Tau and muon pair production cross sections in electron-positron annihilations at $\sqrt{s} = 10.58$ GeV

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The calculational precision of $e^+e^- \rightarrow \tau^+\tau^-$ and $e^+e^- \rightarrow \mu^+\mu^-$ production cross sections in electron-positron annihilations at $\sqrt{s} = 10.58$ GeV is studied for the KKMC Monte Carlo simulation program, modified to include contributions from recent implementation of the hadronic part of vacuum polarization. We determine $\sigma(e^+e^- \rightarrow \tau^+\tau^-) = (0.919 \pm 0.003)$ nb and $\sigma(e^+e^- \rightarrow \mu^+\mu^-) = (1.147 \pm 0.005)$ nb, where the error represents the precision of the calculation.

I. INTRODUCTION

At the present generation *B*-factories lepton pair production, $e^+e^- \rightarrow \tau^+\tau^-$ and $e^+e^- \rightarrow \mu^+\mu^-$, occurs at approximately the same rate as the $e^+e^- \rightarrow B\bar{B}$ process. Given the high statistics of the available data, precise knowledge of the τ-pair and μ-pair production cross sections in e^+e^- annihilations at a center-of-mass energy $\sqrt{s} = 10.58$ GeV is necessary for a number of precision measurements, particularly for measurements of the branching fractions of τ decays and luminosity determinations using the counting of $e^+e^- \rightarrow \mu^+\mu^-$ events. The Belle experiment currently uses a τ-pair cross section of 0.89 nb and quotes a 1.4% error on the number of τ-pairs produced for a given amount of integrated luminosity where the 1.4% is intended to account mainly for the luminosity measurement error [1,2]. The *BABAR* experiment also uses a 0.89 nb cross section but assigns a 2% error to this figure based on a simple comparison of the cross sections from the KKMC [3] and KORALB [4] Monte Carlo simulation programs [5]. Although the 2% figure is adequate in searches for rare or forbidden processes, such a large error currently limits the precision of τ branching fraction measurements at the *B*-factories.[1]

In principle, the cross sections determined with KKMC are the most precise calculations currently available, but the cross sections and their precisions as determined with this code at energies far below the *Z*-pole have not been reported. Only tests for $\sqrt{s} > 40$ GeV have been available in the literature [3,8,9] and the code cautions users of its

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[1]An alternative approach that is insensitive to the τ-pair production cross section is to measure the τ-branching ratios relative to the τ electronic branching fraction which is measured to ~0.3% accuracy, as has been done in a recent publication [6]. However, in such an approach, the corresponding Monte Carlo generators, e.g. TAUOLA [7], would need to be tested at the precision of 0.1% level in the presence of experimental cuts, and such a study is beyond the scope of the present paper.

potential lack of reliability for energies below $\sqrt{s} = 40$ GeV because of insufficient study in that regime.

This unsatisfactory situation should be compared with predictions of KKMC for high energies, where a precision at the 0.1%–0.4% level was achieved [8,9]. In this paper we report on straightforward studies of the μ and τ-pair production cross section calculations of KKMC and KORALB at $\sqrt{s} = 10.58$ GeV with a goal of providing, for the first time, a realistic estimate of the precision of the cross section calculation at energies relevant to the *B*-factories relying on, but not repeating, the detailed high-energy investigations. The cross sections are calculated using code that includes QED and electroweak radiative corrections to different orders in the fine structure constant α and using different approximations for vacuum polarization. KORALB calculates the matrix element to order α including interference between initial (ISR) and final state radiation (FSR). KKMC calculates the cross section to order $\alpha^2 \log(s/m_e^2)$ and includes exponentiation and ISR-FSR interference. Various implementations of the vacuum polarization can be used in both KORALB and KKMC programs.

