $Q = m_t$, but this time including thresholds and the couplings with tildes.[15] We reset the gauge couplings and current-basis Yukawa coupling matrices back to their input values, run back up to M_{GUT}, and iterate the procedure until it converges with the required precision. We can then read off the couplings in any basis at the desired values of Q.

Rather than show numerical results for many cases in many SUSY models that have been considered, we have chosen to illustrate our analysis for a simplified scenario where the sfermions and the electroweak gauginos are all at a mass scale ~ 600 GeV while the heavy Higgs bosons and gluinos have a mass ~ 2 TeV. Thus, our effective theory is supersymmetric for $Q > 2$ TeV, includes only sfermions, charginos, and neutralinos together with SM particles for 600 GeV $< Q <$ 2 TeV, and is the SM for

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[14]This matrix is the KM matrix for the MSSM where one of the Higgs doublets couples to up-type quarks, and the other to down-type quarks. In the general two Higgs doublet model, this is no longer the case.

[15]This procedure would be somewhat modified if, instead of choosing the location of heavy Higgs boson and sparticle thresholds "by hand" as we do here, these were to be obtained using GUT scale boundary conditions for SSB parameters. Universal boundary conditions, such as the ones used in mSUGRA, can of course be used in any sfermion basis, but in all other cases we have to be specific about the choice of basis in which we specify these SSB boundary conditions. This will be discussed in more detail in Ref. [21]. As an aside, we also note that, to get the solutions of the RGEs in this paper, it is only necessary to evolve beyond the highest threshold during the iterations to obtain convergence. We have evolved all the way to M_{GUT} only because we anticipate that this will be the Q value where SSB boundary conditions will be specified in many realistic scenarios.

$Q < 600$ GeV. In the intermediate range between 600 GeV and 2 TeV, we will have to separately examine the couplings $\tilde{\mathbf{g}}^\Phi$ and $\tilde{\mathbf{g}}'^\Phi$ (but not $\tilde{\mathbf{g}}_s^\Phi$) and $\tilde{\mathbf{f}}_{u,d,e}^\Phi$ as these split off from the usual gauge and Yukawa couplings.

A. Quark Yukawa couplings

The values of the running quark mass parameters determine the Yukawa coupling matrices in the mass basis. These can be rotated to an arbitrary current basis using (38), or to the particular basis with diagonal up-type Yukawa couplings by choosing the four matrices suitably. In models where the SSB interactions are flavor-blind (at any one scale), the four matrices are separately unphysical, and physics is determined by just the KM matrix. In more general scenarios though, physical quantities will depend on more than just the KM matrix. In our numerical illustration, we have used the program ISAJET [26] to extract quark Yukawa couplings at the scale $Q = m_t$ in their mass basis.[16] Starting from this boundary condition, we have iteratively solved the system of RGEs for a toy scenario, where the SUSY thresholds are taken to be clustered either at 600 GeV (M_1, M_2, $|\mu|$, and $m_{\tilde{f}}$) or at 2 TeV (m_H, $m_{\tilde{g}}$), as mentioned in the previous paragraph.

The results of this calculation for the elements of $\mathbf{f}_u$ are shown in Fig. 1. Specifically, we show $|(\mathbf{f}_u)_{ij}|$ if $Q > m_H$, and $|(\boldsymbol{\lambda}_u)_{ij}|/\sin\beta$ (which joins continuously to the corresponding element of $\mathbf{f}_u$ at $Q = m_H$) for $Q < m_H$. Since the KM matrix includes a complex phase which we take to be $60°$ [and since the matrices in (38) can themselves be complex], these elements are, in general, complex numbers.[17] In Fig. 1, we have plotted the absolute values of these elements, in the basis where the up-quark Yukawa coupling matrix is diagonal at $Q = m_t$, versus the energy scale Q. In our calculation, we include two-loop terms albeit without threshold corrections which are numerically completely negligible, as even the two-loop corrections have only a small effect on our results. The three roughly horizontal lines show the scale dependence of the diagonal Yukawa couplings. This is largely governed by the strong interaction gauge coupling terms in the RGE, and for large

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[16] ISAJET currently does not include flavor mixing among the quarks. It does, however, include important radiative corrections to the relationship between quark masses and the Yukawa couplings. We expect that, because this flavor mixing is small, its effect on the quark masses will also be small. In any case, we expect to incorporate these RGEs into ISAJET, at which stage flavor-mixing effects can be included in the iterative procedure used to extract the Yukawa couplings. Since our discussion here is meant only to be illustrative, it is not crucial that we use absolutely precise values for the Yukawa couplings.

[17] We have solved these equations in a randomly chosen current basis, i.e. one with a random choice of the matrices in (38), and then rotated back to the basis where up-type quark Yukawa couplings are diagonal at $Q = m_t$. Our results are independent of the initial choice of current basis, as they should be, providing a nontrivial check on the code.
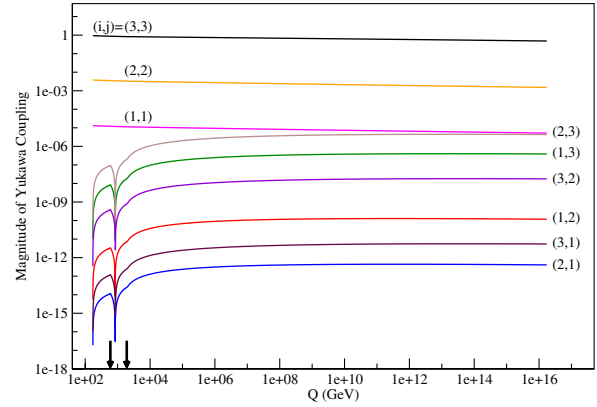

FIG. 1 (color online). Evolution of the magnitudes of the elements of the up-quark Yukawa coupling matrix for the MSSM with thresholds, shown by arrows, clustered at 600 GeV and 2 TeV, as discussed in the text. Above m_H ($= 2$ TeV) we plot $|(\mathbf{f}_u)_{ij}|$, whereas below m_H, where the effective theory includes just one scalar Higgs doublet, we plot $|(\boldsymbol{\lambda}_u)_{ij}|/\sin\beta$ which is equal to $|(\mathbf{f}_u)_{ij}|$ at $Q = m_H = 2$ TeV.

Q values is a calculable power of this gauge coupling.[18] There are kinks in these curves at 600 GeV and 2 TeV, but these are not visible on the scale of this figure: see, however, Fig. 5. The off-diagonal elements of (the scaled) $|\boldsymbol{\lambda}_u|$ all start off at zero at $Q = m_t$, but rapidly increase to values that, though small, may as in the case of $(\mathbf{f}_u)_{23}$ become as large as 10^{-6}. The most striking thing about the figure is the dip in all the off-diagonal elements at a value of Q in the few hundred GeV range, close to the expected value of squark masses in our example. An understanding of this feature is of more than mere academic interest, since we shall see that it also appears in the corresponding $\tilde{\mathbf{f}}$ couplings, which directly enter the amplitudes for flavor-violating squark decays.

To understand this curious feature, let us first consider a further simplification where the two SUSY thresholds coalesce into a single one at $Q = M$. We will, of course, return to the case in Fig. 1 shortly. For $Q > M$, the RGE for $\mathbf{f}_u$ becomes the corresponding MSSM RGE,

$$
\begin{aligned}
(4\pi)^2 \frac{d(\mathbf{f}_u^{\text{diag}})}{dt} &= 3(\mathbf{f}_u^{\text{diag}})(\mathbf{f}_u^{\text{diag}})^\dagger(\mathbf{f}_u^{\text{diag}}) \\
&\quad + \mathbf{K}^*(\mathbf{f}_d^{\text{diag}})(\mathbf{f}_d^{\text{diag}})^\dagger \mathbf{K}^T(\mathbf{f}_u^{\text{diag}}) \\
&\quad + (\mathbf{f}_u^{\text{diag}}) \operatorname{Tr}\{3(\mathbf{f}_u^{\text{diag}})^\dagger(\mathbf{f}_u^{\text{diag}})\} \\
&\quad - (\mathbf{f}_u^{\text{diag}})\left[\frac{13}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2\right], \quad (40)
\end{aligned}
$$

while for $Q < M$, we have the SM RGE for the Yukawa coupling matrix,

---

[18] For the (3,3) element, contributions from the third generation Yukawa coupling are also significant.

$$(4\pi)^2 \frac{d(\boldsymbol{\lambda}_u^{\text{diag}})}{dt} = \frac{3}{2}(\boldsymbol{\lambda}_u^{\text{diag}})(\boldsymbol{\lambda}_u^{\text{diag}})^\dagger(\boldsymbol{\lambda}_u^{\text{diag}})$$

$$- \frac{3}{2}\mathbf{K}^*(\boldsymbol{\lambda}_d^{\text{diag}})(\boldsymbol{\lambda}_d^{\text{diag}})^\dagger\mathbf{K}^T(\boldsymbol{\lambda}_u^{\text{diag}})$$

$$+ (\boldsymbol{\lambda}_u^{\text{diag}})\,\text{Tr}\{3(\boldsymbol{\lambda}_u^{\text{diag}})^\dagger(\boldsymbol{\lambda}_u^{\text{diag}})$$

$$+ 3(\boldsymbol{\lambda}_d^{\text{diag}})^\dagger(\boldsymbol{\lambda}_d^{\text{diag}}) + (\boldsymbol{\lambda}_e^{\text{diag}})^\dagger(\boldsymbol{\lambda}_e^{\text{diag}})\}$$

$$- (\boldsymbol{\lambda}_u^{\text{diag}})\left[\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right]. \quad (41)$$

The KM matrix $\mathbf{K}$ appears in these equations because we have written them in the basis with *all* Yukawa couplings diagonal and real at the scale $Q = m_t$.

That the real or imaginary parts of the off-diagonal elements all have a zero is simple to understand. First, we observe that, since $\boldsymbol{\lambda}_u$ and $\boldsymbol{\lambda}_d$ start off as real and diagonal at $Q = m_t$, off-diagonal elements develop only because the KM matrix has off-diagonal components. Also, the imaginary parts of $\boldsymbol{\lambda}_{u,d}$ develop only via the phase in the KM matrix. Next, notice that the "seed of flavor violation," i.e. the term involving the KM matrix in the RGEs, which is also the seed for the imaginary part of the off-diagonal elements, enters with opposite signs in the SM and in the MSSM RGEs. This means that if this seed term causes the real or imaginary part of any off-diagonal element of, say, the up-type Yukawa coupling to evolve with a negative slope in the SM, it would cause the same element to evolve with a positive slope in the MSSM, and *vice versa*. The evolution for $m_t < Q < M$ is governed by the SM RGE, which causes the real part of an off-diagonal element to evolve to, say, negative values. However, for $Q > M$, the evolution is governed by the MSSM RGE, and the real piece of this Yukawa coupling now begins to evolve in the opposite direction, so that it first becomes less negative, then passes through zero and continues to positive values. Of course, at $Q = M$, we must remember to switch from the SM couplings $\boldsymbol{\lambda}_{u,d,e}$ to the MSSM couplings $\mathbf{f}_{u,d,e}$. The imaginary piece of the off-diagonal Yukawa couplings similarly passes through zero. However, we see from the sharp dips in Fig. 1 that the real and imaginary parts have a zero at (almost) *the same value* of Q, and furthermore, this location appears independent of the choice of the flavor indices i and j. To understand this, we need to analyze the equations further.

In the following we will collectively denote the solutions of these equations by U, D, and E, i.e. $\mathbf{U} = \mathbf{f}_u$ for $Q > M$, while $\mathbf{U} = \boldsymbol{\lambda}_u$ for $Q < M$, and analogously for D and E. For example, the RGE for U in the basis where both up- and down-type quark matrices are diagonal at $Q = m_t$ can be written in the form

$$(4\pi)^2 \frac{d\mathbf{U}_{ij}}{dt} = A_1\mathbf{U}_{ik}\mathbf{U}_{kl}^\dagger\mathbf{U}_{lj} + A_2(\mathbf{K}^*\mathbf{D}\mathbf{D}^\dagger\mathbf{K}^T\mathbf{U})_{ij}$$

$$+ \mathbf{U}_{ij}\,\text{Tr}\{A_3\mathbf{U}^\dagger\mathbf{U} + A_4\mathbf{D}^\dagger\mathbf{D} + A_5\mathbf{E}^\dagger\mathbf{E}\}$$

$$+ \mathbf{U}_{ij}\mathcal{G}(\vec{\alpha}(t)). \quad (42)$$

Here, $\vec{\alpha}(t)$ denotes the collection of the three gauge couplings. Of course, the coefficients A_i and the function $\mathcal{G}$ differ for the SM and the MSSM. As already mentioned, the evolution of the diagonal elements of U and D is dominantly governed by the last term in these equations, which contains the gauge couplings. If we drop all other terms on the right-hand side, the RGE for the diagonal Yukawa terms can be easily integrated, and we find

$$\mathbf{U}_{ii}(t) = \mathbf{U}_{ii}(t_0)F(\vec{\alpha}(t)), \quad \text{no sum on } i. \quad (43)$$

It is important to note that *the function F is independent of the quark flavor*.[19]

