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Supersymmetric giant graviton solutions in AdS_3

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We parametrize all classical probe brane configurations that preserve four supersymmetries in (a) the extremal D1-D5 geometry, (b) the extremal D1-D5-P geometry, (c) the smooth D1-D5 solutions proposed by Lunin and Mathur, and (d) global $AdS_3 \times S_3 \times T^4/K3$. These configurations consist of D1 branes, D5 branes, and bound states of D5 and D1 branes with the property that a particular Killing vector is tangent to the brane world volume at each point. We show that the supersymmetric sector of the D5-brane world volume theory may be analyzed in an effective $1 + 1$ dimensional framework that places it on the same footing as D1 branes. In global AdS and the corresponding Lunin-Mathur solution, the solutions we describe are "bound" to the center of AdS for generic parameters and cannot escape to infinity. We show that these probes only exist on the submanifold of moduli space where the background B_{NS} field and theta angle vanish. We quantize these probes in the near-horizon region of the extremal D1-D5 geometry and obtain the theory of long strings discussed by Seiberg and Witten.

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I. INTRODUCTION

Despite many advances, quantizing string theory in nontrivial spacetime backgrounds remains a difficult task. In the past few years, some progress has been made by approaching this problem using canonical methods [1–8]. The principle behind these studies is that if one can understand a subsector of the *classical* theory well enough it may be possible to quantize it autonomously and obtain a sector of the Hilbert space of the full quantum theory. This procedure can only work if the canonical structure of the classical phase space "decouples" this sector from the rest of the theory. The studies above suggest that supersymmetric sectors, such as the one we will study here, often satisfy this criterion.

Since the space of all classical solutions of a theory is isomorphic to its classical phase space, it is of interest if one can obtain a complete parametrization of even a special subsector of classical solutions. This subsector can then be quantized using the methods of [9] (see [10] for a review). In this paper, we pursue this programme by parametrizing all classical supersymmetric brane probes moving in (a) the extremal D1-D5 background, (b) the extremal D1-D5-P background, (c) the smooth geometries proposed in [11–13] with the same charges as the D1-D5 system, and (d) global $AdS_3 \times S^3 \times T^4/K3$.

The physical significance of these backgrounds is as follows. The AdS/CFT conjecture [14,15] relates type IIB string theory on global AdS_3 to the Neveu-Schwarz (NS) sector of a $1 + 1$ dimensional CFT on its boundary. The solutions in global AdS we find below correspond to the $1/4$ Bogomol'nyi-Prasad-Sommerfield (BPS) sector of the CFT of the Higgs branch. On the boundary, the NS and R sectors are related by an operation called "spectral flow." Performing this operation on the supergravity solution for global AdS yields the near-horizon region of one of the solutions of Lunin and Mathur [12]. This corresponds to the specific Ramond ground state obtained by spectrally flowing the NS vacuum. Other Ramond vacua are described by other solutions in [12]. The zero mass Banados-Teitelboim-Zanelli (BTZ) black hole which is the near horizon of the extremal D1-D5 geometry, on the other hand, has been argued to be an "average" over all Ramond ground states.

The giant graviton brane probes we find comprise D1 branes, D5 branes, and bound states of D1 and D5 branes. As we make more precise in Sec. II B, we find that these supersymmetric probes have the property that a certain Killing vector is tangent to the brane world volume at each point. Hence, given the shape of the brane at any one point of time, one can translate it in time along the integral curves of this Killing vector to obtain the entire brane world volume. The set of all solutions is parametrized by the set of all initial shapes. This simple prescription is sufficient to describe supersymmetric probes in all the backgrounds we mentioned above.

Surprisingly, we find that the symplectic structure on these classical solutions is such that we can describe all the solutions above, including supersymmetric solutions to the Dirac-Born-Infeld (DBI) action on the 6 dimensional D5-brane world volume, in a unified $1 + 1$ dimensional framework. It is well known that the infrared limit of the world-volume theory of a bound state of D1 branes and D5 branes, in flat space, is given by a $1 + 1$ dimensional sigma model. However, our result which we emphasize is classical, is valid in curved backgrounds, and does not rely on taking the infrared limit.

Our probes exist on the submanifold of moduli space where the background NS-NS fluxes and theta angle are set to zero. On this submanifold, the boundary theory is known to be singular because the stack of D1 and D5 branes that make up the background can separate at no cost in energy [16]. One may wonder then whether the probes we find are artifacts of this singularity, i.e., whether they merely represent breakaway D1-D5 subsystems which can escape to infinity. In global AdS, and in the Ramond sector solution dual to global AdS, this is not the case. In these geometries, for generic parameters, the $1/4$ BPS giant gravitons that we describe, are "bound" to the center of AdS and cannot escape to infinity. This indicates that they correspond to discrete states and not to states in a continuum. In the boundary theory this means that they correspond to BPS states that are *not* localized about the singularities of the Higgs branch. Averaging over the Ramond vacua to produce the zero mass BTZ black hole, however, washes out the structure of these discrete bound states and the only solutions we are left with are at the bottom of a continuum of nonsupersymmetric states.

We prove that no BPS probes survive if we turn on a small NS-NS field. This is not a contradiction, for it merely means that the $1/4$ BPS partition function jumps as we move off this submanifold of moduli space. Further investigation of this issue in the quantum theory and of protected quantities, like the elliptic genus and the spectrum of chiral-chiral primaries, is left to [17].

Giant gravitons in AdS_3 have been considered previously [12,18–20], and it was noted that regular $1/2$ BPS brane configurations exist only for specific values of the charges. These are precisely the values at which the giant gravitons we describe can escape to "infinity" in global AdS. The moduli space of $1/4$ BPS giant gravitons, however, is far richer, and this is what we will concern ourselves with in this paper.

A brief outline of this paper is as follows. In Sec. II we perform a Killing spinor and kappa-symmetry analysis to determine the conditions that D brane probes, in the four backgrounds above, must obey in order to be supersymmetric. Using this insight, in Sec. III we explicitly construct supersymmetric D1-brane solutions in these backgrounds and verify that they satisfy the BPS bound. Then, in Sec. IV we show how bound states of D1 and D5 branes (represented by D5 branes with gauge fields turned on in their world volume) can also be described in the framework of Sec. III. In Sec. V we discuss the effect of turning on background NS-NS fluxes. In Sec. VI we discuss the quantization of probes moving in the near-horizon region of the D1-D5 background. In Sec. VII we conclude with a summary of our results and their implications. Appendixes A, B, and C discuss some technical details, while in Appendixes D and E we discuss Killing spinor equations for various D1-D5 geometries and global AdS.

II. KILLING SPINOR AND KAPPA-SYMMETRY ANALYSIS

We consider type IIB superstring theory compactified on $S^1 \times \mathcal{K}$ where $\mathcal{K}$ is T^4 or $K3$. We will concentrate on the case of T^4, unless otherwise stated. Let us parametrize S^1 by the coordinate x_5, T^4 by x^6, x^7, x^8, x^9, and the noncompact spatial directions by x^1, x^2, x^3, x^4. We will use coordinate indices x^M, $M = 0, 1, \ldots, 9$; x^m, $m = 1, 2, 3, 4$; x^a x^i or a, $i = 6, 7, 8, 9$. We will parametrize the 32 supersymmetries of IIB theory by two real constant chiral spinors ϵ_1 and ϵ_2, or equivalently by a single complex chiral spinor $\epsilon = \epsilon_1 + i\epsilon_2$.

In Sec. II A we will review the preserved supersymmetries, or the Killing spinors, of the backgrounds (a) D1-D5, (b) D1-D5-P, (c) Lunin-Mathur geometries, and (d) global $AdS_3 \times S^3$. In Sec. II B we will describe the construction of supersymmetric probe branes, using a kappa-symmetry analysis, which preserve a certain subset of the supersymmetries of the background geometry.

A. Review of supersymmetry of the backgrounds

1. SUSY of D1-D5 and D1-D5-P in the flat space approximation

We first consider the D1-D5 system, which consists of Q_1 D1 branes wrapped on the S^1 and Q_5 D5 branes wrapped on $S^1 \times T^4$. Let us first compute the supersymmetries of the background ignoring backreaction. In this approximation we regard the Q_1 D1 branes and the Q_5 D5 branes as placed in flat space. The residual supersymmetries of the system can be figured out in the following way. A D1 brane wrapped on the S^1 preserves the supersymmetry (SUSY)[1]

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = -i\epsilon^*. \tag{1}$$

Similarly, a D5 brane wrapped on $S^1 \times T^4$ preserves the supersymmetry

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = -i\epsilon^*. \tag{2}$$

The above equations can be derived by considering the BPS relations arising from IIB SUSY algebra or by considering the κ-symmetry condition on the DBI description of a D1 or D5 brane. A combined system of D1 and D5 branes will therefore preserve eight supersymmetries given by ϵ's which satisfy both (1) and (2).

For later reference, we set up some notation. The eight residual supersymmetries of the D1-D5 system can be described as satisfying either

$$\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = \epsilon, \qquad \Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = -\epsilon, \qquad \epsilon = i\epsilon^* \tag{3}$$

or

$$\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\epsilon = \epsilon, \qquad \Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon, \qquad \epsilon = -i\epsilon^*. \tag{4}$$

The two conditions above are called left- and right-moving

[1] We will denote by $\Gamma_{\hat{M}}$ the flat space gamma matrices satisfying $[\Gamma_{\hat{M}}, \Gamma_{\hat{N}}] = 2\eta_{\hat{M},\hat{N}}$. By contrast, gamma matrices in a curved space, Γ_M, will be defined by $\Gamma_M = \Gamma_{\hat{M}} e^{\hat{M}}_M$ where $e^{\hat{M}}$ are the vielbeins. In the flat space approximation, $\Gamma_M = \Gamma_{\hat{M}}$.

supersymmetries, respectively. Thus the D1-D5 system has (4, 4) (left, right) supersymmetries.

D1-D5-P.—If we add to the D1-D5 system P units of left-moving momentum along the S^1, the resulting D1-D5-P system has (0, 4) supersymmetry [defined by (4)], in the notation of the previous paragraph.[2] In the flat space limit and for noncompact x_5, a left-moving momentum can be seen as arising from applying an infinite boost to the D1-D5 system in the t-x_5 plane. It is easy to see that the right-moving supersymmetries are invariant under such a boost, while the left-moving supersymmetries are not. Since the supersymmetry conditions are local, the argument can be extended to the case where x_5 is compact.

2. SUSY of the full D1-D5 and D1-D5-P geometry

It has been assumed above that the Q_1 D1 branes and Q_5 D5 branes are in flat space. For Q_1, Q_5 large, the metric, dilaton, and the Ramond-Ramond (RR) fields get deformed. The modified background geometry, applying standard constructions, is given by the "D1-D5" geometry, described in Table I (in Sec. III B). This geometry should be thought of as describing an "ensemble" rather than any particular microstate of the D1-D5 system. In case of the D1-D5-P, the backreacted metric is given in (50) (the dilaton and RR fields are given by Table I).

To analyze unbroken supersymmetries of these backgrounds and the others to follow, we need to solve the Killing spinor equations in these backgrounds. These Killing spinors were considered, in fact for a much larger class of metrics, in [21,22]. We quote the results of this analysis here, with a very brief introduction, and explain details, for each of the cases, in Appendix D.

In the case of the D1-D5 geometry and the other geometries we consider below, the metric may always be written in terms of vielbeins as

$$ds^2 = -(e^{\hat{t}})^2 + (e^{\hat{5}})^2 + e^{\hat{m}}e^{\hat{m}} + e^{\hat{a}}e^{\hat{a}}. \tag{5}$$

The coordinate indices are as explained in the beginning of Sec. II. The $\hat{()}$ represents a flat space index (vielbein label). Spinors are defined with respect to a specific choice of vielbeins, and they transform in the spinorial representation under an $SO(1,9)$ rotation of the vielbeins. The precise form of the vielbein, in the geometries we consider, may be found in Appendix A 2.

Finding the residual supersymmetries of a particular background amounts to solving the Killing spinor equations which are obtained by setting to zero the dilatino variation (D2) and the gravitino variation (D3). The analysis in Appendix D tells us that (1) and (2) continue to describe the supersymmetries of the D1-D5 geometry, while (4) continues to describe the supersymmetries of the D1-D5-P geometry.

[2] We adopt the slightly unusual terminology that a wave rotating counterclockwise on the S^1 is left moving.

3. SUSY of Lunin-Mathur geometries

It was explained in a sequence of papers [11,13,23,24] that the geometry of Table I should be treated as an "average" over several allowed D1-D5 microstates. The gravity solution dual to any particular Ramond ground state was described by Lunin and Mathur [11,12]. The analysis of [21,22] and Appendix D shows that even these solutions preserve the supersymmetries given by (1) and (2).

4. SUSY of global $\mathrm{AdS}_3 \times S^3 \times T^4$

Type IIB string theory on global AdS_3 is dual to the NS sector of the CFT on the boundary. If we take the geometry to be $\mathrm{AdS}_3 \times S^3 \times T^4$, the boundary CFT has (4, 4) superconformal symmetry. We will describe these supersymmetries below.

Global $\mathrm{AdS}_3 \times S^3$ is described by the metric

$$ds^2 = -\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2 + d\zeta^2. \tag{6}$$

We will find the bulk Killing spinors of this background in two ways. In Appendix E, we will find them by explicitly solving the IIB Killing spinor equations in a manner similar to [25]. Below we will find them in an alternative method, due to Mikhailov [26], which is quite illuminating.

The metric (6) arises by embedding (a) AdS_3 in flat $R^{2,2}$ by the equations $X^{-1} = \cosh\rho\cos t$, $X^0 = \cosh\rho\sin t$, $X^1 = \sinh\rho\cos\theta$, $X^2 = \sinh\rho\sin\theta$ and (b) S^3 in flat R^4 by the equations $Y^1 = \cos\zeta\cos\phi^1$, $Y^2 = \cos\zeta\sin\phi_1$, $Y^3 = \sin\zeta\cos\phi_2$, $Y^4 = \sin\zeta\sin\phi_2$. We can therefore regard $\mathrm{AdS}_3 \times S^3 \times T^4$ as embedded in $R^{2,10}$ as a codimension-two submanifold.

Now consider $R^{2,10}$ spinors that are simultaneously real and chiral. Regard $R^{2,10}$ as a product of $R^{2,2}(\supset \mathrm{AdS}_3)$, $R^4(\supset S^3)$, and R^4 (which we compactify to get the T^4). The spinors now should be regarded as transforming under $SO(2,2) \times SO(4) \times SO(4)$. It is possible to consistently restrict attention to a subclass of these spinors, namely, those that are chiral under the last $SO(4)$ [this is consistent because complex conjugation does not change $SO(4)$ spinor chirality]. We now have a set of 16 real or eight complex spinors. These spinors are chiral in $R^{2,6}$ as well as in R^4. We will denote these spinors by χ.

Let us denote by $\tilde{\Gamma}_A$, $A = -1, 0, 1, \ldots, 10$ the $R^{2,10}$ gamma matrices. We define by N_{AdS} the vector in $R^{2,2}$ which is normal to the AdS_3 submanifold and by N_S the vector in R^4 which is the normal to S^3. The prescription of [26] is that the Killing spinors are given by

$$\epsilon = (1 + (\tilde{\Gamma} \cdot N_{\mathrm{AdS}})(\tilde{\Gamma} \cdot N_S))\chi, \tag{7}$$

where χ are the $R^{2,10}$ spinors constrained as in the previous paragraph. The two normal gamma matrices are explicitly given by $\tilde{\Gamma} \cdot N_{\mathrm{AdS}} = (X^{-1}\tilde{\Gamma}_{-1} + X^0\tilde{\Gamma}_0 + X^1\tilde{\Gamma}_1 + X^2\tilde{\Gamma}_2)$ and $\tilde{\Gamma} \cdot N_S = X^3\tilde{\Gamma}_3 + X^4\tilde{\Gamma}_4 + X^5\tilde{\Gamma}_5 + X^6\tilde{\Gamma}_6$.

In Appendix E we show that the 16 real spinors defined by (7) are the same as the ones obtained from directly solving the IIB Killing spinor equations.

B. Construction of supersymmetric probes

1. D1 probe in D1-D5/D1-D5-P background: Flat space approximation

We first construct supersymmetric D1-brane probes in the D1-D5 background, in the approximation described in Sec. II A 1. Consider a probe D string executing some motion in this background.

In this subsection we demonstrate that this probe preserves all the right-moving supercharges of the background [corresponding to supersymmetry transformations (4)], provided its motion is such that

(1) The vector

$$\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5} \tag{8}$$

is tangent to the brane world volume at every point.

(2) The brane always maintains a positive orientation with respect to the branes that make up the background.

We will first prove these statements, and then return, at the end of this subsection, to an elaboration of their meaning.

According to assumption 1 above, $\mathbf{n}$ is tangent to the world volume at every point. A second, linearly independent, tangent vector may be chosen at each point so that the coefficient of $\frac{\partial}{\partial t}$ is zero. Making this choice, this normalized vector may be written as $\mathbf{v_2} = \sin\alpha \frac{\partial}{\partial x_5} + \cos\alpha \mathbf{u}$ where $\mathbf{u}$ represents a spacelike unit vector orthogonal to x_5. By assumption 2, we have $\sin\alpha > 0$.[3] In general, the direction of $\mathbf{u}$ and the value of α will vary as a function of world-volume coordinates. Although $\mathbf{n}$, $\mathbf{v_2}$ are linearly independent, they are not an orthonormal set since $\mathbf{n}$ is a null vector. We can construct an orthonormal basis of vectors $\mathbf{v_1}$, $\mathbf{v_2}$ at each point of the world volume by the Gram-Schmidt method, yielding

$$\begin{aligned}\mathbf{v_1} &= \mathbf{n}/\sin\alpha - \mathbf{v_2}\\ &= 1/\sin\alpha\left(\frac{\partial}{\partial t} + \cos^2\alpha\frac{\partial}{\partial x_5} - \cos\alpha\sin\alpha\mathbf{u}\right).\end{aligned} \tag{9}$$

For the probe to preserve some supersymmetry ϵ, we must have, at each point of the world volume,

$$\Gamma_{\mathbf{v_1}}\Gamma_{\mathbf{v_2}}\epsilon = -i\epsilon^*. \tag{10}$$

The above equation is equivalent to

$$\left[\Gamma_{\hat{0}}\Gamma_{\hat{5}} - \frac{\Gamma_{\mathbf{u}}}{\sin\alpha}(\cos\alpha\Gamma_{\hat{0}} + (\sin^2\alpha\cos\alpha + \cos^3\alpha)\Gamma_{\hat{5}})\right]\epsilon = -i\epsilon^*. \tag{11}$$

This is clearly satisfied by spinors that satisfy (4) since (4) implies that $\Gamma_0\Gamma_5\epsilon = \epsilon$ which ensures $\Gamma_0\epsilon = -\Gamma_5\epsilon$ and a consequent vanishing of the coefficient of $\Gamma_{\mathbf{u}}$ above. Note that in flat space $\Gamma_{\hat{M}} = \Gamma_M$.[4]

The conditions (1) and (2), listed at the beginning of this subsection, are easily solved by choosing a world-sheet parametrization in terms of coordinates σ, τ, such that

$$\begin{aligned}&x^M = \mathbf{n}^M\tau + x^M(\sigma), \qquad x^0 = \tau, \qquad x_5 = x_5(\sigma) + \tau,\\ &x^q = x^q(\sigma), \qquad q = 1, 2, 3, 4, 6, 7, 8, 9\end{aligned} \tag{12}$$

where $x_5(\sigma)$, $x^q(\sigma)$ are arbitrary functions, except that $\partial_\sigma x_5 > 0$. To connect with the earlier discussion, we identify $\mathbf{v_2}$ as the unit vector along $\mathbf{s}^M \equiv \partial_\sigma x^M$. Note that by condition (2) above, we need $\partial_\sigma x_5 = (\mathbf{n}, \mathbf{s}) > 0$, which is equivalent to our earlier condition $\sin\alpha > 0$. This constraint, together with the periodicity of configurations in σ, implies that $\int d\sigma x_5(\sigma) = 2\pi R w$, where R is the radius of the x_5 circle, and w is a positive integer that we will refer to as the winding number. The configurations described in this paragraph are easy to visualize. They consist of D strings with arbitrary transverse profiles, winding the x_5 direction w times, and moving at the speed of light in the positive x_5 direction.