The results presented here rely on the extensive studies [8,9] used to determine the precision of the calculations in the KKMC as well as KORALZ [10] software. The conclusion of those studies is that cross sections calculated with KKMC at the *Z*-pole and LEP II energies ($\sqrt{s} \sim 200$ GeV) are at the precision level of 0.2%. Fortunately, the physics environment at the higher energies where the detailed tests have been made is far more complex than at the $\Upsilon(4S)$. For example, unlike at the $\Upsilon(4S)$, the higher energy tests had to address issues related to the radiative return to the *Z*-pole, including effects related to initial and final state radiation bremsstrahlung interference; scattering-angle dependencies, of particular interest because of the importance of forward-backward asymmetry measurements at the higher energies; and high-energy electroweak and QCD corrections. Therefore it is only

necessary to verify that the technical precision determined in detail at the higher energies for a wide class of effects can be reliably extended to the lower energies of the B-factories. This has been accomplished using relatively simple tests. In this way we achieve the goal of providing a realistic cross section error without having to repeat the previous high-energy studies. However, because of the different energy scales, the high-energy studies of several effects, such as the vacuum polarization, are not reliably extended to the $\Upsilon(4S)$, and these have been examined separately.

As it turns out, our current knowledge of the lepton pair production cross sections is in fact partially limited by the treatment of vacuum polarization in this calculation. In this paper, we study their contributions using different parametrizations of the hadronic part of the vacuum polarization, as well as our knowledge of the ratio $R = (e^+e^- \rightarrow q\bar{q})/(e^+e^- \rightarrow \mu^+\mu^-)$. Other sources of theoretical uncertainty arise from the implementation of initial and final state radiation including their interference and electroweak corrections, effects from virtual pair-productions of additional fermions, and the impact of low-energy resonances on the cross section.

At the level of precision targeted in this work, cross sections for τ-pair and μ-pair processes are adequately studied with 10^6 generated events. Because of the larger contribution from radiative returns to very low energy in μ-pair events, cross sections are also studied with an angular acceptance in the center of mass for both muons of $[22°, 158°]$ and with effective center-of-mass energy, $\sqrt{s'}$, greater than 10% of 10.58 GeV which corresponds more closely to the actual experimental acceptance.

Our paper is organized as follows. In Sec. II we discuss radiative corrections to the total cross section due to hadronic vacuum polarization. We have found that changes to the public versions of the two programs are necessary in order to achieve the goal of obtaining a reduced systematic error. Section III is devoted to verifying that the high-energy estimates of the systematic error of KKMC photonic (and acceptance dependent) corrections for the total cross section remain valid at our energies as well. Sections IV and V discuss the impact of interference and pair production corrections, respectively. Section VI describes how we address the effects of vector resonances. Section VII discusses the error on the ratio of the $\tau^+\tau^-$ and $\mu^+\mu^-$ cross sections as calculated by KKMC. The estimations of the systematic error of theoretical predictions for the τ-pair and μ-pair cross sections are summarized in Sec. VIII.

II. TREATMENT OF VACUUM POLARIZATION

The particular vacuum polarization implementation is controlled in the KKMC code using the IHVP flag following the DIZET implementation of the calculation [11], with improvements summarized in Sec. 4.1.3 of Ref. [12] and Sec. 4.132 of Ref. [13]. The default modeling of vacuum

polarization in KKMC corresponding to IHVP flag $= 1$ [14], suggested in Ref. [12] was not optimized for the low-energy applications. In the present implementation, the hadronic part of the vacuum polarization is studied using the experimental knowledge of the ratio $R = \sigma(e^+e^- \rightarrow q\bar{q})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ relevant for a precise determination of $\alpha(m_Z)$. The hadronic part of the vacuum polarization is not calculated in this option in DIZET for center-of-mass energies less than 40 GeV. Thus, at $\sqrt{s} = 10.58$ GeV, this option gives a $\sim 3\%$ lower fermion-pair cross section than those obtained from other DIZET options that include the hadronic component of the vacuum polarization. Other available options in DIZET (IHVP flag $= 2$ [15] and $= 3$ [16]) use experimental knowledge of the ratio $R = (e^+e^- \rightarrow q\bar{q})/(e^+e^- \rightarrow \mu^+\mu^-)$ measured more than 15 years ago. Improvements in the measurements of the ratio R from the Crystal Ball experiment [17] in e^+e^- annihilations at $\sqrt{s}$ between 5.0 and 7.4 GeV, and more recent measurements by the BES experiment [18] at $\sqrt{s}$ between 2 and 5 GeV, are not included for these options. Improvements on the calculation of the hadronic part of the vacuum polarization at $\sqrt{s} = m_Z$ obtained by including these new measurements of R have been documented recently [19]. Following these developments, we have introduced a new option in DIZET implemented with IHVP flag $= 4$ to calculate the contribution of vacuum polarization using the routine "REPI" [20], where these improvements on the measurement of R have been incorporated.