Note now that the trace as well as the gauge coupling terms are flavor independent. Assuming that the diagonal Yukawas saturate the trace [remember that we just argued that these diagonal terms are a function only of the gauge couplings $\vec{\alpha}(t)$], we can combine these terms so that the second and third lines on the right-hand side of (42) can be written as

$$\mathbf{U}_{ij}(t)f(t),$$

again with f being independent of flavor. Next, let us look at the term involving the KM matrix. Noting that the diagonal elements of U and D are much larger than the off-diagonal ones, we can write

$$(\mathbf{K}^*\mathbf{D}\mathbf{D}^\dagger\mathbf{K}^T\mathbf{U})_{ij} \simeq \sum_k \mathbf{K}_{ik}^*\mathbf{D}_{kk}(t)\mathbf{D}_{kk}(t)\mathbf{K}_{jk}\mathbf{U}_{jj}(t),$$

where i and j are unsummed, fixed indices. In writing this, we have used the reality of the diagonal elements of the Yukawa matrices under renormalization group evolution (discussed below). From the analogue of (43) for $\mathbf{D}(t)$, we see that this term has the structure

$$\mathbf{M}_{ij}(t_0)G(t),$$

with $\mathbf{M}_{ij}(t_0) = \mathbf{K}_{ik}^*\mathbf{D}_{kk}(t_0)\mathbf{D}_{kk}(t_0)\mathbf{K}_{jk}\mathbf{U}_{jj}(t_0)$, and G a *flavor-independent* function of just the gauge couplings, which themselves depend just on t. Ignoring, for the moment, the first term in the RGE for U (the one cubic in U), we see that we have managed to decouple the RGEs for $\mathbf{U}_{ij}$ which (if the first term is dropped) take the form

$$D\mathbf{U}_{ij} \equiv \frac{d\mathbf{U}_{ij}}{dt} = \mathbf{M}_{ij}(t_0)G(t) + f(t)\mathbf{U}_{ij}(t),$$

where $f(t)$ and $G(t)$ are known functions of t. The solution

---

[19]In the approximation where we retain just the last gauge coupling terms in (40) and (41), these equations take the form

$$(4\pi)^2 \frac{d\mathbf{U}_{ii}}{dt} = \mathbf{U}_{ii}(a_1g_1^2 + a_2g^2 + a_3g_s^2).$$

They can be readily integrated to give

$$\mathbf{U}_{ii}(t) = \mathbf{U}_{ii}(t_0)\Pi_{r=1}^3 (g_r)^{a_r/b_r},$$

where b_r is the coefficient of the one-loop gauge coupling β function, $\beta_r(g_r) = \frac{1}{16\pi^2}b_r g_r^3$.

to this may be written as

$$\mathbf{U}_{ij}(t) = \mathbf{M}_{ij}(t_0)\frac{1}{D-f}G(t) + \mathbf{U}_{ij}^0(t),$$

where $\frac{1}{D-f}$ is the resolvent operator, and

$$\mathbf{U}_{ij}^0(t) = \mathbf{U}_{ij}^0(t_0)\exp\left(\int_{t_0}^t f(t)dt\right)$$

is the solution to the corresponding homogeneous differential equation. For the SM evolution starting at $Q = m_t$, $\mathbf{U}_{ij}^0(t_0)$ vanishes (if $i \neq j$), and we are left with the solution of interest,

$$\mathbf{U}_{ij}(t) = \mathbf{M}_{ij}(t_0)\mathcal{H}(t), \qquad (44)$$

where the function $\mathcal{H}(t) \equiv \frac{1}{D-f}G(t)$ is *independent of flavor*. All the flavor information is contained in the "boundary value," $\mathbf{M}_{ij}(t_0)$. To clarify, we use the boundary conditions on the SM Yukawa couplings to obtain $\mathbf{U}$ up to the (common) SUSY threshold. The values of $\mathbf{U}$ and $\mathbf{D}$ at $Q = 2$ TeV now serve as the boundary condition for MSSM evolution. The central point of this analysis is that the flavor dependence is completely captured in the first factor $\mathbf{M}_{ij}(t_0)$, where t_0 corresponds to $Q = m_t$ for SM evolution, and to the SUSY threshold for MSSM evolution.

To the extent that we can ignore the first "cubic term" in the $\mathbf{U}$ RGEs, we see that the flavor structure of the evolution of $\boldsymbol{\lambda}_{ij}$ is given by (44). We make the following observations.

(i) This structure is independent of the coefficients in the various terms in the RGEs as well as independent of which matrices enter into the trace. The same structure is thus obtained for both SM and MSSM evolution. The important approximation was that the diagonal elements of the Yukawa matrix dominate in the basis where these are diagonal at the low scale.

(ii) Equation (44) was obtained without any assumptions about the hierarchical structure of the KM matrix.

We can immediately obtain two corollaries from (44).

(1) $\frac{\mathbf{U}_{ij}^R(t)}{\mathbf{U}_{ij}^I(t)}$, where the superscripts R and I, respectively, denote the real and imaginary parts, is independent of t.

(2) $\frac{\mathbf{U}_{ij}^{R/I}(t)}{\mathbf{U}_{kl}^{R/I}(t)}$ is independent of t.

Our program is to start at $Q = m_t$ with the "diagonal boundary conditions" for the SM Yukawa couplings and evolve $\boldsymbol{\lambda}_{u,d,e}$ to the scale $Q = m_H$. We then scale the up- and down-type Yukawa couplings by $1/\sin\beta$ and $1/\cos\beta$, respectively, and finally continue this evolution to higher values of t using the MSSM RGEs. By corollary 1 above we see that, when the real part of any off-diagonal Yukawa coupling vanishes (as it must for some value of $Q > M$), the imaginary part of this coupling also vanishes. Corollary 2 then tells us that the other off-diagonal elements likewise

vanish for this same value of Q. This is, of course, exactly the qualitative feature in Fig. 1 that we started out to explain.

Before turning to other issues, let us briefly return to the first term that we have ignored up to now. If we saturate the product $\mathbf{UU}^\dagger\mathbf{U}$ with two diagonal elements in each of the three terms and also remember that the diagonal elements are real, i.e. write $(\mathbf{UU}^\dagger\mathbf{U})_{ij} \simeq \mathbf{U}_{ii}\mathbf{U}_{ii}\mathbf{U}_{ij} + \mathbf{U}_{jj}\mathbf{U}_{jj}\mathbf{U}_{ij} + \mathbf{U}_{ii}\mathbf{U}_{jj}\mathbf{U}_{ji}^*$, we find that, like terms we have considered up to now, two of these terms do not couple $\mathbf{U}_{ij}$ with anything else, while the third term couples $\mathbf{U}_{ij}$ to $\mathbf{U}_{ji}^*$. Equation (44) is thus violated only by this last term. We have indeed checked that the positions of the zeros are not exactly coincident but have a very tiny spread of a few GeV not visible on the figure. We have also checked that increasing the coefficient of the cubic term increases this spread, while altering the other coefficients has no effect on it, in keeping with expectations from our analysis.[20]

Our analysis of Fig. 1, up to now, has assumed a common location for the threshold for all non-SM particles which is, of course, not the case in the figure. It is, however, clear that the bulk of the change in the slope of the curve occurs from the thresholds at 600 GeV where the coefficient of the term involving the KM matrix changes its sign, causing the magnitudes of the off-diagonal couplings to decrease until they pass through zero, beyond which the magnitude starts increasing once again. The kink in the various curves at $Q = 2$ TeV marks the position of the second threshold. It should, therefore, be clear that the location of the zero is largely determined by the spectrum, and is insensitive to other details.[21]

Finally, we remark that the imaginary parts of the off-diagonal couplings (recall that these have their origin in the phase in the KM matrix) may, depending on which matrix element we are considering, be of comparable magnitude to the corresponding real part, or much smaller. The relative size of the imaginary part is largely determined by the relative size of the imaginary part in the corresponding element of the "seed term," $\sim\mathbf{K}^*\mathbf{DD}^\dagger\mathbf{K}^T\mathbf{U}$ in the RGE. It is easy to see that this "seed" is real for the diagonal elements of $\mathbf{U}$ (or $\mathbf{D}$), which, therefore, continue to remain real. For the case study in the figure, the (1,3) element has

---

[20]We have also numerically solved the RGEs using a fictitious KM matrix with large off-diagonal entries. We found the same behavior as in the figure, in keeping with our observation that the analysis does not depend on the hierarchical structure of the KM matrix.

[21]We checked also that using a fictitious KM matrix with large off-diagonal entries changes the evolution of the off-diagonal couplings (for instance, the relative size of the imaginary parts) as expected, but hardly affects the location of the zero which is determined by the mass spectrum. As mentioned previously, the location of the zeros is also largely insensitive to the coefficients of the various terms in the RGEs, unless of course, a coefficient happens to be so large that the solution blows up before the zero can be reached.

the largest imaginary part $\mathcal{O}(10^{-7})$, while the imaginary part of other off-diagonal elements is smaller by 3–6 orders of magnitude over most of the range of Q.

We should mention that in a complete diagrammatic calculation there would be other nonlogarithmic corrections to the couplings not included in our calculation. The corrections to the diagonal elements of the Yukawa coupling matrices are already included in ISAJET using the formulas in Ref. [27], and so will be automatically included when these RGEs are embedded into the ISAJET code. In contrast to a diagrammatic approach, our method (also the one used in ISAJET) automatically sums the potentially large logarithms that arise when the MSSM spectrum is significantly split. We also remark that, in Fig. 1 (as well as in the other figures below), we have ignored any finite shifts in the coupling constants coming from the matching of the two effective theories (with and without the heavy particle) when we decouple particles at their mass scale [28]. This would potentially give jumps in the couplings as we cross the various thresholds, which must be taken into account to achieve true two-loop accuracy. The RGEs that we have derived are, of course, unaffected.

B. Higgsino and gaugino couplings to quarks and squarks

We now turn to a study of the evolution of the couplings of gauginos and higgsinos to the quark-squark system. Above all thresholds, i.e. for $Q > m_{\tilde{g}} = m_H$ in our numerical study, these are equal to the corresponding gauge and quark Yukawa couplings, but differ from them for the range of Q between the lower threshold at the sfermion, higgsino, and electroweak gaugino masses and the high threshold. Of course, for Q values below this lower threshold, the effective theory is the SM and the couplings $\tilde{\mathbf{g}}$, $\tilde{\mathbf{g}}'$, $\tilde{\mathbf{f}}_u$, and $\tilde{\mathbf{f}}_d$ cease to be meaningful quantities in our simple-minded approach with thresholds being incorporated by step functions. Note also that because the gluino is the heaviest SUSY particle in our illustration, there is no $\tilde{\mathbf{g}}_s$ coupling. Above all thresholds, supersymmetry relates gaugino and gauge couplings so that flavor violation via gaugino interactions is not allowed. For $Q < m_H$, the connection between the gauge couplings and gaugino couplings is broken, and flavor-violating interactions of gauginos are no longer forbidden by any symmetry. Although these $\tilde{\mathbf{g}}^{\Phi}$-type couplings start off proportional to unit matrices at $Q = m_H$, contributions to their renormalization group evolution from terms involving Higgs boson and higgsino couplings to quarks and squarks [see, e.g., Eq. (A12) of the Appendix], which cancel out if all thetas are set to unity, cause these to develop off-diagonal elements for $Q < m_H$.

These off-diagonal components of $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{g}}'$ (and also of the corresponding Yukawa couplings) induce flavor-violating decays, $\tilde{q}_i \rightarrow q_j \tilde{Z}_k$, of squarks even if there is no explicit flavor mixing in the SSB sector (i.e. all flavor

violation is induced via superpotential Yukawa couplings as is the case in many models), and hence are phenomenologically relevant. In this case, the partial widths for flavor-violating decays will be small, so that these decays will likely be most important for those squarks for which flavor-conserving, tree-level, two body decays are kinematically forbidden. Thus, the neutralino in the flavor-violating decay is most likely to be the lightest supersymmetric particle (LSP). Since the LSP is bino–like in many models, it would seem that the couplings $\tilde{\mathbf{g}}'$ would be the most relevant, but this would of course depend on the size of the corresponding off-diagonal component, relative to the same component of the other couplings.

We begin by showing the magnitudes of the off-diagonal components of the electroweak gaugino-quark-squark couplings in Fig. 2 for (a) the matrices $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}'^{u_R}$, and (b) the matrices $\tilde{\mathbf{g}}^Q$ (there is no $\tilde{\mathbf{g}}^{u_R}$). We obtain these from the corresponding evolution equations with the boundary condition that these matrices are equal to the corresponding gauge coupling times the unit matrix at $Q = m_H$. On the low Q side, we terminate these curves at the lower SUSY threshold, below which these couplings no longer exist. Also shown for comparison is the (2,3) element, the one with the largest magnitude, of the up-quark Yukawa coupling matrix $\mathbf{f}_u$. Remember that we are plotting these matrix elements in the basis where the up-quark (but not the down-quark) Yukawa couplings are diagonal at $Q = m_t$. There are several features worth remarking about.

  (1) The off-diagonal elements of the gaugino coupling matrices vary over the same broad range as the corresponding elements of the Yukawa coupling matrices shown in Fig. 1. Indeed the magnitude of some of these elements considerably exceeds the largest element of the quark Yukawa coupling matrix. It would, therefore, be dangerous to simply disregard these when discussing sparticle flavor physics, particularly in models where Yukawa couplings are the sole source of flavor violation.

