Comment on "Noncommutative gauge theories and Lorentz symmetry"

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We show that Lorentz symmetry is generally absent for noncommutative (Abelian) gauge theories and obtain a compact formula for the divergence of the Noether currents that allows a thorough study of this instance of symmetry violation. We use that formula to explain why the results of "Noncommutative gauge theories and Lorentz symmetry", Phys. Rev. D **70**, 125004 (2004) by R. Banerjee, B. Chakraborty, and K. Kumar, interpreted there as new criteria for Lorentz *invariance*, are in fact just a particular case of the general expression for Lorentz *violation* obtained here. Finally, it is suggested that the divergence formula should hold in a vast class of cases, such as, for instance, the standard model extension.

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We want to illustrate here why the conclusions of Ref. [1] on the possibility to preserve full Poincaré invariance for Abelian noncommutative gauge theories (NCGTs) *à la* Seiberg-Witten [2], described by a Lagrangian of the form

$$\hat{\mathcal{L}} = -\tfrac{1}{4}\hat{F}^2 = -\tfrac{1}{4}F^2 + \tfrac{1}{8}\theta \cdot FF^2 - \tfrac{1}{2}(F\theta F)\cdot F + \cdots \equiv \hat{\mathcal{L}}|_{O(\theta)} + \cdots, \tag{1}$$

are incorrect. Our notation is standard: $\hat{F}^2 = \hat{F}\cdot\hat{F} = \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$, and so on, where $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i(\hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu)$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) is the noncommutative (commutative) field strength, and $\theta_{\mu\nu} = -i(x_\mu \star x_\nu - x_\nu \star x_\mu)$ is the x-independent antisymmetric matrix encoding noncommutativity of coordinates. Namely, based on the results of Ref. [3], we shall explicitly show that what in Ref. [1] are interpreted as novel criteria for Lorentz invariance—e.g., $\partial_\mu M^{\mu\nu\lambda} = 2[(\delta\hat{\mathcal{L}}/\delta\theta_{\alpha\nu})\times \theta^\lambda_\alpha - (\nu\leftrightarrow\lambda)]$, cf. Eqs. (81) and (82) in [1]—are in fact the opposite.

In Noether's first theorem [4,5] the action $\mathcal{A} = \int d^4x\mathcal{L}(\Phi_i, \partial\Phi_i)$ is said to be *invariant* under the infinitesimal continuous transformation δ_ϵ (or, equivalently, δ_ϵ is said to be a *symmetry* of $\mathcal{A}$)—here $\{\Phi_i(x)\}$ is the set of fields of any spin-type, i is a multi-index, and although the theorem holds for the general case, for the case in point we need only to consider first derivatives of the fields—when, for all field configurations (off-shell), $\delta_\epsilon\mathcal{A} = 0$. If this happens then there is a conservation law,

$$\partial_\mu J^\mu_\epsilon = \sum_{\Phi_i}\Psi[\Phi_i]\delta_\epsilon\Phi_i, \tag{2}$$

when the field configurations respect $\Psi[\Phi_i] = 0$ (on-shell). Here J^μ_ϵ is the current for a rigid gauge transformation, $J^\mu_\epsilon = \sum_{\Phi_i}\Pi^{\mu i}\delta_\epsilon\Phi_i$, or for a spatiotemporal transformation (including supersymmetry), $J^\mu_\epsilon = \sum_{\Phi_i}\Pi^{\mu i}\times \delta_\epsilon\Phi_i - \mathcal{L}\delta_\epsilon x^\mu (+V^\mu)$, $\Pi^{\mu i} = \delta\mathcal{L}/\delta\partial_\mu\Phi_i$, and $\Psi[\Phi_i] = \partial_\mu\Pi^{\mu i} - \delta\mathcal{L}/\delta\Phi_i$ are, in Noether's terminology, the "Lagrange expressions." There are further possibilities for conservation in certain special cases, i.e., when although for some Φ_i $\Psi[\Phi_i]$ is not zero the corresponding $\delta_\epsilon\Phi_i$s on the right side of (2) can be set to zero without this producing a vanishing current J^μ_ϵ on the left side. This happens for special choices of the parameters and only for certain theories, like, e.g., the theory (1) in point. In what follows we shall call these "relic symmetries."

