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Quantization of the $B = 1$ and $B = 2$ Skyrmions

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We propose to set the Skyrme parameters F_π and e such that they reproduce the physical masses of the nucleon and the deuteron. We allow deformation using an axially symmetric solution and simulated annealing to minimize the total energy for the $B = 1$ nucleon and $B = 2$ deuteron. First we find that axial deformations are responsible for a significant reduction (factor of ≈ 4) of the rotational energy but also that it is not possible to get a common set of parameters F_π and e which would fit both nucleon and deuteron masses simultaneously at least for $m_\pi = 138$ MeV, 345 MeV and 500 MeV. This suggests that either $m_\pi > 500$ MeV or additional terms must be added to the Skyrme Lagrangian.

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I. INTRODUCTION

The Skyrme model [1] is a nonlinear theory of pions that admits topological solitons solutions called Skyrmions. These solutions fall into sectors characterized by an integer-valued topological invariant B. In its quantized version, a Skyrmion of topological charge B may be identified with a nucleus with baryon number B.

Since the model is nonrenormalizable, a canonical quantization is not possible and one has to resort to semiclassical quantization of the zero modes of the Skyrmion. This method adds kinematical terms (rotational, vibrational, ...) to the total energy of the Skyrmion. The $B = 1$ Skyrmion was first quantized by Adkins, Nappi, and Witten [2,3]. It provided then a useful mean to set the parameters of the Skyrme model F_π and e by fitting to the proton and delta masses. The experimental value of the pion mass m_π completes the set of input parameters when a pion mass term is added to the Skyrme Lagrangian. The same calibration was assumed by Braaten and Carson [4] in their quantization of the $B = 2$ Skyrmions and their predictions of a rather tightly bound and small sized deuteron. Further analysis by Leese, Manton, and Schroers [5] considering the separation of two single Skyrmions in the most attractive channel led to more accurate predictions for the deuteron.

Yet all these calculations have three caveats: First, they used a rigid-body quantization, i.e. the kinematic term is calculated from the solution which minimizes the static energy neglecting any deformation that could originate form the kinematical terms. This was pointed out by several authors [6,7] who proposed to improve the solutions by allowing the $B = 1$ Skyrmion to deform within the spherically symmetric ansatz. Second, even with such deformations it was noted that the set of parameters mentioned above does not allow for a stable spinning solution for the delta since the rotational frequency Ω would not satisfy the constraint for stability against pion emission, $\Omega^2 \leq \frac{3}{2} m_\pi^2$. These two problems were addressed recently in [8,9]. Assuming an axial symmetry, the calculations were performed using a simulated annealing algorithm allowing for the minimization of the total energy (static and kinetic). It was also shown that the stable spinning nucleon and delta masses could be obtained only if the pion mass is fixed at more than twice its experimental value. One may argue that in this case m_π could be interpreted as a renormalized pion mass which could explain its departure form the experimental value. A third difficulty remains: Fixing the parameters of the Skyrme model still involve the delta which is interpreted as a stable spinning Skyrmion although physically it is an unstable resonance. Recently, Manton and Wood [10] took a different approach and chose data from $B = 6$ Lithium-6 to set the Skyrme parameters. The purpose was to provide for a better description of higher B solutions. Unfortunately, their calculations were based on a two-fold approximation, the rational map ansatz and rigid-body quantization for values of B up to 8.

In this work, we propose to set the Skyrme parameters such that they reproduce the physical masses of the nucleon and the deuteron. We allow deformation within an axially symmetric solution approximation and use simulated annealing to minimize the total energy for the $B = 1$ nucleon and $B = 2$ deuteron. We argue that this procedure provide a solution which is very close to if not the exact solution. Following the procedure in [8], we find the sets of parameters F_π and e that are required to fit the nucleon and deuteron masses, respectively, for $m_\pi = 138$ MeV, 345 MeV, and 500 MeV. The numerical calculations are compared to those obtained with the rational maps ansatz and rigid-body quantization. We find the latter approximation to be misleading since it suggests that it is possible to simultaneously fit the nucleon and deuteron masses which is not the case when we perform our numerical calculation even for larger $m_\pi = 500$ MeV. However for the solution to remains realistic with regard with the size of the nucleon or deuteron, lower values of F_π and e are to be excluded.