  (2) The off-diagonal couplings of the $\tilde{\mathbf{g}}'^Q$ matrix are several orders of magnitude larger than those of $\tilde{\mathbf{g}}'^{u_R}$. The reason for this can be traced to their RGEs, which are listed in the Appendix. We see from (A10) and (A12) that the evolution of the matrices $\tilde{\mathbf{g}}'^Q$ depends on the down-type Yukawa couplings, while that of $\tilde{\mathbf{g}}'^{u_R}$ does not have any such contributions. In the basis that we are working in, these contributions are much larger than those from up-type Yukawa matrices (and larger than those from the gaugino coupling matrices which start off as unit matrices at $Q = m_H$), so that it is no surprise that the off-diagonal elements of $\tilde{\mathbf{g}}'^Q$ attain larger values than those of $\tilde{\mathbf{g}}'^{u_R}$. A similar remark applies to the magnitudes of the elements of $\tilde{\mathbf{g}}^Q$ shown in frame (b).
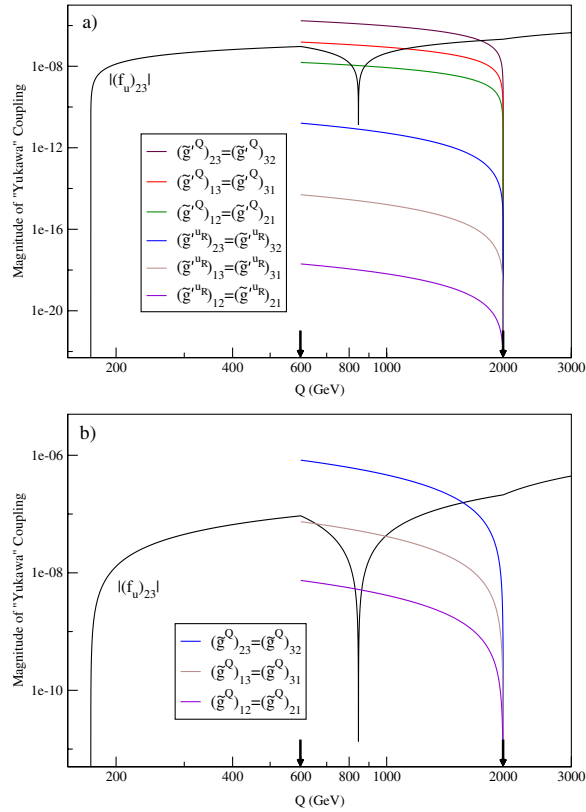
  (3) We also see from the figure that the magnitudes of

FIG. 2 (color online). Evolution of the magnitudes of the complex off-diagonal elements of the gaugino coupling matrices: (a) $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}'^{u_R}$, and (b) $\tilde{\mathbf{g}}^Q$ defined in the text for the MSSM, with the positions of the sparticle and Higgs thresholds (again denoted by arrows) as in Fig. 1. Also shown for comparison is the magnitude of the (2,3) element, the one with the largest magnitude, of the up-quark coupling matrix $\mathbf{f}_u$. The legend is in the same order (from top to bottom) as the curves. The magnitudes of the elements of the gaugino coupling matrices are symmetric in the flavor indices to a good approximation, as explained in the text.

the gaugino coupling matrices are symmetric under the interchange of the two indices. We have checked that this symmetry is not exact but (for our illustration) holds to a few parts per mille. The reason for the symmetry is that in the course of their evolution, starting from a unit matrix at $Q = m_H$, the gaugino coupling matrices remain (approximately) Hermitian. This is simplest to see for the $\tilde{\mathbf{g}}'^Q$ matrices, retaining only the dominant terms involving the down-type Higgs and higgsino coupling matrices on the right-hand side of (A10). If we now ignore the small difference between the Higgs and higgsino coupling matrices in the RGE, and keep in mind that the matrices $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}^Q$ on the right-hand side are almost the unit matrices, it is not difficult to see that the gaugino coupling matrices $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}^Q$ remain Hermitian with, but only with, our approximations. A similar, but somewhat more involved,

argument also holds for the off-diagonal elements of $\tilde{\mathbf{g}}'^{u_R}$, where we have to look at all the terms in (A12) since, in the absence of down-type Yukawa matrices on the right-hand side, no single term dominates. Nonetheless, we have checked that, if the difference between Higgs boson and higgsino coupling matrices on the right-hand side can be ignored, and the gaugino coupling matrices can be approximated by unit matrices, the renormalization group evolution preserves the (approximate) hermiticity of $\tilde{\mathbf{g}}'^{u_R}$, explaining why the magnitudes of the corresponding off-diagonal elements are symmetric in the flavor indices.

Next, we turn to the off-diagonal elements of the $\tilde{\mathbf{f}}$ matrices. We know that these will deviate from the elements of the corresponding Yukawa coupling matrix only below the threshold at $Q = m_H$, but have magnitudes similar to the corresponding Yukawa coupling matrix elements, whose absolute values are shown in Fig. 1. It is the *difference* between the couplings of Higgs bosons and higgsinos that will be the main focus of our attention. In Fig. 3, we show the evolution of the real and imaginary parts of the (1,3) element of (i) the Yukawa coupling matrix, $\mathbf{f}_u$ for $Q > m_H$ and $\boldsymbol{\lambda}_u / \sin\beta$ (which connects continuously to $\mathbf{f}_u$) for $Q < m_H$, and (ii) the higgsino coupling matrices $\tilde{\mathbf{f}}_u^Q$, and $\tilde{\mathbf{f}}_u^{u_R}$ whose evolution is given by (A4) and (A7), respectively, of the Appendix. There is no particular reason for our choice of the (1,3) element (which happens to have a comparable real and imaginary piece) for the illustration in the figure. Here we have focused on the lower end of Q, where the $\mathbf{f}_u$ and $\tilde{\mathbf{f}}_u^\Phi$ couplings are different, while the inset shows the evolution all the way to M_{GUT}. Several points may be
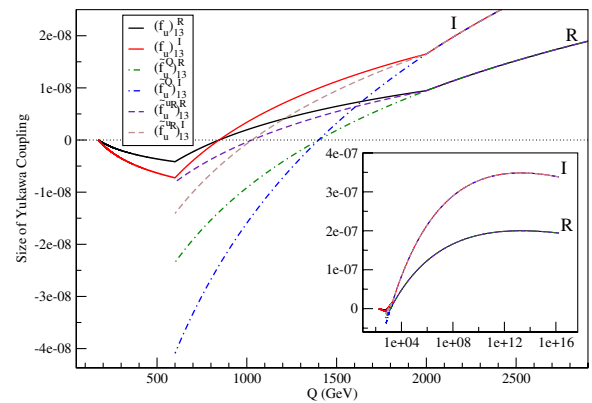

FIG. 3 (color online). The real (R) and imaginary (I) part of the (1,3) element of the up-quark Yukawa coupling matrix, $\mathbf{f}_u$, along with the corresponding elements of the matrices $\tilde{\mathbf{f}}_u^Q$ and $\tilde{\mathbf{f}}_u^{u_R}$ for the MSSM with the spectrum as in Fig. 1. For the solid (black and red) lines, we plot the elements of the matrix $\boldsymbol{\lambda}_u / \sin\beta$ below $Q = m_H$, as in Fig. 1. The main figure zooms in on the low end of the range of Q, where the Higgs boson and higgsino coupling differ from one another, while the inset shows the evolution all the way to M_{GUT}.

worthy of notice.

  (i) For $Q > m_H$ where the effective theory is supersymmetric, we see that the real and imaginary parts of the Higgs boson and higgsino couplings separately come together as expected.

  (ii) The reader can easily check that the ratio of the real and imaginary parts of the Higgs boson Yukawa couplings is independent of the scale, as we had discussed in Sec. VA.

  (iii) For $Q < m_H$, the higgsino couplings are split from the corresponding Higgs boson couplings as well as from one another by a factor of several. For instance, at the scale of squark masses, the real (imaginary) parts of the (1,3) element of both $\tilde{\mathbf{f}}_u^Q$ and $\tilde{\mathbf{f}}_u^{u_R}$ are quite different from the real (imaginary) parts of $(\mathbf{f}_u)_{13}$. It seems to us that the use of the evolved Higgs boson coupling in place of the corresponding higgsino coupling could be a poor approximation.

  (iv) Notice that while the real and imaginary parts of $(\tilde{\mathbf{f}}_u^Q)_{13}$ and $(\tilde{\mathbf{f}}_u^{u_R})_{13}$ come to zero at the same point, the position of the zero differs for the two couplings.

Up to now we have focused our attention on flavor off-diagonal couplings. Before closing our discussion, we briefly consider the effect of the thresholds on the evolution of flavor-diagonal couplings. As an illustration, we show the evolution of the hypercharge gauge coupling g' and the (3,3) elements of the hypercharge gaugino coupling matrices $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}'^{u_R}$ in Fig. 4.

The black solid line denotes the result of our complete calculation of the gauge coupling including both threshold and two-loop effects. Also shown by green dashed and blue dotted lines are the corresponding results obtained at the one- and two-loop levels, but ignoring threshold effects. The evolution of the (3,3) element of $\tilde{\mathbf{g}}'^Q$ breaks away from the evolution of g' at $Q = m_H$, and is shown as the violet dot-dashed curve that terminates at $Q = 600$ GeV. The insets on the left and right show zooms of these curves at the low and high ends, respectively. In the inset on the left, we have also shown the evolution of the (3,3) element of $\tilde{\mathbf{g}}'^{u_R}$ as the orange short-dashed line. We have checked that all the diagonal elements of both $\tilde{\mathbf{g}}'^Q$ and $\tilde{\mathbf{g}}'^{u_R}$ have a similar behavior. Moreover, the (1,1) and (2,2) elements all evolve essentially together with only the (3,3) elements split from these due to top Yukawa couplings.[22] The gauge coupling curves use the measured value of the coupling at the low scale as the boundary condition, while the gaugino coupling curves are obtained assuming that the corresponding matrix equals the hypercharge coupling times

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[22]In our illustration, where we have $\tan\beta = 4$, the curves for the (1,1) and (2,2) elements fall in between the dot-dashed violet curve for $(\tilde{\mathbf{g}}'^Q)_{33}$ and the orange-dashed curve for $(\tilde{\mathbf{g}}'^{u_R})_{33}$ in Fig. 4.
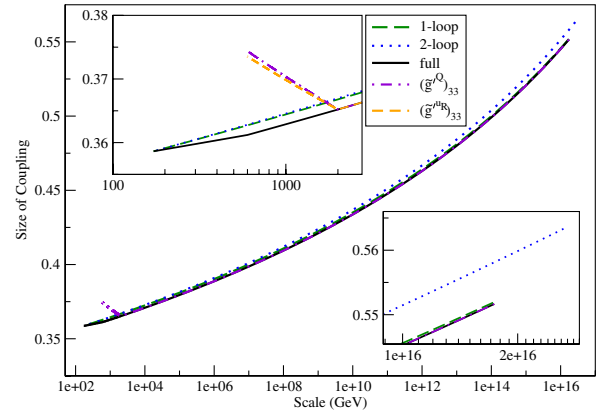

FIG. 4 (color online). Evolution of the hypercharge gauge coupling at the one-loop (green dashed line) and two-loop (blue dotted line) levels (excluding threshold effects), and with the full calculation (solid black line). Also shown is the evolution of the (3,3) element of $\tilde{\mathbf{g}}'^Q$ (and $\tilde{\mathbf{g}}'^{u_R}$ in the left inset). The main figure shows the evolution of these couplings between $Q = m_t$ and $Q = M_{\text{GUT}}$, while the insets on the left and right zoom in on the range of Q near the TeV scale and near M_{GUT}, respectively.

the unit matrix at $Q = m_H$. Several points are worth noting:

(1) Since the value of the gauge coupling is fixed at $Q = m_t$ in our illustration, the difference between one- and two-loop evolution (without threshold effects) is seen at the high Q end where the curves are terminated at $Q = M_{\text{GUT}}$, defined as the point where the $SU(2)$ and the scaled hypercharge couplings meet. Notice that M_{GUT} is different in the two cases, as is the value of the gauge coupling at $Q = M_{\text{GUT}}$. This relative difference is $\mathcal{O}(\frac{1}{16\pi^2})$, roughly the expected magnitude of a higher loop effect. The full calculation of the evolution depicted by the solid black line differs the most from the other two curves at the low end, because we start off with the evolution using the β function of the SM and, as we pass through the various thresholds, join up at $Q = m_H$ to the MSSM evolution. Beyond $Q = m_H$, the full and two-loop curves evolve with the same β function, but the latter ends up lower because threshold effects caused it to start off lower at $Q = m_H$, as can be seen in the inset on the left. That it ends up so close to the one-loop curve is a coincidence, but its proximity reflects the conventional wisdom that two-loop effects in the evolution are numerically comparable in magnitude to the threshold corrections in the evolution at one loop.

(2) Turning to the gaugino couplings at the low end, the most striking feature is that, below $Q = m_H$, these evolve in the opposite direction to the gauge couplings. This behavior can be understood if we recognize that for $Q < m_{\tilde{g}}$ *the evolution of the diagonal gaugino couplings now depends on the much larger*

gluon coupling even at the one-loop level. As we can see from (A10), the terms involving $\tilde{\mathbf{g}}_s^Q$ cancel the terms depending on the QCD coupling constant g_s if $Q > m_{\tilde{g}}$. However, for $Q < m_{\tilde{g}}$ the terms with $\theta_{\tilde{g}}$ are no longer operative, and this cancellation is incomplete, causing a large change in the β function, and hence in the slope of the curve. It is striking to see that, even though we have maintained both thresholds not far from the TeV scale, the corresponding gaugino and gauge couplings can develop a difference of ~4%. The existence of a difference between a gauge boson coupling and the corresponding gaugino coupling has been discussed in Ref. [29] (although without any flavor structure), where it was suggested that its determination at an e^+e^- linear collider [30] would give an idea of the splitting between sparticle masses, even if the heavy sparticles are not kinematically accessible.

Finally, we turn to flavor-conserving Higgs and higgsino interactions. As an illustration, we show the evolution of the (3,3) elements of $\mathbf{f}_u$, of $\tilde{\mathbf{f}}_u^Q$, and of $\tilde{\mathbf{f}}_u^{u_R}$ in Fig. 5. Just as in the previous figure, we show results for the complete calculation at the two-loop level, including threshold effects as well as differences between the couplings of Higgs/gauge bosons and of higgsinos/gauginos by the solid black line. We also show the results that we obtain with just MSSM evolution all the way to $Q = m_t$ using one-loop RGEs (the green dashed curve) and two-loop RGEs (the blue dotted curve). In all three cases we start with the same value for the Yukawa coupling at $Q = m_t$. Also shown by the violet dot-dashed and orange short-dashed lines is the
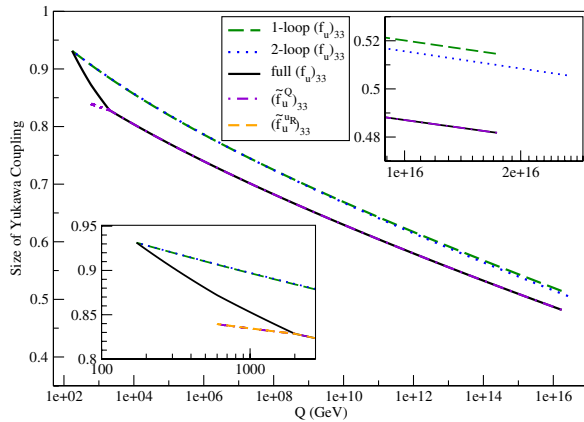

FIG. 5 (color online).   Evolution of the (3,3) element of the Yukawa coupling, $\mathbf{f}_u$, at the one-loop (green dashed line) and two-loop (blue dotted line) levels (excluding threshold effects), and with the full calculation (solid black line). Also shown is the evolution of the (3,3) element of $\tilde{\mathbf{f}}_u^Q$ (and $\tilde{\mathbf{f}}_u^{u_R}$ in the left inset). The main figure shows the evolution of these couplings between $Q = m_t$ and $Q = M_{\text{GUT}}$, while the insets on the left and right zoom in on the range of Q near the TeV scale and near M_{GUT}, respectively. As in Fig. 1, we have plotted $\boldsymbol{\lambda}_u / \sin\beta$ below $Q = m_H$.

evolution of the (3,3) element of the matrices $\tilde{\mathbf{f}}_u^Q$ and $\tilde{\mathbf{f}}_u^{u_R}$, respectively. The two insets are similar to those in the previous figure. We note the following:

(1) We see that the complete calculation of the Yukawa coupling leads to a large difference from the two-loop calculation without thresholds over most of the range of Q. This is not new and is largely due to the difference between the evolution of Yukawa couplings in the SM and in the effective theories that interpolate between the SM and the MSSM.[23] Below the kink at $Q = 600$ GeV in the solid black curve, the evolution of the Yukawa coupling is as in the SM, while above $Q = m_H$ it is as in the MSSM. Notice that in this case, threshold effects are considerably larger than the difference between one- and two-loop evolution.