We evaluate the uncertainties associated with using IHVP flag $= 4$. The routine REPI uses a simple parametrization of the hadronic contribution to the vacuum polarization in the t-channel and can be used safely in the s-channel outside the energy regime dominated by individual resonances. Special care must be taken to account for the large positive and negative fluctuations in the contribution to the hadronic part of the vacuum polarization in the s-channel across the resonance as illustrated in Fig. 1 of Ref. [21]. We have calculated the differences between the s-channel and t-channel contribution of the hadronic part of the vacuum polarization to the τ and μ-pair production cross section, weighted by the distribution of the invariant mass of the lepton-pair system ($\sqrt{s'}$) in bins of 10 MeV. Below and above the resonances, these differences are positive and negative, respectively, and their effects tend to cancel when integrated over $\sqrt{s'}$. At the $\Upsilon(4S)$ these differences cancel partially, and the size of this cancellation is sensitive to the variation of the central value of the center-of-mass energy distribution of colliding beams for different periods of data taking as well as the collider beam energy spread. We assume a beam energy spread of 4.6 MeV (RMS), which is characteristic of the center-of-mass energy spread at *BABAR* experiment at the PEP II *B*-factory [22]. Based on the results of these calculations we assign an uncertainty of 0.18% on the τ-pair cross

section and 0.22% for the μ-pair cross section associated with the implementation of the vacuum polarization in KKMC with IHVP flag = 4 in DIZET. These uncertainties are also valid for the Belle experiment, since the center-of-mass energy spread of the KEK B-factory is similar to that of the PEP II B-factory [23].

In order to check the technical precision of this new implementation of vacuum polarization, we have also calculated the cross section with KKMC using different IHVP options in DIZET as well as with corresponding calculations using KORALB with this new IHVP flag = 4 option implemented. The cross sections for $e^+e^- \rightarrow \tau^+\tau^-$ and $e^+e^- \rightarrow \mu^+\mu^-$ calculated with KKMC are summarized in Table I along with their Monte Carlo statistical errors, which are at the level of 0.02% for τ-pairs and 0.03% for the μ-pairs. The cross sections for μ-pairs with the cuts mentioned in the previous section are denoted with "acc" and "cuts" subscripts, respectively, for the cases when only the angular acceptance cuts are applied and when both the acceptance and the $\sqrt{s'} > 0.1\sqrt{s}$ cuts are applied. For τ-pair production, we report the total cross section.

The τ-pair cross sections from IHVP flags = 2, 3, and 4 agree to within 0.16%. The μ-pair cross sections agree within 0.16% and 0.21% for the no-cuts case, $\sigma(\mu^+\mu^-)$, and all cuts case, $\sigma_{\text{cuts}}(\mu^+\mu^-)$, respectively. As is evident from the distribution of the lepton-pair invariant mass, $\sqrt{s'}$, presented in Fig. 1, the μ-pair cross section has a larger contribution from the vacuum polarization at low $\sqrt{s'}$. Differences in the cross sections are expected because different measurements of R are used in the calculations of the vacuum polarization in Refs. [15,16,20], as described above. Therefore we do not assign any additional uncertainty resulting from this cross check, when IHVP flag = 4 is used as the default option.