In Sec. II, we introduce briefly the Skyrme model and find the static energy for the axially symmetric solution proposed in [8]. The quantization of rotational and isorota-

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tional excitation using this solution leads to an expression for the kinetic energy terms at the end of Sec. III. These expressions suggest that the axial symmetry could be preserved to a large extent. Finally we discuss and compare our numerical results from simulated annealing with an axial solution on a two-dimensional grid with that coming from rational maps with rigid-body approximation in the last section.

II. THE SKYRME MODEL

The $SU(2)$ Skyrme Lagrangian density is

$$\mathcal{L}_S = -\frac{F_\pi^2}{16}\,\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32e^2}\,\mathrm{Tr}[L_\mu, L_\nu]^2 + \frac{m_\pi^2 F_\pi^2}{8}(\mathrm{Tr}U - 2) \tag{1}$$

where L_μ is the left-handed chiral current $L_\mu = U^\dagger \partial_\mu U$ and the parameters F_π and e are, respectively, the pion decay constant and the dimensionless Skyrme constant. U is a $SU(2)$ field associated to the pion field π by

$$U = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} \tag{2}$$

where $\boldsymbol{\tau}$ are the Pauli matrices, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the triplet of pion fields and the scalar meson field σ satisfy $\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi} = 1$. The third term where m_π is the pion mass was first added by Adkins and Nappi [2] to account for the chiral symmetry breaking observed in nature.

The field configurations that satisfy the boundary condition

$$U(\mathbf{r}, t) \to \mathbf{1} \quad \text{for } |\mathbf{r}| \to \infty \tag{3}$$

fall into topological sectors labeled by a topological invariant

$$B = \frac{1}{2\pi^2}\int d^3x\,\det\{L_i^a\} = -\frac{\varepsilon^{ijk}}{48\pi^2}\int d^3x\,\mathrm{Tr}(L_i[L_j, L_k]) \tag{4}$$

taking integral values. Skyrme interpreted this topological invariant as the baryon number. The minimal static energy Skyrmion for $B = 1$ and $B = 2$ turns out to have spherical and axial symmetry, respectively. Since we are only interested by these values of B, the solution will be cast in terms of cylindrical coordinates (ρ, θ, z) in the form

$$\begin{aligned} \sigma &= \psi_3 & \pi_1 &= \psi_1 \cos n\theta \\ \pi_2 &= \psi_1 \sin n\theta & \pi_3 &= \psi_2 \end{aligned} \tag{5}$$

introduced in [11] where $\boldsymbol{\psi}(\rho, z) = (\psi_1, \psi_2, \psi_3)$ is a three-component unit vector that is independent of the azimuthal angle θ. The boundary conditions (3) implies that $\boldsymbol{\psi} \to (0, 0, 1)$ as $\rho^2 + z^2 \to \infty$. Moreover, we must impose that $\psi_1 = 0$ and $\partial_\rho\psi_2 = \partial_\rho\psi_3 = 0$ at $\rho = 0$. The $B = 1$ hedgehog solution appears as a special case of (5) having spherical symmetry and corresponds to

$$(\psi_1, \psi_2, \psi_3) = (\sin F \sin\theta, \sin F\cos\theta, \cos F)$$

where $F = F(r)$ is the profile or chiral angle.

With the axial ansatz (5) and a appropriate scaling[1] the expressions for the static energy and the baryon number become

$$\begin{aligned} E_n &= -\int d^3x\,\mathcal{L}_S \\ &= 2\pi\left(\frac{F_\pi}{2\sqrt{2}e}\right)\int dz d\rho\rho\bigg\{(\partial_\rho\boldsymbol{\psi}\cdot\partial_\rho\boldsymbol{\psi} + \partial_z\boldsymbol{\psi}\cdot\partial_z\boldsymbol{\psi}) \\ &\quad\times\left(1 + n^2\frac{\psi_1^2}{2\rho^2}\right) + \frac{1}{2}|\partial_z\boldsymbol{\psi}\times\partial_\rho\boldsymbol{\psi}|^2 + n^2\frac{\psi_1^2}{\rho^2} \\ &\quad + 2\beta^2(1 - \psi_3)\bigg\} \end{aligned} \tag{6}$$

$$B = \frac{n}{\pi}\int dz d\rho\psi_1|\partial_\rho\boldsymbol{\psi}\times\partial_z\boldsymbol{\psi}| \tag{7}$$

with $\beta = \frac{2\sqrt{2}m_\pi}{eF_\pi}$.