Equation (10) is equivalent to the κ-symmetry projection, which can alternatively be written as

$$\begin{aligned}&\Gamma\epsilon = i\epsilon^*,\\ &\Gamma := \tfrac{1}{2}\Gamma_{MN}\partial_\alpha x^M\partial_\beta x^N\epsilon^{\alpha\beta}/\sqrt{-h}\\ &\quad = \tfrac{1}{2}[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]/\sqrt{-h} = \Gamma_{\mathbf{v_1}}\Gamma_{\mathbf{v_2}},\end{aligned} \tag{13}$$

where h is the determinant of the induced metric on the world volume in the σ, τ coordinates above. In the third line we have used the parametrization (12). This is equivalent to (10) by using $\sqrt{-h} = \sin\alpha|\mathbf{s}|$.

Since all we needed in the above discussion is the (0, 4) supersymmetry (4) of the background, the above discussion goes through unchanged for D1 probes in the D1-D5-P background in the flat space approximation.

2. D1 probe in D1-D5/D1-D5-P background

We now consider the curved D1-D5-P background, described in (50). The specialization to the D1-D5 background is straightforward (we just need to put $r_p = 0$). We will show that (12), or equivalently, the condition that $\mathbf{n} = \partial_t + \partial_5$ is tangent to the world volume, again ensures

[3] When $\sin\alpha$ is less than zero, the $\mathbf{v_1}$ and $\mathbf{v_2}$ are not appropriately oriented. Also $\alpha \neq 0$, because in that case, the determinant of the induced world-sheet metric would vanish.

[4] This derivation does not work for left-moving supercharges where (3) implies $\Gamma_0\epsilon = +\Gamma_5\epsilon$. Left-moving supercharges are symmetries for D1 branes that move at the speed of light to the left [branes whose tangent space includes $(1, -1, 0, \ldots 0)$].

the appropriate supersymmetry of the probe. For this, we need to show that (13) is valid in this background. We find that [see (45)]

$$\sqrt{-h} = \dot{X} \cdot X' \equiv \mathbf{n} \cdot \mathbf{s} = x_5'(g_{05} + g_{55}),$$
$$\Gamma\epsilon = 1/(2\sqrt{-h})[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]\epsilon = \frac{1}{(g_{05} + g_{55})x_5'}(\Gamma_{05}x_5' + (\Gamma_0 + \Gamma_5)\Gamma_q x_q')\epsilon. \tag{14}$$

To show that $\Gamma\epsilon = \epsilon$ we need

$$\Gamma_0\epsilon = -\Gamma_5\epsilon, \qquad (g_{05} + g_{55})^{-1}(\Gamma_0\Gamma_5)\epsilon = \epsilon. \tag{15}$$

The first equation is equivalent to

$$e_0^{\hat{0}}\Gamma_{\hat{0}}\epsilon = -(e_5^{\hat{0}}\Gamma_{\hat{0}} + e_5^{\hat{5}}\Gamma_{\hat{5}})\epsilon. \tag{16}$$

After explicitly inserting the vielbeins using Eqs. (A2) and (A3), we are left with

$$\Gamma_{\hat{0}}\epsilon = -\Gamma_{\hat{5}}\epsilon, \tag{17}$$

which is equivalent to $\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon$. The second equation of (15) gives rise to the same condition,

$$\Gamma_{\hat{0}}\Gamma_{\hat{5}}\epsilon = \epsilon, \tag{18}$$

by using $e_0^{\hat{0}}e_5^{\hat{5}} = g_{05} + g_{55}$.

Thus, we have shown that a D1-brane probe moving such that $\mathbf{n} = \partial_t + \partial_5$ is always tangent to the world volume, or equivalently satisfying Eq. (12), preserves the supersymmetry (4).

3. D1 probe in Lunin-Mathur background

We now show that the same condition as in the previous subsection, namely, that $\mathbf{n}$ should be everywhere tangent to the world volume of the D1 brane [alternatively, that the D1-brane embedding can be expressed as in (12)], is valid for supersymmetry of D1 probes in the background (52), discussed in Sec. II A 3 above. This analysis is fairly similar to the one above. In this case, Eq. (14) changes to

$$\sqrt{-h} = \dot{X} \cdot X' \equiv \mathbf{n} \cdot \mathbf{s} = x_5' g_{55} + x_m'(g_{0m} + g_{5m}). \tag{19}$$

Hence

$$\begin{aligned}\Gamma\epsilon &= 1/(2\sqrt{-h})[\Gamma_{\mathbf{n}}, \Gamma_{\mathbf{s}}]\epsilon \\ &= (x_5' g_{55} + x_m'(g_{0m} + g_{5m}))^{-1} \\ &\quad \times (\Gamma_{05}x_5' + \tfrac{1}{2}x_q'[(\Gamma_0 + \Gamma_5), \Gamma_q])\epsilon \\ &= (x_5' g_{55} + x_m'((g_{0m} + g_{5m}))^{-1}(\Gamma_{\hat{0}\hat{5}}x_5' g_{55} \\ &\quad + x_q'((g_{0q} + g_{5q}) - \Gamma_q(\Gamma_0 + \Gamma_5))\epsilon).\end{aligned} \tag{20}$$

Thus, if $\Gamma_{\hat{0}\hat{5}}\epsilon = \epsilon$, as in (4) [which also implies $(\Gamma_0 + \Gamma_5)\epsilon = 0$, using $e_0^{\hat{0}} = e_5^{\hat{5}}$], the expression (20) evaluates to $\Gamma\epsilon = \epsilon$. For spinors satisfying (4) this also implies $\Gamma\epsilon = i\epsilon^*$, which is the kappa-symmetry projection condition. In the last step of (20) we have used

$$\begin{aligned}\Gamma_{05} &= g_{55}\Gamma_{\hat{0}\hat{5}}, \\ \tfrac{1}{2}[\Gamma_0 + \Gamma_5, \Gamma_m] &= \tfrac{1}{2}\{\Gamma_0 + \Gamma_5, \Gamma_m\} - \Gamma_m(\Gamma_0 + \Gamma_5) \\ &= (g_{0m} + g_{5m}) - \Gamma_m(\Gamma_0 + \Gamma_5).\end{aligned}$$

4. D1 probe in global $\mathrm{AdS}_3 \times S^3$

We will use the description of supersymmetries of the background as in Sec. II A 4. We will show in this section that D1 strings with world volumes, to which

$$\mathbf{n} = \partial_t + \partial_\theta + \partial_{\phi_1} + \partial_{\phi_2} \tag{21}$$

is everywhere tangent, preserve four supercharges.

We will first mention the geometric significance of $\mathbf{n}$. Let us group the $R^{2,6}$ (see Sec. II A 4) coordinates into complex numbers as $X^{-1} + iX^0$, $X^1 + iX^2$, $Y^1 + iY^2$, $Y^3 + iY^4$. This defines a complex structure I on $R^{2,6}$. In Sec. II A 4, we have defined N_{AdS} as the normal to AdS_3 in $R^{2,2}$ and N_S as the normal to S^3 in R^4. It is easy to check that the complex partner of N_{AdS} is $I(N_{\mathrm{AdS}}) = -\partial_t - \partial_\theta$, which generates (twice) the right-moving conformal spin $2h_r$. Similarly, the complex partner of N_S is $I(N_S) = \partial_{\phi^1} + \partial_{\phi_2}$, which generates (twice) the z component of angular momentum in the right-moving $SU(2)$ [out of $SO(4) = SU(2) \times SU(2)$]. The vector $\mathbf{n}$ therefore generates $-2(h_r - J_r)$.[5]

Note, first, that $\mathbf{n}$ is a null vector (its two components are, respectively, unit timelike and unit spacelike vectors). Let $\mathbf{n}_s = K(\partial_\theta + \partial_{\phi_1} + \partial_{\phi_2})$ (the purely spatial component of $\mathbf{n}$) with the normalization K chosen to give $\mathbf{n}_s$ unit norm. Consider a positively oriented purely spatial vector $\mathbf{v_2}$ at a particular point p on the string at constant time. We may decompose $\mathbf{v_2}$ as

$$\mathbf{v_2} = \sin\alpha\,\mathbf{n}_s + \cos\alpha\,\mathbf{u}, \tag{22}$$

where $\mathbf{u}$ is some purely spatial unit vector orthogonal to $\mathbf{n}_s$. Let us assume that the string evolves in time so that the vector $\mathbf{n}$ is always tangent to its world volume. It follows that, at the point P, the world volume of the string is spanned by $\mathbf{n}$ and $\mathbf{v_2}$. These two vectors are not orthogonal, but it is easy to check that, with

$$\mathbf{v_1} = \frac{\mathbf{n}}{\sin\alpha} - \mathbf{v_2}, \tag{23}$$

$\{\mathbf{v_1}, \mathbf{v_2}\}$ form an orthonormal set, with the first vector timelike. The D string preserves those supersymmetries of (7) that satisfy

$$\tilde{\Gamma}_{\mathbf{v_1}}\tilde{\Gamma}_{\mathbf{v_2}}\epsilon = \epsilon. \tag{24}$$

Before proceeding further, let us introduce some terminology. Consider a complex vector u, say $X_1 + iX_2$. A

[5]It is not difficult to check that $2h_L - 2j_L$ is generated by the vector field $\mathbf{n}' = -\partial_t + \partial_\theta - \partial_{\phi_1} + \partial_{\phi_2}$.

spinor that is annihilated by $\tilde{\Gamma}_u$ is said to have spin $(-)$ under rotation in the X_1-X_2 plane, while a spinor annihilated by $\tilde{\Gamma}_{\bar{u}}$ has positive spin (consequently, the spin operator is $i\tilde{\Gamma}_1\tilde{\Gamma}_2$), with similar definitions for the other directions. Let us now consider constant spinors χ whose spins (eigenvalues under this "spin" operator) in $R^{2,2}$ and R^4, respectively, are $(++)(--)$ or $(--)(++)$. The spins in T^4 could be either $(++)$ or $(--)$—this gives a total of four spinors, or two sets of complex conjugate pairs of spinors. We will now demonstrate that any giant graviton whose world volume tangent space contains the vector (21) preserves all four of these supersymmetries.

To avoid cluttering the notation below, we define

$$\begin{aligned} \tilde{\Gamma}_{\mathrm{AdS}} &= \tilde{\Gamma}\cdot N_{\mathrm{AdS}}, & \tilde{\Gamma}_S &= \tilde{\Gamma}\cdot N_S, \\ \tilde{\Gamma}_{I(N_{\mathrm{AdS}})} &= \tilde{\Gamma}\cdot I(N_{\mathrm{AdS}}), & \tilde{\Gamma}_{I(N_S)} &= \tilde{\Gamma}\cdot I(N_S). \end{aligned} \tag{25}$$

Now consider

$$\begin{aligned} A &= (\tilde{\Gamma}_{\mathbf{v}_1}\tilde{\Gamma}_{\mathbf{v}_2} - 1)(1 + \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S)\chi \\ &= \left(\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{n}} - \tilde{\Gamma}_{\mathbf{v}_2}\right)\tilde{\Gamma}_{\mathbf{v}_2}(1 + \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S)\chi - (1 + \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S)\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v}_2}\tilde{\Gamma}_{\mathbf{n}}(1 + \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S)\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v}_2}\tilde{\Gamma}_{I(N_S)}[(1 + \tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{\mathrm{AdS}})})(1 + \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S)]\chi \\ &= -\frac{1}{\sin\alpha}\tilde{\Gamma}_{\mathbf{v}_2}\tilde{\Gamma}_{I(N_S)}(1 + \tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{\mathrm{AdS}})}) \\ &\quad \times [1 + \tilde{\Gamma}_{I(N_S)}\tilde{\Gamma}_{I(N_{\mathrm{AdS}})}\tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_S], \end{aligned} \tag{26}$$

where we have used $\tilde{\Gamma}^2_{I(N_S)} = 1 = -\tilde{\Gamma}^2_{I(N_{\mathrm{AdS}})}$.

It is now relatively simple to check that (26) vanishes when χ is any of the four spinors $(++)(--)(++)$, $(++)(--)(--)$, $(--)(++)(++)$, $(--)(++)(--)$.[6] Recall that a positive spin is annihilated by $\tilde{\Gamma}_S - i\tilde{\Gamma}_{I(N_S)}$ and by the equivalent AdS expression. Using $\tilde{\Gamma}^2_S = -\tilde{\Gamma}^2_{\mathrm{AdS}} = 1$ we find

$$\begin{aligned} \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_{I(N_{\mathrm{AdS}})}\chi_{(++)(\ldots)} &= +i\chi_{(++)(\ldots)}, \\ \tilde{\Gamma}_S\tilde{\Gamma}_{I(N_S)}\chi_{(\ldots)(++)} &= -i\chi_{(\ldots)(++)}, \\ \tilde{\Gamma}_{\mathrm{AdS}}\tilde{\Gamma}_{I(N_{\mathrm{AdS}})}\chi_{(--)(\ldots)} &= -i\chi_{(--)(\ldots)}, \\ \tilde{\Gamma}_S\tilde{\Gamma}_{I(N_S)}\chi_{(\ldots)(--)} &= +i\chi_{(\ldots)(--)}, \end{aligned} \tag{27}$$

from which (24) follows for all the spinors listed above.

We conclude that any D1-brane world volume, to which the vector $\mathbf{n}$ is always tangent, preserves the four supersymmetries listed above. The same is true of a D5-brane world volume that wraps the 4-torus.

[6]The first and second of these spinors are Q's, while the third and fourth of these are complex conjugate S's.

5. D1-D5 bound-state probe

Now, we consider D5 branes that wrap the 4-torus, and move so as to keep the vector $\mathbf{n}$ tangent to their world volume at all points, but also have gauge fields on their world volume. These gauge fields, in a configuration with nonzero instanton number, can represent bound states of D1 and D5 branes. Our analysis here is valid for all four backgrounds considered above.

Consider a D5 brane with a nonzero 2-form Born-Infeld field strength F, that wraps the $S^1 \times T^4$. We denote the world-volume coordinates by $\sigma^\alpha = \sigma^{1,2,6,7,8,9} \equiv \{\tau, \sigma, z^1, z^2, z^3, z^4\}$. The embedding of the world volume, as before, will be denoted by $x^M(\sigma^\alpha)$ and the induced metric by $h_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$. For a nondegenerate world volume ($\det h \neq 0$) the tangent vectors $\partial_\alpha x^M$ are linearly independent and provide a basis for the tangent space at each point of the world volume. It is clearly possible to introduce an orthonormal (in the spacetime metric G_{MN}) basis of six vectors $\mathbf{v}_{\hat{\alpha}}$, related to the $\partial_\alpha x^M$ by $\partial_\alpha x^M = e^{\hat{\alpha}}_\alpha \mathbf{v}_{\hat{\alpha}}$ such that

$$G_{MN}\mathbf{v}^M_{\hat{\alpha}}\mathbf{v}^N_{\hat{\beta}} = \tilde{\eta}_{\hat{\alpha}\hat{\beta}}.$$

The invertible matrix $e^{\hat{\alpha}}_\alpha$ defines 6-beins of the induced metric:

$$h_{\alpha\beta} = G_{MN}\partial_\alpha X^M\partial_\beta X^N = G_{MN}e^{\hat{\alpha}}_\alpha e^{\hat{\beta}}_\beta \mathbf{v}^M_{\hat{\alpha}}\mathbf{v}^N_{\hat{\beta}} = \tilde{\eta}_{\hat{\alpha}\hat{\beta}}e^{\hat{\alpha}}_\alpha e^{\hat{\beta}}_\beta. \tag{28}$$

Here $\tilde{\eta}$ is 6 dimensional and α, β run over the world-volume coordinates. We will define below

$$\gamma_{\hat{\alpha}} = \mathbf{v}^M_{\hat{\alpha}}\Gamma_M.$$

We take $\mathbf{v}_1$, $\mathbf{v}_2$ to be the same as in the previous subsections. The other four vectors point along the internal manifold, $\mathbf{v}_i \propto \frac{\partial}{\partial x^i}$, $i = 6, 7, 8, 9$.

The condition for branes with world-volume gauge fields to be supersymmetric was considered in [27,28]. Using the two-component notation for spinors

$$\epsilon = \begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}, \tag{29}$$

the BPS condition is [see Eq. (13) of [28]]

$$\begin{aligned} &\mathbf{R}\,\gamma_{\hat{1}\hat{2}\hat{6}\hat{7}\hat{8}\hat{9}}\epsilon = \epsilon, \\ &\mathbf{R} = \frac{1}{\sqrt{-\det\{\tilde{\eta}_{\hat{\alpha}\hat{\beta}} + F_{\hat{\alpha}\hat{\beta}}\}}} \\ &\qquad \times \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\gamma^{\hat{\alpha}_1\hat{\beta}_1\ldots\hat{\alpha}_n\hat{\beta}_n}F_{\hat{\alpha}_1\hat{\beta}_1}\ldots F_{\hat{\alpha}_n\hat{\beta}_n}\sigma_3^{n+1}i\sigma_2, \end{aligned} \tag{30}$$

where we have expressed the world-volume gauge fields in the local orthonormal frame: $F_{\alpha\beta} = F_{\hat{\alpha}\hat{\beta}}e^{\hat{\alpha}}_\alpha e^{\hat{\beta}}_\beta$. Note that the product in (30) terminates at $n = 3$ because the indices are antisymmetrized. From the $n = 0$ term we find, using the analysis of the previous subsections, that the condition (30) can be met only for spinors that obey (4). The spinors (4) are eigenspinors of σ_1. Since $i\sigma_2$ appears in the $n = 1$ term, this term must vanish. Hence, the gauge fields must be of the form

$$F_{\hat{1}\hat{2}} = 0, \qquad F_{\hat{1}\hat{i}} = -F_{\hat{2}\hat{i}}, \qquad F_{\hat{i}\hat{j}} = \epsilon_{\hat{i}\hat{j}}^{\hat{k}\hat{l}} F_{\hat{k}\hat{l}}. \tag{31}$$

For a gauge field of this kind, the determinant above is calculated in (75) and

$$\sqrt{-\det\{\tilde{\eta}_{\hat{\alpha}\hat{\beta}} + F_{\hat{\alpha}\hat{\beta}}\}} = 1 + \frac{F_{\hat{i}\hat{j}}F^{\hat{i}\hat{j}}}{4}.$$

The $n = 2$ term gives us the right factor in the numerator to cancel this, and the $n = 3$ term vanishes as a virtue of (31).