As an additional cross check on the introduction of the vacuum polarization into the code, we have studied the technical robustness achieved in the implementation of IHVP flag = 4 by comparing the cross sections calculated with KORALB and KKMC programs using the different configurations given in Table II. Turning off the bremsstrahlung radiative corrections allows us to isolate the vacuum polarization corrections. In the case of KORALB, the vacuum polarization increases the Born-level cross section ($\sigma_{\text{BORN}}^{\text{KORALB}}$) by $2\mathcal{R}e(\Pi_{\gamma\gamma}(s'=s))$, where $\Pi_{\gamma\gamma}$ is the photon self-energy function for the vacuum polarization correction. On the other hand, for KKMC, the vacuum
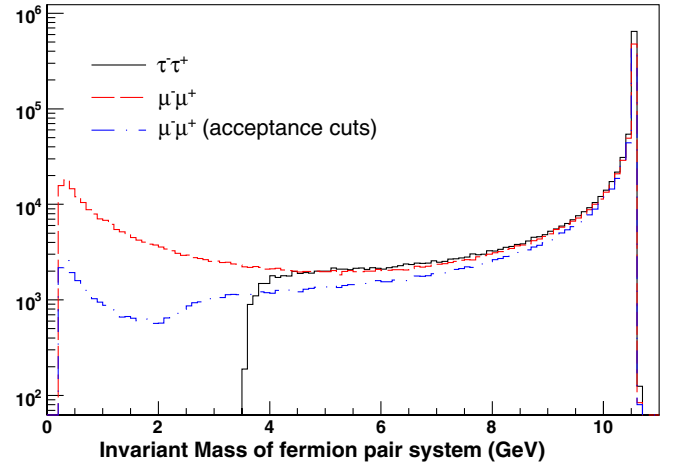

FIG. 1 (color online).   Invariant mass of $\tau^+\tau^-$ and $\mu^+\mu^-$ pairs ($\sqrt{s'}$) with IHVP flag = 4.

polarization factor is $|1 - \Pi_{\gamma\gamma}(s')|^{-2}$, where s' is only determined from ISR, so that for the case of FSR alone, $s = s'$. Therefore, we expect

$$\frac{(\sigma^{\text{KORALB}} - \sigma_{\text{NO VP}}^{\text{KORALB}})}{\sigma_{\text{BORN}}^{\text{KORALB}}} \approx 2\left(1 - \sqrt{\frac{\sigma_{\text{BORN}}^{\text{KK}}}{\sigma_{\text{NO BREM}}^{\text{KK}}}}\right).$$

This relation between KKMC and KORALB is found to be valid to better than the statistical precision of the Monte Carlo calculations for both τ-pairs and μ-pairs with and without additional cuts on acceptance and $\sqrt{s'}$.

These results are consistent with our expectations. However, in general when ISR is turned on in KORALB $s' = s$ is always used for the vacuum polarization calculation, whereas, in KKMC s', more correctly, varies from event to event as expected in the presence of ISR. In the absence of ISR, the above relation comparing KKMC and KORALB cross sections is exactly valid, whereas in the presence of ISR, the difference in treatment of this effect between the two programs is small enough that the relation still provides a useful cross check.

III. INITIAL AND FINAL STATE BREMSSTRAHLUNG

Because bremsstrahlung is calculated to different orders in KKMC and KORALB, comparisons of the cross sections calculated using the two software packages help study

TABLE I. τ-pair and μ-pair cross sections with different IHVP flags using KKMC. Note: IHVP = 1 does not include the hadronic part of the vacuum polarization.

IHVP	$\sigma(\tau^+\tau^-)$ (pb)	$\sigma(\mu^+\mu^-)$ (pb)	$\sigma_{\text{acc}}(\mu^+\mu^-)$ (pb)	$\sigma_{\text{cuts}}(\mu^+\mu^-)$ (pb)
1	891.45 ± 0.20	1117.81 ± 0.35	825.90 ± 0.26	810.75 ± 0.25
2	919.85 ± 0.21	1149.01 ± 0.37	851.47 ± 0.27	836.17 ± 0.27
3	920.17 ± 0.21	1148.56 ± 0.37	851.69 ± 0.27	836.47 ± 0.27
4	918.66 ± 0.21	1146.60 ± 0.36	850.31 ± 0.27	835.15 ± 0.27

TABLE II.   Code configurations. Note: for technical reasons, the KKMC code can only be run with final state radiation on.