However, to describe baryons, Skyrmions must acquire a well-defined spin and isospin state. This is possible only upon proper quantization of the Skyrmions as it will be done in the next section. As we shall see in the next section adding (iso-)rotational energy will in general brake the axial symmetry manifest in $B = 1$ and $B = 2$ static solutions.

III. QUANTIZATION

Since the Skyrme Lagrangian (1) is invariant under rotation and isorotation,[2] the usual method of Skyrmion quantization consist of allowing the zero modes to depend on time and then quantize the resulting dynamical system according to standard semiclassical methods. From this perspective, the dynamical ansatz is assumed to be

$$\tilde{U}(\mathbf{r}, t) = A_1(t)U(R(A_2(t)\mathbf{r})A_1^\dagger(t) \tag{8}$$

where A_1, A_2 are $SU(2)$ matrices and $R_{ij}(A_2) = \frac{1}{2}\,\mathrm{Tr}(\tau_i A_2 \tau_j A_2^\dagger)$ is the associated $SO(3)$ rotation matrix. Introducing this ansatz into the Skyrme Lagrangian (1) one gets the kinematical contribution to the total energy which can be cast in the form

$$T = \tfrac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \tfrac{1}{2}b_i V_{ij} b_j \tag{9}$$

where $a_j = -i\,\mathrm{Tr}(\tau_j A_1^\dagger \dot{A}_1)$, $b_j = i\,\mathrm{Tr}(\tau_j \dot{A}_2 A_2^\dagger)$ and the U_{ij}, V_{ij} and W_{ij} are inertia tensors

[1] We have used $2\sqrt{2}/eF_\pi$ and $F_\pi/2\sqrt{2}$ as units of length and energy, respectively.

[2] Since we are interested only in the computation of the static properties, we will ignore translational modes and quantize the Skyrmions in their rest frame.

$$U_{ij} = -\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\int d^3x \left\{\mathrm{Tr}(T_i T_j) + \frac{1}{8}\mathrm{Tr}([L_k, T_i][L_k, T_j])\right\}, \tag{10}$$

$$V_{ij} = -\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\epsilon_{ikl}\epsilon_{jmn}\int d^3x x_k x_m \times \left\{\mathrm{Tr}(L_l L_n) + \frac{1}{8}\mathrm{Tr}([L_p, L_l][L_p, L_n])\right\}, \tag{11}$$

$$W_{ij} = \left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\epsilon_{jkl}\int d^3x x_k \times \left\{\mathrm{Tr}(T_i L_l) + \frac{1}{8}\mathrm{Tr}([L_m, T_i][L_m, L_n])\right\} \tag{12}$$

where $T_i = iU^\dagger[\frac{\tau_i}{2}, U]$. For the axial ansatz (5), these tensors are all diagonal and satisfy $U_{11} = U_{22}$, $V_{11} = V_{22}$, $W_{11} = W_{22}$ and $U_{33} = \frac{W_{33}}{n} = \frac{V_{33}}{n^2}$. The components of these inertia tensors are

$$U_{11} = 2\pi\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\int dz d\rho \rho \left\{\psi_1^2 + 2\psi_2^2 + \frac{1}{2}\left[\left(\partial_\rho \boldsymbol{\psi}\cdot\partial_\rho\boldsymbol{\psi} + \partial_z\boldsymbol{\psi}\cdot\partial_z\boldsymbol{\psi} + n^2\frac{\psi_1^2}{\rho^2}\right)\psi_2^2 + (\partial_\rho\psi_3)^2 + (\partial_z\psi_3)^2 + n^2\frac{\psi_1^4}{\rho^2}\right]\right\}, \tag{13}$$

$$U_{33} = 2\pi\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\int dz d\rho \rho \psi_1^2(\partial_\rho\boldsymbol{\psi}\cdot\partial_\rho\boldsymbol{\psi} + \partial_z\boldsymbol{\psi}\cdot\partial_z\boldsymbol{\psi} + 2), \tag{14}$$