In any case, invariance of a classical field theory under the continuous transformation δ_ϵ always means

$$\partial_\mu J^\mu_\epsilon = 0. \tag{3}$$

Simply on the basis of this, those results of Ref. [1] that say $\partial_\mu J^\mu_{\text{Lorentz}} \neq 0$ can never be interpreted as an invariance.

Let us now consider the infinitesimal Poincaré transformations as $\delta x_\mu = -f_\mu$, with $f_\mu = a_\mu$ and $f_\mu = \omega_{\mu\nu}x^\nu$ for infinitesimal translations and homogeneous Lorentz transformation, respectively. According to their indices' structure, the fields $\{\Phi_i(x)\}$ respond to those coordinates' changes as $\delta_f\Phi_i = \Phi_i(x) - \Phi'_i(x) = \mathbf{L}_f\Phi_i(x)$ (see, e.g., [6]). Note that the changes are evaluated at the same point x. This is not strictly necessary but simplifies the analysis because $[\partial_\mu, \delta_f] = 0$. Here the Lie derivative along the vector f^μ has the usual expression

$$\mathbf{L}_f\Phi^{\lambda\ldots\kappa}_{\mu\ldots\nu} = f^\alpha\partial_\alpha\Phi^{\lambda\ldots\kappa}_{\mu\ldots\nu} + \Phi^{\lambda\ldots\kappa}_{\alpha\ldots\nu}\partial_\mu f^\alpha + \cdots + \Phi^{\lambda\ldots\kappa}_{\mu\ldots\alpha}\partial_\nu f^\alpha - \Phi^{\alpha\ldots\kappa}_{\mu\ldots\nu}\partial_\alpha f^\lambda - \cdots - \Phi^{\lambda\ldots\alpha}_{\mu\ldots\nu}\partial_\alpha f^\kappa. \tag{4}$$

The ten currents can be written in the compact form

$$J^\mu_f = \sum_{\Phi_i}\Pi^{\mu i}\delta_f\Phi_i - \mathcal{L}f^\mu, \tag{5}$$

where $J^\mu_f = T^{\mu\nu}a_\nu$ and $J^\mu_f = M^{\mu\nu\lambda}\omega_{\nu\lambda}$ for translations and Lorentz transformations, respectively, with $T^{\mu\nu}$ the canonical energy-momentum tensor and $M^{\mu\nu\lambda}$ the angular

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momentum tensor. We can call $\delta_f\Phi_i$ the "algebraic" transformations, i.e., the transformations purely based on the indices' structure of the fields, as opposed to those generated by the Noether charges $\Delta_f\Phi_i = \{\Phi_i, Q_f\}_{\rm Poisson}$, where $Q_f = \int d^3x J_f^0$, which we call "dynamical" transformations. For consistency, the latter can only coincide with the algebraic transformations or be zero [3]: $\Delta_f\Phi_i = \delta_f\Phi_i$ or $\Delta_f\Phi_i = 0$.

Suppose now that there are only two fields, $\{\Phi_i\} = (\phi_j, \chi_k)$, and that the field ϕ_j is dynamical, i.e., the relative $\Pi^{\mu j}$ is nonzero, while the field χ_k is nondynamical, i.e., $\Pi^{\mu k} = 0$ (as before, j and k are to be understood as multi-indices). The currents do not contain the algebraic variations $\delta_f\chi_k$

$$J_f^\mu = \Pi^{\mu j}\delta_f\phi_j - \mathcal{L}f^\mu, \tag{6}$$

thus they cannot depend on whether the field χ_k has been varied in the action. We want to study now the invariance and dynamical consistency properties of theories of this class. We shall do that by studying $\partial_\mu J_f^\mu$.