In the world-volume curved basis, our result implies [see (9) and (23)] that

$$F = F_{\sigma i} d\sigma \wedge dx^i + \tfrac{1}{2} F_{ij} dx^i \wedge dx^j, \tag{32}$$

and is self-dual on the torus, i.e.

$$F_{ij}\epsilon^{ij}_{kl} = F_{kl}. \tag{33}$$

For "wavy instantons" where the gauge fields depend on σ and the field strength is of the form (32), the Gauss law and Eq. (33) are enough to guarantee that F solves the equations of motion [29].

The form of F in (32) is adequate to guarantee supersymmetry in all four backgrounds considered previously. For the sake of completeness, we mention that the explicit embedding of the D5 brane in spacetime is described by the functions $X^M(\tau, \sigma, z^{1\ldots4})$ satisfying

$$\frac{\partial X^M(\tau, \sigma, z^{1\ldots4})}{\partial \tau} = \mathbf{n}^M. \tag{34}$$

In the coordinate systems that we will discuss, $\mathbf{n}^M$ is a constant, and in such a coordinate system we again have

$$X^M(\tau, \sigma, z^{1\ldots4}) = X^M(\sigma) + \mathbf{n}^M\tau. \tag{35}$$

Using the value of $\mathbf{n}$ (8) in the D1-D5, D1-D5-P, and Lunin-Mathur geometries, the above equation translates to

$$t = \tau, \qquad x_5 = x_5(\sigma) + \tau, \qquad x^m = x^m(\sigma), \qquad x^6 = z^1, \ldots, x^9 = z^4, \tag{36}$$

while, in global $\mathrm{AdS}_3 \times S^3 \times T^4$, using (21), the brane motion is

$$t = \tau, \qquad \theta = \theta(\sigma) + \tau, \qquad \rho = \rho(\sigma), \qquad \zeta = \zeta(\sigma), \qquad \phi_1 = \phi_1(\sigma) + \tau, \qquad \phi_2 = \phi_2(\sigma) + \tau, \qquad x^6 = z^1, \ldots, x^9 = z^4. \tag{37}$$

We are assuming, in the embedding above, that the brane wraps the internal manifold only once. The case of multiple wrapping is identical to the case of multiple brane probes, each wrapping the internal manifold once, and is discussed in more detail in Sec. IV C.

The field strength above gives rise to an induced D1 charge, p, on the D5-brane world volume, which is proportional to the second Chern class and is given by

$$p = \frac{1}{(2\pi\sqrt{\alpha'})^4}\int_{T^4} \frac{\mathrm{Tr}(F\wedge F)}{2}, \tag{38}$$

and also to an induced D3 brane charge on the two-cycles of the T^4 (which we denote by C_2 below), proportional to the first Chern class, given by

$$p^3_{C_2} = \frac{1}{(2\pi\sqrt{\alpha'})^2}\int_{C_2} \mathrm{Tr}(F). \tag{39}$$

This D5-brane configuration with world-volume gauge fields then represents a D1-D3-D5 bound state. This bound state has the property that, whenever we wrap a D3 brane on a two-cycle, we need to put an equal amount of D3 brane charge on the dual two-cycle. It may be surprising that a probe of this kind, with induced D3 brane charge, is mutually supersymmetric with the D1-D5 background.

However, this fact may be familiar to the reader from another perspective. Consider a configuration of Q_1 D1 branes, Q_5 D5 branes, Q_3 D3 branes, and Q_3' D3$'$ branes, wrapping the 5, 56 789, 567, 589 directions, respectively. Following the standard BPS analysis of, say, Chapter 13 in [30], the BPS bound for this configuration is

$$M \geq \sqrt{(Q_1 + Q_5)^2 + (Q_3 - Q_3')^2}. \tag{40}$$

When $Q_3 = Q_3'$, this bound becomes $M \geq Q_1 + Q_5$ and it may further be shown that this configuration preserves the same supersymmetries as the D1-D5 system.

Nevertheless, we will not be interested in probes with a nonvanishing first Chern class in this paper. The AdS/CFT conjecture requires us to sum over all geometries with fixed boundary conditions for the fields at ∞. When we consider a D1 or D5 probe, we can reduce the D1 or D5 charge in the background so that the total D1 and D5 charge remains constant at ∞. A probe with nonvanishing $p^3_{C_2}$ will lead to some finite D3 charge at ∞, and turning on an anti-D3 charge in the background will render the probe nonsupersymmetric. So, such probes must be excluded from a consideration of the supersymmetric excitations of the pure D1-D5 system. Henceforth, we will set $p^3_{C_2}$ to zero on all two-cycles C_2 of the T^4.

III. CHARGE ANALYSIS: D STRINGS

From the Killing spinor analysis above, we conclude that, in all four different backgrounds we will consider, D strings that move so as to keep a particular null Killing vector field tangent to their world volume at each point preserve four supersymmetries. This means, as we mentioned, that given the initial shape of the D string we can translate it along the integral curves of this vector field to generate the entire world volume. In this section, we will use this fact to explicitly parametrize all supersymmetric D string probes in terms of their initial profile functions. We will then use the DBI action to calculate the spacetime momenta of these configurations and verify the saturation of the BPS bound.

In the first subsection below, we present a general formalism that is applicable to all the examples we consider. We then proceed to apply this formalism to the extremal D1-D5 background, the D1-D5-P background, the smooth geometries of [11], and finally global AdS.

A. Supersymmetric D1 probe solutions

We introduce coordinates τ and σ on the D1-brane world volume. We use $X^M(\sigma, \tau)$ to describe the embedding of the world sheet in spacetime, with $t \equiv X^0$ denoting time. We will use $\dot{X}^M \equiv \frac{\partial X^M}{\partial \tau}$ and $(X^M)' \equiv \frac{\partial X^M}{\partial \sigma}$. The special null vector, discussed above, is denoted by $\mathbf{n}^M$ (see also Sec. II B). We will always work with the string-frame metric G_{MN}. This is the metric we use while calculating dot products. For example, $X' \cdot X' = G_{MN} X'^M X'^N$. The Ramond-Ramond 3 form field strength is denoted by $G^{(3)}_{MNP}$, and the 2 form potential is denoted by $C^{(2)}_{MN}$. The dilaton is ϕ. The induced world-sheet metric is $h_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$. In all the cases that we consider in this section, the NS-NS 2-form is set to zero.

With this notation, the bosonic part of the D1-brane action is

$$\begin{aligned} S &= \int \mathcal{L}_{\text{brane}} d\sigma d\tau \\ &= -\frac{1}{2\pi\alpha'}\int e^{-\phi}\sqrt{-h}d\sigma d\tau + \frac{1}{2\pi\alpha'} \\ &\quad \times \int C^{(2)}_{MN}\partial_\alpha X^M \partial_\beta X^N \frac{\epsilon^{\alpha\beta}}{2} d\sigma d\tau, \end{aligned} \tag{41}$$

where

$$h = \det[h_{\alpha\beta}] = (X' \cdot X')(\dot{X} \cdot \dot{X}) - (X' \cdot \dot{X})^2. \tag{42}$$

We take $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = +1$. In line with the analysis presented above, we take our solutions to have the property

$$\frac{\partial X^M(\sigma, \tau)}{\partial \tau} = \mathbf{n}^M. \tag{43}$$

In the examples in this section, we will be using a coordinate system where $\mathbf{n}^M$ is constant. When this happens, we may solve (43) via [see (12) and (37)]

$$X^M(\sigma, \tau) = X^M(\sigma) + \mathbf{n}^M \tau. \tag{44}$$

As we explained above, the set of supersymmetric world volumes is parametrized by the set of initial shapes $X^M(\sigma)$.

On these solutions, we find

$$\sqrt{-h} = |X' \cdot \dot{X}|. \tag{45}$$

From the action (41), we can then derive the momenta

$$\begin{aligned} P_M &= \frac{\partial \mathcal{L}_{\text{brane}}}{\partial \dot{X}^M} \\ &= \frac{-e^{-\phi}}{2\pi\alpha'}\left[(G_{MN} - e^{\phi}C^{(2)}_{MN})X'^N - \mathbf{n}_M \frac{(X' \cdot X')}{X' \cdot \dot{X}}\right]. \end{aligned} \tag{46}$$

Since these momenta are independent of τ, the equations of motion reduce to

$$\begin{aligned} -\frac{\partial \mathcal{L}_{\text{brane}}}{\partial X^P} &= \left(\frac{\partial(e^{-\phi}G_{MN})}{\partial X^P} + \frac{\partial C^{(2)}_{MN}}{\partial X^P}\right) \\ &\quad \times \left(X'^M \dot{X}^N - \dot{X}^M \dot{X}^N \frac{X' \cdot X'}{X' \cdot \dot{X}}\right) = 0. \end{aligned} \tag{47}$$

Before we apply this general formalism to specific cases, we would like to make two comments.

(1) First, as noted above, we find that $\sqrt{-h} = +|X' \cdot \dot{X}|$. Without the absolute value sign, a world sheet that folds on itself could have zero area. If we now work out the equations of motion carefully, taking into account that no such absolute value sign occurs in the coupling to the RR 2-form, then we find that, unless $X' \cdot \dot{X}$ maintains a constant sign, our configurations are not solutions to the equations of motion. Here, we have taken $|X' \cdot \dot{X}| = +X' \cdot \dot{X}$. The other choice of sign would have led to antibranes which would not be supersymmetric in the backgrounds we consider.

(2) The world sheet may be parametrized by two coordinates, σ and τ. In many of the examples that we will consider, the vector $\mathbf{n}$ is a constant in our preferred coordinate system (see Tables I and II). In such cases, we may take $t = \tau$. Now, given the profile of the string at any fixed τ, we can translate each point on that profile by the integral curves of $\mathbf{n}$, to obtain the entire world sheet. We may then use σ to label these various integral curves of $\mathbf{n}$.

B. Supersymmetric solutions in the D1-D5 background

Consider Q_1 D1 branes and Q_5 D5 branes wrapping an internal T^4 with sides of length $2\pi(\alpha')^{1/2}v^{1/4}$ and an S^1 of length 2π that we take to be along x_5. Table I describes the geometry of this background. Notice that the 3-form fluxes are normalized so that

$$\frac{1}{2\pi}\int_{S^3}\frac{G^{(3)}}{\alpha'} = 2\pi Q_5, \qquad \frac{1}{2\pi}\int_{S_3\times M_{\text{int}}}\frac{\star_{10}G^{(3)}}{\alpha'} = 2\pi Q_1. \tag{48}$$

If we take the near-horizon limit of the solution above, we find the geometry of AdS_3 in the Poincare patch, with x_5

TABLE I. D1-D5 system.

Geometry
$ds^2 = f_1^{-(1/2)} f_5^{-(1/2)}(-dt^2 + (dx_5)^2) + f_1^{1/2} f_5^{1/2}(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2)) + \frac{e^\phi}{g} ds_{\text{int}}^2$
$e^{-2\phi} = \frac{1}{g^2}\frac{f_5}{f_1}, \quad f_1 = 1 + \frac{g\alpha' Q_1}{v r^2}, \quad f_5 = 1 + \frac{g\alpha' Q_5}{r^2}, \quad v = \frac{V}{(2\pi)^4 \alpha'^2}$
$\frac{G^{(3)}}{\alpha'} = Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_1}{v f_1^2 r^3} dr \wedge dt \wedge dx_5$
$\frac{C^{(2)}}{\alpha'} = -\frac{Q_5}{2}(\cos 2\zeta + b)\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_1 \alpha'} dt \wedge dx_5$
BPS condition
$E - L = -\int P_t d\sigma - \int P_5 d\sigma = 0$
Null vector tangent to world volume
$\mathbf{n}^M = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5}$
Solution
$t = \tau x_5 = x_5(\sigma) + \tau r = r(\sigma)$
$\zeta = \zeta(\sigma)\phi_1 = \phi_1(\sigma)\phi_2 = \phi_2(\sigma)$
$z_{\text{int}}^a = z_{\text{int}}^a(\sigma)$
Momenta
$P_t = \frac{1}{2\pi\alpha' g}[\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}]$
$P_5 = -\frac{1}{2\pi\alpha' g}[\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}]$
$P_r = -\frac{1}{2\pi\alpha'}[\frac{f_5}{g} r']$
$P_\zeta = -\frac{1}{2\pi\alpha'}[\frac{f_5 r^2 \zeta'}{g}]$
$\tilde{P}_{\phi_1} = -\frac{1}{2\pi\alpha'}[\frac{f_5 r^2 \cos^2\zeta \phi_1'}{g} + \frac{Q_5\alpha'}{2}[\cos(2\zeta) - 1]\phi_2']$
$\tilde{P}_{\phi_2} = -\frac{1}{2\pi\alpha'}[\frac{f_5 r^2 \sin^2\zeta \phi_2'}{g} - \frac{Q_5\alpha'}{2}[\cos(2\zeta) + 1]\phi_1']$
$P_{z^a} = -\frac{1}{2\pi\alpha' g}[g_{ab}^{\text{int}} z^{b'}]$ (internal manifold)

identified on a circle. This is nothing but the zero mass BTZ black hole. Although the probe solutions we present below are valid in the entire D1-D5 geometry, it will turn out that quantization of these solutions in Sec. VI is only tractable when the probe branes are in the near-horizon region.

The equations of motion (47) reduce, on the solutions of (44), to

$$\frac{\partial(e^{-\phi}G_{55} + C_{5t}^{(2)})}{\partial X^P} = 0, \tag{49}$$

and these are manifestly satisfied since $e^{-\phi}G_{55} + C_{5t}^{(2)} = 0$.

Table I explicitly lists the solutions (44) and the conserved charges. The RR 2-form potential in Table I has a gauge ambiguity (the coefficient b). The canonical momenta $P_{\phi_{1,2}}$, to begin with, depend on b; however, the momenta $\tilde{P}_{\phi_{1,2}}$ appearing in both Table I and Table II (which deals with probe D strings in global AdS) are the gauge-invariant momenta which figure in the BPS relations and do not have a gauge ambiguity. This issue is discussed in detail in Appendix C. Note that the gauge ambiguity is only in the magnetic part and not in the case of the electric part. The reason is that it is possible to have a globally defined electric part of the potential while it is impossible to do so for the magnetic part (for reasons similar to the case of the Dirac monopole).

We now apply the general analysis presented above to obtain Table I.

C. Supersymmetric solutions in the D1-D5-P background

The D1-D5 system above may be generalized by adding a third charge using purely left-moving excitations, which gives the "D1-D5-P" system. The field strengths and dilaton are exactly as in Table I, but the metric is altered as follows:

$$\begin{aligned} ds^2 = {} & f_1^{-(1/2)} f_5^{-(1/2)}\Big(-dt^2 + dx_5^2 + \frac{r_p^2}{r^2}(dt - dx_5)^2\Big) \\ & + f_1^{1/2} f_5^{1/2}(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2)) \\ & + \frac{e^\phi}{g} ds_{\text{int}}^2. \end{aligned} \tag{50}$$

Here $r_p^2 = c_p g^2 P$, where P is the quantized momentum along x_5 and c_p is a numerical constant which is not important for our purpose here.

It is easy to repeat the supersymmetry analysis above, for this background. In particular, we find that

$$P_t = \frac{1}{2\pi\alpha' g}\left[\left(1+\frac{r_p^2}{r^2}\right)\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right],$$
$$P_5 = -\frac{1}{2\pi\alpha' g}\left[\left(1+\frac{r_p^2}{r^2}\right)\frac{x_5'}{f_1} - \sqrt{\frac{f_5}{f_1}}\frac{X'\cdot X'}{x_5'}\right], \quad (51)$$
$$P_t + P_5 = 0.$$

The rest of Table I remains valid.

D. Supersymmetric solutions in the Lunin-Mathur geometries

In this subsection, we describe supersymmetric D-string probes in the smooth 2-charge geometries of Lunin and Mathur [11,31]. The geometry is as follows:

$$ds^2 = \sqrt{\frac{H}{1+K}}[-(dt - A_m dx^m)^2 + (dx_5 + B_m dx^m)^2] + \sqrt{\frac{1+K}{H}} d\vec{x}\cdot d\vec{x} + \sqrt{H(1+K)} d\vec{z}\cdot d\vec{z},$$
$$e^{2\phi} = H(1+K), \qquad C^{(2)}_{tm} = \frac{-B_m}{1+K}, \qquad C^{(2)}_{t5} = \frac{1}{1+K},$$
$$C^{(2)}_{m5} = \frac{A_m}{1+K}, \qquad C^{(2)}_{mn} = C_{mn} + \frac{A_m B_n - A_n B_m}{1+K},$$
$$dB = -*dA, \qquad dC = -*dH^{-1}, \quad (52)$$

where $H = H(\vec{x})$, $A = A(\vec{x})$, and $K = K(\vec{x})$ are three harmonic functions that are determined by four "string-profile" functions $F_m(v)$ as follows:

$$H^{-1} = 1 + \frac{1}{2\pi}\int_0^{2\pi Q_5} \frac{dv}{|x - F(v)|^2},$$
$$K = \frac{1}{2\pi}\int_0^{2\pi Q_5} \frac{|\dot{F}|^2 dv}{|x - F(v)|^2}, \quad (53)$$
$$A_m = -\frac{1}{2\pi}\int_0^{2\pi Q_5} \frac{\dot{F}_m dv}{|x - F(v)|^2}.$$

We have added 1 to $C^{(2)}_{t5}$ to be consistent with our conventions where the energy of a probe D string infinitely far away from the parent stack of D1-D5 branes is zero. Comparing conventions with Table I, we see that the parameter g has been absorbed into an additive shift of the dilaton and is set to 1.

The vector $\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_5}$ is null and we choose our solutions so that this vector is always tangent to the D-string world volume. We may apply the formalism of Sec. III A here to obtain

$$P_t = -\frac{1}{2\pi\alpha'}(e^{-\phi}G_{tM} - C^{(2)}_{tM})(X^M)' - \mathbf{n}_t\gamma,$$
$$P_5 = -\frac{1}{2\pi\alpha'}(e^{-\phi}G_{5M} - C^{(2)}_{5M})(X^M)' - \mathbf{n}_5\gamma, \quad (54)$$

where we have defined $\gamma = \frac{(X')^2}{X'\cdot\dot{X}}$. We now only need to notice that $\mathbf{n}_t + \mathbf{n}_5 = 0$, $e^{-\phi}G_{55} - C^{(2)}_{t5} = 0$, $e^{-\phi}(G_{tm} + G_{5m}) + (C^{(2)}_{tm} + C^{(2)}_{5m}) = 0$ to see that the BPS condition $P_t + P_5 = 0$ is satisfied.

