Absence of vortex condensation in a two dimensional fermionic *XY* model

D. J. Cecile and Shailesh Chandrasekharan

Department of Physics, Box 90305, Duke University, Durham, North Carolina 27708, USA
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Motivated by a puzzle in the study of two-dimensional lattice quantum electrodynamics with staggered fermions, we construct a two-dimensional fermionic model with a global $U(1)$ symmetry. Our model can be mapped into a model of closed packed dimers and plaquettes. Although the model has the same symmetries as the *XY* model, we show numerically that the model lacks the well-known Kosterlitz-Thouless phase transition. The model is always in the gapless phase showing the absence of a phase with vortex condensation. In other words the low energy physics is described by a noncompact $U(1)$ field theory. We show that by introducing an even number of layers one can introduce vortex condensation within the model and thus also induce a Kosterlitz-Thouless transition.

I. MOTIVATION

Two-dimensional lattice quantum electrodynamics continues to be of interest today as a test bed for ideas and algorithms for lattice QCD [1–5]. In this work we focus on the formulation with staggered fermions, and refer to it as LQED2 [6]. In the continuum limit, the theory is expected to describe the two-flavor Schwinger model [7]. With massless fermions the two-flavor Schwinger model contains an $SU(2) \times SU(2)$ chiral symmetry. Away from the continuum limit, finite lattice spacing effects in LQED2 break the chiral symmetry to a $U(1)$ subgroup. In the mean field approximation this symmetry is spontaneously broken. However, since in two dimensions strong infrared fluctuations forbid spontaneous symmetry breaking, the mean field result is modified [8,9]: Instead the theory develops critical long range (gapless) correlations, which can be detected through the chiral condensate susceptibility

$$\chi = \sum_i \langle \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \rangle \sim AL^{2-\eta} \qquad (1)$$

where ψ_i and $\bar{\psi}_i$ are the staggered fermion fields at the site i on a square lattice. In general one expects $\chi \sim AL^{2-\eta}$ in the gapless phase and $\chi \sim B$ when the theory develops a mass gap, where A and B are constants. In the case of a $U(1)$ symmetric theory, the mass gap can be generated only due to vortex condensation. In LQED2, at strong couplings one finds that Eq. (1) holds with $\eta = 0.5$ [10]. On the other hand in the continuum limit, using the result in the two-flavor Schwinger model, one expects $\eta = 1$ [11]. Since previous studies have shown no evidence of a phase transition from strong to weak gauge couplings one might conjecture that the theory is always in the gapless phase with η varying smoothly from 0.5 to 1.0 as a function of the gauge coupling. As far as we know this has not yet been shown analytically or observed numerically.

Although at first glance there does not seem to be anything strange with the above scenario, a closer examination reveals an interesting puzzle. Away from the continuum

limit we expect the low energy physics of LQED2 to be described by a $U(1)$ symmetric bosonic field theory since the fermions are confined. Is this a compact or a noncompact $U(1)$ field theory? A compact $U(1)$ theory will contain vortices, just like the *XY* model, and these vortices can condense. Thus, it can undergo the famous Kosterlitz-Thouless (KT) phase transition, due to the presence of vortices, from a gapless phase to a massive phase. Further, although the susceptibility χ is governed by Eq. (1) in the gapless phase, η satisfies the constraint $0 < \eta \leq 0.25$ [12]. Thus $0.5 \leq \eta \leq 1$, expected in LQED2, appears inconsistent with the physics of a compact $U(1)$ field theory. On the other hand it was shown recently that when one studies LQED2 at strong couplings in a slab geometry with four two-dimensional layers, one finds a KT transition separating a gapless phase and a gapped phase. Further in the gapless phase Eq. (1) holds and indeed one finds $0 < \eta \leq 0.25$ as expected [10]. Thus, we conclude that LQED2 behaves like a noncompact $U(1)$ theory on a square lattice, but behaves like a compact $U(1)$ theory on a lattice where four two-dimensional layers are coupled to each other. *How is this possible*? The motivation behind our work is to shed some light on this question by constructing a simpler model that can be easily studied numerically and that clearly exhibits all the above features.

