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Compact oscillons in the signum-Gordon model

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We present explicit solutions of the signum-Gordon scalar field equation which have finite energy and are periodic in time. Such oscillons have strictly finite size. They do not emit radiation.

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I. INTRODUCTION

Dynamics of self-interacting scalar fields is still a subject of very interesting and important research. Because such fields are ubiquitous in physics, from condensed matter to cosmology, the results can have a wide impact on our understanding of the material world. Nonlinearity of pertinent field equations is the reason for the existence of very nontrivial and often unexpected phenomena. In particular, much attention has been paid to so-called oscillons, extremely long-lived scalar field configurations which in many respects behave like solitons, but which finally decay into radiation, see e.g. recent papers [1]. The nonlinearity can have various forms. In the current literature dominate models with polynomial Lagrangians which are smooth in a vicinity of the pertinent ground state field. However, in a recent paper [2] we have pointed out a rather interesting class of scalar field models with non polynomial field potentials (i.e., interaction terms) which are V-shaped at their minima and which have another kind of nonlinearity. The simplest example of such a model is the signum-Gordon model.

The signum-Gordon model (s-G) has the following Lagrangian [3]

$$L = \frac{1}{2}\frac{\partial \varphi}{\partial x^{\mu}}\frac{\partial \varphi}{\partial x_{\mu}} - g|\varphi|, \qquad (1)$$

where φ is a real (1 + 1)-dimensional scalar field, $\mu = 0$, 1 and x^0, x^1 are dimensionless variables obtained by the appropriate choice of units for the physical time and position coordinates. $g > 0$ is a dimensionless coupling constant. The pertinent field potential has the form $U(\varphi) = g|\varphi|$. Its left and right derivatives at $\varphi = 0$, where it has the absolute minimum, do not vanish—they are equal to $\mp g$. Hence, in the case of the s-G model the second derivative $U''(0)$ is (in a sense) infinite. In the majority of field theoretic models, the field potential $U(\varphi)$ is smooth at its absolute minimum at $\varphi = \varphi_0$, and $\lambda_0 = (U''(\varphi_0))^{-1/2}$ is recognized as the fundamental length scale in the model. In the case of massless models λ_0 is infinite, while finite $\lambda_0 > 0$ characterizes massive models with the finite mass parameter $m_0^2 = 1/\lambda_0^2$. In this classification, the s-G model corresponds to $\lambda_0 = 0$ and infinite m_0^2—for this reason it can be called a supermassive model.

Several rather interesting features of the models with V-shaped field potentials have been pointed out in our previous papers: the existence of static, compact solitons (topological compactons) [4], the presence of a scaling symmetry [2], and the variety of exact self-similar solutions [3,5]. In these papers we have also presented a sound physical motivation for considering such models. It includes the description of an infinite chain of harmonically coupled classical pendulums, and the dynamics of an elastic string (a vortex) pinned with a constant force to a line. Because of such a down-to-earth physics behind the s-G model, it is not surprising that the model is perfectly well behaved from the physical viewpoint. In particular, the conserved energy is bounded from below, and the field equation is of the standard hyperbolic type with the usual causality features. In the papers [3,4] examples of spontaneous symmetry breaking with the corresponding topological defects have been discussed. V-shaped field potentials should be considered as a viable alternative to smooth potentials in Ginzburg-Landau type models. On the other hand, the dynamics of the scalar field in this model is of course influenced by the fact that the field potential is not smooth at the minimum. Moreover, the model cannot be linearized even if the amplitude of the scalar field is arbitrarily small, contrary to the models with smooth field potentials. Perhaps such a mathematical obstacle is one of the of the reasons for which the models with V-shaped field potentials were not discussed in the literature[1]

In the present paper we show that the s-G model possesses solutions with finite energy which are periodic in time—the oscillons. These oscillons are interesting for the following reasons. First, they are given explicitly as exact solutions of the s-G field equation. Moreover, they do not lose their energy because they do not emit any radiation—the field oscillations are strictly periodic in time. They have a compact support in the space, i.e., a strictly finite size. In these respects they differ from the oscillons of the φ^4 model which are known only as approximate numerical solutions and seem to contain a radiative component.