We comment on the relation of these geometries to global AdS in Sec. III E 1.

E. Supersymmetric solutions in global AdS

We now consider a probe D1 string propagating in global $AdS_3 \times S^3 \times M_{\rm int}$. This geometry is described in Table II. In particular, the metric is

$$ds^2 = G_{MN}dx^M dx^N = g\sqrt{\frac{Q_1 Q_5}{v}}\alpha'[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2] + \sqrt{\frac{Q_1}{Q_5 v}}\alpha' ds^2_{\rm int}. \quad (55)$$

$ds^2_{\rm int}$ is the metric on the internal manifold. g, v, Q_1, Q_5 are parameters that determine the string coupling constant, volume of the internal manifold, and the electric and magnetic parts of the 3-form RR field strength according to the formulas summarized in Table II below. We are following the notation of [32]. We parametrize the internal manifold using the coordinate $z^{1\ldots4}$.

In terms of this coordinate system, the Killing spinor analysis of Sec. II B 4 tell us that probe branes that preserve the Killing vector

$$\mathbf{n} = \frac{\partial}{\partial t} + \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}$$

(i.e. branes that have $\mathbf{n}$ everywhere tangent to their world volume) will preserve 4 of the background 16 supersymmetries.

We can now proceed as above to obtain Table II.

1. Spectral flow

The global AdS geometry above corresponds to the NS vacuum of the boundary CFT. The geometries considered in Sec. III D correspond, on the other hand, to the different Ramond ground states of this CFT. Now, the NS sector and Ramond sector in CFT with at least (2, 2) supersymmetry are related by an operation called spectral flow, where the Virasoro generators L_n and R-symmetry current modes J_n change as follows (see, e.g., [33] for a review):

$$L_n^{\rm NS} = L_n^R + J_n^R + \frac{c}{24}\delta_{n,0}, \qquad J_n^{\rm NS} = J_n^R + \frac{c}{12}\delta_{n,0}, \quad (56)$$

and the moding of the fermions changes from integral to half-integral. c is the central charge of the theory which, for the boundary CFT, is $6Q_1Q_5$.

TABLE II. D branes in global AdS.

Geometry
$\frac{ds^2}{\alpha'} = l^2[-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2] + \sqrt{\frac{Q_1}{Q_5 v}}\frac{ds_{\mathrm{int}}^2}{\alpha'}$
$e^{-2\phi} = \frac{Q_5 v}{g^2 Q_1}, \quad l^2 = \frac{g}{\sqrt{v}}\sqrt{Q_1 Q_5}$
$\frac{G^{(3)}}{\alpha'} = \frac{*G^{(7)}}{\alpha'} = \frac{dC^{(2)}}{\alpha'} = Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 + Q_5 \sinh(2\rho) d\rho \wedge dt \wedge d\theta$
$\frac{C^{(2)}}{\alpha'} = -\frac{Q_5}{2}[(\cos 2\zeta + b) d\phi_1 \wedge d\phi_2 - (\cosh(2\rho) - 1) dt \wedge d\theta]$
BPS condition
$E - L - J_1 - J_2 = -\int(P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2}) d\sigma = 0$
Null vector tangent to world volume
$\mathbf{n}^M = \frac{\partial}{\partial t} + \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}$
Solution
$t = \tau\theta = \theta(\sigma) + \tau\rho = \rho(\sigma)$
$\zeta = \zeta(\sigma)\phi_1 = \phi_1(\sigma) + \tau\phi_2 = \phi_2(\sigma) + \tau$
$z_{\mathrm{int}}^a = z_{\mathrm{int}}^a(\sigma)$
Momenta
$\gamma = \frac{\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \zeta'^2 + \rho'^2 + \frac{1}{g\alpha' Q_5} g_{ab}^{\mathrm{int}} z^{a'} z^{b'}}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}$
$P_t = \frac{Q_5}{2\pi}[-\gamma\cosh^2\rho + \sinh^2\rho\theta']$
$P_\theta = \frac{-Q_5}{2\pi}[(-\gamma + \theta')\sinh^2\rho]$
$\tilde{P}_{\phi_1} = \frac{-Q_5}{2\pi}[(-\gamma + \phi_1')\cos^2\zeta + \frac{1}{2}(\cos 2\zeta - 1)\phi_2']$
$\tilde{P}_{\phi_2} = \frac{-Q_5}{2\pi}[(-\gamma + \phi_2')\sin^2\zeta - \frac{1}{2}(\cos 2\zeta + 1)\phi_1']$
$P_\rho = \frac{-Q_5}{2\pi}\rho'$
$P_\zeta = \frac{-Q_5}{2\pi}\zeta'$
$P_{z^a} = \frac{-1}{2\pi\alpha' g}[g_{ab}^{int} z^{b'}]$ (internal manifold)

Under spectral flow, the NS vacuum maps to the Ramond vacuum with the smallest possible $U(1)$ charge of $J_0^R = -\frac{Q_1 Q_5}{2}$. It was shown in [12] that, in the set of solutions (52), this corresponds to the profile function $F_1(v) = a\sin(wv)$, $F_2(v) = -a\cos(wv)$, $F_3(v) = F_4(v) = 0$. In our conventions, $a = \sqrt{Q_1 Q_5}$, $w = \frac{1}{Q_5}$. After choosing this profile function, we make the coordinate redefinitions

$$x_1 = a\cosh\rho\sin\zeta\cos\phi_1, \qquad x_2 = a\cosh\rho\sin\zeta\sin\phi_1,$$
$$x_3 = a\sinh\rho\cos\zeta\cos\phi_2, \qquad x_4 = a\sinh\rho\cos\zeta\sin\phi_2, \tag{57}$$

and take the near-horizon limit (i.e. drop the 1 in the harmonic functions) to obtain the metric and 3-form field strength:

$$ds^2 = \sqrt{Q_1 Q_5}[-\cosh^2\rho dt^2 + \sinh^2\rho dx_5^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta(d\phi_1 + dx_5)^2 + \sin^2\zeta(d\phi_2 + t)^2] + \sqrt{\frac{Q_1}{Q_5}} dz^i dz^i, \tag{58}$$

$$G^3 = Q_5\sinh(2\rho) dt \wedge d\theta \wedge d\rho + Q_5\sin(2\zeta) d\zeta \wedge (d\phi_1 + dx_5) \wedge (d\phi_2 + dt).$$

The dual of the "spectral flow" (56) on the boundary in supergravity is the coordinate redefinition [12]

$$t_{\mathrm{NS}} = t_{\mathrm{R}}, \qquad \theta_{\mathrm{NS}} = (x_5)_{\mathrm{R}}, \qquad (\phi_1)_{\mathrm{NS}} = (\phi_1)_{\mathrm{R}} + (x_5)_{\mathrm{R}},$$
$$(\phi_2)_{\mathrm{NS}} = (\phi_2)_{\mathrm{R}} + t_{\mathrm{R}}. \tag{59}$$

Under this mapping the solution above turns into global AdS. Moreover, going around the θ circle, once in the NS sector, causes us to also go around the $(\phi_1)_{\mathrm{NS}}$ circle to stay at constant $(\phi_1)_{\mathrm{R}}$. Hence, fermions which are antiperiodic in the NS sector become periodic in the R sector. One may also check that the coordinate transformation above takes

$$\frac{\partial}{\partial t_{\mathrm{R}}} + \frac{\partial}{\partial(x_5)_{\mathrm{R}}} = \frac{\partial}{\partial t_{\mathrm{NS}}} + \frac{\partial}{\partial\theta_{\mathrm{NS}}} + \frac{\partial}{\partial(\phi_1)_{\mathrm{NS}}} + \frac{\partial}{\partial(\phi_2)_{\mathrm{NS}}}. \tag{60}$$

Thus this mapping maps the null Killing vector $\mathbf{n}$ of the Ramond sector to the special null Killing vector $\mathbf{n}$ of the NS sector. It also takes us from solutions that satisfy $E - L = 0$ to solutions that satisfy $E - L - (J_1 + J_2) = 0$.

This one-to-one mapping between global AdS and the corresponding Lunin-Mathur solution implies that everything that we say below regarding probes in global AdS is also true (with appropriate redefinitions) for probes in this Lunin-Mathur geometry.

2. Bound states

The probe solutions in global AdS above have a salient feature that we wish to point out. Consider a D string near the boundary of AdS. Such a string can have finite energy only if the flux through the string almost cancels its tension. Hence, it must wrap the θ direction, and we can use our freedom to redefine σ to set $\theta' = w$. For such a string, if we take the strict $\rho \to \infty$ limit, we obtain

$$E - L = \frac{Q_5}{2\pi}\int \gamma d\sigma = \frac{Q_5}{2\pi}\int\left[\frac{\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \rho'^2 + G_{ab}X^{a'}X^{b'}}{\cos^2\zeta\phi_1' + \sin^2\zeta\phi_2' + \sinh^2\rho\theta'}\right]d\sigma = Q_5 w. \tag{61}$$

Thus, we notice that, for strings stretched close to the boundary, the quantity $E - L$ must be quantized in units of Q_5. If we wish to have intermediate values of $E - L$, our strings are "bound" to the center of AdS. In other words, the moduli space of solutions with a value of $E - L$ other than $Q_5 w$ does not include these long strings. This leads us to believe that, quantum mechanically, the quantization of these solutions would lead to discrete states and not states in a continuum. This expectation is validated by the analysis of [17].

The "spectral flow" operation discussed above tells us that a similar statement holds in the geometry described by (58). There, what must be quantized in units of Q_5 is the quantity $J_1 + J_2$. On the other hand, if we consider the near horizon of the D1-D5 geometry [see (121)], which is the zero mass BTZ black hole, we find that the various momenta become independent of the radial direction. This means that, in that background, all probes can escape to infinity. This implies that "averaging" over different Ramond vacua to obtain the zero mass BTZ black hole washes out the interesting structure of "bound states" that we see above.

Returning now to probes in global AdS, those probes that do not wrap the θ direction cannot go to $\rho \to \infty$, yet their energy shows an interesting ρ dependence. Consider the following solution (parametrized by $w, \rho_0, \zeta_0, \phi_{1_0}, \theta_0$):

$$t = \tau, \qquad \theta(\sigma) = \theta_0, \qquad \rho(\sigma) = \rho_0, \qquad \zeta(\sigma) = \zeta_0, \\ \phi_1(\sigma) = \phi_{1_0}, \qquad \phi_2(\sigma) = w\sigma. \tag{62}$$

For this solution (using $w > 0$ which is necessary for supersymmetry)

$$E = Q_5 w\cosh^2(\rho_0), \qquad L = Q_5 w\sinh^2(\rho_0), \\ P_{\phi_1} = Q_5 w, \qquad P_{\phi_2} = 0. \tag{63}$$

In this subsector, a given set of charges fixes ρ_0:

$$\sinh^2\rho_0 = \frac{L}{wQ_5}. \tag{64}$$

The fact that the size of the bound state is larger for smaller w is intuitively obvious; e.g. the size of an electron orbit is inversely proportional to its mass.

Equation (64) leads to an interesting result. The extremal BTZ black hole [34] has a horizon radius:

$$\sinh^2\rho_h = 4MG = 4JG/l. \tag{65}$$

Using the values of various constants appearing in the above equation (cf. [35], p. 8),

$$l = 2\pi\alpha'\sqrt{g}(Q_1Q_5)^{1/4}V^{-1/4}, \\ G^{-1} = 2(Q_1Q_5)^{3/4}V^{1/4}/(\pi\alpha'\sqrt{g}), \tag{66}$$

we get for the radius of the horizon

$$\sinh^2\rho_h = \frac{J}{Q_1Q_5}. \tag{67}$$

We now make the following identifications:

Probe configuration	BTZ
L	J
w	Q_1
E	$lM + 1$

We find that the horizon radius (67) exactly coincides with the size of the bound state, (64), under the above identifications (the third identification, of energies, follows from the second one; the extra "1" on the BTZ side owes to the mass convention used by [34] in which AdS_3 space has mass $-1/l$).

The above agreement would appear to suggest an interpretation of the BTZ black hole as an ensemble of bound states of Q_1 D-string probes rotating around the center of the global AdS_3 background at a coordinate distance ρ_h, given by (67). Since the AdS_3 background itself is "made of" Q_1 D strings and Q_5 D5 branes, the above configuration is well beyond the domain of validity of the probe approximation,[7] and the above interpretation should be regarded as tentative. Note that probe configurations with $w < Q_1$ have a size *larger than* the black-hole radius

[7]This is similar to the situation with N dual giant gravitons in $AdS_5 \times S^5$ background, at a fixed value of the global radius ρ.

$$w < Q_1 \Rightarrow \rho_0 > \rho_h, \tag{68}$$

and, therefore, do not form a black hole.[8] The backreacted geometry corresponding to such probe configurations is likely to be some smooth nonsingular configuration. The maximum allowed value of $w(= Q_1)$ corresponds precisely to a threshold for black-hole formation ($\rho_0 = \rho_h$).

3. Classical lower bound of energy

It can be shown (see Appendix B) that, in global AdS, the set of solutions that we have described above has an "energy gap."

$$E = -\int P_t d\sigma \geq Q_5. \tag{69}$$

IV. CHARGE ANALYSIS: D1-D5 BOUND-STATE PROBES

We now consider D5 branes with gauge fields on their world volume. Supersymmetric probes of this kind were discussed in Sec. II B 5. The embedding for such branes is given by (35) and the gauge fields $A_i(\sigma)$ are of the form that gives rise to (32),

$$F = F_{\sigma i} d\sigma \wedge dz^i + \tfrac{1}{2} F_{ij} dz^i \wedge dz^j, \tag{70}$$

with the self-duality requirement (33)

$$F_{ij} = \epsilon^{kl}{}_{ij} F_{kl}. \tag{71}$$

In this section we will obtain two results. First, we will verify the analysis of Sec. II B 5 by a charge analysis and confirm that the above configurations are indeed supersymmetric. Next, we will show that the canonical structure on the space of supersymmetric solutions of the $5 + 1$ dimensional world-volume theory of coincident D5 branes is identical to the canonical structure on the set of supersymmetric solutions to a $1 + 1$ dimensional theory. For a probe comprising p D1 branes and q D5 branes, this effective $1 + 1$ dimensional theory is the theory of a D string propagating in the geometries discussed above but with the internal manifold T^4 or $K3$ replaced by the instanton moduli space of p instantons in a $U(q)$ theory on T^4 (or $K3$). This is similar to the result [16,36,37] (see, e.g., [33] for a review) that the world-volume theory of supersymmetric D5 branes in flat space flows, in the IR, to the sigma model on the instanton moduli space. However, our result here is for D5 branes in *curved backgrounds* (discussed in Sec. II B 5) and, furthermore, the result holds (as we will see below) as long as the DBI description is valid and *we do not need to go to the IR fixed point*.

[8]This is similar to the situation with a star, e.g. the sun, whose size is larger than its Schwarzschild radius and hence does not form a black hole.

A. Classical supersymmetric bound-state solutions

We consider, first, a single D5 brane.[9] Our background has both a 3-form flux $G^{(3)} = dC^{(2)}$ and a 7-form flux $G^{(7)} = *G^{(3)} = dC^{(6)}$. In all the examples we will consider, it is possible to define a new 2-form $C'^{(2)}$ such that

$$C^{(6)} = C'^{(2)} \wedge dz^1 \wedge \ldots \wedge dz^4. \tag{72}$$

Using this notation, the DBI action becomes

$$\begin{aligned} S &= \int \mathcal{L} d\sigma d\tau \prod_i dz^i \\ &= -\frac{1}{(2\pi)^5 \alpha'^3} \int e^{-\phi} \sqrt{-\det[D_{\alpha\beta}]} \\ &\quad + \frac{1}{(2\pi)^5 \alpha'^3} \bigg[\int C^{(2)} \wedge \frac{1}{2!} F \wedge F \\ &\quad + \int C'^{(2)} \wedge dz^1 \wedge \ldots \wedge dz^4 \bigg], \\ D_{\alpha\beta} &= h_{\alpha\beta} + F_{\alpha\beta}, \end{aligned} \tag{73}$$

where as usual $h_{\alpha\beta}$ is the pullback of the string-frame metric to the world volume, $F_{\alpha\beta} = \partial_{[\alpha} A_{\beta]}$ is the 2-form field strength, and A_α is the gauge potential. It is important to note that we have normalized F unconventionally, which accounts for the absence of the usual $2\pi\alpha'$ factor. We have written the action in terms of forms to lighten the notation, but in indices $C^{(2)} = \frac{1}{2} C^{(2)}_{MN} dX^M \wedge dX^N$.

We will now formally assume that F is of the form (70) and write

$$D_{\alpha\beta} = \begin{pmatrix} 0 & h_{\tau\sigma} & 0 & 0 & 0 & 0 \\ h_{\tau\sigma} & h_{\sigma\sigma} & F_{\sigma 1} & F_{\sigma 2} & F_{\sigma 3} & F_{\sigma 4} \\ 0 & -F_{\sigma 1} & e^\phi/g & F_{12} & F_{13} & F_{14} \\ 0 & -F_{\sigma 2} & -F_{12} & e^\phi/g & F_{14} & -F_{13} \\ 0 & -F_{\sigma 3} & -F_{13} & -F_{14} & e^\phi/g & F_{12} \\ 0 & -F_{\sigma 4} & -F_{14} & +F_{13} & -F_{12} & e^\phi/g \end{pmatrix}, \tag{74}$$

where we have assumed an internal T^4 with a metric $ds^2_{T^4} = \frac{e^\phi}{g} \sum_i dz^i dz^i$ and the embedding (36) or (37).

The determinant of this matrix is

$$\begin{aligned} \sqrt{-|D|} &= h_{t\sigma}\left(\beta^2 + \frac{F_{ij}F^{ij}}{4}\right) \equiv h_{t\sigma}\left(\beta^2 + \frac{|F|^2}{2}\right), \\ \beta &= \frac{e^\phi}{g}. \end{aligned} \tag{75}$$

[9]We will eventually be interested in the instanton moduli space only for $q > 1$ D5 branes since the $q = 1$ case is rather subtle [16]. However, we include the calculations for $q = 1$ here for simplicity. The generalization to $q > 1$, which is straightforward, is left to Sec. IV C.

Note that

$$|F|^2 dz^1 \wedge \ldots \wedge dz^4 = F \wedge F. \tag{76}$$

The field strength F is derived from the gauge fields A_i via $F_{\alpha\beta} = \partial_{[\alpha} A_{\beta]}$. Note that the A_i have components only along the internal manifold. Let us suppose that there are solutions to (71) characterized by "moduli" ζ^a (the solutions we are interested in exist, actually, for $q > 1$, so the calculations in this section and the next are to be understood in a formal sense until we apply these to $q > 1$ in Sec. IV C). We can assign σ dependence to these moduli consistent with Gauss's law [29] and supersymmetry, and thus

$$A_i(\sigma) = A_i(\zeta^a(\sigma)). \tag{77}$$

Although the moduli can vary as functions of σ, supersymmetry implies that they cannot depend on τ.