(2) It can be clearly seen from the left inset that the higgsino couplings evolve quite differently from the corresponding Higgs boson couplings once Q is below m_H. Once again, this difference is largely due to incomplete cancellations in terms involving "strong" interaction couplings of gluons and gluinos, as may be seen from (A4) and (A7) of the Appendix.

VI. CONCLUDING REMARKS

RGEs provide the bridge that allows us to extract predictions of theories with simple physical principles valid at very high energy scales, many orders of magnitude larger than energies accessible in experiments. Because of renormalization effects, these same simple principles lead to complex predictions at accessible energies. Since supersymmetric theories allow sensible extrapolation to high energy, RGEs have played a central role in the analysis of many supersymmetric models, generally assumed to reduce to the MSSM (possibly augmented by right-handed neutrino superfields) in the range between the weak and GUT or Planck scales.

In this article, the first of a series of two papers, we have reexamined the threshold corrections to the RGEs for the dimensionless couplings of the MSSM, incorporating also the effects from flavor mixing of quarks and squarks. Above the scale of all new particle thresholds, the effective theory is the MSSM, with just three gauge couplings and three different Yukawa coupling matrices (that specify the interactions of matter fermions with Higgs bosons) being the independent dimensionless couplings of the theory. All

---

[23]Notice in the left inset that the solid black line "curves" significantly between $Q = 600$ GeV and $Q = m_t$. This highlights the advantage of our approach (also used in the event generator ISAJET), which "sums logs of the ratio of the low and high thresholds by solving the RGEs," over that sometimes used in the literature where MSSM evolution is used all the way down to $Q = m_t$ and then corrected for "via a single step" evolution to take into account the difference between the running in the MSSM and in the SM.

other dimensionless couplings, for instance those of gauginos or higgsinos to the fermion-sfermion system, or quartic scalar couplings, are related to these by supersymmetry. These relations are, however, no longer valid once supersymmetry is broken, so that then the couplings to gauginos and higgsinos will renormalize differently from gauge and Yukawa couplings, respectively. In a consistent treatment of threshold corrections, the RGEs for these couplings will therefore differ from the RGEs for the gauge and quark Yukawa couplings that are available in the literature.

We have adapted the RGEs for a general (i.e. nonsupersymmetric) field theory [14,22] and rewritten them in four-component spinor notation that we use when we obtain the RGEs of the MSSM. The details of our procedure may be found in Secs. II and III, and the application to the MSSM in Sec. IV. The complete set of RGEs for the dimensionless couplings between SM particles and their superpartners (but not for the quartic couplings of scalars) is given in the Appendix. These quartic couplings are less important from a phenomenological perspective, though some of these do enter the squark and slepton mass matrices. The important thing for the present discussion is that these do not enter into the RGEs for the fermion-fermion-scalar couplings, which can then be evolved independently from the quartic couplings.

We have presented some sample numerical results for the evolution of various couplings in Sec. V. The analysis here is meant only to give a sampling of effects that, to our knowledge, have not been previously pointed out in the literature. Since flavor physics in the sparticle sector has been the main motivation for our analysis, we point out that when threshold effects are included, gauginos are not different from higgsinos in that neutral gauginos also develop flavor-changing couplings to quarks and squarks. As seen from Fig. 2, while these flavor-violating gaugino couplings vary over several orders of magnitude, they can be comparable, or even larger, than the corresponding couplings to Higgs bosons. An illustration of flavor-violating higgsino couplings is shown in Fig. 3. The scale dependence of these is quite different from that of the usual Yukawa couplings of quarks to Higgs bosons. We stress that the induced flavor-violating couplings of gauginos and higgsinos, evaluated at the scale of any particular squark mass, will contribute to the amplitude for the flavor-violating decay of this squark. Such a contribution is distinct from the usually evaluated contribution from the induced flavor violation in the squark mass matrix. Both types of contributions need to be included. Finally, we point out one last striking feature of our analysis. We see from Figs. 4 and 5 that the inclusion of SUSY threshold corrections leads to a difference of a few percent between the flavor-diagonal electroweak couplings of gauginos and gauge bosons, and of higgsinos and Higgs bosons, traced to SUSY QCD contributions in even the one-loop RGEs. This could have a noticeable effect on the evaluation of sparticle masses as well as of thermal relic densities of neutralino dark matter created in the big bang.

In summary, we have presented the RGEs of the dimensionless couplings of the MSSM in this paper. In a follow-up paper under preparation [21], we will present the RGEs for the dimensionful parameters, again including threshold and flavor-mixing effects. This complete set will facilitate the examination of flavor phenomenology in SUSY models with arbitrary ansätze for flavor in the SSB sector.

ACKNOWLEDGMENTS

APPENDIX

$$
\begin{aligned}
(4\pi)^2 \frac{d(s\mathbf{f}_u)_{ij}}{dt} =\ & \frac{s}{2}\{3[s^2\theta_h + c^2\theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik} + [c^2\theta_h + s^2\theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik} + 4c^2[-\theta_h + \theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik}\}(\mathbf{f}_u)_{kj} \\
& + s(\mathbf{f}_u)_{ik}\left[\theta_{\tilde h}\theta_{\tilde Q_l}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_u^Q)_{lj} + \frac{4}{9}\theta_{\tilde B}\theta_{\tilde u_l}(\tilde{\mathbf{g}}^{\prime u_R})_{kl}^*(\tilde{\mathbf{g}}^{\prime u_R})_{lj}^T + \frac{4}{3}\theta_{\tilde g}\theta_{\tilde u_l}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^*(\tilde{\mathbf{g}}_s^{u_R})_{lj}^T \right] \\
& + \frac{s}{4}\left[2\theta_{\tilde h}\theta_{\tilde u_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^\dagger + 2\theta_{\tilde h}\theta_{\tilde d_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^\dagger + 3\theta_{\tilde W}\theta_{\tilde Q_k}(\tilde{\mathbf{g}}^Q)_{ik}^T(\tilde{\mathbf{g}}^Q)_{kl}^* + \frac{1}{9}\theta_{\tilde B}\theta_{\tilde Q_k}(\tilde{\mathbf{g}}^{\prime Q})_{ik}^T(\tilde{\mathbf{g}}^{\prime Q})_{kl}^* \right. \\
& \left. + \frac{16}{3}\theta_{\tilde g}\theta_{\tilde Q_k}(\tilde{\mathbf{g}}_s^Q)_{ik}^T(\tilde{\mathbf{g}}_s^Q)_{kl}^* \right](\mathbf{f}_u)_{lj} + s\theta_{\tilde h}\theta_{\tilde Q_k}\left[-3\theta_{\tilde W}(\tilde g^{h_u})^*(\tilde{\mathbf{g}}^Q)_{ik}^T + \frac{1}{3}\theta_{\tilde B}(\tilde g^{\prime h_u})^*(\tilde{\mathbf{g}}^{\prime Q})_{ik}^T \right](\tilde{\mathbf{f}}_u^Q)_{kj} \\
& - \frac{4}{3}s\theta_{\tilde B}\theta_{\tilde h}\theta_{\tilde u_k}(\tilde g^{\prime h_u})^*(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{g}}^{\prime u_R})_{kj}^T + s(\mathbf{f}_u)_{ij}[(s^2\theta_h + c^2\theta_H)\,\mathrm{Tr}\{3\mathbf{f}_u^\dagger\mathbf{f}_u\} + c^2(\theta_h - \theta_H)\,\mathrm{Tr}\{3\mathbf{f}_d^\dagger\mathbf{f}_d + \mathbf{f}_e^\dagger\mathbf{f}_e\}] \\
& + \frac{s}{2}\theta_{\tilde h}(\mathbf{f}_u)_{ij}\{3\theta_{\tilde W}[|\tilde g^{h_u}|^2(s^2\theta_h + c^2\theta_H) + |\tilde g^{h_d}|^2(c^2\theta_h - c^2\theta_H)] \\
& + \theta_{\tilde B}[|\tilde g^{\prime h_u}|^2(s^2\theta_h + c^2\theta_H) + |\tilde g^{\prime h_d}|^2(c^2\theta_h - c^2\theta_H)]\} - s(\mathbf{f}_u)_{ij}\left[\frac{17}{12}g^{\prime 2} + \frac{9}{4}g_2^2 + 8g_s^2 \right],
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(c\mathbf{f}_d)_{ij}}{dt} = {}& \frac{c}{2}\{3[c^2\theta_h + s^2\theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik} + [s^2\theta_h + c^2\theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik} + 4s^2[-\theta_h + \theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik}\}(\mathbf{f}_d)_{kj} \\
& + c(\mathbf{f}_d)_{ik}\left[\theta_{\tilde{h}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_d^Q)_{lj} + \frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}'^{d_R})_{kl}^*(\tilde{\mathbf{g}}'^{d_R})_{lj}^T + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^*(\tilde{\mathbf{g}}_s^{d_R})_{lj}^T\right] \\
& + \frac{c}{4}\left[2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^\dagger + 2\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^\dagger + 3\theta_{\tilde{W}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^Q)_{ik}^T(\tilde{\mathbf{g}}^Q)_{kl}^* + \frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}'^Q)_{ik}^T(\tilde{\mathbf{g}}'^Q)_{kl}^*\right. \\
& + \left.\frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)_{ik}^T(\tilde{\mathbf{g}}_s^Q)_{kl}^*\right](\mathbf{f}_d)_{lj} + c\theta_{\tilde{h}}\theta_{\tilde{Q}_k}\left[-3\theta_{\tilde{W}}(\tilde{g}^{h_d})^*(\tilde{\mathbf{g}}^Q)_{ik}^T - \frac{1}{3}\theta_{\tilde{B}}(\tilde{g}'^{h_d})^*(\tilde{\mathbf{g}}'^Q)_{ik}^T\right](\tilde{\mathbf{f}}_d^Q)_{kj} \\
& - \frac{2}{3}c\theta_{\tilde{B}}\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{g}'^{h_d})^*(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{g}}'^{d_R})_{kj}^T + c(\mathbf{f}_d)_{ij}[s^2(\theta_h - \theta_H)\mathrm{Tr}\{3\mathbf{f}_u^\dagger\mathbf{f}_u\} + (c^2\theta_h + s^2\theta_H)\mathrm{Tr}\{3\mathbf{f}_d^\dagger\mathbf{f}_d + \mathbf{f}_e^\dagger\mathbf{f}_e\}] \\
& + \frac{c}{2}\theta_{\tilde{h}}(\mathbf{f}_d)_{ij}\{3\theta_{\tilde{W}}[|\tilde{g}^{h_u}|^2(s^2\theta_h - s^2\theta_H) + |\tilde{g}^{h_d}|^2(c^2\theta_h + s^2\theta_H)] \\
& + \theta_{\tilde{B}}[|\tilde{g}'^{h_u}|^2(s^2\theta_h - s^2\theta_H) + |\tilde{g}'^{h_d}|^2(c^2\theta_h + s^2\theta_H)]\} - c(\mathbf{f}_d)\left[\frac{5}{12}g'^2 + \frac{9}{4}g_2^2 + 8g_s^2\right], \quad\quad (\mathrm{A2})
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(c\mathbf{f}_e)_{ij}}{dt} = {}& \frac{3}{2}c[c^2\theta_h + s^2\theta_H](\mathbf{f}_e\mathbf{f}_e^\dagger\mathbf{f}_e)_{ij} + c(\mathbf{f}_e)_{ik}[\theta_{\tilde{h}}\theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)_{kl}^\dagger(\tilde{\mathbf{f}}_e^L)_{lj} + \theta_{\tilde{B}}\theta_{\tilde{e}_l}(\tilde{\mathbf{g}}'^{e_R})_{kl}^*(\tilde{\mathbf{g}}'^{e_R})_{lj}^T] \\
& + \frac{c}{4}[2\theta_{\tilde{h}}\theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{ik}(\tilde{\mathbf{f}}_e^{e_R})_{kl}^\dagger + 3\theta_{\tilde{W}}\theta_{\tilde{L}_k}(\tilde{\mathbf{g}}^L)_{ik}^T(\tilde{\mathbf{g}}^L)_{kl}^* + \theta_{\tilde{B}}\theta_{\tilde{L}_k}(\tilde{\mathbf{g}}'^L)_{ik}^T(\tilde{\mathbf{g}}'^L)_{kl}^*](\mathbf{f}_e)_{lj} \\
& + c\theta_{\tilde{h}}\theta_{\tilde{L}_k}[-3\theta_{\tilde{W}}(\tilde{g}^{h_d})^*(\tilde{\mathbf{g}}^L)_{ik}^T + \theta_{\tilde{B}}(\tilde{g}'^{h_d})^*(\tilde{\mathbf{g}}'^L)_{ik}^T](\tilde{\mathbf{f}}_e^L)_{kj} - 2c\theta_{\tilde{B}}\theta_{\tilde{h}}\theta_{\tilde{e}_k}(\tilde{g}'^{h_d})^*(\tilde{\mathbf{f}}_e^{e_R})_{ik}(\tilde{\mathbf{g}}'^{e_R})_{kj}^T \\
& + c(\mathbf{f}_e)_{ij}[s^2(\theta_h - \theta_H)\mathrm{Tr}\{3\mathbf{f}_u^\dagger\mathbf{f}_u\} + (c^2\theta_h + s^2\theta_H)\mathrm{Tr}\{3\mathbf{f}_d^\dagger\mathbf{f}_d + \mathbf{f}_e^\dagger\mathbf{f}_e\}] \\
& + \frac{c}{2}\theta_{\tilde{h}}(\mathbf{f}_e)_{ij}\{3\theta_{\tilde{W}}[|\tilde{g}^{h_u}|^2(s^2\theta_h - s^2\theta_H) + |\tilde{g}^{h_d}|^2(c^2\theta_h + s^2\theta_H)] \\
& + \theta_{\tilde{B}}[|\tilde{g}'^{h_u}|^2(s^2\theta_h - s^2\theta_H) + |\tilde{g}'^{h_d}|^2(c^2\theta_h + s^2\theta_H)]\} - c(\mathbf{f}_e)_{ij}\left[\frac{15}{4}g'^2 + \frac{9}{4}g_2^2\right], \quad\quad (\mathrm{A3})
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_u^Q)_{ij}}{dt} = {}& [s^2\theta_h + c^2\theta_H](\tilde{\mathbf{f}}_u^Q)_{ik}(\mathbf{f}_u^\dagger\mathbf{f}_u)_{kj} + (\tilde{\mathbf{f}}_u^Q)_{ik}\left[\frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}'^{u_R})_{kl}^*(\tilde{\mathbf{g}}'^{u_R})_{lj}^T + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^*(\tilde{\mathbf{g}}_s^{u_R})_{lj}^T + \theta_{\tilde{h}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_u^Q)_{lj}\right] \\
& + \frac{3}{2}\theta_{\tilde{h}}(\tilde{\mathbf{f}}_u^Q)_{ij}[\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^\dagger(\tilde{\mathbf{f}}_u^{u_R})_{lk} + \theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_u^Q)_{lk}] + \frac{1}{4}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H](\tilde{\mathbf{f}}_u^Q)_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_u}|^2 + \theta_{\tilde{B}}|\tilde{g}'^{h_u}|^2\} \\
& + [s^2\theta_h + c^2\theta_H]\left\{-3\theta_{\tilde{W}}\tilde{g}^{h_u}(\tilde{\mathbf{g}}^Q)_{ik}^* + \frac{1}{3}\theta_{\tilde{B}}\tilde{g}'^{h_u}(\tilde{\mathbf{g}}'^Q)_{ik}^*\right\}(\mathbf{f}_u)_{kj} - \frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}'^Q)_{ik}^*(\tilde{\mathbf{f}}_u^{u_R})_{kl}(\tilde{\mathbf{g}}'^{u_R})_{lj}^T \\
& - \frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_l}(\tilde{\mathbf{g}}_s^Q)_{ik}^*(\tilde{\mathbf{f}}_u^{u_R})_{kl}(\tilde{\mathbf{g}}_s^{u_R})_{lj}^T + \theta_{\tilde{Q}_l}\left[\frac{3}{2}\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ik}^*(\tilde{\mathbf{g}}^Q)_{kl}^T + \frac{1}{18}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ik}^*(\tilde{\mathbf{g}}'^Q)_{kl}^T + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^Q)_{ik}^*(\tilde{\mathbf{g}}_s^Q)_{kl}^T\right](\tilde{\mathbf{f}}_u^Q)_{lj} \\
& + \theta_{\tilde{h}}\theta_{\tilde{Q}_l}[(\tilde{\mathbf{f}}_u^Q)_{ik}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger + (\tilde{\mathbf{f}}_d^Q)_{ik}(\tilde{\mathbf{f}}_d^Q)_{kl}^\dagger](\tilde{\mathbf{f}}_u^Q)_{lj} - (\tilde{\mathbf{f}}_u^Q)_{ij}\left[\frac{25}{12}g'^2 + \frac{9}{4}g_2^2 + 4g_s^2\right], \quad\quad (\mathrm{A4})
\end{aligned}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_d^Q)_{ij}}{dt} = {}& [c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_d^Q)_{ik}(\mathbf{f}_d^\dagger\mathbf{f}_d)_{kj} + (\tilde{\mathbf{f}}_d^Q)_{ik}\left[\frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}'^{d_R})_{kl}^*(\tilde{\mathbf{g}}'^{d_R})_{lj}^T + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^*(\tilde{\mathbf{g}}_s^{d_R})_{lj}^T + \theta_{\tilde{h}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_d^Q)_{lj}\right] \\
& + \frac{1}{2}\theta_{\tilde{h}}(\tilde{\mathbf{f}}_d^Q)_{ij}[3\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^\dagger(\tilde{\mathbf{f}}_d^{d_R})_{lk} + \theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{kl}^\dagger(\tilde{\mathbf{f}}_e^{e_R})_{lk} + 3\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)_{kl}^\dagger(\tilde{\mathbf{f}}_d^Q)_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)_{kl}^\dagger(\tilde{\mathbf{f}}_e^L)_{lk}] \\
& + \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_d^Q)_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}'^{h_d}|^2\} + [c^2\theta_h + s^2\theta_H] \\
& \times \left\{-3\theta_{\tilde{W}}\tilde{g}^{h_d}(\tilde{\mathbf{g}}^Q)_{ik}^* - \frac{1}{3}\theta_{\tilde{B}}\tilde{g}'^{h_d}(\tilde{\mathbf{g}}'^Q)_{ik}^*\right\}(\mathbf{f}_d)_{kj} + \frac{2}{9}\theta_{\tilde{B}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}'^Q)_{ik}^*(\tilde{\mathbf{f}}_d^{d_R})_{kl}(\tilde{\mathbf{g}}'^{d_R})_{lj}^T \\
& - \frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{d}_l}(\tilde{\mathbf{g}}_s^Q)_{ik}^*(\tilde{\mathbf{f}}_d^{d_R})_{kl}(\tilde{\mathbf{g}}_s^{d_R})_{lj}^T + \theta_{\tilde{Q}_l}\left[\frac{3}{2}\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ik}^*(\tilde{\mathbf{g}}^Q)_{kl}^T + \frac{1}{18}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ik}^*(\tilde{\mathbf{g}}'^Q)_{kl}^T + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^Q)_{ik}^*(\tilde{\mathbf{g}}_s^Q)_{kl}^T\right](\tilde{\mathbf{f}}_d^Q)_{lj} \\
& + \theta_{\tilde{h}}\theta_{\tilde{Q}_l}[(\tilde{\mathbf{f}}_u^Q)_{ik}(\tilde{\mathbf{f}}_u^Q)_{kl}^\dagger + (\tilde{\mathbf{f}}_d^Q)_{ik}(\tilde{\mathbf{f}}_d^Q)_{kl}^\dagger](\tilde{\mathbf{f}}_d^Q)_{lj} - (\tilde{\mathbf{f}}_d^Q)_{ij}\left[\frac{13}{12}g'^2 + \frac{9}{4}g_2^2 + 4g_s^2\right], \quad\quad (\mathrm{A5})
\end{aligned}
$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_e^L)_{ij}}{dt} = [c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_e^L)_{ik}(\mathbf{f}_e^\dagger \mathbf{f}_e)_{kj} + (\tilde{\mathbf{f}}_e^L)_{ik}[\theta_{\tilde{B}}\theta_{\tilde{e}_l}(\tilde{\mathbf{g}}^{\prime e_R})^*_{kl}(\tilde{\mathbf{g}}^{\prime e_R})^T_{lj} + \theta_{\tilde{h}}\theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)^\dagger_{kl}(\tilde{\mathbf{f}}_e^L)_{lj}]$$