In general, we cannot say what $\partial_\mu J_f^\mu$ looks like. This depends on the way χ_k appears in the action. For instance, it could be fully decoupled from the dynamical field ϕ_j, in which case no sign of it would be found in J_f^μ, hence in $\partial_\mu J_f^\mu$ (see, e.g., [7]). Let us consider instead the Lagrangian for two vector fields $\phi_j = B_\mu$ and $\chi_k = P_\mu$

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu B^\alpha\partial^\mu B_\alpha - V(B) + B_\alpha P^\alpha, \tag{7}$$

where $V(B) = aB^2 + bB^4 + \cdots$ and the nondynamical field is indeed coupled to the dynamical one to form what *would be a scalar* if both fields are transformed according to their indices' structure (algebraically). We have $\Pi_B^{\mu\nu} \equiv \Pi^{\mu\nu} = \partial^\mu B^\nu$, $\Pi_P^{\mu\nu} = 0$, $\Psi[B_\nu] = \partial_\mu\Pi^{\mu\nu} + \delta V/\delta B_\nu - P^\nu$, $\Psi[P^\nu] = B_\nu$, $\delta_f B_\nu = f^\alpha\partial_\alpha B_\nu + B_\alpha\partial_\nu f^\alpha$, and $\delta_f P^\nu = f^\alpha\partial_\alpha P^\nu - P^\alpha\partial_\alpha f^\nu$. The Poincaré currents are $J_f^\mu = \Pi^{\mu\nu}\delta_f B_\nu - \mathcal{L}f^\mu$ and (using $\partial_\mu f^\mu = 0$ and $\partial^2 f^\mu = 0$)

$$\begin{aligned}\partial_\mu J_f^\mu &= (\partial_\mu\Pi^{\mu\nu})\delta_f B_\nu + \Pi^{\mu\nu}\partial_\mu\delta_f B_\nu - f^\mu\partial_\mu\mathcal{L}\\ &= (P^\nu - \delta V/\delta B_\nu)(f^\alpha\partial_\alpha B_\nu + B_\alpha\partial_\nu f^\alpha)\\ &\quad + \Pi^{\mu\nu}[(\partial_\mu f^\alpha)\partial_\alpha B_\nu + f^\alpha\partial_\mu\partial_\alpha B_\nu\\ &\quad + (\partial_\mu B_\alpha)\partial_\nu f^\alpha] - f^\mu[(P^\alpha - \delta V/\delta B_\alpha)\partial_\mu B_\alpha\\ &\quad + \Pi^{\alpha\beta}\partial_\mu\partial_\alpha B_\beta + B_\alpha\partial_\mu P^\alpha]\\ &= B_\alpha(-f^\mu\partial_\mu P^\alpha + P^\nu\partial_\nu f^\alpha)\end{aligned} \tag{8}$$

$$\begin{aligned}&+ (\partial^\mu B^\nu)(\partial^\alpha B_\nu)\partial_\mu f_\alpha + (\partial^\mu B^\nu)(\partial_\mu B^\alpha)\partial_\nu f_\alpha\\ &- (\delta V/\delta B_\nu)B_\alpha\partial_\nu f_\alpha.\end{aligned} \tag{9}$$

Each one of the three terms in (9) is separately zero: for translations this is simply due to $\partial f_\mu = 0$, while for Lorentz transformations each term is a product of a symmetric expression and the antisymmetric $\omega_{\mu\nu}$. What is left is then the expression in (8) which reads

$$\partial_\mu J_f^\mu = B_\alpha(-\mathbf{L}_f P^\alpha) = \Psi[P^\alpha](-\delta_f P^\alpha). \tag{10}$$

Let us make here several comments:

(I) In general the Poincaré symmetry is broken because we cannot implement the constraint $\Psi[P^\alpha] = 0$ unless we want the theory to become trivial, $B_\alpha = 0$. One may argue that it never seems meaningful to require $\Psi[\chi_k] = 0$, but it is not so and this is at the heart of what in [3] is called dynamical consistency. The (counter-)example one could consider is that of the dummy fields in supersymmetric theories, as we shall show in some details later. There is still room for dynamical consistency, though, for noninvariant theories as the theory (7). In this case the charges are in general not conserved because $\Psi[\chi_k] = 0$ does not make sense, but they still generate the Δs and one has to demand that $\Delta\phi_j = \delta\phi_j$ while $\Delta\chi_k = 0$.