Our paper is organized as follows: In Sec. II we construct a new realization of the lattice *XY* model with fermionic composites. This model is easier to study than LQED2 but captures the essential physics. In Sec. III we show that our simpler model, on a square lattice, is always in the gapless phase and does not contain the Kosterlitz-Thouless (KT) phase transition. In other words the condensation of vortices is eliminated completely suggesting the long distance theory is similar to noncompact $U(1)$ field theory. In Sec. IV we show that adding one extra two-dimensional layer leads to a gapped (massive) phase. This means the extra layer introduces vortex condensation. With four or more layers the theory contains both the gapless and the gapped phases separated by the usual KT transition. In

Sec. V we present our conclusions and suggest some directions for the future.

II. A FERMIONIC *XY* MODEL

The action of the conventional lattice bosonic-*XY* (b*XY*) model is given by

$$S = -\frac{1}{2}\sum_{\langle ij \rangle} z_i^* z_j + z_j^* z_i. \tag{2}$$

$\langle ij \rangle$ stands for the sum over nearest neighbor sites and $z_i = \exp(i\phi_i)$ is a complex variable of unit magnitude at site i on a two-dimensional $L \times L$ square lattice with periodic boundary conditions. The partition function of the model is given by

$$Z = \int [d\phi] \exp\left(-\frac{1}{T}S\right). \tag{3}$$

Clearly the action and the measure of the partition function are invariant under the $U(1)$ symmetry $z \to \exp(i\theta)z$. The b*XY* model is known to contain vortices which leads to the existence of two phases as a function of T: a gapless phase where vortices are confined and a gapped phase where they are free and condense. The phase transition separating them is the well-known KT transition and occurs at some critical temperature T_c. For $T < T_c$ for large L one expects

$$\chi = \sum_i \langle z_i z_j^* + z_j z_i^* \rangle \sim AL^{2-\eta}, \tag{4}$$

where η depends on T. One of the important predictions of the KT transition is that $\eta \to 0.25$ as $(T_c - T) \to 0$.

Here we construct and study a new type of fermionic-*XY* (f*XY*) model constructed with Grassmann variables. The action of this model is given by

$$S_1 = -\sum_{\langle ij \rangle} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j - \beta \sum_{\langle ijkl \rangle} \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \bar{\psi}_k \psi_k \bar{\psi}_l \psi_l, \tag{5}$$

where ψ_i and $\bar{\psi}_i$ are two independent Grassmann variables on the site i and $\langle ijkl \rangle$ stands for all plaquettes made up of sites i, j, k, l in a cyclic order. The partition function is given by

$$Z = \int [d\psi d\bar{\psi}] \exp(-S_1). \tag{6}$$

Note that an overall multiplicative constant $1/T$, like in Eq. (3), drops out of all observables up to multiplicative constants, if β is redefined as $T\beta$. The action and the measure in Eq. (6) are again invariant under the $U(1)$ symmetry $\psi \to \exp(i\sigma_i\theta)\psi$ and $\bar{\psi} \to \exp(i\sigma_i\theta)\bar{\psi}$ where $\sigma_i = 1$ on all even sites and -1 on all odd sites. Here we ask if the low energy theory of the f*XY*-model resembles that of the b*XY*-model. In particular, as a function of β, are there two phases separated by the KT phase transition?

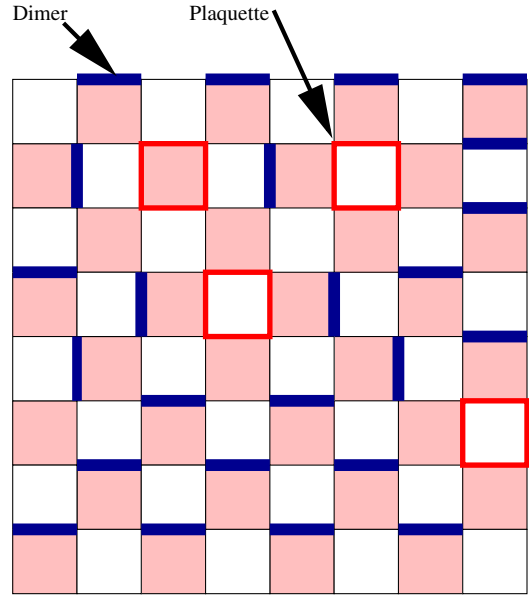It is possible to integrate out the Grassmann variables and rewrite the f*XY*-model as a statistical mechanics of