	Software package	Bremsstrahlung configuration	Electroweak and vacuum polarization corrections
$\sigma_{\text{BORN}}^{\text{KORALB}}$	KORALB	ISR off, FSR off	off
$\sigma_{\text{BORN}}^{\text{KK}}$	KKMC	ISR off, FSR on	off
σ^{KORALB}	KORALB	on with interference	on
$\sigma_{\text{NO VP}}^{\text{KORALB}}$	KORALB	on with interference	off
$\sigma_{\text{NO BREM}}^{\text{KK}}$	KKMC	ISR off, FSR on	on
$\sigma_{\text{NO VP}}^{\text{KK}}$	KKMC	on with interference	off
σ^{KK}	KKMC	on with interference	on
$\sigma_{\text{NO INT}}^{\text{KK}}$	KKMC	ISR/FSR on but interference off	on

properties of the corrections, and as a consequence, establish the precision for the KKMC cross sections. To that end, we compare results from Born-level calculations (KKMC or KORALB), first-order radiative effects (KORALB), and second-order calculations with exponentiation (KKMC). For these studies, the electroweak and vacuum polarization corrections are initially switched off.

As a technical benchmark, we initially compare the Born-level $e + e- \rightarrow \mu + \mu$- cross sections calculated at $\sqrt{s} = 10.58$ GeV using KKMC and KORALB when both electroweak and radiative corrections are switched off. For technical reasons [24], the final state radiation in KKMC is not switched off and therefore it is expected that the ratio $\sigma_{\text{BORN}}^{\text{KK}}/\sigma_{\text{BORN}}^{\text{KORALB}}$ is equal to $1 + (\frac{3}{4})(\frac{\alpha}{\pi})$ when no experimental cuts are applied. To better than 0.01%, we find this to be the case.

Having established that the Born-level cross sections are in good agreement, we proceed to use the KORALB cross sections with ISR-FSR switched on and with electroweak and vacuum polarization corrections switched off ($\sigma_{\text{NO VP}}^{\text{KORALB}}$). The quantity $(1 - \sigma_{\text{NO VP}}^{\text{KORALB}}/\sigma_{\text{BORN}}^{\text{KORALB}})$ is a measure of the contribution of first-order bremsstrahlung to the cross section. The magnitude of the bremsstralung effects at the second and third orders in the exponential expansion can be estimated as $(1 - \sigma_{\text{NO VP}}^{\text{KORALB}}/\sigma_{\text{BORN}}^{\text{KORALB}})^2/2$ and $(1 - \sigma_{\text{NO VP}}^{\text{KORALB}}/\sigma_{\text{BORN}}^{\text{KORALB}})^3/6$, respectively. For $\sigma_{\text{cuts}}(\mu^+\mu^-)$, $\sigma_{\text{NO VP}}^{\text{KORALB}}/\sigma_{\text{BORN}}^{\text{KORALB}} = 1.1126$ whereas for $\sigma(\tau^+\tau^-)$, $\sigma_{\text{NO VP}}^{\text{KORALB}}/\sigma_{\text{BORN}}^{\text{KORALB}} = 1.1054$. Taking 11% as the size of first-order correction from bremsstrahlung the second order in the exponential expansion series is $\sim 0.11^2/2 = 0.0061$. Therefore, since KKMC includes the second order, we expect a difference between the KKMC and KORALB cross sections to be $\lesssim 1\%$. This 1% estimate is consistent with the value of 1.0072 for $\sigma_{\text{NO VP}}^{\text{KK}}/\sigma_{\text{NO VP}}^{\text{KORALB}}$ for $\sigma_{\text{cuts}}(\mu^+\mu^-)$, and somewhat larger than the value of 1.0012 for $\sigma(\tau^+\tau^-)$, which also receives contributions from other effects, e.g. mass corrections. The next order of the leading log terms, which is not fully controlled in KKMC, can be estimated to be $\sim 0.11^3/6 = 0.0002$. These tests confirm that KKMC and KORALB are yielding the expected relative cross section behavior near $\sqrt{s} = 10.58$ GeV. We can estimate the relative error associated with the treatment of ISR and FSR to be equal to the size of

the last term fully controlled in the approximation implemented in KKMC: $\alpha^2 \log(s/m_e^2) = 0.0011$. This next-to-leading-log term is larger than the third-order leading log term, which is expected to be of order of $\sim 0.11^3/6$, by about a factor of 5. The above estimates are rather naive as they assume an exponentiation pattern for all QED effects. Nevertheless, in our case the resulting relations hold rather well, pointing to a simple pattern of QED effects, and at the same time to the technical correctness of our calculations. This is to be compared with the 0.2% uncertainty assigned for the KKMC treatment of bremsstrahlung at LEP II energies [9], where further complications arise due to significant contributions from the radiative return to Z.