$$V_{11} = 2\pi\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\int dz d\rho \rho \left\{|\rho\partial_z\boldsymbol{\psi} - z\partial_\rho\boldsymbol{\psi}|^2\left(1 + n^2\frac{\psi_1^2}{2\rho^2}\right) + z^2 n^2\frac{\psi_1^2}{\rho^2} + \frac{1}{2}(\rho^2 + z^2)|\partial_\rho\boldsymbol{\psi}\times\partial_z\boldsymbol{\psi}|^2\right\}, \tag{15}$$

$$W_{11} = 2\pi\left(\frac{2\sqrt{2}}{e^3 F_\pi}\right)\int dz d\rho \rho \left\{[\psi_1(\rho\partial_z\psi_2 - z\partial_\rho\psi_2) - \psi_2(\rho\partial_z\psi_1 - z\partial_\rho\psi_1)] \times \left(1 + \frac{1}{2}\left[(\partial_z\psi_3)^2 + (\partial_\rho\psi_3)^2 + \frac{\psi_1^2}{\rho^2}\right]\right) + \frac{\psi_3}{2}(z\partial_z\psi_3 + \rho\partial_\rho\psi_3)[\partial_\rho\psi_2\partial_z\psi_1 - \partial_\rho\psi_1\partial_z\psi_2] + \frac{z\psi_1\psi_2}{2\rho}(2 + \partial_\rho\boldsymbol{\psi}\cdot\partial_\rho\boldsymbol{\psi} + \partial_z\boldsymbol{\psi}\cdot\partial_z\boldsymbol{\psi})\right\}. \tag{16}$$

Let us note that $W_{11} \neq 0$ only for $n = 1$. In order to obtain the energy corresponding to the nucleons and the deuteron, we must compute the Hamiltonian for the rotational and isorotational degrees of freedom in term of the inertia tensors (13) to (16) as well as its eigenvalues for the states corresponding to these particles.

The body-fixed isospin and angular momentum canonically conjugate to $\mathbf{a}$ and $\mathbf{b}$ are respectively

$$K_i = \frac{\partial T}{\partial a_i} = U_{ij}a_j - W_{ij}b_j, \tag{17}$$

$$L_i = \frac{\partial T}{\partial b_i} = -W_{ij}^T a_j + V_{ij}b_j. \tag{18}$$

These momenta are related to the usual space-fixed isospin ($\mathbf{I}$) and spin ($\mathbf{J}$) by the orthogonal transformations

$$I_i = -R(A_1)_{ij}K_j, \tag{19}$$

$$J_i = -R(A_2)_{ij}^T L_j. \tag{20}$$

We can now write the Hamiltonian for the rotational and isorotational degrees of freedom as

$$H = \mathbf{K}\cdot\mathbf{a} + \mathbf{L}\cdot\mathbf{b} - T$$

where T is the kinematical energy (9).

The quantization procedure consists of promoting four-momenta as quantum operators that satisfy each one of the $SU(2)$ commutation relations. According to (19) and (20), we see that the Casimir invariants satisfy $\mathbf{I}^2 = \mathbf{K}^2$ and $\mathbf{J}^2 = \mathbf{L}^2$. Then the operators form a $O(4)_{\mathrm{I,K}} \otimes O(4)_{\mathrm{L,J}}$ Lie algebra. The physical states on which these operators act are the states formed in the base $|\Psi\rangle = |ii_3k_3\rangle|jj_3l_3\rangle$ where $-i < i_3$, $-j < j_3$ and $l_3 < j$ that satisfy the constraints formulated by Finkelstein and Rubinstein [12]. One of these constraints, namely,

$$e^{2\pi i\mathbf{n}\cdot\mathbf{L}}|\Psi\rangle = e^{2\pi i\mathbf{n}\cdot\mathbf{K}}|\Psi\rangle = (-1)^B|\Psi\rangle, \tag{21}$$

implies that the spin and isospin must be an integer for even B or a half-integer for odd baryon numbers. Another constraint

$$(nK_3 + L_3)|\Psi\rangle = 0 \tag{22}$$

comes from the axial symmetry imposed on the solution here.