[1]Let us stress that it is not a fundamental difficulty. It should rather be regarded only as a certain inconvenience—the signum-Gordon model is perfectly sound from both physical and mathematical viewpoints. Nevertheless, it has the implication that one cannot base a perturbative expansion on the free field.

II. THE SIGNUM-GORDON EQUATION

The signum-Gordon equation is the Euler-Lagrange equation corresponding to Lagrangian (1). It can be written in the following form

$$\partial_t^2 \varphi(x,t) = \partial_x^2 \varphi(x,t) - \mathrm{sign}(\varphi(x,t)), \tag{2}$$

where $t = \sqrt{g} x^0$, $x = \sqrt{g} x^1$. The sign function has the values ± 1 when $\varphi \neq 0$ and 0 for $\varphi = 0$. Because of the $-\mathrm{sign}(\varphi)$ term Eq. (1) is nonlinear in a rather special way. Mathematical aspects of this equation are discussed in [3]. Generally speaking, physically relevant are solutions which are continuous functions of x and t, but only piecewise smooth. They belong to the class of so-called weak solutions of partial differential equations. The reader not familiar with the mathematics of such solutions should consult the literature, e.g., [6,7].

The novel feature of Eq. (2) is that the nonlinear term $-\mathrm{sign}(\varphi)$ remains finite for arbitrarily small $|\varphi|$ provided that $\varphi \neq 0$. This fact has profound influence on the dynamics of weak fields. In other models nonlinear terms vanish when fields become arbitrarily small, cf. the $-\varphi^3$ nonlinear term in the case of the nonlinear Klein-Gordon equation. Consider, for example, the Gaussian wave packet $\varphi(x, t=0) = \exp(-x^2)$ at the initial time $t = 0$. Let us assume for simplicity that $\partial_t \varphi(x, t=0) = 0$. The field acceleration $\partial_t^2 \varphi(x, t=0)$ is obtained from the field equations. In the case of the $-\varphi^3$ nonlinearity the dominating at large $|x|$ contribution to the acceleration comes from the $\partial_x^2 \varphi(x, t=0)$ term and it is positive, hence the Gaussian wave packet will spread out. In the case of the s-G equation the $-\mathrm{sign}(\varphi)$ term dominates at large $|x|$ and it is negative, hence the wave packet will shrink for certain time. Heuristically, both "forces" $-\varphi^3$ and $-\mathrm{sign}(\varphi)$ push the field toward the equilibrium value $\varphi = 0$, but the $-\mathrm{sign}(\varphi)$ force is much more effective in this respect.

The signum term in Eq. (2) remains constant until φ becomes equal to zero. Therefore, on each interval on the x axis such that φ has a constant sign on it, one can use the well-known formula for the general solution of the one-dimensional wave equation, suitably modified to incorporate the constant $+1$ or -1 term in the equation:

$$\varphi(x,t) = h(x-t) + g(x+t) + c_1 t^2, \tag{3}$$

where $2c_1 = \pm 1 (= -\mathrm{sign}(\varphi))$. The functions h, g and the constant c_1 are determined from the initial and boundary conditions.

III. THE OSCILLON

Motivated by the heuristic considerations about the wave packets we shall try to construct a solution of the s-G equation (2) which does not spread out at least during a certain time interval. We start by specifying simple initial data at $t = 0$:

$$\varphi(x,0) = 0, \qquad \partial_t \varphi(x,0) = \begin{cases} 0 & \text{if } x \le 0 \quad \text{or} \quad x \ge 1, \\ v(x) & \text{if } 0 < x < 1, \end{cases} \tag{4}$$

where the initial field velocity $v(x)$ is assumed to be negative. Its exact form will be determined later. Moreover, we impose the following boundary conditions

$$\varphi(x=0,t) = 0, \qquad \varphi(x=1,t) = 0. \tag{5}$$

The hope is that the field will remain localized in the interval $0 \le x \le 1$ for arbitrary times $t > 0$.