To calculate the momenta, we will need the inverse of D. We have listed the relevant components of the inverse in Appendix A 1. Using these, we find

$$\begin{aligned}
P_M &= \frac{\delta \mathcal{L}}{\delta \dot{X}^M} \\
&= \frac{-e^{-\phi}}{(2\pi)^5 \alpha'^3} \bigg(\sqrt{-D} \frac{D^{\tau\beta} + D^{\beta\tau}}{2} G_{MN} \partial_\beta X^N \\
&\quad - e^{\phi} \partial_\sigma X^N \bigg(C^{(2)}_{MN} \frac{|F|^2}{2} + C'^{(2)}_{MN} \bigg) \bigg) \\
&= \frac{-e^{-\phi}}{(2\pi)^5 \alpha'^3} \bigg[\bigg(\bigg(\beta^2 + \frac{|F|^2}{2} \bigg) G_{MN} - \frac{e^{\phi} C^{(2)}_{MN} |F|^2}{2} \\
&\quad - e^{\phi} C'^{(2)}_{MN} \bigg) \partial_\sigma X^N - \frac{\beta F_{\sigma i} F^i_\sigma + h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{h_{\tau\sigma}} \mathbf{n}_M \bigg], \\
P_{Ai} &= \frac{\delta \mathcal{L}}{\delta \partial_\tau A_i} = -\frac{e^{-\phi}}{(2\pi)^5 \alpha'^3} \sqrt{-D} \frac{D^{\tau i} - D^{i\tau}}{2} = \frac{e^{-\phi} \beta F_{\sigma i}}{(2\pi)^5 \alpha'^3} \\
&= \frac{1}{(2\pi)^5 \alpha'^3 g} \frac{\partial A_i}{\partial \zeta^\alpha} \frac{\partial \zeta^\alpha}{\partial \sigma}.
\end{aligned} \tag{78}$$

In the equation above, M, N run over $0 \ldots 5$. To obtain the conserved charges of the action (73), we need to integrate the momenta above over all six world-volume coordinates. We now proceed to show that a D5 brane that keeps the vector $\mathbf{n}^M$ of Sec. II tangent to its world volume at all points and has a world-volume field strength of the form (32) is supersymmetric in the four backgrounds that we have discussed.

1. D1-D5 background

We will discuss the D1-D5 background in some detail. The calculations required to verify supersymmetry in other backgrounds are almost identical, so we will be brief in later subsections.

In the D1-D5 background of Table I,

$$\begin{aligned}
\frac{G^{(3)}}{\alpha'} &= Q_5 \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_1}{v f_1^2 r^3} dr \wedge dt \wedge dx_5, \\
\frac{C^{(2)}}{\alpha'} &= -\frac{Q_5}{2} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_1 \alpha'} dt \wedge dx_5, \\
\frac{G^{(7)}}{\alpha'} &= \bigg(\frac{Q_1}{v} \sin 2\zeta d\zeta \wedge d\phi_1 \wedge d\phi_2 - \frac{2Q_5}{f_5^2 r^3} dr \wedge dt \wedge dx_5 \bigg) \\
&\quad \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4, \\
\frac{C^{(6)}}{\alpha'} &= \bigg(\frac{-Q_1}{2v} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_5 \alpha'} dt \wedge dx_5 \bigg) \\
&\quad \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4.
\end{aligned} \tag{79}$$

With the definition of $C'^{(2)}$ above, we have

$$\frac{C'^{(2)}}{\alpha'} = \bigg(\frac{-Q_1}{2v} \cos 2\zeta d\phi_1 \wedge d\phi_2 + \frac{1}{g f_5 \alpha'} dt \wedge dx_5 \bigg). \tag{80}$$

Notice that, in the near-horizon limit, we find $C'^{(2)} = \frac{e^{2\phi}}{g^2} C^{(2)}$.

To check the supersymmetry condition, we explicitly calculate P_t and P_5 using (78).

$$\begin{aligned}
(2\pi)^5 \alpha'^3 P_t &= -\frac{F_{\sigma i} F^i_\sigma}{g x'_5} - \frac{e^{-\phi} h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} \\
&\quad - C^{(2)}_{5t} \bigg(\beta^2 + \frac{|F|^2}{2} \bigg) x'_5, \\
(2\pi)^5 \alpha'^3 P_5 &= \frac{F_{\sigma i} F^i_\sigma}{g x'_5} + \frac{e^{-\phi} h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} \\
&\quad - \bigg(\beta^2 + \frac{|F|^2}{2} \bigg) e^{-\phi} G_{55} x'_5,
\end{aligned} \tag{81}$$

where we have used that

$$C'^{(2)}_{5t} = \beta^2 C^{(2)}_{5t}. \tag{82}$$

Using $G_{00} = -G_{55}$ and $e^{-\phi} G_{55} + C^{(2)}_{5t} = 0$ (see Table I), we see that

$$E - L = \int (P_t + P_5) d\tau d\sigma dz^1 \ldots dz^4 = 0, \tag{83}$$

and hence, the BPS relation is satisfied.

If we integrate (78) to obtain the conserved charges, we see that in the near-horizon limit, where $C'^{(2)} = \frac{e^{2\phi}}{g^2} C^{(2)}$, the formulas for the energy, angular momentum, and other charges are almost identical in structure to Table I except that

$$\frac{1}{2\pi\alpha'} \to \frac{1}{2\pi\alpha'} \bigg(\beta^2 v + \frac{1}{32\pi^4 \alpha'^2} \int |F|^2 d^4 z^i \bigg). \tag{84}$$

Hence, turning on the gauge fields simply renormalizes the tension according to the "instanton number" (38).[10] This equation is the precursor to the more general (102).

2. D1-D5-P geometry

The discussion for the D1-D5-P geometry specified by Eq. (50) is almost identical to the one above. The only modification is that we fincd

$$\begin{aligned}(2\pi)^5\alpha'^3 P_t &= -\frac{F_{\sigma i}F^i_\sigma}{g x'_5} - \frac{e^{-\phi}h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} \\ &\quad - (C^{(2)}_{5t} + e^{-\phi}G_{5t})\Big(\beta^2 + \frac{|F|^2}{2}\Big)x'_5, \\ (2\pi)^5\alpha'^3 P_5 &= \frac{F_{\sigma i}F^i_\sigma}{g x'_5} + \frac{e^{-\phi}h_{\sigma\sigma}(\beta^2 + \frac{|F|^2}{2})}{x'_5} \\ &\quad - \Big(\beta^2 + \frac{|F|^2}{2}\Big)e^{-\phi}G_{55}x'_5.\end{aligned} \tag{85}$$

In the new background (50), we have $e^{-\phi}(G_{55} + G_{5t}) + C^{(2)}_{5t} = 0$. Hence, the BPS relation follows.

3. Lunin-Mathur geometries

To check the BPS condition for bound-state probes in the Lunin-Mathur geometries, we need to derive an expression for $C'^{(2)}$ which is defined by (72). At first sight, this may seem a formidable task, but the result is quite intuitive. In Appendix D 4 we show that $C'^{(2)}$ is obtained by taking $C^{(2)}$ in (52) and performing the substitution $H \leftrightarrow \frac{1}{1+K}$. So

$$\begin{gathered} C'^{(2)}_{tm} = -B_m H, \qquad C'^{(2)}_{t5} = H, \qquad C'^{(2)}_{m5} = HA_m, \\ C'^{(2)}_{mn} = C'_{mn} + H(A_m B_n - A_n B_m), \qquad dB = -*dA, \\ dC' = -*d(1+K). \end{gathered} \tag{86}$$

Now, we only need to notice that $C'^{(2)}_{tM} = \beta^2 C^{(2)}_{tM}$, $C'(2)_{5M} = \beta^2 C^{(2)}_{5M}$, $\forall M$[11] and repeat the argument for the D1-D5 system above to see that $P_t + P_5 = 0$.

4. Global AdS

The analysis with gauge fields turned on in the D5-brane world volume is almost identical to the analysis in the full D1-D5 background. Here, we find

$$\begin{aligned}\frac{C'^{(2)}_{\rm global}}{\alpha'} &= \frac{e^{2\phi}}{g^2}\frac{C^{(2)}_{\rm global}}{\alpha'} \\ &= -\frac{Q_1}{2v}[\cos 2\zeta d\phi_1 \wedge d\phi_2 - (\cosh(2\rho) - 1)dt \wedge d\theta].\end{aligned} \tag{87}$$

To check the BPS condition, let us use formula (78) to write down the momenta in the t, θ, ϕ_1, ϕ_2 directions. In analogy to the analysis for the D string, we define

$$\gamma_1 = \frac{\frac{1}{g}F_{\sigma i}F^i_\sigma + Q_5\alpha'(\beta^2 + \frac{|F|^2}{2})(\sinh^2\rho\theta'^2 + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 + \zeta'^2 + \rho'^2)}{\cos^2\zeta\phi'_1 + \sin^2\zeta\phi'_2 + \sinh^2\rho\theta'}. \tag{88}$$

With this definition, we find the momenta

$$\begin{aligned} (2\pi)^5\alpha'^3 P_t &= -\gamma_1\cosh^2(\rho) + Q_5\alpha'\theta'\sinh^2(\rho)(\beta^2 + \tfrac{1}{2}|F|^2), \\ (2\pi)^5\alpha'^3 P_\theta &= \gamma_1\sinh^2(\rho) - Q_5\alpha'\theta'\sinh^2(\rho)(\beta^2 + \tfrac{1}{2}|F|^2), \\ (2\pi)^5\alpha'^3 \tilde{P}_{\phi_1} &= \gamma_1\cos^2\zeta - Q_5\alpha'(\beta^2 + \tfrac{1}{2}|F|^2)(\cos^2\zeta\phi'_1 - \sin^2\zeta\phi'_2)\phi'_2, \\ (2\pi)^5\alpha'^3 \tilde{P}_{\phi_2} &= \gamma_1\sin^2\zeta + Q_5\alpha'(\beta^2 + \tfrac{1}{2}|F|^2)(\cos^2\zeta\phi'_1 - \sin^2\zeta\phi'_2)\phi'_1, \\ P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2} &= 0, \end{aligned} \tag{89}$$

which verifies the BPS relation.

B. Obtaining an effective 2 dimensional action

The space of supersymmetric solutions above gives us a description of the supersymmetric sector of the classical phase space of the world-volume theory defined by the action (73). Each solution corresponds to a point in this phase space. Now, the action (73) gives rise to a canonical symplectic structure on this phase space. This structure may be encapsulated in terms of a *symplectic form*. See, for example, [38] for details of this construction. We will return to this formalism again in Sec. VI. We will now show that the classical symplectic structure on the space of supersymmetric solutions above is identical to the symplectic structure on the space of supersymmetric solutions of a 1 + 1 dimensional theory. This 1 + 1 dimensional theory will be like the theory of the D string studied in Sec. III but propagating on a different space, where the internal manifold has been replaced by the instanton moduli space. Furthermore, we will find that the tension of this

[10]This will become the *real* instanton number for $q > 1$ in Sec. IV C.

[11]As we mentioned earlier, the conventions of [12] differ slightly from [32] and g has been absorbed into a shift of ϕ. So, here $\beta = e^{\phi}$.

string is renormalized by a factor determined by the instanton number.

First consider the gauge fields. Recall that, in (78), we found that

$$p_{Ai} = \frac{1}{(2\pi)^5 \alpha'^3 g} \frac{\partial A_i}{\partial \zeta^\alpha} \frac{\partial \zeta^\alpha}{\partial \sigma}. \tag{90}$$

The *symplectic structure* on the manifold of solutions may be written in terms of the symplectic form:

$$\Omega = \int \delta p_{Ai} \wedge \delta A_i d\sigma d^4 z^i, \tag{91}$$

where δ is an exterior derivative on the space of all solutions. δA_i is then a 1-form in the cotangent space at the point in phase space specified by the function A_i, and the wedge product is taken in this cotangent space.

The A_i are given as a function of the moduli ζ^a by (77). We can then rewrite (91) as

$$\Omega = \frac{1}{(2\pi)^5 \alpha'^3 g} \int \delta\Big(\int d^4 z^i \frac{\partial A_i}{\partial \zeta^a} \frac{\partial A_i}{\partial \zeta^b} \zeta'^a\Big) \wedge \delta \zeta^b. \tag{92}$$

If we define a metric on instanton moduli space,

$$g_{ab}^{\mathrm{inst}} = \frac{1}{(2\pi\sqrt{\alpha'})^4} \int d^4 z^i \frac{\partial A_i}{\partial \zeta^a} \frac{\partial A_i}{\partial \zeta^b}, \tag{93}$$

then this is exactly the symplectic structure of the left-moving sector $[(\zeta^a)'(\sigma,\tau) = \dot{\zeta}^a(\sigma,\tau)]$ of the nonlinear sigma model on the instanton moduli space defined by

$$S_{\mathrm{inst}} = \frac{1}{4\pi\alpha' g} \int g_{ab}^{\mathrm{inst}} (\dot{\zeta}^a \dot{\zeta}^b - (\zeta^a)'(\zeta^b)') d\sigma d\tau. \tag{94}$$

What about the contribution of the gauge fields to the spacetime Hamiltonian? From formula (78) and the expressions in (A1), we see that the gauge field momenta enter the expression for the spacetime energy only through

$$\frac{1}{(2\pi)^5 \alpha'^3} \int d^4 z^i d\sigma \frac{F_{\sigma i} F_\sigma^i}{g} = \frac{1}{2\pi\alpha' g} \int d\sigma g_{ab}^{\mathrm{inst}} \zeta'^a \zeta'^b.$$

This is exactly the Hamiltonian of the "left-moving" sector of the nonlinear sigma model (94).

Finally, we would like to write down an effective action that generates the symplectic structure above, both in the D1-D5 system and in global AdS. To do this, first we formally extend our spacetime, by excising the coordinates on the internal manifold and including coordinates on the instanton moduli space. We now define a metric and B field on this extended space as follows:

$$\begin{aligned}
\chi^m &= \begin{pmatrix} X^M \\ \zeta^a \end{pmatrix}, \\
\mathcal{G}^1_{mn} &= \begin{pmatrix} e^{-\phi}(\beta^2 v + \int d^4 z^i \frac{|F|^2}{8\pi^2 (2\pi\alpha')^2}) G_{MN} & 0 \\ 0 & \frac{g_{ab}^{\mathrm{inst}}}{g} \end{pmatrix}, \\
\mathcal{B}^1 &= \Big(C'^{(2)}_{MN} v + C^{(2)}_{MN} \int d^4 z^i \frac{|F|^2}{8\pi^2 (2\pi\alpha')^2}\Big) dX^M \wedge dX^N, \\
\mathcal{H}^1_{\alpha\beta} &= \mathcal{G}^1_{mn} \partial_\alpha \chi^m \partial_\beta \chi^n.
\end{aligned} \tag{95}$$

In the equation above, M, N run over $0\ldots5$; a, b run over the coordinates of the instanton moduli space; m, n run over both these ranges; and α, β range over σ, τ. Now, consider a sector with a fixed value of the "instanton number" $\int d^4 z^i \frac{|F|^2}{8\pi^2 (2\pi\alpha')^2}$ [see (38), and also footnote [10]]. In this sector, consider the action

$$S^1_{\mathrm{eff}} = \frac{1}{2\pi\alpha'} \int (-\det[\mathcal{H}^1])^{1/2} d\sigma d\tau + \frac{1}{2\pi\alpha'} \int \mathcal{B}^1. \tag{96}$$

If we look for supersymmetric solutions to the action above, we will find that they too have the property that

$$\frac{\partial \chi^m}{\partial \tau} = \mathbf{n}^m, \tag{97}$$

where we have extended the Killing vector field $\mathbf{n}^M$ of the previous section to this extended space in the natural way by setting its components along $\frac{\partial}{\partial \zeta^a}$ to zero. On *these* solutions, the spacetime momenta derived from the action above reproduce the momenta (78). Together with (92) this tells us the symplectic structure on supersymmetric solutions to the action (73) is the same as the symplectic structure on supersymmetric solutions to the action (96). The superscript 1 above indicates that this analysis is valid for a single D5 brane. The formula above is very suggestive and has a natural non-Abelian extension that we now proceed to discuss.

C. Non-Abelian extensions

The analysis in the last two subsections was valid for a single D5 brane. It is easy to generalize the salient results to q D5 branes for $q > 1$. Again, we consider a sector with fixed

$$p = \frac{1}{(2\pi\sqrt{\alpha'})^4} \int_{T^4} \frac{\mathrm{Tr}(F \wedge F)}{2}. \tag{98}$$

p is now a bona-fide instanton number. In this sector consider the following natural extension to the effective quantities above given by (95):

$$\chi^m = \begin{pmatrix} X^M \\ \zeta^a \end{pmatrix},$$
$$\mathcal{G}^{p,q}_{mn} = \begin{pmatrix} e^{-\phi}(q\beta^2 v + p)G_{MN} & 0 \\ 0 & \frac{g^{\rm inst}_{ab}}{g} \end{pmatrix}, \qquad (99)$$
$$\mathcal{B}^{p,q} = (qC'^{(2)}_{MN} v + C^{(2)}_{MN} p) dX^M \wedge dX^N,$$
$$\mathcal{H}^{p,q}_{\alpha\beta} = \mathcal{G}^{p,q}_{mn}\partial_\alpha \chi^m \partial_\beta \chi^n.$$

ζ^a span the moduli space of p instantons in a $U(q)$ theory. We can define an effective 2 dimensional action for each such value of p, q as

$$S^{p,q}_{\rm eff} = \frac{1}{2\pi\alpha'}\int (-\det[\mathcal{H}^{p,q}])^{1/2} + \frac{1}{2\pi\alpha'}\int \mathcal{B}^{p,q}. \qquad (100)$$

Remarkably, we have found that we can now apply the entire machinery of Sec. III (which we developed for D1 branes) to bound states of D1 and D5 branes.

This result takes an especially pretty form in the near horizon of the D1-D5 and D1-D5-P system and global AdS. Recall that, for these scenarios,

$$C'^{(2)}_{MN} = \beta^2 C^{(2)}_{MN} = \frac{Q_1}{Q_5 v} C^{(2)}_{MN}. \qquad (101)$$

The formula (99) then tells us that, in the near horizon of the D1-D5 system and in global AdS (and in the corresponding Ramond sector, LM geometry), the formulas for the canonical momenta in Tables I and II are *quantitatively* correct with the following substitutions:

(1) The internal manifold is replaced by the instanton moduli space of p instantons in a $U(q)$ theory.

(2) The tension of the "string" is renormalized by $Q_5 \to pQ'_5 + qQ'_1$. Here Q'_5 is the D5 charge of the background in Tables I and II which must be taken to be $Q_5 - q$ in case the D5 charge of the probe is q (so that the total charge at the boundary is kept fixed at Q_5). Similarly $Q'_1 = Q_1 - p$. Thus

$$Q_5 \to p(Q_5 - q) + q(Q_1 - p). \qquad (102)$$

V. MOVING OFF THE SPECIAL POINT IN MODULI SPACE

We can generalize the simplest D1-D5 system that we have been discussing by turning on a bulk anti-self-dual $B_{\rm NS}$ field in the background geometry.[12] This is like turning on some dissolved D3 brane charge in the background that we have taken, until now, to have only D1 and D5 charges. We should expect that the BPS solutions we have been discussing above no longer remain BPS, since a D1 or a D5 probe is not, in general, mutually supersymmetric with a D1-D3-D5 bound state (the exception is the system considered in Sec. II B 5). In this section, we will verify the expectation above by first performing a Killing spinor analysis and then by verifying our results using the DBI action.