This formalism allows one to obtain the total energy in terms of the inertia tensors of (13)–(16) for the nucleon [9]

$$E_N = E_1 + \frac{1}{4}\left[\frac{(1 - \frac{W_{11}}{U_{11}})^2}{V_{11} - \frac{W_{11}^2}{U_{11}}} + \frac{1}{U_{11}} + \frac{1}{2U_{33}}\right] \tag{23}$$

and for the deuteron [4]

$$E_D = E_2 + \frac{1}{V_{11}}. \tag{24}$$

For the $B = 1$ solution the minimization of the static energy E_1 leads to a spherically symmetric solution. The nucleon mass in (23) however receives (iso-)rotational energy contributions from (iso-)rotations around the three principal axis among which two are equivalent (direction 1 and 2). The deformation should then lead to a axially symmetric solution along the z-axis (direction 3) as argued in Refs. [8,9].

The situation is somewhat different for $B = 2$ which is known to have a toroidal solution upon minimization of the static energy E_2. In our scheme it corresponds to the axially symmetric solution with respect to the z-axis. Assuming such a solution the deuteron mass in (24) gets kinetic energy contributions which may be recast in the form

$$E_{\rm rot} = \frac{1}{2V_{11}} + \frac{1}{2V_{22}} = \frac{1}{V_{11}}, \tag{25}$$

which means that the contributions come from rotations perpendicular to the axis of symmetry. Unfortunately there is no guarantee that in a nonrigid rotator approach the axial symmetry will be preserved when the rotational terms are added. However the deformations are not expected to be very large since the magnitude rotational energy only accounts for less than 4% of the total mass of the deuteron in the rigid-body approximation [4]. Allowing deformations should increase the moments of inertia and bring the relative contribution of rotational energy to an even lower value. Clearly large changes in the configuration are prohibited by an eventual increase in the static energy E_2 so, from that argument alone we can infer that deformations from axial symmetry are bound to be contained into a 4% effect. Our results will show that nonaxial deformations represents less than 1% of the deuteron mass if they contribute at all. This justifies the use of the axial solution in (5).

In order to obtain the configurations $\boldsymbol{\psi}(\rho, z) = (\psi_1, \psi_2, \psi_3)$ that minimize the energies defined in (23) and (24), we use a two-dimensional version of a simulated annealing algorithm used in [13]. The minimization is carried out on a grid in a plane (ρ, z) made up of 250×500 points with a spacing of 0.042.[3] The algorithm starts with an initial configuration $\boldsymbol{\psi}_0(\rho, z)$ on the grid and evolves towards the exact solution. Here $\boldsymbol{\psi}_0(\rho, z)$ is generated using the suitable rational map ansatz [14] to ensure that the initial solution have the appropriate baryon number, $B = 1$ or 2, with a profile function of the form

$$F(r) = 4\arctan(e^{-\alpha r}) \tag{26}$$

as inspired from Ref. [15]. Here α is a parameter chosen so that the entire baryon number density (such that $B = 1$ or 2, respectively, for the nucleon or the deuteron) fits into the grid.

IV. RESULTS AND DISCUSSION

There are several ways to fix the parameters of the Skyrme model e, F_π, and m_π and the authors of Ref. [8] showed that one must take some precautions in order that the spinning solutions for the nucleon and the delta remain stable against pion emission. And indeed it was found that, in order to achieve a fit for the energies of spinning Skyrmions to the masses of the nucleon and delta, one must impose a value for the pion mass that is larger than its experimental value. Differences between the fitted and experimental values of F_π and m_π are not proscribed since after all, the values that enters the Lagrangian (1) are the unrenormalized parameters which could differ from the physical ones.

Even so it remains that this procedure still assumes that a physically unstable particle, the delta resonance, is described as a stable spinning Skyrmion and to be perfectly consistent one should instead rely on stable particles. For these reasons we chose to carry out calculations using data from two stable particles, the nucleon and the deuteron. We proceed as follows: Assuming a value for the pion mass and e, and an initial value for the Skyrme parameter F_π we compute the lowest energy solution for the spinning Skyrmion corresponding to the nucleon using simulated annealing which leads to a mass prediction. We iterate the procedure adjusting values of F_π until the predicted mass fits that of the nucleon. The same procedure is repeated for the $B = 2$ deuteron as well as for several values of e. A set of points requires about two weeks of computer calculations on a regular PC. Performing an equivalent calculation on a 3D grid with similar spacing for a nonaxial solution, for example, would be prohibitive. This explain in parts why we use the axially symmetric ansatz.