Because $v(x)$ is negative, for small enough $t > 0$ the field φ will be negative or equal to zero. Therefore, in the region $0 < x < 1$ we use formula (3) in which $c_1 = 1/2$. The functions h and g are determined from conditions (4) and (5). For a certain reason which will become clear later we denote the solution given below by φ^-. Simple calculations give the following results

$$\varphi^-(x,t) = \begin{cases} \varphi_1(x,t) & \text{if } 0 \le x \le t, \\ \varphi_2(x,t) & \text{if } t \le x \le 1-t, \\ \varphi_3(x,t) & \text{if } 1-t \le x \le 1, \end{cases} \tag{6}$$

where

$$\varphi_1(x,t) = -\frac{x^2}{2} + tx + \frac{1}{2}\int_{t-x}^{t+x} ds\, v(s), \tag{7}$$

$$\varphi_2(x,t) = \frac{t^2}{2} + \frac{1}{2}\int_{x-t}^{t+x} ds\, v(s), \tag{8}$$

$$\varphi_3(x,t) = -\frac{x^2}{2} - \frac{1}{2} - xt + x + t + \frac{1}{2}\int_{x-t}^{2-t-x} ds\, v(s). \tag{9}$$

The reason for splitting the unit interval $0 \le x \le 1$ into the three parts is that the boundary conditions (5) modify the field at the points which are causally connected with the points $x = 0$ and $x = 1$: such points form the two subintervals $0 \le x \le t$, $1 - t \le x \le 1$. These subintervals meet at the time $t = 1/2$. Hence, the solution given above is valid for t in the interval $[0, 1/2]$, and for x in the interval [0,1].

Note that at the initial time $t = 0$ only φ_2 is present because the spatial supports of φ_1, φ_3 are shrunk to the points $x = 0$, $x = 1$, respectively. The functions φ_1, φ_2 match each other at the point $x = t$, similarly as φ_2, φ_3 at $x = 1 - t$, so that φ^- as well as $\partial_t \varphi^-$, $\partial_x \varphi^-$ are continuous functions of x, t.

In the next step we extend the solution φ^- to the whole x axis. Of course we just would like to put $\varphi(x,t) = 0$ for $x < 0$ and for $x > 1$. This is possible provided that

$$\partial_x \varphi_1(x=0,t) = 0, \qquad \partial_x \varphi_3(x=1,t) = 0, \tag{10}$$

because otherwise $\partial_x \varphi$ would have discontinuities at the points $x = 0$, $x = 1$ Such discontinuities, if present, would have to move with "the velocity of light" ± 1 because the s-G equation belongs to the class of hyperbolic partial differential equations. Conditions (10) give the following equations

$$v(t) = -t, \qquad v(1-t) = -t,$$

where $0 \le t \le 1/2$. It follows that the initial field velocity $v(x)$ has the following simple form

$$v(x) = \begin{cases} -x & \text{if } 0 \le x \le \frac{1}{2}, \\ x-1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \qquad (11)$$

Now we can compute the integrals present in formulas (7)–(9). It is convenient to split the time interval $[0, 1/2]$ into two parts. It turns out that when t has values from the interval $[0, 1/4]$ then

$$\varphi_1(x,t) = -\frac{x^2}{2}, \qquad (12)$$

$$\varphi_2(x,t) = \begin{cases} \frac{t^2}{2} - xt & \text{if } t \le x \le \frac{1}{2} - t, \\ \frac{x^2}{2} + t^2 - \frac{x}{2} - \frac{t}{2} + \frac{1}{8} & \text{if } \frac{1}{2} - t \le x \le \frac{1}{2} + t, \\ \frac{t^2}{2} + t(x-1) & \text{if } \frac{1}{2} + t \le x \le 1 - t, \end{cases} \qquad (13)$$

$$\varphi_3(x,t) = -\frac{(1-x)^2}{2}, \qquad (14)$$

and for t from the interval $[1/4, 1/2]$

$$\varphi_1(x,t) = \begin{cases} -\frac{x^2}{2} & \text{if } 0 \le x \le \frac{1}{2} - t, \\ \frac{t^2}{2} + tx - \frac{x}{2} - \frac{t}{2} + \frac{1}{8} & \text{if } \frac{1}{2} - t \le x \le t, \end{cases} \qquad (15)$$

$$\varphi_2(x,t) = t^2 + \frac{x^2}{2} - \frac{x}{2} - \frac{t}{2} + \frac{1}{8}, \qquad (16)$$

$$\varphi_3(x,t) = \begin{cases} \frac{t^2}{2} - tx + \frac{x}{2} + \frac{t}{2} - \frac{3}{8} & \text{if } 1 - t \le x \le \frac{1}{2} + t, \\ -\frac{(1-x)^2}{2} & \text{if } \frac{1}{2} + t \le x \le 1. \end{cases} \qquad (17)$$

Note that $\partial_t\varphi(x,t) = 0$ at the time $t = 1/4$.