Final state bremsstrahlung has mass-term dependencies that potentially are more significant for τ-pair production. These would be expected to contribute a maximum of $\frac{\alpha}{\pi} \times \frac{4m_\tau^2}{s} = 0.03\%$ to the relative systematic error and therefore are negligible for this level of study and not considered further.

As these tests assume KORALB is correctly calculating the matrix element to order α, it is useful to ensure that its behavior associated with the technical parameter XK0, the minimal energy for the real bremsstrahlung photon to be explicitly generated, is insensitive to the actual choice of this parameter [25]. To that end, we have verified that reducing XPAR11 = XK0 from its default value of 0.01 by a factor of 2 changes total KORALB cross section by considerably less than 0.1%.

Finally, as a technical cross check on the final results, we also compare the KKMC cross section calculation with all radiative corrections turned on, σ^{KK}, to a cross section calculation assuming naive factorization of the bremsstrahlung and non-bremsstrahlung corrections: $\sigma^{\text{KK}} = \sigma_{\text{NO VP}}^{\text{KK}} \times (\sigma_{\text{NO BREM}}^{\text{KK}}/\sigma_{\text{BORN}}^{\text{KK}})$. For μ-pairs with $\sqrt{s'} > 0.9\sqrt{s}$, which is in the regime of very small radiation, we find that this factorization holds to better than the statistical precision of the Monte Carlo calculations.

These studies verify that the 0.1%–0.2% error tag assigned for photonic corrections in Refs. [8,9] for higher energies is also valid at $\sqrt{s} = 10.58$ GeV. We adopt the higher, 0.20%, value as the uncertainty on the implementation of ISR/FSR bremsstrahlung at the $\Upsilon(4S)$ and thereby avoid having to address the dependence on selection cuts.

IV. INTERFERENCE EFFECTS

Two kinds of interference effects are considered: the electroweak interference between γ and $Z^\star$ propagators, and the QED interference between initial and final state radiation arising from $\gamma\gamma$ box diagrams. The contribution from the γ-$Z^\star$ interference is smaller than the Monte Carlo statistical precision. Although the QED interference results in a forward-backward asymmetry of a few percent, the contribution to the total cross section is considerably smaller. The magnitude of the contribution of interference to the total cross section can be estimated from the ratio of the cross section with all corrections on, σ^{KK}, to the cross section calculated with all corrections on but with interference switched off, $\sigma^{\mathrm{KK}}_{\mathrm{NO\ INT}}$. For both μ-pairs and τ-pairs, $\sigma^{\mathrm{KK}}/\sigma^{\mathrm{KK}}_{\mathrm{NO\ INT}} = 1.0004$ and is relatively insensitive to whether or not geometrically symmetric cuts are applied. We conclude that the largest contribution to uncertainties from these interference effects are of the order of 0.04%.

V. PAIR PRODUCTION AND VERTEX CORRECTION UNCERTAINTY

The default calculations in KKMC do not include the effects of the emission of an extra fermion pair accompanying the main process and the virtual fermion loop vertex correction. As these two effects are of opposite sign and comparable magnitude they will largely cancel each other and the error introduced by neglecting them both in the calculation is small. This large cancellation is a well-known feature of QED and has been numerically verified by studies at higher energies [13].