(II) The algebraic transformations of the nondynamical field appear on the right side of (10) regardless of whether this field has been varied or not in the action to obtain the current. They are produced by the combination of $(\Psi[\phi_j] = 0) \times \delta_f\phi_j$ (term $B_\alpha P^\nu\partial_\nu f^\alpha$ here) and of $f^\mu\partial_\mu\mathcal{L}$ (term $-B_\alpha f^\mu\partial_\mu P^\alpha$ here).

(III) It is possible in this case to have relic symmetries, i.e., to set $\delta_f\chi_k$ to zero without making J_f^μ trivially vanishing. Thus there is a subset of the parameters f_μ for which there is invariance, namely, the solutions to $\mathbf{L}_f P^\alpha = 0$. For translations $\mathbf{L}_f P^\alpha = a^\mu\partial_\mu P^\alpha = 0$, i.e., the directional derivative along a^μ of P^α must vanish. For nonconstant P^α only those translations are symmetries, hence, in general not even $T^{\mu\nu}$ is always conserved. To have at least general energy and momentum conservation one chooses a constant P^α which gives as conditions for relic Lorentz symmetry $\omega^\alpha_\mu P^\mu = 0$. This gives $\vec{\zeta}\cdot\vec{P} = 0$ and $\vec{\omega}\times\vec{P} = P_0\vec{\zeta}$, where $\omega^{0i} = \zeta^i$ and $\omega^{ij} = \epsilon^{ijk}\omega_k$, with $\vec{\zeta}$ the rapidity vector and $\vec{\omega}$ identifying the axis of rotation. For $P_0 = 0$ all boosts in the plane orthogonal to $\vec{P}$ and all rotations around $\vec{P}$ are solutions, thus the subgroup of $SO(3,1)$ they identify is $SO(2,1)$.

(IV) $\delta_f P^\alpha$ enter the expression for the conservation of the current with the minus sign. As it does not make sense to set $\Psi[P^\alpha]$ to zero, the flux of the current is, in general, nonzero and proportional to the variations of the background field seen from the point of view of the transforming field, i.e., transforming with the opposite sign. If, for instance, the dynamical field rotates of an angle ϑ the relative angular momentum has a net flux given by ($\Psi[P^\alpha]$ times) a rotation of the background field of an angle $-\vartheta$.

This mechanism gives a precise meaning to what in literature is sometimes referred to as the "decoupling" between "particle" and "observer" transformations occurring in certain (Lorentz) noninvariant models, such as the standard model extension (SME) [8]: From the point of view of the Noether currents there is no ambiguity and always the invariance is broken with the exception of the relic symmetries. Hence *only one kind* of transformation is generated by the Noether charges (the particle transformations) while the other transformations manifest themselves as the terms breaking the invariance in the way described above.

Let us explain now in more details why for dummy fields in supersymmetric theories the constraint $\Phi[\chi_k] = 0$ makes sense. Although this is a general result, let us consider the simple case of the massive free Wess-Zumino theory. The Lagrangian is

$$\mathcal{L}_{WZ} = -\partial_\mu \varphi \partial^\mu \varphi^\dagger + DD^\dagger + \left[\left(-\frac{i}{2}\psi \partial \bar{\psi} + m\varphi D - \frac{m}{2}\psi^2\right) + (\text{H.c.})\right], \quad (11)$$

and the algebraic supersymmetry transformations that leave it invariant are

$$\delta\varphi = \sqrt{2}\epsilon\psi \qquad \delta\varphi^\dagger = \sqrt{2}\bar{\epsilon}\,\bar{\psi} \quad (12)$$

$$\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar{\epsilon})_\alpha \partial_\mu \varphi + \sqrt{2}\epsilon_\alpha D$$
$$\delta\bar{\psi}^{\dot{\alpha}} = i\sqrt{2}(\bar{\sigma}^\mu \epsilon)^{\dot{\alpha}} \partial_\mu \phi^\dagger + \sqrt{2}\bar{\epsilon}^{\dot{\alpha}} D^\dagger \quad (13)$$

$$\delta D = i\sqrt{2}\bar{\epsilon}\,\bar{\not{\partial}}\,\psi \qquad \delta D^\dagger = i\sqrt{2}\epsilon \not{\partial}\bar{\psi}, \quad (14)$$