$$+ \frac{1}{2}\theta_{\tilde{h}}(\tilde{\mathbf{f}}_e^L)_{ij}[3\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl}(\tilde{\mathbf{f}}_d^{d_R})_{lk} + \theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})^\dagger_{kl}(\tilde{\mathbf{f}}_e^{e_R})_{lk} + 3\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)^\dagger_{kl}(\tilde{\mathbf{f}}_d^Q)_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)^\dagger_{kl}(\tilde{\mathbf{f}}_e^L)_{lk}]$$

$$+ \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_e^L)_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_d}|^2\} + [c^2\theta_h + s^2\theta_H]$$

$$\times \{-3\theta_{\tilde{W}}\tilde{g}^{h_d}(\tilde{\mathbf{g}}^L)^*_{ik} + \theta_{\tilde{B}}\tilde{g}^{\prime h_d}(\tilde{\mathbf{g}}^{\prime L})^*_{ik}\}(\mathbf{f}_e)_{kj} - 2\theta_{\tilde{B}}\theta_{\tilde{e}_l}(\tilde{\mathbf{g}}^{\prime L})^*_{ik}(\tilde{\mathbf{f}}_e^{e_R})_{kl}(\tilde{\mathbf{g}}^{\prime e_R})^T_{lj}$$

$$+ \frac{1}{2}\theta_{\tilde{L}_l}[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^L)^*_{ik}(\tilde{\mathbf{g}}^L)^T_{kl} + \theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime L})^*_{ik}(\tilde{\mathbf{g}}^{\prime L})^T_{kl}](\tilde{\mathbf{f}}_e^L)_{lj} + \theta_{\tilde{h}}\theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)_{ik}(\tilde{\mathbf{f}}_e^L)^\dagger_{kl}(\tilde{\mathbf{f}}_e^L)_{lj} - (\tilde{\mathbf{f}}_e^L)_{ij}\left[\frac{15}{4}g^{\prime 2} + \frac{9}{4}g_2^2\right],$$

$$(\mathrm{A6})$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_u^{u_R})_{ij}}{dt} = \frac{3}{2}\theta_{\tilde{h}}[\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{lk}(\tilde{\mathbf{f}}_u^{u_R})^\dagger_{kl} + \theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_u^Q)_{lk}(\tilde{\mathbf{f}}_u^Q)^\dagger_{kl}](\tilde{\mathbf{f}}_u^{u_R})_{ij} + \frac{1}{4}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H](\tilde{\mathbf{f}}_u^{u_R})_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_u}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_u}|^2\}$$

$$+ \frac{1}{2}\{[s^2\theta_h + c^2\theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik} + [c^2\theta_h + s^2\theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik}\}(\tilde{\mathbf{f}}_u^{u_R})_{kj}$$

$$+ \frac{1}{2}\left[\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})^\dagger_{kl} + \theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^Q)^T_{ik}(\tilde{\mathbf{g}}^Q)^*_{kl} + \frac{1}{18}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^{\prime Q})^T_{ik}(\tilde{\mathbf{g}}^{\prime Q})^*_{kl}\right.$$

$$+ \left.\frac{8}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)^T_{ik}(\tilde{\mathbf{g}}_s^Q)^*_{kl}\right](\tilde{\mathbf{f}}_u^{u_R})_{lj} - \frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^{\prime Q})^T_{ik}(\tilde{\mathbf{f}}_u^Q)_{kl}(\tilde{\mathbf{g}}^{\prime u_R})^*_{lj} - \frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)^T_{ik}(\tilde{\mathbf{f}}_u^Q)_{kl}(\tilde{\mathbf{g}}_s^{u_R})^*_{lj}$$

$$- \frac{4}{3}\theta_{\tilde{B}}[s^2\theta_h + c^2\theta_H]\tilde{g}^{\prime h_u}(\mathbf{f}_u)_{ik}(\tilde{\mathbf{g}}^{\prime u_R})^*_{kj} + 2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})^\dagger_{kl}(\tilde{\mathbf{f}}_u^{u_R})_{lj}$$

$$+ \theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}\left[\frac{8}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime u_R})^T_{kl}(\tilde{\mathbf{g}}^{\prime u_R})^*_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{u_R})^T_{kl}(\tilde{\mathbf{g}}_s^{u_R})^*_{lj}\right] - (\tilde{\mathbf{f}}_u^{u_R})_{ij}\left[\frac{5}{6}g^{\prime 2} + \frac{9}{2}g_2^2 + 4g_s^2\right],$$

$$(\mathrm{A7})$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_d^{d_R})_{ij}}{dt} = \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{lk}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl} + 3\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)_{lk}(\tilde{\mathbf{f}}_d^Q)^\dagger_{kl} + \theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{lk}(\tilde{\mathbf{f}}_e^{e_R})^\dagger_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)_{lk}(\tilde{\mathbf{f}}_e^L)^\dagger_{kl}](\tilde{\mathbf{f}}_d^{d_R})_{ij}$$

$$+ \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_d^{d_R})_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_d}|^2\} + \frac{1}{2}\{[s^2\theta_h + c^2\theta_H](\mathbf{f}_u\mathbf{f}_u^\dagger)_{ik}$$

$$+ [c^2\theta_h + s^2\theta_H](\mathbf{f}_d\mathbf{f}_d^\dagger)_{ik}\}(\tilde{\mathbf{f}}_d^{d_R})_{kj} + \frac{1}{2}\left[\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{f}}_u^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})^\dagger_{kl} + \theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^Q)^T_{ik}(\tilde{\mathbf{g}}^Q)^*_{kl}\right.$$

$$+ \left.\frac{1}{18}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^{\prime Q})^T_{ik}(\tilde{\mathbf{g}}^{\prime Q})^*_{kl} + \frac{8}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)^T_{ik}(\tilde{\mathbf{g}}_s^Q)^*_{kl}\right](\tilde{\mathbf{f}}_d^{d_R})_{lj} + \frac{2}{9}\theta_{\tilde{B}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}^{\prime Q})^T_{ik}(\tilde{\mathbf{f}}_d^Q)_{kl}(\tilde{\mathbf{g}}^{\prime d_R})^*_{lj}$$