Initially, the first set of calculations was performed with the pion mass equal to its experimental value $m_\pi = 138$ MeV. Since Ref. [8] suggests that a pion mass value of $m_\pi = 345$ MeV or more is necessary to avoid instability due to pion emission we repeated the calculations and evaluated the Skyrme parameters using $m_\pi = 345$ MeV. The results are displayed in Fig. 1. Although it may seem interesting here to consider values of e lower than those illustrated on Fig. 1, the physical relevance of these values is questionable. Below a certain value, the rotational energy of the Skyrmion is larger than the contribution coming from the pion mass term. As was highlighted in Refs. [6,7], this leads to an unstable Skyrmion with respect to emission of pions.

Our results for $B = 1$ reproduce the same behavior as in Refs. [8,9], i.e. the deformation of the Skyrmion becomes relatively important only for larger values of m_π and e. However, the results for $B = 2$ are far more interesting with respect to deformation. Indeed, as we can see directly from Fig. 1, there is a noticeable difference between our numerical results (squares) which allow for deformations as long as they preserve axial symmetry and that of the rigid-body approximation (dashed lines). Note that our implementation of the rigid-body approximation shown here relies on the rational map ansatz which is known to be accurate to a few percent and so it could not alone be

[3]These parameters were adopted in order to be similar with those used in [8,9].

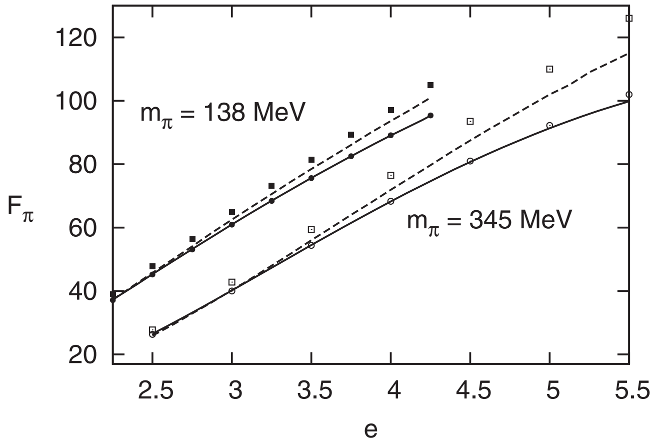

FIG. 1. F_π as a function of e for which $M + E_{\rm rot}$ is equal to the nucleon mass (circles) and that of the deuteron (squares). The solid and dashed lines correspond to the results obtained from the rigid-body approach for the nucleon and deuteron, respectively. The set of data and lines at the upper left are for $m_\pi = 138$ MeV whereas that at the bottom right are for $m_\pi = 345$ MeV.

responsible for such a large difference in energy. This difference indeed corresponds to a surprisingly much smaller energy for the deformed spinning Skyrmion than what is obtained from a the rigid-body approximation wether it is based on the rational map ansatz or not.

To illustrate this difference, we carried out our minimization procedure using the set of parameters $F_\pi = 108$ MeV and $e = 4.84$ for the deformed deuteron. The results, set SA, are listed in of Table I. For comparison we also present three rigid-body calculations: set SA-RB performed with our simulated annealing algorithm, set Ref. [4]-RB from Braaten *et al.* and finally set RM-RB obtained through the rational maps approximation. Note that our result SA-RB agrees fairly well with that of Ref. [4]. Clearly the mass of the deuteron E_D must be larger than $E_2^{\rm min} = 1655$ MeV, the minimum static energy for this choice of parameters. Computing the rotational energy for this solution leads to the rigid-body approximation result (SA-RB) for the deuteron mass of $E_D =$

TABLE I. Total energy, rotational energy, and charge radius of the deuteron using parameters $F_\pi = 108$ MeV and $e = 4.84$. The results are presented for our simulated annealing calculations with axial deformation (SA) along with three rigid-body calculations: with simulated annealing (SA-RB), from Braaten *et al.* (Ref. [4]-RB) and with the rational maps approximation (RM-RB), respectively. The last column (Exp.) shows the experimental values where the charge radius comes from [16].