where φ is the dynamical complex scalar field, ψ is its (Weyl) partner, and D is the nondynamical complex scalar field. Here the implementation of the constraint $\Psi[D] = D^\dagger + m\varphi = 0$ (and its H.c.) gives a perfectly meaningful theory, namely, the free, massive Wess-Zumino Lagrangian

$$\mathcal{L}_{WZ} = -\partial_\mu \varphi \partial^\mu \varphi^\dagger + m^2 \varphi\varphi^\dagger - \frac{i}{2}(\psi \not{\partial} \bar{\psi} - \bar{\psi}\,\bar{\not{\partial}}\,\psi) - \frac{m}{2}(\psi^2 + \bar{\psi}^2). \quad (15)$$

Furthermore, the supercurrent, $J^\mu_{\text{susy}} = \sqrt{2}(\bar{\psi}\bar{\sigma}^\mu \sigma^\nu \bar{\epsilon}\partial_\nu\varphi - im\epsilon\sigma^\mu\bar{\psi}\varphi^\dagger + \text{H.c.})$, is conserved and the relative charge $Q_{\text{susy}} = \int d^3x J^0_{\text{susy}}$ generates on-shell also the transformations of D, even though there is no associate momentum Π_D to D. This is easily seen by considering that $\Psi[D^\dagger] = 0$ means $D = -m\varphi^\dagger$, while $\Psi[\bar{\psi}^{\dot{\alpha}}] = 0$ means $i(\bar{\not{\partial}}\psi)_{\dot{\alpha}} = -m\bar{\psi}_{\dot{\alpha}}$, thus acting on-shell with Q_{susy} on D gives $\Delta_{\text{susy}} D = \{D, Q_{\text{susy}}\} = -m\sqrt{2}\bar{\epsilon}\,\bar{\psi}\{\varphi^\dagger, \Pi_{\varphi^\dagger}\} = -m\sqrt{2}\bar{\epsilon}\,\bar{\psi}$, with $\Pi_{\varphi^\dagger} = \partial_0\varphi$. Using the on-shell expression for $\bar{\psi}_{\dot{\alpha}}$ gives $\Delta_{\text{susy}} D = i\sqrt{2}\bar{\epsilon}\,\bar{\not{\partial}}\,\psi$, which coincides with the algebraic transformation. This is an illuminating instance of dynamical consistency: supposing $\Psi[\phi_j] = 0$ is always implemented, $\Psi[\chi_k] = 0$ on one side gives conservation, and on the other side gives an expression for the nondynamical field in terms of dynamical ones $\chi_k(\phi_j)$ that, when acted upon with the charge, gives back precisely the algebraic transformation, $\Delta = \delta$.

With the help of the previous considerations, to treat the case in point of the NCGT (1) is now fairly easy. The current has the form $J^\mu_f = \Pi^{\mu\nu}\delta_f A_\nu - \hat{\mathcal{L}} f^\mu$, and the divergence can be written as follows:

$$\partial_\mu J^\mu_f = \Pi^{\mu\nu} F_{\alpha\nu}\partial_\mu f^\alpha = \left(\frac{\delta\hat{\mathcal{L}}}{\delta F_{\mu\nu}} F_{\alpha\nu}\right) 2\partial_\mu f^\alpha \quad (16)$$

$$= \left(\theta^{\mu\beta}\frac{\delta\hat{\mathcal{L}}}{\delta\theta^{\alpha\beta}}\right) 2\partial_\mu f^\alpha = \frac{\delta\hat{\mathcal{L}}}{\delta\theta^{\alpha\beta}}(\theta^{\mu\beta} 2\partial_\mu f^\alpha) \quad (17)$$

$$= \Psi[\theta^{\alpha\beta}](-\mathbf{L}_f\theta^{\alpha\beta}) = \Psi[\theta^{\alpha\beta}](-\delta_f\theta^{\alpha\beta}). \quad (18)$$