A. Killing spinor analysis

The explicit extremal D1-D5 supergravity background with a nonzero $B_{\rm NS}$ field turned on was calculated in [39,40]. We will follow [39] here. In addition to this $B_{\rm NS}$ field and the usual 3-form RR field strength G, this background also has a 5-form field strength $G^{(5)}$. This solution depends on a single parameter φ that determines the strength of the anti-self-dual $B_{\rm NS}$ field. The metric, dilaton, and field strengths (adapted to our conventions regarding "self-duality," and with $\alpha' = 1$ for simplicity) may be written as follows:

$$\begin{aligned} ds^2 &= (f_1 f_5)^{-1/2}[-dt^2 + dx_5^2] \\ &\quad + (f_1 f_5)^{+1/2}(dr^2 + r^2(d\zeta^2 + \cos^2\zeta d\phi_1^2 \\ &\quad + \sin^2\zeta d\phi_2^2)) + (f_1 f_5)^{+1/2} Z^{-1}[(dx_6^2 + dx_8^2) \\ &\quad + (dx_7^2 + dx_9^2)], \\ e^{2\phi} &= \frac{f_1 f_5}{Z^2}, \\ H &= dB_{\rm NS}, \\ B^{(2)}_{\rm NS} &= \Big(Z^{-1}\sin(\varphi)\cos(\varphi)(f_1 - f_5) \\ &\quad + \frac{(\mu_5 - \mu_1)\sin\varphi\cos\varphi}{\mu_5\cos^2\varphi - \mu_1\sin^2\varphi}\Big)(dx^6\wedge dx^8 + dx^7\wedge dx^9), \\ G^{(3)} &= \cos^2(\varphi)\tilde{K}^{(3)} - \sin^2(\varphi)K^{(3)}, \\ G^{(5)} &= Z^{-1}\cos\varphi\sin\varphi(+f_5 K^{(3)} + f_1\tilde{K}^3) \\ &\quad \wedge (dx^6\wedge dx^8 + dx^7\wedge dx^9), \end{aligned} \qquad (103)$$

where we defined

$$\begin{aligned} f_1 &= 1 + \frac{\mu_1}{r^2} \\ f_5 &= 1 + \frac{\mu_5}{r^2}, \\ \tilde{K}^{(3)} &= -\frac{f_1'}{f_1^2} dr\wedge dx^0 \wedge dx_5 + \mu_5 \sin(2\zeta) d\zeta\wedge d\phi_1\wedge d\phi_2, \\ K^{(3)} &= -\frac{f_5'}{f_5^2} dr\wedge dx^0 \wedge dx_5 + \mu_1 \sin(2\zeta) d\zeta\wedge d\phi_1\wedge d\phi_2, \\ Z &= 1 + \frac{\mu_1\sin^2(\varphi) + \mu_5\cos^2\varphi}{r^2}. \end{aligned} \qquad (104)$$

μ_1, μ_5 are parameters that determine the charges of the system according to the formulas in [39]. We alert the reader that our normalizations for μ_1, μ_5 differ from that paper by a factor of 2.

[12] Our conventions regarding "self-dual" and "anti-self-dual" are the opposite of [16,37,39].

We start by calculating the bulk Killing spinors that this geometry preserves. As explained earlier, the supersymmetries of the type IIB theory may be written in terms of a two-component spinor

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{105}$$

which satisfies $\Gamma^{11}\epsilon = -\epsilon$. The dilatino Killing spinor equation is (see [41] and references therein)

$$\Big[\partial_M \phi \Gamma^M + \frac{1}{12} H_{MAB}\Gamma^{MAB} \otimes \sigma_3 + \frac{1}{4} e^{\phi} \sum_{n=1}^{5} \frac{(-1)^{n-1}(n-3)}{(2n-1)!} G_{A_1\ldots A_{2n-1}}\Gamma^{A_1\ldots A_{2n-1}} \otimes \lambda_n\Big]\epsilon = 0, \tag{106}$$

where $\lambda_n = \sigma_1$ for n even, and $\lambda_n = i\sigma_2$ for n odd. The $\{\sigma_i\}$, $i = 1, 2, 3$ are the Pauli matrices. H and G are the NS-NS and RR field strengths, and ϕ denotes the dilaton. Our conventions are slightly different from [41] because the solution of (103) has $G_7 = *G_3$ and $G_5 = - * G_5$.

The spinors above are defined with respect to a particular local Lorentz frame. In our case, a convenient basis is defined by the following 1-forms.

$$e^{\hat t} = (f_1 f_5)^{-(1/4)} dt, \qquad e^{\hat 5} = (f_1 f_5)^{-(1/4)} dx_5,$$
$$e^{\hat r} = (f_1 f_5)^{1/4} dr, \qquad e^{\hat \zeta} = (f_1 f_5)^{1/4} r d\zeta,$$
$$e^{\hat\phi_1} = (f_1 f_5)^{1/4} r \cos\zeta d\phi_1,$$
$$e^{\hat\phi_2} = (f_1 f_5)^{1/4} r \cos\zeta d\phi_2, \qquad e^{\hat a} = (f_1 f_5)^{1/4} Z^{-(1/2)} dx^a.$$

Defining spinors with respect to this local Lorentz frame, we find that the dilatino equation becomes

$$\Big[f_1^{-5/4} f_5^{-1/4} f_1' \Gamma^{\hat r}\Big(\Big(1 - 2\frac{f_1}{f_5}\frac{\sin^2(\varphi)}{\alpha}\Big)\mathbb{1} - \Gamma^{\hat 0 \hat 5} \otimes \sigma_1 - B(\Gamma^{\hat 6 \hat 8} + \Gamma^{\hat 7 \hat 9}) \otimes \sigma_3\Big)\Big]\epsilon + \Big[f_5^{-5/4} f_1^{-1/4} f_5' \Gamma^{\hat r}\Big(-\Big(1 - 2\frac{f_1}{f_5}\frac{\sin^2(\varphi)}{\alpha}\Big)\mathbb{1} - \Gamma^{\hat r \hat\zeta \hat\phi_1 \hat\phi_2} \otimes \sigma_1 + B(\Gamma^{\hat 6 \hat 8} + \Gamma^{\hat 7 \hat 9}) \otimes \sigma_3\Big)\Big]\epsilon = 0, \tag{108}$$

where we defined $\alpha \equiv \cos^2(\varphi) + \frac{f_1}{f_5}\sin^2(\varphi)$, $B \equiv \sqrt{\frac{f_1}{f_5}}\frac{1}{\alpha} \times \sin(\varphi)\cos(\varphi) = \frac{\sqrt{f_1 f_5}\sin(\varphi)\cos(\varphi)}{f_5\cos^2(\varphi) + f_1 \sin^2(\varphi)}$. All products of gamma matrices above can be simultaneously diagonalized. We will denote the eigenvalues of $\Gamma^{\hat 0 \hat 5}$, $\Gamma^{\hat 6 \hat 8}$, $\Gamma^{\hat 7 \hat 9}$, $\Gamma^{\hat r \hat\zeta \hat\phi_1 \hat\phi_2}$ by $\pm n_1$, $\pm i n_2$, $\pm i n_3$, $\pm n_4$, respectively. The condition $\Gamma^{11}\epsilon = -\epsilon$ subjects these to the constraint $\prod n_1 n_2 n_3 n_4 = -1$.

Diagonalizing the matrix above is then equivalent to diagonalizing the two matrices

$$M_1 = n_1 \sigma_1 - iB(n_2 + n_3)\sigma_3,$$
$$M_2 = n_4 \sigma_1 + iB(n_2 + n_3)\sigma_3. \tag{109}$$

Both these matrices have eigenvalues $\pm\sqrt{1 - B^2(n_2 + n_3)^2}$. In particular, when $n_2 n_3 = 1 = -n_1 n_4$, there are eight spinors that simultaneously satisfy the two equations

$$(\Gamma^{\hat 0 \hat 5} \otimes \sigma_1 + B(\Gamma^{\hat 6 \hat 8} + \Gamma^{\hat 7 \hat 9}) \otimes \sigma_3)\epsilon = \frac{f_5\cos^2\varphi - f_1 \sin^2\varphi}{f_5\cos^2\varphi + f_1\sin^2\varphi}\epsilon,$$
$$(\Gamma^{\hat r \hat\zeta \hat\phi_1 \hat\phi_2} \otimes \sigma_1 - B(\Gamma^{\hat 6 \hat 8} + \Gamma^{\hat 7 \hat 9}) \otimes \sigma_3)\epsilon = -\frac{f_5\cos^2\varphi - f_1 \sin^2\varphi}{f_5\cos^2\varphi + f_1\sin^2\varphi}\epsilon. \tag{110}$$

These two equations are consistent with $\Gamma^{11}\epsilon = -\epsilon$ and satisfy Eq. (108). They also imply $\Gamma^{6789}\epsilon = \epsilon$.

Hence, we have shown that the background defined by (103) preserves eight supersymmetries that are parametrized by the projection conditions above. Notice that *none* of these spinors can be preserved by a probe D1 brane or a probe D5 brane. For arbitrary unit tangent vectors of the world volume $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, a probe D1 brane preserves the spinors that have $\Gamma_{\hat{\mathbf{v}}_1}\Gamma_{\hat{\mathbf{v}}_2} \otimes \sigma_1 \psi = \psi$. In the 2 dimensional space specified by (105) these spinors are eigenspinors of σ_1. Hence none of them coincide with the spinors that are preserved in the background above that are eigenspinors of $\sigma_1 \pm 2iB\sigma_3$. The same argument works to show that no probe D5 branes or bound states of D1 and D5 branes can be supersymmetric in this background.

Now, consider the near-horizon limit of the geometry (103). In this limit, the equation above simplifies dramatically, and it is easy to convince oneself that the only projection that survives above is $\Gamma^{6789}\epsilon = \epsilon$. There are 16 spinors that satisfy this equation. Hence, this is consistent with the "doubling" of supersymmetries that is associated with the appearance of a conformal symmetry in the near-horizon limit. One may now naively suspect that in the near horizon a probe D string could maintain some supersymmetries.

In the superconformal algebra, there are two types of supercharges. Conventionally, these are denoted by Q, with a charge under dilatation of $+\frac{1}{2}$, and S, with a dilatation charge $-\frac{1}{2}$. Now, to be BPS, a brane must preserve some Q charges (in the superconformal algebra all primary states, whether of short representations or not are annihilated by the S's). To determine which supercharges are Q and which are S in the near horizon, we consider the $\hat r$ component of the gravitino equation in the near-horizon limit.

The gravitino equation reads

$$\left[\partial_M + \frac{1}{4}w_M^{BC}\Gamma_{BC} + \frac{1}{8}H_{MAB}\Gamma^{AB}\otimes\sigma_3 + \frac{1}{16}e^{\phi}\sum_{n=1}^{5}\frac{(-1)^{n-1}}{(2n-1)!}G_{A_1\ldots A_{2n-1}}\Gamma^{A_1\ldots A_{2n-1}}\Gamma_M\otimes\lambda_n\right]\epsilon = 0, \tag{111}$$

where w_M^{BC} is the spin connection. In the near horizon the r component of this equation is, for the background above,

$$\frac{\partial\epsilon}{\partial r} - \frac{1}{2r}\left[\Gamma^{\hat{0}\hat{5}}\frac{(\mu_5\cos^2\varphi - \mu_1\sin^2\varphi)\sigma_1 - \sqrt{\mu_5\mu_1}\cos\varphi\sin\varphi(\Gamma^{\hat{6}\hat{8}} + \Gamma^{\hat{7}\hat{9}})\otimes(i\sigma_2)}{\mu_5\cos^2\varphi + \mu_1\sin^2\varphi}\right]\epsilon = 0. \tag{112}$$

If we impose $n_2n_3 = 1$ (as the dilatino equation tells us to), the square bracket on the right has eigenvalues ±1. Somewhat more remarkably, the eigenvalue $+1$ occurs when the projection condition (110) is satisfied. This means that the Q's in the near horizon are the same as the Q's in the bulk. The new supercharges are the S's. From the argument above, we now see that a D string or a D5 brane cannot be BPS even in the near horizon. The argument for global AdS is very similar to the near-horizon argument above, and instead of repeating it here, we will proceed to verify our results using a charge analysis.

B. Charge analysis

In this section, we will use the DBI action to verify the results that we obtained above. For global AdS, we find the interesting result that there are still solutions to the equations of motion that preserve the Killing vector $\mathbf{n}$, but these solutions are no longer BPS.

We start by considering the extremal D1-D5 geometry. From the formulas in (103), we see that

$$C_{t5}^{(2)} = \frac{f_5\cos^2\varphi - f_1\sin^2\varphi}{f_1f_5}, \qquad e^{-\phi}G_{55} = \frac{Z}{f_1f_5} = \frac{f_5\cos^2\varphi + f_1\sin^2\varphi}{f_1f_5}. \tag{113}$$

We see that the ratio between the components of the $C^{(2)}$ field and the metric has been spoilt. This effect is quite general and is the same as what we should expect if we turn on a theta angle. Now, the equation of motion (47) for r receives contributions from the following terms: (1) $X^M = x_5$, $X^N = x_5$ and (2) $X^M = x_5$, $X^N = \tau$. Since, now $e^{-\phi}G_{55} + C_{5t}^{(2)} \neq 0$, the only way to force our solutions to obey these equations is to set $(x_5)' = 0$. This confirms the expectation that, in the D1-D5 geometry, the supersymmetric brane probe solutions vanish if we move on the moduli space. It is easy to repeat the argument above to show that the same result also holds true in the D1-D5-P geometry.

The situation in global AdS is more interesting. When we take the near-horizon limit of (103) and translate to global coordinates, we find the metric

$$e^{-\phi}G_{MN}dx^Mdx^N = Q_5'(-\cosh^2\rho dt^2 + \sinh^2\rho d\theta^2 + d\rho^2 + d\zeta^2 + \cos^2\zeta d\phi_1^2 + \sin^2\zeta d\phi_2^2) + dz^idz^i, \tag{114}$$

and RR 2-form components

$$C_{\phi_1\phi_2}^{(2)} = -Q_5'(1-\epsilon^2)\frac{\cos(2\zeta)}{2}, \qquad C_{t\theta}^{(2)} = Q_5'(1-\epsilon^2)\frac{\cosh(2\rho)-1}{2}, \tag{115}$$

where

$$Q_5' = \mu_5\cos^2\varphi + \mu_1\sin^2\varphi, \qquad \epsilon^2 = \frac{2\mu_1\sin^2\varphi}{\mu_5\cos^2\varphi + \mu_1\sin^2\varphi}. \tag{116}$$

The equation of motion for ρ now receives contributions from (1) $X^M = \theta, X^N = \theta$ and (2) $X^M = \theta, X^N = \tau$, while the equation of motion for ζ receives contributions from (1) $X^M = \phi_1$, $X^N = \phi_1$, (2) $X^M = \phi_2$, $X^N = \phi_2$, (3) $X^M = \phi_1$, $X^N = \phi_2$, and (4) $X^M = \phi_2$, $X^N = \phi_1$. The identities we need are

$$\begin{aligned} e^{-\phi}G_{\theta\theta} + C_{\theta t}^{(2)} &= \epsilon^2 G_{\theta\theta} = Q_5'\epsilon^2\sinh^2\rho, \\ e^{-\phi}G_{\phi_1\phi_1} + C_{\phi_1\phi_2}^{(2)} &= \frac{Q_5'}{2}(1+\epsilon^2\cos(2\zeta)), \\ e^{-\phi}G_{\phi_2\phi_2} + C_{\phi_2\phi_1}^{(2)} &= \frac{Q_5'}{2}(1-\epsilon^2\cos(2\zeta)). \end{aligned} \tag{117}$$

The equations of motion are then satisfied if

$$\sinh 2\rho\theta' = 0, \qquad \sin(2\zeta)(\phi_1' - \phi_2') = 0. \tag{118}$$

The first equation requires us to stay at a constant point in θ. The second equation requires $\phi_1' = \phi_2'$. With these constraints, one can find solutions of the form (44) to the equations of motion.

Unfortunately, these solutions do not maintain the BPS bound. Generalizing the formulas of Table II, we find that

$$P_t = \frac{-Q_5'}{2\pi}\gamma\cosh^2\rho, \qquad P_\theta = \frac{Q_5'}{2\pi}\gamma\sinh^2\rho,$$
$$\tilde{P}_{\phi_1} = \frac{Q_5'}{2\pi}\Big(\gamma\cos^2\zeta - \phi_1'\cos^2\zeta - \frac{1-\epsilon^2}{2}\cos(2\zeta)\phi_2' + \frac{1-\epsilon^2}{2}\phi_2'\Big),$$
$$\tilde{P}_{\phi_2} = \frac{Q_5'}{2\pi}\Big(\gamma\sin^2\zeta - \phi_2'\sin^2\zeta + \frac{1-\epsilon^2}{2}\cos(2\zeta)\phi_1' + \frac{1-\epsilon^2}{2}\phi_1'\Big). \tag{119}$$

Substituting $\phi_1' = w = \phi_2'$, we find that

$$E - L - J_1 - J_2 = -\int (P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2})d\sigma = Q_5'\epsilon^2 w. \tag{120}$$

So, the energy of these solutions increases as we move off the special submanifold in moduli space where the anti-self-dual NS-NS fluxes and theta angles are set to zero. Equation (120) tells us how this happens as a function of the distance in moduli space from the special submanifold.

VI. SEMICLASSICAL QUANTIZATION

The phase space of a theory is isomorphic to the space of all its classical solutions. Using the Lagrangian, we can equip this space with a symplectic form that we can invert to calculate Dirac brackets. Then, by promoting Dirac brackets to commutators, we can use the set of classical solutions to canonically quantize the theory. The advantage of this approach is that it is covariant and that it allows us to restrict attention to special sectors of phase space by identifying the corresponding sector of classical solutions.[13] This technique has a long history, and the first published reference to it, known to us, is by Dedecker [9]. Later, this was studied in [42–46] and then brought back into use in the 1980s by [38,47]. We refer the reader to [10] for a nice exposition of this method.

In this section, we will show how this procedure can be implemented for supersymmetric brane probes propagating in the *near-horizon* region of the D1-D5 system. As we explained earlier, this study has limited physical relevance because it has been argued that the extremal D1-D5 geometry is not the dual to any particular Ramond vacuum of the boundary CFT but should be thought of as an average over all Ramond vacua. In fact, even classically, we see that our probes in global AdS have the striking feature that they are generically bound to the center of AdS. On quantization we would expect these to give rise to "discrete" states. This is in sharp contrast to what we find by quantizing probes in the extremal D1-D5 background where all the states that we obtain are at the bottom of a continuum. Since the Ramond and NS sectors of the boundary theory are related by "spectral flow" on the boundary, this bolsters the argument above that the extremal D1-D5 geometry is only an "average" geometry and that we should really consider probes about the geometries described in [11–13].