	SA	SA-RB	Ref. [4]-RB	RM-RB	Exp.
E_D (MeV)	1679	1716	1720	1750	1876
$E_{\rm rot}$ (MeV)	13.3	60.6	61.2	55.2	—
$\langle r^2\rangle_D^{1/2}$ (fm)	0.94	0.93	0.92	0.94	2.095

1716 MeV. As expected the results for the deformed deuteron (SA) $E_D = 1679$ MeV lies between these two values. Moreover our axial solution brings the relative contribution of the rotational term to about 1% of the minimum static energy $E_2^{\rm min}$. So allowing for axial deformation reduces by about a factor of 4 the rotational term going from 60.6 MeV to 13.3 MeV. As for nonaxial deformations that might be present in a completely general solution, they must at the very most represent a 1% correction to the deuteron mass since it is bounded by $E_2^{\rm min}(= 1655 \text{ MeV}) < E_D^{\rm exact} \leq E_D(= 1679 \text{ MeV})$. On the other hand they may still represent a significant portion of the remaining 13.3 MeV rotational energy. We conclude nonetheless that the axial symmetry ansatz represents a very good approximation of the deuteron configuration. In addition, since these bounds are both based on axially symmetric solutions and their energy only differ by 1%, one can even contemplate the possibility that the exact solution may have axial symmetry and therefore would correspond to our solution, contrary to what physical intuition might suggest. This has yet to be proven and needless to say that such a demonstration would require a 3D calculation with a level of precision less than 1%.

Note that, both for $m_\pi = 138$ MeV and 345 MeV, the solid and dashed lines intersect for a relatively low value of e. However despite what this rigid-body calculations suggests, there is no set F_π and e that leads simultaneously to the masses of the nucleon and the deuteron. However, the data of Fig. 1 would indicate that the gap between the values of F_π decreases for a larger pion mass. For that reason, we repeated the calculations with a relatively large $m_\pi = 500$ MeV. Some of the computations required that we adjust the spacing of our grid to 0.0057 for smaller e since the configurations turned out to be much smaller in size. The results for $m_\pi = 500$ MeV are presented in Fig. 2. The main conclusion one can draw is that even for this pion mass, no common set of Skyrme parameters can be found.

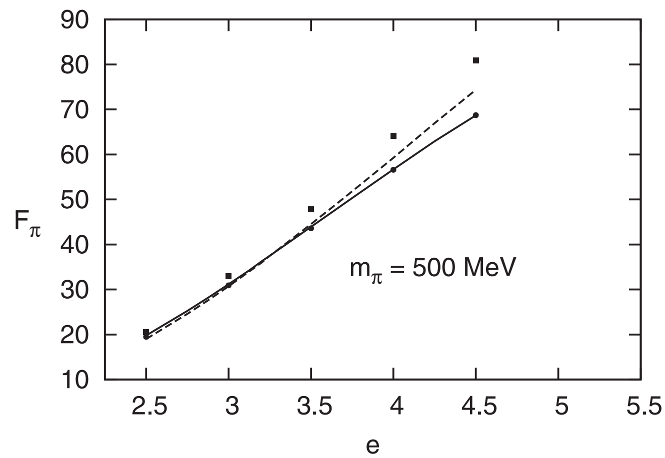

FIG. 2. Same as Fig. 1 for $m\pi = 500$ MeV.

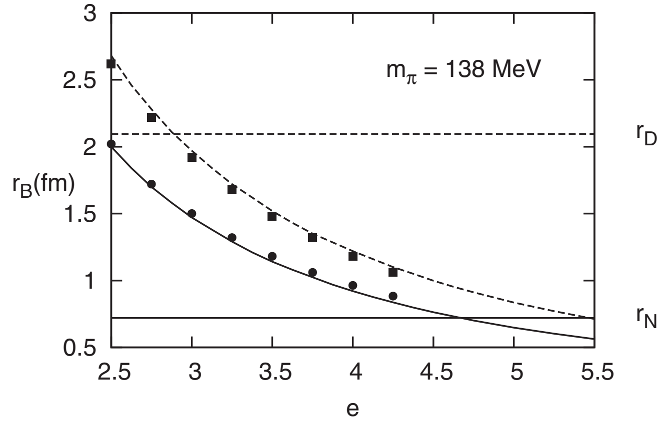

FIG. 3. Charge radius of the nucleon (bold circles) and deuteron (bold squares) for the set of parameters of Fig. 1 with $m\pi = 138$ MeV. The curves and the horizontal lines, solid for the nucleons and dashed for the deuteron, correspond, respectively, to the results obtained from the rigid-body approach and to the experimental data.