Let us prove it. It was shown in [9] that, after partial integration, no derivatives of $F_{\mu\nu}$ appear in the expansion hence one can write symbolically $\hat{\mathcal{L}} \sim \sum_n \theta^n F^{n+2}$, i.e., the Lagrangian is a homogeneous polynomial in θ and F. Furthermore, only two things can happen: either one given θ is coupled to one F (i) or to two Fs (ii). Notice now that deriving $\hat{\mathcal{L}}$ with respect to F and then multiplying by F produces precisely the same result as multiplying by θ and then deriving with respect to θ (in reverse order) because: in case (i) the $\theta^{\mu\nu}$ singled out from the derivation $\delta/\delta F_{\mu\nu}$ contracts (with the μ of $\partial_\mu f^\alpha$ and) with the ν of the outcome of the derivation with $\delta/\delta\theta^{\alpha\nu}$, i.e., $F_{\alpha\nu}$ times the same terms multiplying $\theta^{\mu\nu}$; in case (ii) when the free index of θ left out of the derivation is ν then the contribution is zero for a mechanism of cancellation we shall soon describe, while when the free index is μ (say $\theta^{\mu\beta}$) then the ν that contracts with the $F_{\alpha\nu}$ must be on one F, and together with the $F_{\alpha\nu}$ gives what would be obtained by deriving with $\delta/\delta\theta^{\alpha\beta}$. There is still need to address the apparent mismatch between the number of terms produced by deriving $\sum_n \theta^n F^{n+2}$ with respect to F and the number of terms obtained by deriving it with respect to θ. They are in fact the same. For translations everything vanishes. For Lorentz transformations $\partial_\mu f^\alpha = \omega^\alpha_\mu$. Let us consider this case. The extra two terms one obtains by deriving with respect to F vanish because: either one gets $\sim 2(\theta^n F^n) \times (F^{\mu\nu}F_{\alpha\nu})\omega^\alpha_\mu$ (case (i) above) or one gets $\sim[(\theta^n F^{n+2})^\mu{}_\alpha + (\theta^n F^{n+2})_\alpha{}^\mu]\omega^\alpha_\mu$ (case (ii) above), i.e., always a symmetric expression times ω^α_μ. The latter cancellation is also responsible for the matching $(\delta/\delta F)F \sim \theta\delta/\delta\theta$ in case (ii) above. Finally, notice that for constant $\theta^{\alpha\beta}$: $A_{\alpha\beta}\mathbf{L}_f\theta^{\alpha\beta} = A_{\alpha\beta}\theta^{\mu\alpha}2\partial_\mu f^\beta$ for any antisymmetric $A_{\alpha\beta}$. Collecting all this information gives the result (18).

The discussion on invariance and dynamical consistency goes along the lines of the previous discussion. The theory is in general *not invariant* under Poincaré transformations because it does not make sense to set $\Psi[\theta^{\alpha\beta}]$ to zero: on the one hand this constraint would make the theory trivial (e.g., at first order we would get $F_{\mu\nu} = 0$), on the other hand it does not allow to express $\theta(F)$ in a meaningful way. That is why the only way left for dynamical consistency is $\Delta_f \theta_{\mu\nu} = 0$ and $\Delta_f A_\mu = \delta_f A_\mu$, as proved already in [3]. There is room for relic symmetries found by solving $\mathbf{L}_f \theta^{\mu\nu} = 0$, which is satisfied for all translations. For Lorentz transformations we have to solve $\theta^{\mu\beta}\omega_\mu^\alpha = 0$. For $\beta = 0$ we get $\vec{\tilde{\theta}} \times \vec{\omega} = 0$, while for $\beta = j$ and $\alpha = 0$ we get $\vec{\theta} \times \vec{\zeta} = 0$, where $\vec{\omega}$ and $\vec{\zeta}$ have been already defined, $\vec{\tilde{\theta}} = (\theta^{01}, \theta^{02}, \theta^{03})$ and $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$, with $\theta^{ij} = \epsilon^{ijk}\theta_k$. Furthermore, by taking the equation for $\beta = j$ and $\alpha = k$, $\tilde{\theta}^j \zeta_k = -\delta_k^j \vec{\theta} \cdot \vec{\omega} + \theta_k \omega^j$, and contracting it with ω_k we get $\vec{\zeta} \cdot \vec{\omega} = 0$, while contracting it with θ_j we get $\vec{\tilde{\theta}} \cdot \vec{\theta} = 0$. Thus, rotations around $\vec{\tilde{\theta}}$ and boosts along $\vec{\theta}$ are still symmetries, provided $\vec{\tilde{\theta}} \cdot \vec{\theta} = 0$. The group is $SO(2) \times SO(1,1)$, modulo some discrete symmetries, and it has been considered in great detail in [10].