To estimate the error this introduces on the total cross section, we activate the appropriate second order contribution to the vertex in KKMC as explained in Sec. 4.54 of Ref. [13], and supplement the simulation with the additional sample of the four fermion final states, e.g. with the help of KORALW [26]. The effect of the virtual fermion loop vertex correction at the $\Upsilon(4S)$, using appropriate options in KKMC, is 0.3%. However, because emission of real pairs largely cancels contribution from this vertex correction, we can conservatively assign half of this 0.3% as the error associated with ignoring both the pair production and vertex virtual fermion loop corrections. We therefore assign a contribution of 0.15% to the uncertainty on the cross sections.

VI. IMPACT OF RESONANCES

All of the studies so far discussed have assumed $e^+e^- \to \tau^+\tau^-$ and $e^+e^- \to \mu^+\mu^-$ production to proceed via γ and $Z^\star$ propagators. However, at $\sqrt{s} = 10.58$ GeV it is also possible for a non-negligible fraction of the final state lepton-pair yield to arise from the production and decay of intermediate vector meson resonances, J/Ψ, $\Psi(2S)$, $\Upsilon(3S)$, $\Upsilon(4S)$, etc., when accompanied by hard ISR: e.g. $e^+e^- \to J/\Psi\gamma \to \mu^+\mu^-\gamma$. We estimate the potential size of these contributions and add a systematic error that accounts for the fact that they are not included in the KKMC program.

We calculate the cross section associated with the radiative return to each resonance below the $\Upsilon(4S)$ using the narrow width approximation and a radiator function calculated to order α^2 [27,28]. Updated resonance parameters and leptonic branching ratios of Ref. [29] are employed. In the case of the J/Ψ, we can cross-check the calculation against the *BABAR* experiment's measurement of $e^+e^- \to J/\Psi\gamma \to \mu^+\mu^-\gamma$ [30] and find they agree to 7%. Table III lists the vector meson resonances that give a small contribution to the cross section. Of the other vector meson resonances of potential interest ($\Psi(4415)$, $\Psi(4160)$, $\Psi(4039)$, $\Psi(3770)$, $\rho(2150)$, $\rho(1900)$, $\rho(1700)$, $\phi(1680)$, $\omega(1650)$, $\pi_1(1600)$, $\rho(1450)$, $\omega(1420)$, $\pi_1(1400)$, $\phi(1020)$, $\omega(782)$, and $\rho(770)$) none contribute to the $\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$ or $\sigma(\tau^+\tau^-)$ cross sections at the levels of interest because they are either below the μ-pair $\sqrt{s'}$ cut or τ-pair threshold, or because they have a negligible leptonic branching ratio. For the $\Upsilon(4S)$, we use the measured cross section of 1.101 nb at the peak of the resonance [22]. The calculation of the contribution of the resonances on $\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$ employ the KKMC Monte Carlo simulation to estimate the impact of the cuts.[2]

From this study it is evident that the vector meson resonances contribute at the 0.16% to the τ-pair cross section and the 0.28% level to $\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$. We assign these as conservative uncertainties on the KKMC cross section calculation because they are not included in the KKMC code. Note that if an s' cut is applied to a μ-pair selection that removes the $J/\Psi(1S)$ and $\Psi(2S)$ resonances, e.g. $\sqrt{s'} > 0.4\sqrt{s}$, then this component of the error on $\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$ is reduced to 0.15%, otherwise there will be an additional contribution of 0.12% from $J/\Psi(1S)$ contribution to $\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$. If no cuts are applied, the error on $\sigma(\mu^+\mu^-)$ from this source is 0.33%.

VII. UNCERTAINTY ON THE RATIO $\sigma(\tau^+\tau^-)/\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$

One of the means of determining the integrated luminosity of a data set at the *B*-factories is to count μ-pair events. If this luminosity is used to establish the number of produced τ-pair events in the sample, then the ratio $\sigma(\tau^+\tau^-)/\sigma_{\mathrm{cuts}}(\mu^+\mu^-)$ appears in the denominator of the branching fraction calculation. In this case, the error on this ratio contributes a theoretical systematic error associated with the overall normalization of the data sample. One can expect that some of the errors discussed in this paper will cancel in the ratio.

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[2]Note that the method employed to estimate the contributions to the resonances has been improved compared to that used in the first version of this paper (arXiv:0706.3235v1 [hep-ph]) and results in a slightly larger determination of the total uncertainty.