FIG. 1 (color online). A example of a dimer-plaquette configuration of the fermionic *XY* model.

bond variables (referred to as dimers) b and plaquette variables p. In this representation, the partition function is given by

$$Z = \sum_{[b,p]} \prod_{\langle ijkl \rangle} \beta^{p_{\langle ijkl \rangle}}, \tag{7}$$

where $b_{\langle ij \rangle} = 0, 1$ and $p_{\langle ijkl \rangle} = 0, 1$ are the allowed values with the constraint that only one of the four b's or p's associated with a given lattice site must be nonzero. Physically, this constraint means that each site be connected to only one dimer or one plaquette. An example of the dimer-plaquette configuration is shown in Fig. 1.

When $\beta = 0$ the f*XY*-model defined above is exactly solvable and was first studied in [13]. It was found that the chiral condensate susceptibility

$$\chi = \sum_i \langle \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \rangle \tag{8}$$

satisfies Eq. (1) with $\eta = 0.5$. Since η does not lie between 0 and 0.25, the f*XY* model is clearly different from the b*XY* model. Below we will argue this difference in behavior results from elimination of vortex condensation completely.

III. ABSENCE OF VORTEX CONDENSATION

In order to study the f*XY*-model we have developed a directed path algorithm which is a straightforward extension of the ideas presented in [14]. Here we focus on the results and postpone the discussion of the algorithm to another publication. One of the features of the algorithm is that it allows us to measure χ efficiently. In Table I we

TABLE I.   Comparison of χ obtained using the directed path algorithm with exact results. The algorithm appears to remain efficient for relatively large values of β on small lattices.

Lattice size	β	Exact	Algorithm
6×6	0.0	3.336 40...	3.3366(9)
4×4	0.3	1.431 03...	1.4307(4)
4×4	5.5	0.346 15...	0.3461(2)
6×6	10.0	0.306 63...	0.3068(5)
6×6	100.0	0.028 88...	0.0288(2)

compare the results of the algorithm with exact calculations on small lattices for various values of β.

In Fig. 2 we plot χ as a function of L for various values of β ranging from 0 to 7.5. We believe the algorithm remains stable for these values of β. We find that the behavior of χ as a function of L fits well to the form $AL^{2-\eta}$, expected in the gapless phase, for all values of β. The values of A and η obtained from the fits are tabulated in Table II. As Fig. 2 and the fits in Table II clearly show, there is no sign of a phase transition as a function of β on lattices up to $L = 1024$. The value of η slowly rises with β. Based on locality of the theory we expect that in the $\beta \to \infty$ limit we should get $\eta = 2$. Thus,
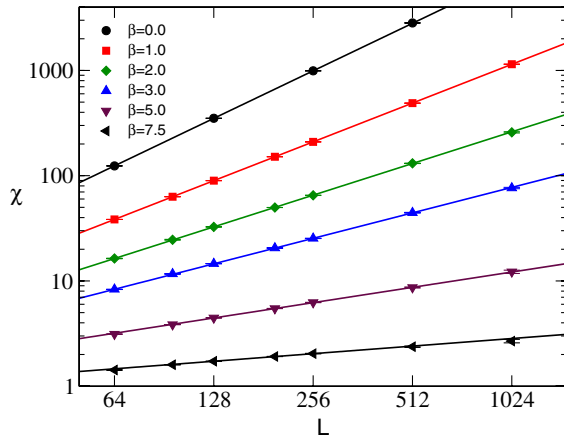

FIG. 2 (color online).   Chiral susceptibility as a function of L for a variety of $\beta \leq 7.5$. For larger β we find it difficult to reach the thermodynamic limit even though the algorithm may be efficient.

TABLE II.   Values of A and η obtained by fitting χ to the form $AL^{2-\eta}$.

β	A	η	$\chi^2/$DOF
0.0	0.242(1)	0.5	1.3
1.0	0.236(1)	0.776(1)	0.5
2.0	0.256(3)	1.000(2)	0.7
3.0	0.294(3)	1.196(2)	1.3
5.0	0.428(8)	1.517(4)	0.3
7.5	0.54(1)	1.765(5)	1.3

there is no gapped phase in the fXY model as a function of β: Vortices do not condense and the model behaves like a noncompact $U(1)$ field theory.