To summarize, we have found a solution of the s-G equation such that $\varphi = 0$ outside the interval [0,1] on the x-axis, and inside this interval it has the form given by formulas (6) and (12)–(17), provided that $0 \le t \le 1/2$. Snapshots of the field configurations at the times $t = 1/8$ and $t = 1/4$ are presented in Figs. 1 and 2. For times t from the interval $(1/4, 1/2)$ the solution has the same shape as presented in Fig. 1, but now the two rectilinear segments expand and move until they cover the intervals $[0, 1/2]$, $[1/2, 1]$ on the x-axis.

In order to find φ for times larger than $1/2$ we compute φ^- and $\partial_t\varphi^-$ at $t = 1/2$ and we take these functions of x as the new initial data for the s-G equation. The calculation is trivial, but one should remember about watching the domains of the involved functions because they are given by inequalities which also depend on time. The result is quite surprising and crucial for further progress:

$$\varphi^-(x, t = 1/2) = 0, \qquad \partial_t\varphi^-(x, t = 1/2) = -v(x), \quad (18)$$

where $v(x)$ is given by formula (11). These formulas imply that $\varphi(x,t)$ for $t > 1/2$ can be obtained with the help of symmetries of the s-G equation: the time translation $t \to t - 1/2$ and the change of sign of the field $\varphi \to -\varphi$. The solution obtained in this way is denoted by φ^+. Thus,

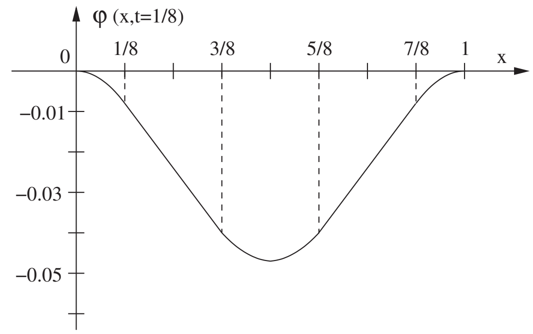

FIG. 1. Picture of the oscillon solution at the time $t = 1/8$. For x from the intervals $[1/8, 3/8]$ and $[5/8, 7/8]$ the function $\varphi(x, 1/8)$ is represented by rectilinear segments which at the time $t = 1/4$ shrink to the points $1/4$ and $3/4$, respectively, see the first and third lines of formula (13).

$$\varphi^+(x,t) = -\varphi^-(x, t - \tfrac{1}{2}). \qquad (19)$$

The solution $\varphi^+(x,t)$ holds for t in the interval $[1/2, 1]$. At the time $t = 1$ the field φ and its time derivative $\partial_t\varphi$ return to their initial values (4). Thus, the period of the oscillon is equal to its spatial size, i.e., to 1.

Note that the oscillon is spatially symmetric: the solution is invariant under the spatial reflection at the point $x = 1/2$, that is with respect to the substitution $x \to 1 - x$. The point $x = 1/2$ is the center of the oscillon.