TABLE III. Estimates of intermediate vector resonance contributions to the cross sections.

Vector resonance	Γ_{total} (MeV)	BF($\mu^+\mu^-$) (%)	Contribution to $\sigma_{\text{cuts}}(\mu^+\mu^-)$ (%)	BF($\tau^+\tau^-$) (%)	Contribution to $\sigma(\tau^+\tau^-)$ (%)
$\Upsilon(4S)(10580)$	20.5	0.0016	<0.01	0.0016	<0.01
$\Upsilon(3S)(10355)$	0.020	2.18	0.07	2.29	0.07
$\Upsilon(2S)(10023)$	0.032	1.93	0.04	2.00	0.03
$\Upsilon(1S)(9460)$	0.054	2.48	0.05	2.60	0.05
$\Psi(2S)(3686)$	0.327	0.74	<0.01	0.30	<0.01
$J/\Psi(1S)(3097)$	0.093	5.93	0.12	...	...
Total			0.28		0.16

For the vacuum polarization uncertainty, the only component of uncertainty that survives in the ratio is associated with the contributions to $\sigma(\mu^+\mu^-)$ below the $\tau^+\tau^-$ threshold because above the $\tau^+\tau^-$ threshold the vacuum polarization contributes to both τ-pairs and μ-pairs in the same way, apart from small effects related to the exact description of the s' dependence above the threshold. We apply the same technique for evaluating the systematic error as described in Sec. II except we only consider the contributions up to the $\tau^+\tau^-$ threshold. This yields an uncertainty of less than 0.05%. The uncertainty of the determination of the hadronic vacuum polarization from the dispersion integral varies between 0.01% and 0.02% below the $\tau^+\tau^-$ threshold. These two uncertainties are independent and we assign the total uncertainty coming from the vacuum polarization in the ratio $\sigma(\tau^+\tau^-)/\sigma_{\text{cuts}}(\mu^+\mu^-)$ to be 0.05%.

For the treatment of initial and final state bremsstrahlung, we conservatively take as the error on the ratio the same value of 0.2% assigned for the uncertainty on the absolute cross section. Almost certainly there are cancellations that further reduce this and which can be determined with additional study. Similarly, we assume the interference error on the ratio to be 0.04%. However, the pair-production and vertex correction uncertainties are 100% correlated and largely cancel in the ratio. Assuming the μ-pair selection employed for a luminosity determination applies a cut of $\sqrt{s'} > 0.4\sqrt{s}$, the error associated with the resonances will also cancel in the ratio, otherwise there will be an additional error of 0.12%.

The total error we assign to the ratio $\sigma(\tau^+\tau^-)/\sigma_{\text{cuts}}(\mu^+\mu^-) = 1.1000$ is $(0.05 \oplus 0.20 \oplus 0.04 \oplus 0.12)\% = 0.24\%$.

VIII. SUMMARY

Adding the uncertainties from the effects discussed in this paper in quadrature we conclude that KKMC calculates the τ-pair and μ-pair production cross sections at $\sqrt{s} = 10.58$ GeV with a $(0.18 \oplus 0.20 \oplus 0.04 \oplus 0.15 \oplus 0.16)\% = 0.35\%$ and $(0.22 \oplus 0.20 \oplus 0.04 \oplus 0.15 \oplus 0.28)\% = 0.44\%$ relative calculational uncertainty, respectively. Using the IHVP flag = 4 option with the KKMC program as the central values, we obtain: $\sigma(e^+e^- \to \tau^+\tau^-) = (0.919 \pm 0.003)$ nb, $\sigma(e^+e^- \to \mu^+\mu^-) = (1.147 \pm 0.005)$ nb, $\sigma_{\text{cuts}}(e^+e^- \to \mu^+\mu^-) = (0.835 \pm 0.004)$ nb, and $\sigma(\tau^+\tau^-)/\sigma_{\text{cuts}}(\mu^+\mu^-) = 1.100 \pm 0.003$. Given the approach taken here to estimate the cross section error, it is evident that, if required, the quoted precision can be further improved in the future by undertaking a more detailed, though well-defined, analysis.

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