IV. INTRODUCING VORTICES

How can we introduce vortex condensation in the fXY model? This is indeed possible if we consider $N > 1$ two-dimensional layers of square lattices, coupled through a local interaction. The action of the N-layered model is given by

$$S_N = S_1 - t \sum_{i,\langle ll' \rangle} \bar{\psi}_{i,l} \psi_{i,l} \bar{\psi}_{i,l'} \psi_{i,l'}, \qquad (9)$$

where, in addition to the action given in Eq. (5), we have added a term that couples neighboring layers l and l' represented by $\langle ll' \rangle$ at each site i. In the dimer-plaquette language the extra term introduces additional dimers that connect two neighboring layers at a site. We will refer to the fermionic model with N layers as fXY_N. Thus the model f$XY_1 \equiv$ fXY is the model we considered above which does not contain vortex condensation. Here we will show that fXY_2 and fXY_4 contain a gapped phase unlike the fXY_1 model. Here we focus at $\beta = 0$ and study the models as a function of t and N.

Let us first consider the model with two layers ($N = 2$). Figure 3 shows χ as a function of L for various values of t. Table III gives the value of A and η obtained from a fit of the data to the form $AL^{2-\eta}$. The fits begin to fail for $t \geq 0.25$ as shown by the large $\chi^2/$DOF in Table III, while they are good for $t \leq 0.1$. The saturation of χ for large L at $t = 0.5$ is consistent with the presence of a finite correlation length which implies the existence of a gapped phase due to vortex condensation. As t decreases the correlation length increases. Is there a phase transition at a finite t? Note that the value of η approaches 0.25 as t is lowered to zero. Interestingly, as discussed earlier, $\eta = 0.25$ is a
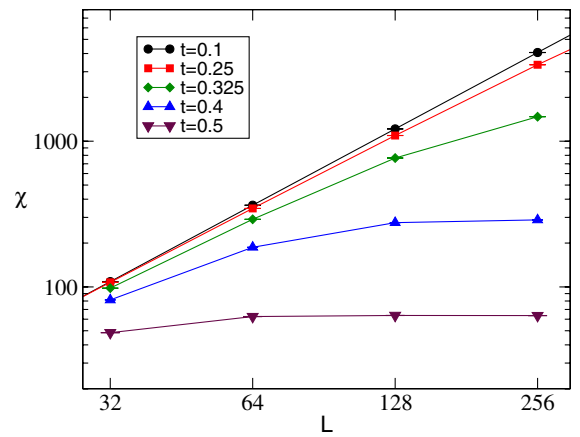

FIG. 3 (color online). χ as a function of L for $N = 2$. The saturation of χ for large t implies the presence of a finite correlation length. The fits are consistent with a diverging correlation length as $t \to 0$.

TABLE III. Values of A and η obtained from fitting the behavior of χ as a function of L, in the range $32 \leq L \leq 256$, to the form $AL^{2-\eta}$ for different values of t but $N = 2$. The fit fails for $t \geq 0.25$.

t	A	η	χ^2/DOF
0.01	0.185(1)	0.252(1)	2.2
0.1	0.261(1)	0.259(1)	0.4
0.25	0.337(1)	0.335(1)	419

universal result at the critical point in a KT phase transition. Thus, it is likely that the correlation length diverges only at $t \to 0$ and not at any finite t. The reason for a good fit for $t \leq 0.1$ could be due to the fact that the correlation lengths at these values of t are much larger than the lattice sizes explored.

Next we consider $N = 4$, which was studied earlier in [10]. In Fig. 4 we plot χ as a function of L for various values of t. Table IV contains the values of A and η obtained from fits to the data as in the $N = 2$ case. The evidence from the fits is consistent with the following scenario: For $t \leq 1.0$ the model is in a gapless phase
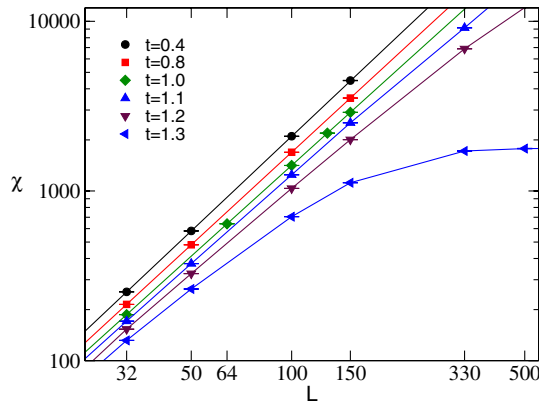