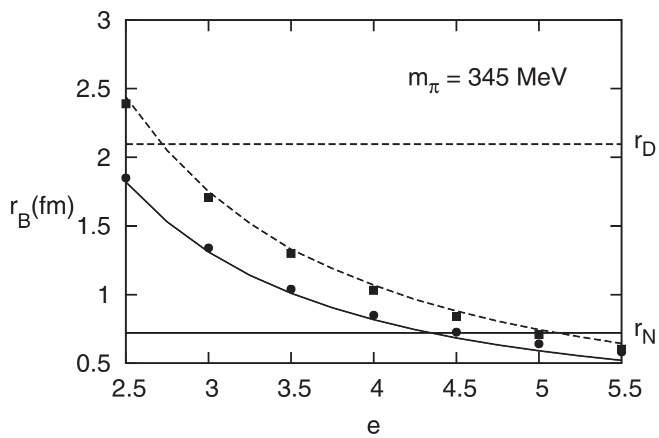

FIG. 4. Same as Fig. 3 for $m\pi = 345$ MeV.

It is also interesting to note that the gap between the values of F_π decrease as e decreases. This would suggest that at very low values of e a fit is possible. However computing the charge radius, i.e. the square root of

$$r_n^2 = \frac{8}{e^2 F_\pi^2 \pi} \int dz d\rho(\rho^2 + z^2)\psi_1 |\partial_\rho \boldsymbol{\psi} \times \partial_z \boldsymbol{\psi}| \qquad (27)$$

leads to an significant increase for the radius for small values of e and F_π as illustrated in Figs. 3–5. So lower values of e are incompatible with the physical size of the deuteron and nucleon and may be discarded. The results also indicate that the best fit for the radius of the nucleon and deuteron would favor intermediate values of e around $e \simeq 3.5$ while it looks fairly insensitive to large changes in m_π.

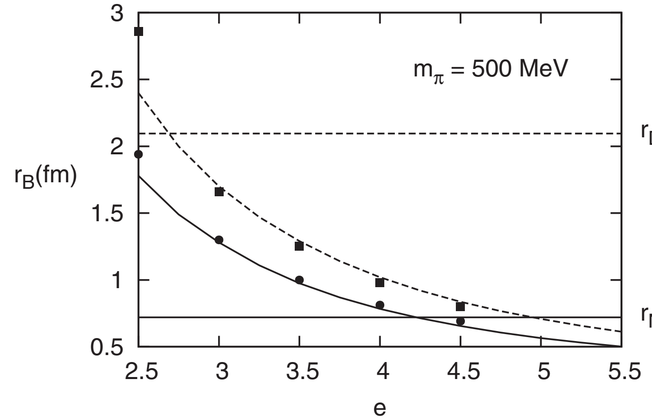

FIG. 5. Same as Fig. 3 for $m\pi = 500$ MeV.

To summarize, our calculations showed that the axial symmetry ansatz is a very good approximation of the exact solution for the deuteron. This hints at the possibility that it may even represent the exact solution. This remains to be proven with a general 3D calculation. We also found that allowing for axial deformation reduces the rotational energy by a significant factor. On the other hand we found that it is not possible to get a common set of parameters F_π and e which would fit both nucleon and deuteron masses simultaneously at least for $m_\pi = 138$ MeV, 345 MeV, and 500 MeV. This conclusion should hold even for the exact $B = 2$ solution since if the solution was allowed to adjust free of any symmetry constraints it would achieve a configuration with lower total energy which would require larger values of F_π for the same set of e. This suggests that either $m_\pi > 500$ MeV or additional terms must be added to the Skyrme Lagrangian. We also observed an increase in the deformations due to the spinning of the $B = 2$ Skyrmion (deuteron) especially for larger values of e and F_π so the rigid-body approximation may not be appropriate in that case.

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