Nevertheless, we include this study as an example of how these supersymmetric solutions may be quantized. A detailed study of the quantization of probes in global AdS is left to [17].

Consider the near-horizon limit of the D1-D5 system. Let us define $y = \frac{\alpha' l^2}{r}$ where l^2 is a constant defined in the next equation. In the near horizon our background is

$$ds^2 = l^2\alpha'\Big(\frac{-dt^2 + dx^2}{y^2} + \frac{dy^2}{y^2} + d\omega_3^2\Big) + \sqrt{\frac{Q_1}{Q_5 v}}ds^2_{\rm int},$$
$$e^{-2\phi} = \frac{Q_5 v}{g^2 Q_1},$$
$$G^{(3)} = Q_5\alpha'\sin(2\zeta)d\zeta\wedge d\phi_1\wedge d\phi_2 - \frac{2Q_5\alpha'}{y^3}dy\wedge dt\wedge dx_5,$$
$$C^{(2)} = \frac{-Q_5\alpha'}{2}\cos 2\zeta\, d\phi_1\wedge d\phi_2 + \frac{Q_5\alpha'}{y^2}dt\wedge dx_5,$$
$$l^2 = \frac{g}{\sqrt{v}}\sqrt{Q_1 Q_5}. \tag{121}$$

The momentum conjugate to y is

$$P_y = -\frac{Q_5}{2\pi}\frac{y'}{y^2}. \tag{122}$$

The near-horizon geometry of the background described above would have been AdS_3 in Poincare coordinates, had the D1 branes and D5 branes not been on a circle. Adding in the circle identification, we simply get the orbifold of AdS_3 by a (Poincare) shift, i.e. the zero mass BTZ black hole.

Recall from Sec. IV that we can treat all probes, D strings, or bound states of p D1 branes and q D5 branes on the same footing by performing the replacements (102)

$$Q_5 \to k = p(Q_5 - q) + q(Q_1 - p), \qquad M_{\rm int} \to \mathcal{M}_{p,q}, \tag{123}$$

where $\mathcal{M}_{p,q}$ is the instanton moduli space of p instantons in an $SU(q)$ theory.

The symplectic form, Ω, on the space of solutions is given by

$$\Omega = \int \delta P_M \wedge \delta X^M d\sigma, \tag{124}$$

where δ may be thought of as an exterior derivative in the space of solutions. Recall the discussion in Sec. II B. Apart from fixing $t = \tau$ we can use diffeomorphism invariance to set

[13] This is valid only if the symplectic form does not mix a solution that belongs to this subset with a solution that does not.

$$x_5 = w\sigma. \tag{125}$$

The formula for the spacetime energy becomes

$$\begin{aligned} E &= \frac{k}{w}\int \frac{d\sigma}{2\pi}\Big(\frac{y'^2}{y^2} + \cos^2\zeta\phi_1'^2 + \sin^2\zeta\phi_2'^2 \\ &\quad + \zeta'^2 + \frac{g^{\rm int}_{ab}(z^a)'(z^b)'}{kg\alpha'}\Big) \\ &= \frac{E_y + E_{S^3} + E_{\rm int}}{w}. \end{aligned} \tag{126}$$

Since we have fixed both t and x_5, the $\delta P_5 \wedge \delta x_5 + \delta P_t \wedge \delta t$ terms drop out of the symplectic form, which then becomes

$$\begin{aligned} \Omega &= \int (\delta P_y \wedge \delta y + \delta P_{\phi_1} \wedge \delta \phi_1 + \delta P_{\phi_2} \wedge \delta\phi_2 \\ &\quad + \delta P_\zeta \wedge \delta\zeta + \delta P^{\rm int}_i \wedge \delta x^i) d\sigma \\ &= \Omega_y + \Omega_{S^3} + \Omega_{\rm int}. \end{aligned} \tag{127}$$

Now, if we define $y = e^\rho$, we find that

$$\delta P_y \wedge \delta y = \frac{-k}{2\pi}\delta\rho' \wedge \delta\rho, \qquad E_y = \frac{k}{2\pi}\int (\rho')^2 d\sigma. \tag{128}$$

We can now expand ρ in modes

$$\rho = \frac{1}{\sqrt{2k|n|}}\rho_n \exp in\sigma. \tag{129}$$

This leads to the Dirac brackets and Hamiltonian

$$\{\rho_n, \rho_{-n}\}_{\rm D.B.} = i, \qquad n > 0, \qquad E_y = \sum_{n\in\mathbb{Z}} \frac{1}{2} n|\rho_n|^2. \tag{130}$$

We can promote these Dirac brackets to commutators to get an infinite sequence of harmonic oscillators. We can think of these oscillators as coming from the left-moving part of a free boson. Roughly, the antiholomorphic oscillators have been set to zero by supersymmetry. Moreover, the zero modes that tie the left and right movers together are also absent from the expression (130).

Now we turn to Ω_{S^3}. We can map the S^3 into an $SU(2)$ group element using

$$g = e^{i[(\phi_1-\phi_2)/2]\sigma_3} e^{i\zeta\sigma_2} e^{i[(\phi_1+\phi_2)/2]\sigma_3}. \tag{131}$$

Now, introduce light-cone coordinates on the world sheet $x^\pm = \tau \pm \sigma$. Consider the Wess-Zumino-Witten (WZW) action

$$S = \frac{-k}{4\pi}\int d^2x \, {\rm Tr}\{(g^{-1}\partial_M g)^2\} + k\Gamma^{SU(2)}_{\rm WZ}, \tag{132}$$

where $\Gamma^{SU(2)}_{\rm WZ}$ is the standard Wess Zumino term for the $SU(2)$ model [48]. The symplectic form and energy obtained from the action above by restricting to solutions that satisfy $\partial_+ g = 0$ coincide with Ω_{S^3} and E_{S^3}. Roughly speaking, we have the "left-moving" part of the $SU(2)$ WZW model.

The quantum WZW model has a current algebra, and states in its Hilbert space break up into representations of this algebra. Each representation is identified by its affine primary $[j]$ [49]. The number of affine primaries is finite and $j \in \{0, \frac{1}{2}, \ldots, \frac{k}{2}\}$. What primaries occur in the spectrum above? If we consider the limit of large k, the WZW model describes three free bosons. If we were to quantize three bosons, $X^i(\sigma, \tau)$, using the symplectic form $\int d(X^i)' \wedge dX^i$, we would project out all right-moving oscillators and all zero mode motion. This suggests that the only affine primary in the spectrum is $[0]$.

We can obtain this result another way by using the fact that the spectrum of the $SU(2)$ model comprises the affine primaries $\sum_{j=0}^{k/2}[j]_{\rm left} \times [j]_{\rm right}$. Since here we have restricted the right-moving sector to be trivial, the only left-moving primary that can occur is $[0]$.

Finally, we turn to the internal degrees of freedom that correspond to fluctuations on the internal manifold. Just as above, the symplectic form $\Omega_{\rm int}$ and energy $E_{\rm int}$ give rise to the left-moving sector of the nonlinear sigma model on $\mathcal{M}_{p,q}$. We will denote this Hilbert space, which corresponds to the holomorphic part of the trivial zero mode sector of the sigma model on $\mathcal{M}_{p,q}$ by $H^0(\mathcal{M}_{p,q})$.

To conclude, we have found that the quantization of D strings in the near horizon of the D1-D5 system yields the left-moving part of the $R \times SU(2) \times \mathcal{M}_{p,q}$ sigma model defined on a circle of length $2\pi w$. We need to sum over all w to obtain the physical spectrum.

The theory above is the Ramond sector of the theory of long strings studied in [16,50,51]. (A closely related theory was studied in [52–54].) There, it is shown how the $R \times SU(2)$ theory on the world sheet may be embedded into a *spacetime* $N = 4$ superconformal algebra with central charge $6(k-1)$. The $N = 4$ superconformal algebra on $\mathcal{M}_{p,q}$ carries over to spacetime.

It is important to note that we do not sum over spin structures in the world-sheet theory. The fermions are always in the Ramond sector. The second important feature of the spectrum above is that it is at the bottom of a continuum of nonsupersymmetric states. We can always move infinitesimally away from supersymmetry by turning on the continuous momentum modes of ρ. This means that the Hilbert space we obtained above is of measure "zero" in the full quantum theory.

VII. RESULTS AND DISCUSSION

In this paper we studied brane probes in (a) the extremal D1-D5 background, (b) the extremal D1-D5-P background, (c) the smooth geometries of Lunin and Mathur with the same charges as the D1-D5 background, and (d) global $AdS_3 \times S^3 \times T^4/K3$. In the first three backgrounds, states that satisfy $E - L = 0$ preserve the right-moving supercharges. The charge $-(E - L)$ is generated by the vector $\frac{\partial}{\partial t} + \frac{\partial}{\partial x_5}$ and we found that D strings that maintained this vector tangent to their world volume at all points preserved all right-moving supersymmetries. The three backgrounds above preserve eight supersymmetries and the supersymmetric probes preserve $\frac{1}{2}$ of these. In global $\mathrm{AdS}_3 \times S^3 \times T^4/K3$, the right-moving BPS relation is $-(E - L - J_1 - J_2) = 0$. This combination of charges is generated by the vector $\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}$ and we found that D strings that keep this vector tangent to their world volume at all points preserve four right-moving supersymmetries (this makes them 1/4 BPS in this background). This fact allowed us to parametrize all supersymmetric D-string probes in these backgrounds by their initial profiles. This result is summarized in Eq. (44).

D5 branes with self-dual gauge fields on their world volumes, that preserve the Killing vector above, are also supersymmetric. These gauge fields correspond to a dissolved D1 charge on the D5 world volume, so we interpreted supersymmetric probes of this kind as supersymmetric bound states of D1 and D5 branes. We found that these bound-state probes could be described in a unified $1 + 1$ dimensional framework described by Eqs. (99) and (100). This allowed us to treat them on the same footing as D1 branes.

In global AdS, and the corresponding Lunin-Mathur solution, the probes we found could not escape to infinity for a generic assignment of charges. This indicates that upon quantization they give rise to discrete bound states that contribute to the BPS partition function of string theory on this background. A detailed investigation of this is left to [17]. The fact that this structure of classical bound states is not seen in the extremal D1-D5 geometry provides further evidence for the argument that this background is not the correct dual to any Ramond vacuum in the boundary CFT.

In Sec. V, we showed that these supersymmetric probes vanished if we turned on an anti-self-dual NS-NS field or theta angle. This means that the BPS partition function jumps as we move off the special point in moduli space where these background moduli are set to zero. This issue is discussed further in [17]. We note that this result is similar to the result that the $\frac{1}{8}$ and $\frac{1}{16}$ BPS partition functions of $\mathcal{N} = 4$ super-Yang-Mills theory on $S^3 \times R$ jump as soon as we turn on a 't Hooft coupling but are not further renormalized [55]. Finally, in Sec. VI, we quantized the supersymmetric probes above in the near horizon of the extremal D1-D5 geometry to obtain long-string states at the bottom of a continuum of nonsupersymmetric states.

It would be interesting to find smooth supergravity solutions that correspond to the probes above. It is possible that these solutions could be generated by using the profiles we find in the programme of [11,12]. An ensemble of energetic spinning probes may be a useful representation of the BTZ black hole. An indication of this was seen in Sec. III. Now, in the probe approximation, we can have many probes moving in AdS_3 that are simultaneously supersymmetric. In global AdS our analysis indicates that these probes would all be bound to AdS and hence exist at a finite distance determined by their charges. If these probes have large values of p, q, they have many internal degrees of freedom that could give rise to a macroscopically measurable degeneracy. This suggests the interesting possibility that there may be multi-black-hole solutions in global $\mathrm{AdS}_3 \times S^3 \times T4/K3$. Similar ideas have been proposed by de Boer *et al.* [56] and Sundborg [57].

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APPENDIX A: MISCELLANEOUS TECHNICAL DETAILS

1. Inverse of the Born-Infeld matrix

The matrix D in (74) is simple to invert. We will only be interested in the first row and column, so we list those below:

$$\begin{aligned}\sqrt{-|D|}D^{\tau\alpha} = \Big\{&-\frac{\beta F_{\sigma i}F^i_\sigma + h_{\sigma\sigma}(\beta^2 + \frac{1}{2}|F|^2)}{h_{\tau\sigma}}, \beta^2 + \frac{1}{2}|F|^2, -\beta F_{\sigma 1} - F_{12}F_{\sigma 2} - F_{13}F_{\sigma 3} - F_{14}F_{\sigma 4}, F_{12}F_{\sigma 1} - \beta F_{\sigma 2} \\ &- F_{14}F_{\sigma 3} + F_{13}F_{\sigma 4}, F_{13}F_{\sigma 1} + F_{14}F_{\sigma 2} - \beta F_{\sigma 3} - F_{12}F_{\sigma 4}, F_{14}F_{\sigma 1} - F_{13}F_{\sigma 2} + F_{12}F_{\sigma 3} - \beta F_{\sigma 4}\Big\}, \\ \sqrt{-|D|}D^{\alpha\tau} = \Big\{&-\frac{\beta F_{\sigma i}F^i_\sigma + h_{\sigma\sigma}(\beta^2 + \frac{1}{2}|F|^2)}{h_{\tau\sigma}}, \beta^2 + \frac{1}{2}|F|^2, \beta F_{\sigma 1} - F_{12}F_{\sigma 2} - F_{13}F_{\sigma 3} - F_{14}F_{\sigma 4}, F_{12}F_{\sigma 1} + \beta F_{\sigma 2} \\ &- F_{14}F_{\sigma 3} + F_{13}F_{\sigma 4}, F_{13}F_{\sigma 1} + F_{14}F_{\sigma 2} + \beta F_{\sigma 3} - F_{12}F_{\sigma 4}, F_{14}F_{\sigma 1} - F_{13}F_{\sigma 2} + F_{12}F_{\sigma 3} + \beta F_{\sigma 4}\Big\}. \end{aligned} \tag{A1}$$

2. Vielbeins

In this subsection, we list our vielbein conventions for the backgrounds considered above.

a. D1-D5

The metric is given in Table I. The vielbein is defined by

$$\begin{gathered} e^{\hat{t}} = (f_1 f_5)^{-(1/4)} dt, \qquad e^{\hat{5}} = (f_1 f_5)^{-(1/4)} dx_5, \\ e^{\hat{r}} = (f_1 f_5)^{1/4} dr, \qquad e^{\hat{\zeta}} = (f_1 f_5)^{1/4} r d\zeta, \\ e^{\hat{\phi}_1} = (f_1 f_5)^{1/4} r \cos\zeta, \qquad e^{\hat{\phi}_2} = (f_1 f_5)^{1/4} r \sin\zeta, \\ e^a = \frac{e^{\phi/2}}{\sqrt{g}} dz^a. \end{gathered} \tag{A2}$$

b. D1-D5-P

The metric is given in Eq. (50). The vielbein is defined by

$$\begin{gathered} e^{\hat{t}} = (f_1 f_5)^{-1/4}\left(\left(1 - \frac{r_p^2}{r^2}\right)^{1/2} dt - \frac{\frac{r_p^2}{r^2}}{\sqrt{1 - \frac{r_p^2}{r^2}}} dx_5\right), \\ e^{\hat{5}} = (f_1 f_5)^{-1/4}\left(1 - \frac{r_p^2}{r^2}\right)^{-1/2} dx_5, \qquad e^{\hat{r}} = (f_1 f_5)^{1/4} dr, \\ e^{\hat{\zeta}} = (f_1 f_5)^{1/4} r d\zeta, \qquad e^{\hat{\phi}_1} = (f_1 f_5)^{1/4} r \cos\zeta, \\ e^{\hat{\phi}_2} = (f_1 f_5)^{1/4} r \sin\zeta, \qquad e^a = \frac{e^{\phi/2}}{\sqrt{g}} dz^a. \end{gathered} \tag{A3}$$

c. Lunin-Mathur

The metric is given by (52). The vielbein is defined by

$$\begin{aligned} e^{\hat{t}} &= \left(\frac{H}{1+K}\right)^{1/4} (dt - A_{\hat{i}} dx^{\hat{i}}), \\ e^{\hat{5}} &= \left(\frac{H}{1+K}\right)^{1/4} (dx_5 + B_{\hat{i}} dx^{\hat{i}}), \\ e^{\hat{m}} &= \left(\frac{H}{1+K}\right)^{-1/4} dx^{\hat{m}}, \\ e^{\hat{a}} &= \{H(1+K)\}^{1/4} dx^{\hat{a}}. \end{aligned} \tag{A4}$$

d. Global AdS

The metric is defined in Table II. The vielbein is defined by

$$\begin{gathered} e^{\hat{t}} = l \cosh\rho dt, \qquad e^{\hat{\theta}} = l \sinh\rho d\theta, \qquad e^{\hat{\zeta}} = l d\zeta, \\ e^{\hat{\phi}_1} = l \cos\zeta, \qquad e^{\hat{\phi}_2} = l \sin\zeta, \qquad e^{\hat{a}} = \sqrt{\frac{Q_1}{Q_5 v}} dz^a. \end{gathered} \tag{A5}$$

APPENDIX B: PROOF OF THE CLASSICAL ENERGY BOUND

We will use the notation

$$\begin{gathered} \theta' = w, \qquad \phi_1' = w_1, \qquad \phi_2' = w_2, \qquad x = \sinh^2\rho, \\ s = \sin^2\zeta, \qquad \mathcal{A}^2 = \rho'^2 + \zeta'^2 + X'^2. \end{gathered} \tag{B1}$$

In general, these quantities depend on σ.

Note that

$$E = \frac{Q_5}{2\pi} \int d\sigma f, \tag{B2}$$

where

$$\begin{aligned} f &\equiv \frac{asx + a_1 x + a_2 s + b}{c_1 x + c_2 s + d}, \\ a &= w_2^2 - w_1^2 - w(w_2 - w_1), \\ a_1 &= \mathcal{A}^2 + w^2 + w_1^2 - w w_1, \\ a_2 &= w_2^2 - w_1^2, \qquad b = \mathcal{A}^2 + w_1^2, \\ c_1 &= w, \qquad c_2 = w_2 - w_1, \qquad d = w_1. \end{aligned} \tag{B3}$$

The variables $(s, x), 0 \le s \le 1, 0 \le x < \infty$ span the rectangle $ABCD$, where

$$\begin{gathered} A = (s, x) = (0, 0), \qquad B = (1, 0), \\ C = (1, \infty), \qquad D = (0, \infty). \end{gathered} \tag{B4}$$

It is possible to prove that a function f of the form (B3) attains its minimum (with respect to the variables s, x) at one of the four vertices A, B, C, or D.

Hence the minimum value of f is

$$f_{\min} = \min\{f_A, f_B, f_C, f_D\}. \tag{B5}$$

We will assume that the w's (w, w_1, w_2) are non-negative (consistent with supersymmetry as discussed in the previous subsections). We will also assume that not all w's are simultaneously zero (so that the induced metric in (42) is nonsingular); $\mathcal{A}^2$ can be zero or nonzero.