Note that the expression (17) coincides, at first order in θ, with what is obtained in [1] (use $\hat{\mathcal{L}}|_{O(\theta)}$ in (1), compute the derivatives and rearrange the terms as discussed) and interpreted there as "the criterion for Lorentz invariance in the case $\theta_{\mu\nu}$ transforms like a tensor," which is evidently an incorrect interpretation (cf. Eqs. (81) and (82) and also Eq. (87) and the following discussion).

We conclude that for (Abelian) NCGTs of the kind in (1) only relic symmetries are present and they are translations in any direction and the subgroup of the Lorentz group compatible with $\theta_{\mu\nu}$, i.e., whose parameters satisfy $\mathbf{L}_f \theta_{\mu\nu} = 0$ ($SO(2) \times SO(1,1)$ for the case $\vec{\tilde{\theta}} \cdot \vec{\theta} = 0$). There is no choice on whether to transform or not transform $\theta_{\mu\nu}$: for dynamical consistency $\Delta_f \theta_{\mu\nu} = 0$ for all f^μ, while the $\delta_f \theta_{\mu\nu}$ are precisely the terms breaking the invariance in general (as $\Psi[\theta_{\mu\nu}] = 0$ cannot be implemented) and they are produced, with the minus sign, in $\partial_\mu J^\mu$ regardless of whether $\theta_{\mu\nu}$ has been varied in the action due to the particular coupling of these fields with the dynamical field $F_{\mu\nu}$. This means that the system undergoing a Lorentz transformation with parameter ω_μ^α, not belonging to the relic symmetries, sees the $\theta_{\mu\nu}$ transforming with the parameter $-\omega_\mu^\alpha$ as the cause of nonconservation of the $M^{\mu\nu\lambda}$. This also solves the ambiguity of the particle and observer transformations in this context, as the Noether (in general not conserved) charges only can generate one kind of transformation (particle) while the other kind (observer) are seen to appear in the way illustrated above. Thus, there is only one criterion for Lorentz invariance of NCGTs and it is the usual one, $\partial_\mu M^{\mu\nu\lambda} = 0$.

We hope that this Comment will ultimately clarify that for NCGTs of the kind discussed here *standard* Lorentz symmetry is not present and that statements like "NGCT has Lorentz invariance only when $\theta_{\mu\nu}$ transforms like a tensor" are simply wrong. We did not consider here noncommutative *modifications* of the Lorentz algebra—as, e.g., that proposed in [11] (the twisted coproduct approach) or in [12] (the *θ-deformed* transformations approach)—where the transformations themselves are modified hence a different meaning must be ascribed to Lorentz invariance. In [1] and in [3] the same approach is used of standard Lorentz transformations hence there is no room for opposite conclusions on Lorentz invariance.

One further development of this analysis is to prove under which conditions the following conjecture is true: When in the action the nondynamical $\chi_{k_1}^1, \ldots, \chi_{k_n}^n$ are coupled to the dynamical $\phi_{j_1}^1, \ldots, \phi_{j_m}^m$ (and/or their derivatives) to obtain what *would be a scalar* if both sets of fields are transformed *algebraically* then the result

$$\partial_\mu J_f^\mu = \sum_{i=1}^{n} \Psi[\chi_{k_i}^i](-\delta_f \chi_{k_i}^i), \quad (19)$$

holds. Here $\sum_{i=1}^m \Psi[\phi_{j_i}^i] = 0$ has been used, and let us stress again that the expression of the current on the left side is independent of whether the nondynamical fields in the action have been varied or not. We expect that this is the case of the SME [8]. Finally, it would be interesting to study within this approach the supercurrents of the Lorentz-violating Wess-Zumino model proposed in the SME context in Ref. [13].

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