$$- \frac{16}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{g}}_s^Q)^T_{ik}(\tilde{\mathbf{f}}_d^Q)_{kl}(\tilde{\mathbf{g}}_s^{d_R})^*_{lj} - \frac{2}{3}\theta_{\tilde{B}}[c^2\theta_h + s^2\theta_H]\tilde{g}^{\prime h_d}(\mathbf{f}_d)_{ik}(\tilde{\mathbf{g}}^{\prime d_R})^*_{kj} + 2\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl}(\tilde{\mathbf{f}}_d^{d_R})_{lj}$$

$$+ \theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{ik}\left[\frac{2}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime d_R})^T_{kl}(\tilde{\mathbf{g}}^{\prime d_R})^*_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{d_R})^T_{kl}(\tilde{\mathbf{g}}_s^{d_R})^*_{lj}\right] - (\tilde{\mathbf{f}}_d^{d_R})_{ij}\left[\frac{5}{6}g^{\prime 2} + \frac{9}{2}g_2^2 + 4g_s^2\right],$$

$$(\mathrm{A8})$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{f}}_e^{e_R})_{ij}}{dt} = \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{d}_k}(\tilde{\mathbf{f}}_d^{d_R})_{lk}(\tilde{\mathbf{f}}_d^{d_R})^\dagger_{kl} + 3\theta_{\tilde{Q}_l}(\tilde{\mathbf{f}}_d^Q)_{lk}(\tilde{\mathbf{f}}_d^Q)^\dagger_{kl} + \theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{lk}(\tilde{\mathbf{f}}_e^{e_R})^\dagger_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)_{lk}(\tilde{\mathbf{f}}_e^L)^\dagger_{kl}](\tilde{\mathbf{f}}_e^{e_R})_{ij}$$

$$+ \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H](\tilde{\mathbf{f}}_e^{e_R})_{ij}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_d}|^2\} + \frac{1}{2}[c^2\theta_h + s^2\theta_H](\mathbf{f}_e\mathbf{f}_e^\dagger)_{ik}(\tilde{\mathbf{f}}_e^{e_R})_{kj}$$

$$+ \frac{1}{2}\left[\theta_{\tilde{h}}\theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{ik}(\tilde{\mathbf{f}}_e^{e_R})^\dagger_{kl} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{L}_k}(\tilde{\mathbf{g}}^L)^T_{ik}(\tilde{\mathbf{g}}^L)^*_{kl} + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{L}_k}(\tilde{\mathbf{g}}^{\prime L})^T_{ik}(\tilde{\mathbf{g}}^{\prime L})^*_{kl}\right](\tilde{\mathbf{f}}_e^{e_R})_{lj}$$

$$- 2\theta_{\tilde{B}}\theta_{\tilde{L}_k}(\tilde{\mathbf{g}}^{\prime L})^T_{ik}(\tilde{\mathbf{f}}_e^L)_{kl}(\tilde{\mathbf{g}}^{\prime e_R})^*_{lj} - 2\theta_{\tilde{B}}[c^2\theta_h + s^2\theta_H]\tilde{g}^{\prime h_d}(\mathbf{f}_e)_{ik}(\tilde{\mathbf{g}}^{\prime e_R})^*_{kj} + 2\theta_{\tilde{h}}\theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{ik}(\tilde{\mathbf{f}}_e^{e_R})^\dagger_{kl}(\tilde{\mathbf{f}}_e^{e_R})_{lj}$$

$$+ 2\theta_{\tilde{B}}\theta_{\tilde{e}_k}(\tilde{\mathbf{f}}_e^{e_R})_{ik}(\tilde{\mathbf{g}}^{\prime e_R})^T_{kl}(\tilde{\mathbf{g}}^{\prime e_R})^*_{lj} - (\tilde{\mathbf{f}}_e^{e_R})_{ij}\left[\frac{3}{2}g^{\prime 2} + \frac{9}{2}g_2^2\right],$$

$$(\mathrm{A9})$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}'^Q)_{ij}}{dt} = \frac{1}{2}\theta_{\tilde{B}}\bigg[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)_{lk}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)_{lk}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{lk}(\tilde{\mathbf{g}}'^{u_R})^{\dagger}_{kl} + \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}'^{d_R})_{lk}(\tilde{\mathbf{g}}'^{d_R})^{\dagger}_{kl}$$

$$+ 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}'^{e_R})_{lk}(\tilde{\mathbf{g}}'^{e_R})^{\dagger}_{kl}\bigg](\tilde{\mathbf{g}}'^Q)_{ij} + \frac{1}{2}\theta_{\tilde{h}}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}'^{h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}'^{h_d}|^2\}$$

$$+ \frac{1}{2}(\tilde{\mathbf{g}}'^Q)_{ik}\{[s^2\theta_h + c^2\theta_H](\mathbf{f}_u^*\mathbf{f}_u^T)_{kj} + [c^2\theta_h + s^2\theta_H](\mathbf{f}_d^*\mathbf{f}_d^T)_{kj}\}$$

$$+ \frac{1}{2}(\tilde{\mathbf{g}}'^Q)_{ik}\bigg[\theta_{\tilde{h}}\theta_{\tilde{u}_l}(\tilde{\mathbf{f}}_u^{u_R})^*_{kl}(\tilde{\mathbf{f}}_u^{u_R})^T_{lj} + \theta_{\tilde{h}}\theta_{\tilde{d}_l}(\tilde{\mathbf{f}}_d^{d_R})^*_{kl}(\tilde{\mathbf{f}}_d^{d_R})^T_{lj} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}^Q)_{lj}$$

$$+ \frac{1}{18}\theta_{\tilde{B}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^Q)_{lj} + \frac{8}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}_s^Q)_{lj}\bigg]$$

$$+ 4\theta_{\tilde{h}}[-2\theta_{\tilde{u}_l}(\tilde{\mathbf{f}}_u^Q)^*_{ik}(\tilde{\mathbf{g}}'^{u_R})_{kl}(\tilde{\mathbf{f}}_u^{u_R})^T_{lj} + \theta_{\tilde{d}_l}(\tilde{\mathbf{f}}_d^Q)^*_{ik}(\tilde{\mathbf{g}}'^{d_R})_{kl}(\tilde{\mathbf{f}}_d^{d_R})^T_{lj}]$$

$$+ 6\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]\tilde{g}'^{h_u}(\tilde{\mathbf{f}}_u^Q)^*_{ik}(\mathbf{f}_u)^T_{kj} - [c^2\theta_h + s^2\theta_H]\tilde{g}'^{h_d}(\tilde{\mathbf{f}}_d^Q)^*_{ik}(\mathbf{f}_d)^T_{kj}\}$$

$$+ \frac{1}{2}\theta_{\tilde{Q}_l}\bigg[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ik}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \frac{1}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ik}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl} + \frac{16}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^Q)_{ik}(\tilde{\mathbf{g}}_s^Q)^{\dagger}_{kl}\bigg](\tilde{\mathbf{g}}'^Q)_{lj}$$

$$+ \theta_{\tilde{h}}\theta_{\tilde{Q}_l}[(\tilde{\mathbf{f}}_u^Q)^*_{ik}(\tilde{\mathbf{f}}_u^Q)^T_{kl} + (\tilde{\mathbf{f}}_d^Q)^*_{ik}(\tilde{\mathbf{f}}_d^Q)^T_{kl}](\tilde{\mathbf{g}}'^Q)_{lj} - (\tilde{\mathbf{g}}'^Q)_{ij}\bigg[\frac{1}{12}g'^2 + \frac{9}{4}g_2^2 + 4g_s^2\bigg], \tag{A10}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}'^L)_{ij}}{dt} = \frac{1}{2}\theta_{\tilde{B}}\bigg[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)_{lk}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)_{lk}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{lk}(\tilde{\mathbf{g}}'^{u_R})^{\dagger}_{kl} + \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}'^{d_R})_{lk}(\tilde{\mathbf{g}}'^{d_R})^{\dagger}_{kl}$$

$$+ 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}'^{e_R})_{lk}(\tilde{\mathbf{g}}'^{e_R})^{\dagger}_{kl}\bigg](\tilde{\mathbf{g}}'^L)_{ij} + \frac{1}{2}\theta_{\tilde{h}}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^L)_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}'^{h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}'^{h_d}|^2\}$$

$$+ \frac{1}{2}(\tilde{\mathbf{g}}'^L)_{ik}[c^2\theta_h + s^2\theta_H](\mathbf{f}_e^*\mathbf{f}_e^T)_{kj} + \frac{1}{2}(\tilde{\mathbf{g}}'^L)_{ik}\bigg[\theta_{\tilde{h}}\theta_{\tilde{e}_l}(\tilde{\mathbf{f}}_e^{e_R})^*_{kl}(\tilde{\mathbf{f}}_e^{e_R})^T_{lj} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}(\tilde{\mathbf{g}}^L)_{lj}$$

$$+ \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^L)_{lj}\bigg] - 4\theta_{\tilde{h}}\theta_{\tilde{e}_l}(\tilde{\mathbf{f}}_e^L)^*_{ik}(\tilde{\mathbf{g}}'^{e_R})_{kl}(\tilde{\mathbf{f}}_e^{e_R})^T_{lj} + 2\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]\tilde{g}'^{h_d}(\tilde{\mathbf{f}}_e^L)^*_{ik}(\mathbf{f}_e)^T_{kj}$$

$$+ \frac{1}{2}\theta_{\tilde{L}_l}[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^L)_{ik}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl} + \theta_{\tilde{B}}(\tilde{\mathbf{g}}'^L)_{ik}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl}](\tilde{\mathbf{g}}'^L)_{lj} + \theta_{\tilde{h}}\theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)^*_{ik}(\tilde{\mathbf{f}}_e^L)^T_{kl}(\tilde{\mathbf{g}}'^L)_{lj} - (\tilde{\mathbf{g}}'^L)_{ij}\bigg[\frac{3}{4}g'^2 + \frac{9}{4}g_2^2\bigg], \tag{A11}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}'^{u_R})_{ij}}{dt} = [s^2\theta_h + c^2\theta_H](\mathbf{f}_u^T\mathbf{f}_u^*)_{ik}(\tilde{\mathbf{g}}'^{u_R})_{kj} + \bigg[\frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}(\tilde{\mathbf{g}}'^{u_R})^{\dagger}_{kl} + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{ik}(\tilde{\mathbf{g}}_s^{u_R})^{\dagger}_{kl} + \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)^T_{ik}(\tilde{\mathbf{f}}_u^Q)^*_{kl}\bigg]$$

$$\times (\tilde{\mathbf{g}}'^{u_R})_{lj} + \frac{1}{2}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})_{ij}\bigg[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^Q)_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^L)_{lk} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})^{\dagger}_{kl}(\tilde{\mathbf{g}}'^{u_R})_{lk}$$

$$+ \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}'^{d_R})^{\dagger}_{kl}(\tilde{\mathbf{g}}'^{d_R})_{lk} + 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}'^{e_R})^{\dagger}_{kl}(\tilde{\mathbf{g}}'^{e_R})_{lk}\bigg] + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{h}}(\tilde{\mathbf{g}}'^{u_R})_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}'^{h_u}|^2$$

$$+ [c^2\theta_h + s^2\theta_H]|\tilde{g}'^{h_d}|^2\} - 3\theta_{\tilde{B}}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H]\tilde{g}'^{h_u}(\mathbf{f}_u)^T_{ik}(\tilde{\mathbf{f}}_u^{u_R})^*_{kj} - \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)^T_{ik}(\tilde{\mathbf{g}}'^Q)_{kl}(\tilde{\mathbf{f}}_u^{u_R})^*_{lj}$$

$$+ 2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})^T_{kl}(\tilde{\mathbf{f}}_u^{u_R})^*_{lj} + \theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}\bigg[\frac{8}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})^{\dagger}_{kl}(\tilde{\mathbf{g}}'^{u_R})_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{u_R})^{\dagger}_{kl}(\tilde{\mathbf{g}}_s^{u_R})_{lj}\bigg]$$

$$- (\tilde{\mathbf{g}}'^{u_R})_{ij}\bigg[\frac{4}{3}g'^2 + 4g_s^2\bigg], \tag{A12}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}^{\prime d_R})_{ij}}{dt} = [c^2\theta_h + s^2\theta_H](\mathbf{f}_d^T \mathbf{f}_d^*)_{ik}(\tilde{\mathbf{g}}^{\prime d_R})_{kj} + \left[\frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})_{ik}(\tilde{\mathbf{g}}^{\prime d_R})_{kl}^\dagger + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{d_R}_s)_{ik}(\tilde{\mathbf{g}}^{d_R}_s)_{kl}^\dagger + \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}^Q_d)^T_{ik}(\tilde{\mathbf{f}}^Q_d)^*_{kl}\right]$$

$$\times (\tilde{\mathbf{g}}^{\prime d_R})_{lj} + \frac{1}{2}\theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime d_R})_{ij}\left[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^{\prime Q})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime Q})_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^{\prime L})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime L})_{lk} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}^{\prime u_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime u_R})_{lk}\right.$$

$$\left.+ \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime d_R})_{lk} + 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime e_R})_{lk}\right] + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{h}}(\tilde{\mathbf{g}}^{\prime d_R})_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{\prime h_u}|^2$$

$$+ [c^2\theta_h + s^2\theta_H]|\tilde{g}^{\prime h_d}|^2\} - 6\theta_{\tilde{B}}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]\tilde{g}^{\prime h_d}(\mathbf{f}_d)^T_{ik}(\tilde{\mathbf{f}}^{d_R}_d)^*_{kj} + 2\theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}^Q_d)^T_{ik}(\tilde{\mathbf{g}}^{\prime Q})_{kl}(\tilde{\mathbf{f}}^{d_R}_d)^*_{lj}$$

$$+ 2\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})_{ik}(\tilde{\mathbf{f}}^{d_R}_d)^T_{kl}(\tilde{\mathbf{f}}^{d_R}_d)^*_{lj} + \theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})_{ik}\left[\frac{2}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime d_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime d_R})_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}^{d_R}_s)^\dagger_{kl}(\tilde{\mathbf{g}}^{d_R}_s)_{lj}\right]$$