The s-G equation is dilation invariant [3]: if $\varphi(x,t)$ is a solution of it, then so is also $\varphi_l(x,t) = l^2\varphi\left(\frac{x}{l}, \frac{t}{l}\right)$, where l is an arbitrary real, positive constant. Applying this symmetry to the oscillon solution presented above we generate a one-parameter family of oscillons of the length $l/\sqrt{g}$ with the period equal to the length (let us recall that we use the dimensionless variables x^μ). The amplitude of the oscillations of the scalar field is equal to $l^2/16$. The total energy of the oscillons is given by the formula obtained from Lagrangian (1)

$$E = \frac{1}{2}\sqrt{g}\int_{-\infty}^{\infty} dx[(\partial_t\varphi)^2 + (\partial_x\varphi)^2] + \sqrt{g}\int_{-\infty}^{\infty} dx|\varphi| = \frac{\sqrt{g}}{24} l^3.$$

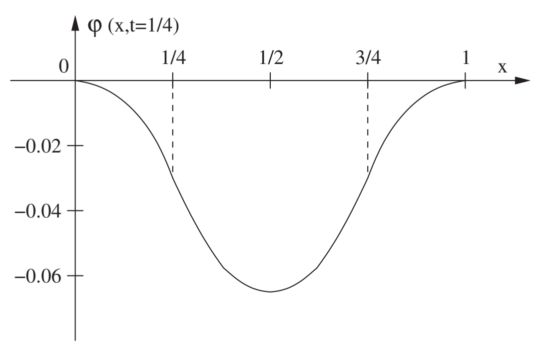

FIG. 2. Picture of the oscillon solution at the time $t = 1/4$. This configuration is "the turning point" because precisely at this time $\partial_t\varphi$ changes its sign from negative to positive.

The frequency of the oscillations of the field is $\sim l^{-1}$, hence the energy E vanishes in the limit of high frequencies.

The s-G equation is also invariant with respect to Poincaré transformations. In particular, boosts provide oscillons moving with an arbitrary constant velocity v such that $|v| < 1$.

IV. REMARKS

(1) Note that the s-G oscillons have a strictly finite size, and that the field approaches its vacuum value $\varphi = 0$ in the parabolic manner. They share these features with static topological compactons discussed in [4]. The compactness and the parabolic approach to the vacuum field seem to be generic features of the models with V-shaped field potentials.

(2) We have attempted to investigate the stability of the s-G oscillon under small perturbations. Our numerical simulations did not show any instability. On the analytic side, the problem turns out to be quite difficult. The main reason is that there is no linear approximation to the function sign(φ) around $\varphi = 0$. This fact seems to render the standard linear stability theory useless. Note however that sign($\varphi + \varepsilon$) = sign(φ) for $\varphi \neq 0$ and a small enough perturbation ε. Therefore, in such cases the s-G equation implies the simple massless wave equation for the perturbation $\partial_\mu \partial^\mu \varepsilon = 0$. It follows that the perturbation does not grow: it splits into left- and right-movers which just travel with the velocities ± 1 until they reach a point at which the function sign($\varphi + \varepsilon$) changes its sign. It remains to be investigated what happens at such points. Generally, one could expect complicated dynamical processes including an emission of radiation or of small oscillons.

(3) The s-G equation admits explicit solutions which describe arbitrary chains of static, nonoverlapping oscillons of arbitrary sizes. The multioscillon solutions are obtained trivially by adding appropriately translated in space single oscillon solutions. Such oscillons do not interact with each other because they have strictly finite sizes.
We expect that the s-G oscillons will likely have to be taken into account when considering various dynamical processes with the s-G field. The oscillons with arbitrarily small energies can probably be emitted in such processes, thus forming a kind of "infrared" cloud consisting of a number of small, compact oscillons.

(4) There is a remote analogy between our oscillons and the breather known from the sine-Gordon model. The breather is a soliton-antisoliton bound state, where the soliton as well as the antisoliton alone are represented by static, finite energy solutions of the sine-Gordon field equation. In the case of the signum-Gordon model we do not expect any static, finite energy solutions. However, there exist static, infinite energy solutions of the form $\pm(x - x_0)^2/2$, where x_0 is a free parameter present because of translational invariance of the model. The left- and right-hand ends of the oscillon are given by pieces of these static solutions, see, e.g., formulas (12) and (14). Therefore, one may regard the oscillon as a finite energy bound state of the two infinite energy configurations represented by the static solutions.

(5) The present work can be continued in several directions. For example, it is not clear at all what would happen if our oscillons collide with each other. Another interesting question is about existence of oscillons in the signum-Gordon model in two and three spatial directions—work in this direction is in progress.

ACKNOWLEDGMENTS

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