FIG. 4 (color online). Chiral susceptibility as a function of L for $N = 4$. The solid lines for $t = 0.4$, 0.8, and 1.0 are fits to $AL^{2-\eta}$ given in the text. This power-law fit begins to fails for $t = 1.1$ and above. In particular, a KT transition is expected at $t = t_c \sim 1.05$ [10].

TABLE IV. Values of fitting coefficients A and η obtained by fitting $\chi = AL^{2-\eta}$ for different values of t. The range $32 \leq L \leq 150$ was used in the fit.

t	A	η	χ^2/DOF
0.4	0.412(2)	0.146(1)	1.3
0.8	0.402(2)	0.188(1)	0.1
1.0	0.396(1)	0.224(1)	1.1
1.1	0.410(1)	0.259(1)	2.8
1.2	0.487(1)	0.338(1)	124
1.3	1.004(5)	0.586(1)	1500

(absence of vortex condensation), while for $t > 1.1$ the model is in a gapped phase (existence of vortex condensation). Note again the large χ^2/DOF for $t \geq 1.1$ in Table IV, which show that the fits to the form $AL^{2-\eta}$ begin to fail. The value of η close to the critical point, $t_c \sim 1.05(5)$, is again about 0.25, consistent with a KT phase transition. Thus, for $N = 4$ the model contains a KT phase transition. For even $N > 4$ we have evidence, from other unpublished studies, that the behavior is similar to that of $N = 4$.

V. CONCLUSIONS

In this work we have constructed and studied a fermionic XY model. When the model is studied on a square lattice it contains no evidence for a gapped phase with vortex condensation. For this reason we believe it gives a realization of a two-dimensional noncompact $U(1)$ field theory. Introducing an even number of layers, leads to a gapped phase which must be accompanied by vortex condensation. LQED2, away from the continuum limit, gives another realization of the long distance physics of our fermionic XY model physics. Our work sheds light on why the low energy physics of LQED2 cannot be described by the bXY model even though the symmetries are the same.

There are several questions that may be of further interest. For example:

(1) Is it possible to characterize a vortex in the dimer-plaquette configuration? We conjecture that a vortex core may be identified by the existence of a site such that, on that site all the layers are connected by dimers to the neighboring layers. An example of such a site is shown in Fig. 5 for two layers.

(2) Is there some difference about even and odd values of N since our definition of the vortex core does not work if N is odd? In one layer we have already seen lack of vortex condensation. So what happens in $N = 3, 5, \ldots$?
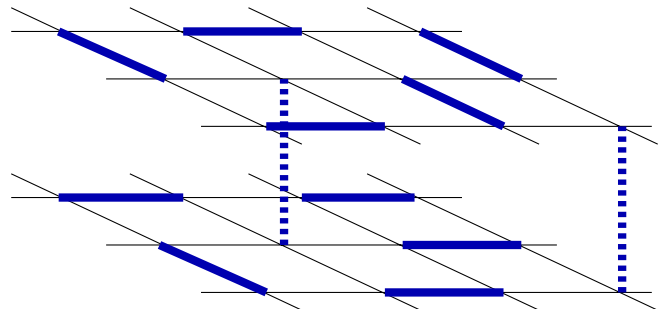
(3) Does the two layered model approach a KT critical

FIG. 5 (color online). A dimer confinguration in two layers. The sites with *dashed* dimers connecting the layers is conjectured to be the location of the vortex. Note that for all $t > 0$ the density of such sites will be nonzero leading to a condensation of vortices [10].

point as $t \to 0$ as conjectured here? Can this result be understood through analytic means?

(4) What is the effect of $\beta > 0$ in the layered model? Hopefully, this will only introduce more disorder in the system. This may be worth studying further since a new type of order may be established for large β.

The answers to these questions could shed more light on the field theories described by the dimer-plaquette models.

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