$$- (\tilde{\mathbf{g}}^{\prime d_R})_{ij}\left[\frac{1}{3}g^{\prime 2} + 4g_s^2\right], \tag{A13}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}^{\prime e_R})_{ij}}{dt} = [c^2\theta_h + s^2\theta_H](\mathbf{f}_e^T \mathbf{f}_e^*)_{ik}(\tilde{\mathbf{g}}^{\prime e_R})_{kj} + [\theta_{\tilde{B}}\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})_{ik}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl} + \theta_{\tilde{h}}\theta_{\tilde{L}_k}(\tilde{\mathbf{f}}^L_e)^T_{ik}(\tilde{\mathbf{f}}^L_e)^*_{kl}](\tilde{\mathbf{g}}^{\prime e_R})_{lj}$$

$$+ \frac{1}{2}\theta_{\tilde{B}}(\tilde{\mathbf{g}}^{\prime e_R})_{ij}\left[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^{\prime Q})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime Q})_{lk} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^{\prime L})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime L})_{lk} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}^{\prime u_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime u_R})_{lk}\right.$$

$$\left.+ \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime d_R})_{lk} + 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime e_R})_{lk}\right] + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{h}}(\tilde{\mathbf{g}}^{\prime e_R})_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{\prime h_u}|^2$$

$$+ [c^2\theta_h + s^2\theta_H]|\tilde{g}^{\prime h_d}|^2\} - 2\theta_{\tilde{B}}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]\tilde{g}^{\prime h_d}(\mathbf{f}_e)^T_{ik}(\tilde{\mathbf{f}}^{e_R}_e)^*_{kj} - 2\theta_{\tilde{h}}\theta_{\tilde{L}_k}(\tilde{\mathbf{f}}^L_e)^T_{ik}(\tilde{\mathbf{g}}^{\prime L})_{kl}(\tilde{\mathbf{f}}^{e_R}_e)^*_{lj}$$

$$+ 2\theta_{\tilde{h}}\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})_{ik}(\tilde{\mathbf{f}}^{e_R}_e)^T_{kl}(\tilde{\mathbf{f}}^{e_R}_e)^*_{lj} + 2\theta_{\tilde{B}}\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})_{ik}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl}(\tilde{\mathbf{g}}^{\prime e_R})_{lj} - 3(\tilde{\mathbf{g}}^{\prime e_R})_{ij}g^{\prime 2}, \tag{A14}$$

$$(4\pi)^2 \frac{d(s\tilde{g}^{\prime h_u})}{dt} = \frac{2}{4}\theta_{\tilde{B}}\left[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^{\prime Q})_{lk}(\tilde{\mathbf{g}}^{\prime Q})^\dagger_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^{\prime L})_{lk}(\tilde{\mathbf{g}}^{\prime L})^\dagger_{kl} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}^{\prime u_R})_{lk}(\tilde{\mathbf{g}}^{\prime u_R})^\dagger_{kl} + \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})_{lk}(\tilde{\mathbf{g}}^{\prime d_R})^\dagger_{kl}\right.$$

$$\left.+ 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})_{lk}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl}\right]s\tilde{g}^{\prime h_u} + \frac{1}{2}\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{\prime h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{\prime h_d}|^2\}s\tilde{g}^{\prime h_u}$$

$$+ \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{u}_k}s\tilde{g}^{\prime h_u}(\tilde{\mathbf{f}}^{u_R}_u)^*_{lk}(\tilde{\mathbf{f}}^{u_R}_u)^T_{kl} + 3\theta_{\tilde{Q}_l}s\tilde{g}^{\prime h_u}(\tilde{\mathbf{f}}^Q_u)^*_{lk}(\tilde{\mathbf{f}}^Q_u)^T_{kl}] + \frac{1}{4}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H]s\tilde{g}^{\prime h_u}\{3\theta_{\tilde{W}}|\tilde{g}^{h_u}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_u}|^2\}$$

$$+ 2\theta_{\tilde{Q}_l}s(\tilde{\mathbf{g}}^{\prime Q})_{lk}(\mathbf{f}_u)^*_{km}(\tilde{\mathbf{f}}^Q_u)^T_{ml} - 8\theta_{\tilde{u}_k}s(\tilde{\mathbf{f}}^{u_R}_u)^T_{km}(\mathbf{f}_u)^*_{ml}(\tilde{\mathbf{g}}^{\prime u_R})_{lk} + s\tilde{g}^{\prime h_u}[(s^2\theta_h + c^2\theta_H)\text{Tr}\{3\mathbf{f}_u^\dagger \mathbf{f}_u\}$$

$$+ c^2(\theta_h - \theta_H)\text{Tr}\{3\mathbf{f}_d^\dagger \mathbf{f}_d + \mathbf{f}_e^\dagger \mathbf{f}_e\}] + \frac{1}{2}\theta_{\tilde{h}}s\tilde{g}^{\prime h_u}\{3\theta_{\tilde{W}}[(s^2\theta_h + c^2\theta_H)|\tilde{g}^{h_u}|^2 + c^2(\theta_h - \theta_H)|\tilde{g}^{h_d}|^2]$$

$$+ \theta_{\tilde{B}}[(s^2\theta_h + c^2\theta_H)|\tilde{g}^{\prime h_u}|^2 + c^2(\theta_h - \theta_H)|\tilde{g}^{\prime h_d}|^2]\} - s\tilde{g}^{\prime h_u}\left[\frac{3}{4}g^{\prime 2} + \frac{9}{4}g_2^2\right], \tag{A15}$$

$$(4\pi)^2 \frac{d(c\tilde{g}^{\prime h_d})}{dt} = \frac{2}{4}\theta_{\tilde{B}}\left[\frac{1}{3}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^{\prime Q})_{lk}(\tilde{\mathbf{g}}^{\prime Q})^\dagger_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^{\prime L})_{lk}(\tilde{\mathbf{g}}^{\prime L})^\dagger_{kl} + \frac{8}{3}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}^{\prime u_R})_{lk}(\tilde{\mathbf{g}}^{\prime u_R})^\dagger_{kl} + \frac{2}{3}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}^{\prime d_R})_{lk}(\tilde{\mathbf{g}}^{\prime d_R})^\dagger_{kl}\right.$$

$$\left.+ 2\theta_{\tilde{e}_k}(\tilde{\mathbf{g}}^{\prime e_R})_{lk}(\tilde{\mathbf{g}}^{\prime e_R})^\dagger_{kl}\right]c\tilde{g}^{\prime h_d} + \frac{1}{2}\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{\prime h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{\prime h_d}|^2\}c\tilde{g}^{\prime h_d}$$

$$+ \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{d}_k}c\tilde{g}^{\prime h_d}(\tilde{\mathbf{f}}^{d_R}_d)^*_{lk}(\tilde{\mathbf{f}}^{d_R}_d)^T_{kl} + \theta_{\tilde{e}_k}c\tilde{g}^{\prime h_d}(\tilde{\mathbf{f}}^{e_R}_e)^*_{lk}(\tilde{\mathbf{f}}^{e_R}_e)^T_{kl} + 3\theta_{\tilde{Q}_l}c\tilde{g}^{\prime h_d}(\tilde{\mathbf{f}}^Q_d)^*_{lk}(\tilde{\mathbf{f}}^Q_d)^T_{kl} + \theta_{\tilde{L}_l}c\tilde{g}^{\prime h_d}(\tilde{\mathbf{f}}^L_e)^*_{lk}(\tilde{\mathbf{f}}^L_e)^T_{kl}]$$

$$+ \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]c\tilde{g}^{\prime h_d}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}^{\prime h_d}|^2\} + 2[-\theta_{\tilde{Q}_l}c(\tilde{\mathbf{g}}^{\prime Q})_{lk}(\mathbf{f}_d)^*_{km}(\tilde{\mathbf{f}}^Q_d)^T_{ml}$$

$$+ \theta_{\tilde{L}_l}c(\tilde{\mathbf{g}}^{\prime L})_{lk}(\mathbf{f}_e)^*_{km}(\tilde{\mathbf{f}}^L_e)^T_{ml}] - 4[\theta_{\tilde{d}_k}c(\tilde{\mathbf{f}}^{d_R}_d)^T_{km}(\mathbf{f}_d)^*_{ml}(\tilde{\mathbf{g}}^{\prime d_R})_{lk} + \theta_{\tilde{e}_k}c(\tilde{\mathbf{f}}^{e_R}_e)_{km}(\mathbf{f}_e)^\dagger_{ml}(\tilde{\mathbf{g}}^{\prime e_R})_{lk}]$$

$$+ c\tilde{g}^{\prime h_d}[s^2(\theta_h - \theta_H)\text{Tr}\{3\mathbf{f}_u^\dagger \mathbf{f}_u\} + (c^2\theta_h + s^2\theta_H)\text{Tr}\{3\mathbf{f}_d^\dagger \mathbf{f}_d + \mathbf{f}_e^\dagger \mathbf{f}_e\}]$$

$$+ \frac{1}{2}\theta_{\tilde{h}}c\tilde{g}^{\prime h_d}\{3\theta_{\tilde{W}}[s^2(\theta_h - \theta_H)|\tilde{g}^{h_u}|^2 + (c^2\theta_h + s^2\theta_H)|\tilde{g}^{h_d}|^2] + \theta_{\tilde{B}}[s^2(\theta_h - \theta_H)|\tilde{g}^{\prime h_u}|^2$$

$$+ (c^2\theta_h + s^2\theta_H)|\tilde{g}^{\prime h_d}|^2]\} - c\tilde{g}^{\prime h_d}\left[\frac{3}{4}g^{\prime 2} + \frac{9}{4}g_2^2\right], \tag{A16}$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(\tilde{\mathbf{g}}^Q)_{ij}}{dt} &= \frac{1}{2}\theta_{\tilde{W}}[3\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)_{lk}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)_{lk}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}](\tilde{\mathbf{g}}^Q)_{ij} + \frac{1}{2}\theta_{\tilde{h}}\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{h_u}|^2 \\
&\quad + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{h_d}|^2\} + \frac{1}{2}(\tilde{\mathbf{g}}^Q)_{ik}\{[s^2\theta_h + c^2\theta_H](\mathbf{f}_u^*\mathbf{f}_u^T)_{kj} + [c^2\theta_h + s^2\theta_H](\mathbf{f}_d^*\mathbf{f}_d^T)_{kj}\} \\
&\quad + \frac{1}{2}(\tilde{\mathbf{g}}^Q)_{ik}\Big[\theta_{\tilde{h}}\theta_{\tilde{u}_l}(\tilde{\mathbf{f}}_u^{u_R})^*_{kl}(\tilde{\mathbf{f}}_u^{u_R})^T_{lj} + \theta_{\tilde{h}}\theta_{\tilde{d}_l}(\tilde{\mathbf{f}}_d^{d_R})^*_{kl}(\tilde{\mathbf{f}}_d^{d_R})^T_{lj} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}^Q)_{lj} + \frac{1}{18}\theta_{\tilde{B}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^Q)_{lj} \\
&\quad + \frac{8}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^Q)^{\dagger}_{kl}(\tilde{\mathbf{g}}_s^Q)_{lj}\Big] - 2\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]\tilde{g}^{h_u}(\tilde{\mathbf{f}}_u^Q)^*_{ik}(\mathbf{f}_u)^T_{kj} + [c^2\theta_h + s^2\theta_H]\tilde{g}^{h_d}(\tilde{\mathbf{f}}_d^Q)^*_{ik}(\mathbf{f}_d)^T_{kj}\} \\
&\quad + \frac{1}{2}\theta_{\tilde{Q}_l}\Big[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ik}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \frac{1}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ik}(\tilde{\mathbf{g}}'^Q)^{\dagger}_{kl} + \frac{16}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^Q)_{ik}(\tilde{\mathbf{g}}_s^Q)^{\dagger}_{kl}\Big](\tilde{\mathbf{g}}^Q)_{lj} + \theta_{\tilde{h}}\theta_{\tilde{Q}_l}[(\tilde{\mathbf{f}}_u^Q)^*_{ik}(\tilde{\mathbf{f}}_u^Q)^T_{kl} \\
&\quad + (\tilde{\mathbf{f}}_d^Q)^*_{ik}(\tilde{\mathbf{f}}_d^Q)^T_{kl}](\tilde{\mathbf{g}}^Q)_{lj} - (\tilde{\mathbf{g}}^Q)_{ij}\Big[\frac{1}{12}g'^2 + \frac{33}{4}g_2^2 + 4g_s^2\Big],
\end{aligned}
\tag{A17}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(\tilde{\mathbf{g}}^L)_{ij}}{dt} &= \frac{1}{2}\theta_{\tilde{W}}[3\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)_{lk}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)_{lk}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}](\tilde{\mathbf{g}}^L)_{ij} + \frac{1}{2}\theta_{\tilde{h}}\theta_{\tilde{W}}(\tilde{\mathbf{g}}^L)_{ij}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{h_u}|^2 \\
&\quad + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{h_d}|^2\} + \frac{1}{2}[c^2\theta_h + s^2\theta_H](\tilde{\mathbf{g}}^L)_{ik}(\mathbf{f}_e^*\mathbf{f}_e^T)_{kj} \\
&\quad + \frac{1}{2}(\tilde{\mathbf{g}}^L)_{ik}\Big[\theta_{\tilde{h}}\theta_{\tilde{e}_l}(\tilde{\mathbf{f}}_e^{e_R})^*_{kl}(\tilde{\mathbf{f}}_e^{e_R})^T_{lj} + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}(\tilde{\mathbf{g}}^L)_{lj} + \frac{1}{2}\theta_{\tilde{B}}\theta_{\tilde{L}_l}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl}(\tilde{\mathbf{g}}'^L)_{lj}\Big] \\
&\quad - 2\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]\tilde{g}^{h_d}(\tilde{\mathbf{f}}_e^L)^*_{ik}(\mathbf{f}_e)^T_{kj} + \frac{1}{2}\theta_{\tilde{L}_l}[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^L)_{ik}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl} + \theta_{\tilde{B}}(\tilde{\mathbf{g}}'^L)_{ik}(\tilde{\mathbf{g}}'^L)^{\dagger}_{kl}](\tilde{\mathbf{g}}^L)_{lj} \\
&\quad + \theta_{\tilde{h}}\theta_{\tilde{L}_l}(\tilde{\mathbf{f}}_e^L)^*_{ik}(\tilde{\mathbf{f}}_e^L)^T_{kl}(\tilde{\mathbf{g}}^L)_{lj} - (\tilde{\mathbf{g}}^Q)_{ij}\Big[\frac{3}{4}g'^2 + \frac{33}{4}g_2^2\Big],
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(s\tilde{g}^{h_u})}{dt} &= \frac{1}{2}\theta_{\tilde{W}}[3\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)_{lk}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)_{lk}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}]s\tilde{g}^{h_u} + \frac{1}{2}\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{h_d}|^2\}s\tilde{g}^{h_u} \\
&\quad + \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{u}_k}s\tilde{g}^{h_u}(\tilde{\mathbf{f}}_u^{u_R})^*_{lk}(\tilde{\mathbf{f}}_u^{u_R})_{kl} + 3\theta_{\tilde{Q}_l}s\tilde{g}^{h_u}(\tilde{\mathbf{f}}_u^Q)^*_{lk}(\tilde{\mathbf{f}}_u^Q)^T_{kl}] + \frac{1}{4}\theta_{\tilde{h}}[s^2\theta_h + c^2\theta_H]s\tilde{g}^{h_u}\{3\theta_{\tilde{W}}|\tilde{g}^{h_u}|^2 + \theta_{\tilde{B}}|\tilde{g}'^{h_u}|^2\} \\
&\quad - 6\theta_{\tilde{Q}_l}s(\tilde{\mathbf{g}}^Q)_{lk}(\mathbf{f}_u)^*_{km}(\tilde{\mathbf{f}}_u^Q)^T_{ml} + s\tilde{g}^{h_u}[(s^2\theta_h + c^2\theta_H)\operatorname{Tr}\{3\mathbf{f}_u^{\dagger}\mathbf{f}_u\} + c^2(\theta_h - \theta_H)\operatorname{Tr}\{3\mathbf{f}_d^{\dagger}\mathbf{f}_d + \mathbf{f}_e^{\dagger}\mathbf{f}_e\}] \\
&\quad + \frac{1}{2}\theta_{\tilde{h}}s\tilde{g}^{h_u}\{3\theta_{\tilde{W}}[(s^2\theta_h + c^2\theta_H)|\tilde{g}^{h_u}|^2 + c^2(\theta_h - \theta_H)|\tilde{g}^{h_d}|^2] + \theta_{\tilde{B}}[(s^2\theta_h + c^2\theta_H)|\tilde{g}'^{h_u}|^2 \\
&\quad + c^2(\theta_h - \theta_H)|\tilde{g}'^{h_d}|^2]\} - s\tilde{g}^{h_u}\Big[\frac{3}{4}g'^2 + \frac{33}{4}g_2^2\Big],
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
(4\pi)^2 \frac{d(c\tilde{g}^{h_d})}{dt} &= \frac{1}{2}\theta_{\tilde{W}}[3\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)_{lk}(\tilde{\mathbf{g}}^Q)^{\dagger}_{kl} + \theta_{\tilde{L}_l}(\tilde{\mathbf{g}}^L)_{lk}(\tilde{\mathbf{g}}^L)^{\dagger}_{kl}]c\tilde{g}^{h_d} + \frac{1}{2}\theta_{\tilde{h}}\{[s^2\theta_h + c^2\theta_H]|\tilde{g}^{h_u}|^2 + [c^2\theta_h + s^2\theta_H]|\tilde{g}^{h_d}|^2\}c\tilde{g}^{h_d} \\
&\quad + \frac{1}{2}\theta_{\tilde{h}}[3\theta_{\tilde{d}_k}c\tilde{g}^{h_d}(\tilde{\mathbf{f}}_d^{d_R})^*_{lk}(\tilde{\mathbf{f}}_d^{d_R})^T_{kl} + \theta_{\tilde{e}_k}c\tilde{g}^{h_d}(\tilde{\mathbf{f}}_e^{e_R})^*_{lk}(\tilde{\mathbf{f}}_e^{e_R})^T_{kl} + 3\theta_{\tilde{Q}_l}c\tilde{g}^{h_d}(\tilde{\mathbf{f}}_d^Q)^*_{lk}(\tilde{\mathbf{f}}_d^Q)^T_{kl} + \theta_{\tilde{L}_l}c\tilde{g}^{h_d}(\tilde{\mathbf{f}}_e^L)^*_{lk}(\tilde{\mathbf{f}}_e^L)^T_{kl}] \\
&\quad + \frac{1}{4}\theta_{\tilde{h}}[c^2\theta_h + s^2\theta_H]c\tilde{g}^{h_d}\{3\theta_{\tilde{W}}|\tilde{g}^{h_d}|^2 + \theta_{\tilde{B}}|\tilde{g}'^{h_u}|^2\} - 2[3\theta_{\tilde{Q}_l}c(\tilde{\mathbf{g}}^Q)_{lk}(\mathbf{f}_d)^*_{km}(\tilde{\mathbf{f}}_d^Q)^T_{ml} + \theta_{\tilde{L}_l}c(\tilde{\mathbf{g}}^L)_{lk}(\mathbf{f}_e)^*_{km}(\tilde{\mathbf{f}}_e^L)^T_{ml}] \\
&\quad + c\tilde{g}^{h_d}[s^2(\theta_h - \theta_H)\operatorname{Tr}\{3\mathbf{f}_u^{\dagger}\mathbf{f}_u\} + (c^2\theta_h + s^2\theta_H)\operatorname{Tr}\{3\mathbf{f}_d^{\dagger}\mathbf{f}_d + \mathbf{f}_e^{\dagger}\mathbf{f}_e\}] + \frac{1}{2}\theta_{\tilde{h}}c\tilde{g}^{h_d}\{3\theta_{\tilde{W}}[s^2(\theta_h - \theta_H)|\tilde{g}^{h_u}|^2 \\
&\quad + (c^2\theta_h + s^2\theta_H)|\tilde{g}^{h_d}|^2] + \theta_{\tilde{B}}[s^2(\theta_h - \theta_H)|\tilde{g}'^{h_u}|^2 + (c^2\theta_h + s^2\theta_H)|\tilde{g}'^{h_d}|^2]\} - c\tilde{g}^{h_d}\Big[\frac{3}{4}g'^2 + \frac{33}{4}g_2^2\Big],
\end{aligned}
\tag{A20}
$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}_s^Q)_{ij}}{dt} = \frac{1}{2}\theta_{\tilde{g}}[2\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^Q)_{lk}(\tilde{\mathbf{g}}_s^Q)_{kl}^\dagger + \theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{lk}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^\dagger + \theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{lk}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^\dagger](\tilde{\mathbf{g}}_s^Q)_{ij} + \frac{1}{2}(\tilde{\mathbf{g}}_s^Q)_{ik}\{[s^2\theta_h + c^2\theta_H](\mathbf{f}_u)_{kl}^*(\mathbf{f}_u)_{lj}^T$$

$$+ [c^2\theta_h + s^2\theta_H](\mathbf{f}_d)_{kl}^*(\mathbf{f}_d)_{lj}^T\} + \frac{1}{2}(\tilde{\mathbf{g}}_s^Q)_{ik}\Big[\theta_{\tilde{h}}\theta_{\tilde{u}_l}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^*(\tilde{\mathbf{f}}_u^{u_R})_{lj}^T + \theta_{\tilde{h}}\theta_{\tilde{d}_l}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^*(\tilde{\mathbf{f}}_d^{d_R})_{lj}^T + \frac{3}{2}\theta_{\tilde{W}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}^Q)_{kl}^\dagger(\tilde{\mathbf{g}}^Q)_{lj}$$

$$+ \frac{1}{18}\theta_{\tilde{B}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}'^Q)_{kl}^\dagger(\tilde{\mathbf{g}}'^Q)_{lj} + \frac{8}{3}\theta_{\tilde{g}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^Q)_{kl}^\dagger(\tilde{\mathbf{g}}_s^Q)_{lj}\Big] - 2\theta_{\tilde{h}}[\theta_{\tilde{u}_l}(\tilde{\mathbf{f}}_u^Q)_{ik}^*(\tilde{\mathbf{g}}_s^{u_R})_{kl}(\tilde{\mathbf{f}}_u^{u_R})_{lj}^T + \theta_{\tilde{d}_l}(\tilde{\mathbf{f}}_d^Q)_{ik}^*(\tilde{\mathbf{g}}_s^{d_R})_{kl}(\tilde{\mathbf{f}}_d^{d_R})_{lj}^T]$$

$$+ \frac{1}{2}\theta_{\tilde{Q}_l}\Big[3\theta_{\tilde{W}}(\tilde{\mathbf{g}}^Q)_{ik}(\tilde{\mathbf{g}}^Q)_{kl}^\dagger + \frac{1}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^Q)_{ik}(\tilde{\mathbf{g}}'^Q)_{kl}^\dagger + \frac{16}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^Q)_{ik}(\tilde{\mathbf{g}}_s^Q)_{kl}^\dagger\Big](\tilde{\mathbf{g}}_s^Q)_{lj}$$

$$+ \theta_{\tilde{h}}\theta_{\tilde{Q}_l}[(\tilde{\mathbf{f}}_u^Q)_{ik}^*(\tilde{\mathbf{f}}_u^Q)_{kl}^T + (\tilde{\mathbf{f}}_d^Q)_{ik}^*(\tilde{\mathbf{f}}_d^Q)_{kl}^T](\tilde{\mathbf{g}}_s^Q)_{lj} - (\tilde{\mathbf{g}}_s^Q)_{ij}\Big[\frac{1}{12}g'^2 + \frac{9}{4}g_2^2 + 13g_s^2\Big], \tag{A21}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}_s^{u_R})_{ij}}{dt} = [s^2\theta_h + c^2\theta_H](\mathbf{f}_u^T\mathbf{f}_u^*)_{ik}(\tilde{\mathbf{g}}_s^{u_R})_{kj} + \Big[\frac{4}{9}\theta_{\tilde{B}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}'^{u_R})_{ik}(\tilde{\mathbf{g}}'^{u_R})_{kl}^\dagger + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{ik}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^\dagger + \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)_{ik}^T(\tilde{\mathbf{f}}_u^Q)_{kl}^*\Big]$$

$$\times (\tilde{\mathbf{g}}_s^{u_R})_{lj} + \theta_{\tilde{g}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^{u_R})_{ij}(\tilde{\mathbf{g}}_s^Q)_{kl}^\dagger(\tilde{\mathbf{g}}_s^Q)_{lk} + \frac{1}{2}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{u_R})_{ij}[\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{u_R})_{lk} + \theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{d_R})_{lk}]$$

$$- 4\theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_u^Q)_{ik}^T(\tilde{\mathbf{g}}_s^Q)_{kl}(\tilde{\mathbf{f}}_u^{u_R})_{lj}^* + 2\theta_{\tilde{h}}\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{ik}(\tilde{\mathbf{f}}_u^{u_R})_{kl}^T(\tilde{\mathbf{f}}_u^{u_R})_{lj}^* + \theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{ik}$$

$$\times \Big[\frac{8}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{u_R})_{kl}^\dagger(\tilde{\mathbf{g}}'^{u_R})_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{u_R})_{lj}\Big] - (\tilde{\mathbf{g}}_s^{u_R})_{ij}\Big[\frac{4}{3}g'^2 + 13g_s^2\Big], \tag{A22}$$

$$(4\pi)^2 \frac{d(\tilde{\mathbf{g}}_s^{d_R})_{ij}}{dt} = [c^2\theta_h + s^2\theta_H](\mathbf{f}_d^T\mathbf{f}_d^*)_{ik}(\tilde{\mathbf{g}}_s^{d_R})_{kj} + \Big[\frac{1}{9}\theta_{\tilde{B}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}'^{d_R})_{ik}(\tilde{\mathbf{g}}'^{d_R})_{kl}^\dagger + \frac{4}{3}\theta_{\tilde{g}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{ik}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^\dagger + \theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_d^Q)_{ik}^T(\tilde{\mathbf{f}}_d^Q)_{kl}^*\Big]$$

$$\times (\tilde{\mathbf{g}}_s^{d_R})_{lj} + \theta_{\tilde{g}}\theta_{\tilde{Q}_l}(\tilde{\mathbf{g}}_s^{d_R})_{ij}(\tilde{\mathbf{g}}_s^Q)_{kl}^\dagger(\tilde{\mathbf{g}}_s^Q)_{lk} + \frac{1}{2}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{d_R})_{ij}[\theta_{\tilde{u}_k}(\tilde{\mathbf{g}}_s^{u_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{u_R})_{lk} + \theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{d_R})_{lk}]$$

$$- 4\theta_{\tilde{h}}\theta_{\tilde{Q}_k}(\tilde{\mathbf{f}}_d^Q)_{ik}^T(\tilde{\mathbf{g}}_s^Q)_{kl}(\tilde{\mathbf{f}}_d^{d_R})_{lj}^* + 2\theta_{\tilde{h}}\theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{ik}(\tilde{\mathbf{f}}_d^{d_R})_{kl}^T(\tilde{\mathbf{f}}_d^{d_R})_{lj}^* + \theta_{\tilde{d}_k}(\tilde{\mathbf{g}}_s^{d_R})_{ik}$$

$$\times \Big[\frac{2}{9}\theta_{\tilde{B}}(\tilde{\mathbf{g}}'^{d_R})_{kl}^\dagger(\tilde{\mathbf{g}}'^{d_R})_{lj} + \frac{8}{3}\theta_{\tilde{g}}(\tilde{\mathbf{g}}_s^{d_R})_{kl}^\dagger(\tilde{\mathbf{g}}_s^{d_R})_{lj}\Big] - (\tilde{\mathbf{g}}_s^{d_R})_{ij}\Big[\frac{1}{3}g'^2 + 13g_s^2\Big]. \tag{A23}$$

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