In the generic case when the w's (w, w_1, w_2) as well as $\mathcal{A}^2$ are nonvanishing, the values of f at the four vertices are

$$\begin{aligned} f_A &= \frac{b}{d} = w_1 + \frac{\mathcal{A}^2}{w_1}, \\ f_B &= \frac{a_2 + b}{c_2 + d} = w_2 + \frac{\mathcal{A}^2}{w_2}, \\ f_C &= \frac{a + a_1}{c_1} = \frac{3w}{4} + \frac{1}{w}\left[\mathcal{A}^2 + \left(w_2 - \frac{w}{2}\right)^2\right], \\ f_D &= \frac{a_1}{c_1} = \frac{3w}{4} + \frac{1}{w}\left[\mathcal{A}^2 + \left(w_1 - \frac{w}{2}\right)^2\right]. \end{aligned} \tag{B6}$$

Note that for w, w_1, w_2, $\mathcal{A}^2$ all nonvanishing,

$$f_A \geq w_1, \qquad f_B \geq w_2, \qquad f_C \geq \tfrac{3}{2}w_2, \qquad f_D \geq \tfrac{3}{2}w_1. \tag{B7}$$

The minimum value of f_C is obtained for $w = 2w_2$, and that of f_D is obtained for $w = w_1$.

In the above discussion we worked at a fixed σ. If w, w_1, w_2, $\mathcal{A}^2$, s, x are independent of σ, the above bounds (B5) and (B7) for the function imply similar bounds for $-P_t$ [see (B2)]. Thus, suppose that the minimum value of f is f_A. In that case we get

$$E \geq w_1 Q_5. \tag{B8}$$

Now since $w_1 \equiv \phi_1' > 0$ is independent of σ, it has to be a positive integer, since

$$\int_0^{2\pi} d\sigma w_1 = \phi_1(\sigma = 2\pi) - \phi_1(\sigma = 0) = n_1 2\pi, \qquad n_1 \in Z_+. \tag{B9}$$

Note that we are considering all w's to be positive at the moment.

Thus (B8) is consistent with the bound (69) we found in the special cases.

The special cases in which some of the quantities w, w_1, w_2, $\mathcal{A}^2$ vanish can be understood as limits of (B6) or can be dealt with separately. The conclusion about the bound remains the same.

Dependence on σ.—In the most general case, w, w_1, w_2, $\mathcal{A}^2$, s, x depend on σ. It can be shown that even in this case the bound (69) for $-P_t$ is satisfied. As an example, suppose that the minimum value of f occurs at the point A for some subset I_1 of $0 \leq \sigma < 2\pi$ and the minimum switches to B in the remaining part I_2 of $0 \leq \sigma < 2\pi$. Thus

$$\frac{E}{Q_5/(2\pi)} \geq \int_{I_1} d\sigma w_1 + \int_{I_2} d\sigma w_2 \geq \int_{I_1+I_2} d\sigma w_1 = 2\pi n_1 \tag{B10}$$

since by hypothesis $w_2 > w_1$ in I_2. Here n_1 is the integer winding number of the string around ϕ_1.

Summary

We have proved in this section that the classical energy of an arbitrary supersymmetric configuration satisfies the lower bound (69). The essential reason why the bound exists, as clear from the proof above, is that supersymmetry allows only non-negative winding of the string along ϕ_1, ϕ_2, θ. Furthermore, we do not allow all the winding numbers to be zero simultaneously (so that det h remains nonzero).

APPENDIX C: GAUGE-INVARIANT NOETHER CHARGES

In this section, we address the issue of the apparent dependence of the Noether charges in Table II on the gauge choice of the 2-form potential B.

Note that, like in the case of the Dirac monopole potential A_i on S^2, the magnetic part of the 2-form potential B,

$$\frac{B_{\mathrm{mag}}}{\alpha'} = -\frac{1}{2}Q_5(\cos 2\zeta + b)d\phi_1 \wedge d\phi_2, \qquad b = \text{constant},$$

cannot be globally defined with a fixed value of b on S^3. For B to be nonsingular, we must have $b = -1$ in a neighborhood of $\zeta = \pi/2$, and $b = 1$ in a neighborhood of $\zeta = 0$.

In an overlap of such neighborhoods, we have an ambiguity in the choice of b, and we must ensure that Noether charges and BPS relations are gauge invariant.

We find below that the BPS relations are indeed written in terms of gauge-invariant Noether charges (obtained from the "gauge-invariant momenta" $\tilde{P}$ below) which are defined as follows.

$$\begin{aligned}
E - L - J_1 - J_2 &= -\int d\sigma[P_t + P_\theta + \tilde{P}_{\phi_1} + \tilde{P}_{\phi_2}] = 0, \\
\tilde{P}_{\phi_1} &:= P_{\phi_1} + \frac{(b+1)Q_5}{4\pi}\phi_2' = \left\{P_{\phi_1} - \frac{1}{2\pi\alpha'}C^{(2)}_{\phi_1\phi_2}\phi_2'\right\} - \frac{Q_5}{4\pi}[\cos 2\zeta - 1]\phi_2', \\
\tilde{P}_{\phi_2} &:= P_{\phi_2} - \frac{(b-1)Q_5}{4\pi}\phi_1' = \left\{P_{\phi_2} + \frac{1}{2\pi\alpha'}C^{(2)}_{\phi_1\phi_2}\phi_1'\right\} + \frac{Q_5}{4\pi}[\cos 2\zeta + 1]\phi_1'.
\end{aligned} \tag{C1}$$

Here the expressions in {} are the so-called "mechanical momenta" (cf. $p_i - A_i$) which are also gauge invariant, but are different from the ones ($\tilde{P}$) entering the BPS relation.

Derivation of the gauge-invariant "momenta" from a Bogomol'nyi relation

We will consider the Bogomol'nyi bound for D1 branes in global coordinates. Consider the following motion of the D1 brane (this is sufficiently general for our purposes here):

$$t = \tau, \qquad \theta = \tau, \qquad \zeta = \text{const}, \qquad \rho = \text{const}, \qquad \phi_{1,2} = w_{1,2}\sigma + \phi_{1,2}(\tau). \tag{C2}$$

We will show that the Bogomol'nyi bound involves the "gauge-invariant momenta" $\tilde{P}_{\phi_{1,1}}$, thus justifying their definition which we introduced above.

We list below, for such motion, the Lagrangian, the canonical momenta, and the canonical Hamiltonian,

$$c = \cos\zeta, \qquad s = \sin\zeta, \qquad v = w_2\dot{\phi}_1 - w_1\dot{\phi}_2,$$
$$\lambda = Q_5/(2\pi), \qquad \beta = c^2w_1^2 + s^2w_2^2,$$
$$\Delta = \beta - c^2s^2v^2, \qquad L = -\lambda\left[\sqrt{\Delta} + \frac{1}{2}v(\cos 2\zeta + b)\right],$$
$$P_{\phi_1} = \lambda w_2\left[\frac{vc^2s^2}{\sqrt{\Delta}} - \frac{1}{2}(\cos 2\zeta + b)\right],$$
$$P_{\phi_2} = \lambda w_1\left[-\frac{vc^2s^2}{\sqrt{\Delta}} + \frac{1}{2}(\cos 2\zeta + b)\right], \qquad H_{\text{can}} = \frac{\lambda\beta}{\sqrt{\Delta}}. \tag{C3}$$

It is easy to see that the momenta satisfy a constraint:

$$w_1P_{\phi_1} + w_2P_{\phi_2} = 0. \tag{C4}$$

In the expression for the canonical Hamiltonian (C3), Δ depends on the velocity combination v which is to be expressed in terms of the (constrained) momenta P_{ϕ_1}, P_{ϕ_2}.

The gauge-invariant momenta, (C1), are given by

$$\tilde{P}_{\phi_1} = P_{\phi_1} + \frac{\lambda}{2}w_2(b+1) = \lambda w_2 s^2\left[\frac{vc^2}{\sqrt{\Delta}} + 1\right],$$
$$\tilde{P}_{\phi_2} = P_{\phi_2} - \frac{\lambda}{2}w_1(b-1) = \lambda w_1 c^2\left[-\frac{vs^2}{\sqrt{\Delta}} + 1\right]. \tag{C5}$$

We now proceed with our analysis of the Bogomol'nyi relation.

(i) Case: One of w_1, w_2 vanishes.
We have written the constraint equation (C4) for w_1, w_2 nonzero. The analysis becomes significantly simpler when either of them vanishes. We will consider the case $w_1 = 0$. Equation (C4) becomes

$$\tilde{P}_{\phi_2} = 0 (= P_{\phi_2}). \tag{C6}$$

The expressions for the other momentum and the canonical Hamiltonian are

$$\tilde{P}_{\phi_1} = \lambda s w_2\left[\frac{c^2\dot{\phi}_1}{\sqrt{1 - c^2\dot{\phi}_1^2}} + s\right],$$
$$H_{\text{can}} = \lambda s w_2 \frac{1}{\sqrt{1 - c^2\dot{\phi}_1^2}}. \tag{C7}$$

Eliminating $\dot{\phi}_1$ between $\tilde{P}_{\phi_1}$, H_{can}, we get

$$\left(\frac{H_{\text{can}}}{\lambda w_2}\right)^2 = s^2 + \frac{1}{c^2}\left(\frac{\tilde{P}_{\phi_1}}{\lambda w_2} - s^2\right)^2$$
$$= 2\frac{\tilde{P}_{\phi_1}}{\lambda w_2} - 1 + \frac{1}{c^2}\left(\frac{\tilde{P}_{\phi_1}}{\lambda w_2} - 1\right)^2. \tag{C8}$$

In a sector with a given gauge-invariant "charge" $\tilde{P}_{\phi_1}$, the minimum value of H_{can} is obtained for

$$\frac{\tilde{P}_{\phi_1}}{\lambda w_2} = 1, \tag{C9}$$

where

$$H_{\text{can,BPS}} = \lambda w_2 = \tilde{P}_{\phi_1}. \tag{C10}$$

Note that it is the gauge-invariant $\tilde{P}_{\phi_1}$ that appears in the BPS relation, as promised.
The case $w_2 = 0$ can be similarly computed. Again, it is the gauge-invariant $\tilde{P}_{\phi_2}$ that appears in the BPS relation.

(ii) Case: Both w_1, w_2 are nonzero.
We define a canonical transformation

$$\phi = w_2\phi_1 - w_1\phi_2, \qquad \Phi = \frac{\phi_2 - \phi_1}{w_2 - w_1},$$
$$p = \frac{P_{\phi_1} + P_{\phi_2}}{w_2 - w_1}, \qquad P = w_1P_{\phi_1} + w_2P_{\phi_2}. \tag{C11}$$

The constraint, encountered in (C4), becomes $P = 0$. The gauge transformation generated by the constraint can be fixed by putting $\Phi = 0$. Note that, although Φ is not a periodic coordinate, the constraint $\Phi = 0$ is well defined. We have assumed here $w_1 \neq w_2$; the case $w_1 = w_2$ can be dealt with similarly by an appropriate canonical transformation.
We denote $v \equiv \dot{\phi}$. From (C3) we get

$$p = \lambda\left[\frac{c^2s^2v}{\sqrt{\beta - c^2s^2v^2}} - \frac{1}{2}(b + \cos 2\zeta)\right],$$
$$H = \lambda\frac{\beta}{\sqrt{\beta - c^2s^2v^2}}. \tag{C12}$$

The Bogomol'nyi bound must be saturated when $v = w_2 - w_1$ (which follows from $\dot{\phi}_1 = \dot{\phi}_2 = 1$). When we substitute this in the above equation, we get

$$H = (w_2 - w_1)p',$$
$$p' := p + \frac{\lambda}{2}\left[b + \frac{w_2 + w_1}{w_2 - w_1}\right]. \tag{C13}$$

The top line is the BPS relation and the second line defines the gauge-invariant momentum. The definition of p' agrees with the gauge-invariant momenta (C5), in the sense that if we replace P_{ϕ_i} by $\tilde{P}_{\phi_i}$ in the definition of p in (C11) we recover the expression for p' as given above.
This proves the expression for the gauge-invariant momenta (C1) for nonzero w_1, w_2.

APPENDIX D: KILLING SPINOR EQUATIONS FOR D1-D5 SYSTEMS

In this appendix, we write down and solve Killing spinor equations for the naive D1-D5 system. The general Killing spinor equations are shown in Appendix D 1. We then proceed to write down and solve the dilatino equation, both in the bulk and in the near-horizon limit, in Appendix D 2 a. The gravitino equation is similarly written down and solved in the bulk and in the near-horizon limit in Appendix D 2 b.

We then go on to investigate what happens when we add momentum to the D1-D5 system in Appendix D 3. Finally, we write down and solve the dilatino Killing spinor equation for the Lunin-Mathur geometries in Appendix D 4.

Although the results we derive below are quite well known, we found it surprisingly difficult to find their explicit derivations in the supergravity literature. So, we hope that these explicit calculations will be useful for the reader. The reader may also find Refs. [21,22,25,28,41,58] useful.

1. Killing spinor equations

In this section, we write down the Killing spinor equations for type IIB string theory. Consider a type IIB two-component spinor

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{D1}$$

which satisfies $\Gamma^{11}\epsilon = -\epsilon$. The dilatino Killing spinor equation is

$$\Big[\partial_M \phi \Gamma^M + \frac{1}{12} H_{MAB}\Gamma^{MAB} \otimes \sigma_3 + \frac{1}{4} e^{\phi} \sum_{n=1}^{5} \frac{(-1)^{n-1}(n-3)}{(2n-1)!} G_{A_1 \ldots A_{2n-1}} \Gamma^{A_1 \ldots A_{2n-1}} \otimes \lambda_n \Big]\epsilon = 0, \tag{D2}$$

where $\lambda_n = \sigma_1$ for n even, and $\lambda_n = i\sigma_2$ for n odd. The $\{\sigma_i\}$, $i = 1, 2, 3$ are the Pauli matrices. H and G are the NS-NS and RR field strengths, and ϕ denotes the dilaton. Similarly, the gravitino Killing spinor equation is

$$\Big[\partial_M + \frac{1}{4} w_M^{BC}\Gamma_{BC} + \frac{1}{8} H_{MAB}\Gamma^{AB} \otimes \sigma_3 + \frac{1}{16} e^{\phi} \sum_{n=1}^{5} \frac{(-1)^{n-1}}{(2n-1)!} G_{A_1 \ldots A_{2n-1}} \Gamma^{A_1 \ldots A_{2n-1}} \Gamma_M \otimes \lambda_n \Big]\epsilon = 0. \tag{D3}$$

Throughout this appendix, we will find it useful to divide the coordinates $M = 0, \ldots, 9$ into $\mu = 0, 5$ (x^0 and x_5), $m = 1, 2, 3, 4$ (r, ζ, ϕ_1, ϕ_2), and $a = 6, 7, 8, 9$ (the torus directions). We also need to define $r^2 \equiv x_m x^m$.

2. The D1-D5 system

a. The dilatino equation

In this section, we write down the dilatino equation for the D1-D5 system, and solve it to find the corresponding projection conditions. We do this both in the bulk and in the near-horizon limit. This system is defined by the following metric, dilaton, and RR background[14]:

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{f_1 f_5}} dx_\mu dx^\mu + \sqrt{f_1 f_5} dx_m dx^m + \sqrt{\frac{f_1}{f_5}} dx_a dx^a, \\ e^{2\phi} &= \frac{f_1}{f_5}, \\ G^{(3)} &= -\frac{f_1'}{f_1^2} dr \wedge dx^0 \wedge dx_5 + Q_5 \sin(2\zeta) d\zeta \wedge d\phi_1 \wedge d\phi_2, \\ G^{(7)} &= Q_1 \sin(2\zeta) d\zeta \wedge d\phi_1 \wedge d\phi_2 \wedge \prod d\Sigma_a \\ &\quad - f_5' f_5^{-2} dr \wedge dx^0 \wedge dx_5 \wedge \prod d\Sigma_a, \end{aligned} \tag{D4}$$

where we used the definitions

$$f_1 = 1 + \frac{Q_1}{r^2}, \qquad f_5 = 1 + \frac{Q_5}{r^2}. \tag{D5}$$

Hence, the dilatino equation (D2) in the bulk is

$$\begin{aligned} &[f_1^{-5/4} f_5^{-1/4} f_1' \Gamma^{\hat r}(\mathbb{1} - \Gamma^{\hat 0 \hat 5} \otimes \sigma_1) \\ &\quad + f_5^{-5/4} f_1^{-1/4} f_5' \Gamma^{\hat r}(-\mathbb{1} - \Gamma^{\hat r \hat \zeta \hat \phi_1 \hat \phi_2} \otimes \sigma_1)]\epsilon = 0 \\ &\Rightarrow \Gamma^{\hat 6 \hat 7 \hat 8 \hat 9}\epsilon = +\epsilon, \qquad \Gamma^{\hat 0 \hat 5} \otimes \sigma_1 \epsilon = \epsilon. \end{aligned} \tag{D6}$$

Note that we find the expected projection conditions (1) and (2) for this background.

We now want to investigate what happens to the dilatino equation (D6) in the near-horizon limit $r \longrightarrow 0$. In this limit, Eqs. (D5) become

$$f_1 \longrightarrow \frac{Q_1}{r^2}, \qquad f_5 \longrightarrow \frac{Q_5}{r^2}. \tag{D7}$$

Consequently, some terms in (D6) cancel, and we are left with

$$[\Gamma^{\hat r \hat 0 \hat 5} + \Gamma^{\hat \zeta \hat \phi_1 \hat \phi_2}] \otimes \sigma_1 \epsilon = 0 \Rightarrow \Gamma^{\hat 6 \hat 7 \hat 8 \hat 9}\epsilon = +\epsilon, \tag{D8}$$

which is the dilatino equation in the near-horizon limit. Note that one of the projection conditions has dropped out; i.e. we get the expected doubling of supersymmetries in the near-horizon geometry.

b. The gravitino equation

In this section, we find and solve the gravitino equation for the background (D4). We will again do this both for the bulk and in the near-horizon limit, beginning with the bulk. First, note that we can define the vielbeins

[14] Note that, to avoid cluttering the notation, here we are using a form of the metric and the dilaton in which constant factors have been scaled away. This convention is used throughout Appendixes D and E.

$$e^{\hat{\mu}} = (f_1 f_5)^{-1/4} dx^{\mu}, \qquad e^{\hat{m}} = (f_1 f_5)^{+1/4} dx^m, \qquad e^{\hat{a}} = \left(\frac{f_1}{f_5}\right)^{+1/4} dx^a. \tag{D9}$$

Thus, the corresponding spin connections are

$$\begin{aligned} w^{\hat{\mu}\,\hat{n}} &= -\frac{1}{4r}(f_1 f_5)^{-3/2}(f_1 f_5)'[x^n dx^{\mu}], \\ w^{\hat{m}\,\hat{n}} &= +\frac{1}{4r}(f_1 f_5)^{-1}(f_1 f_5)'[x^n dx^m - x^m dx^n], \\ w^{\hat{a}\,\hat{n}} &= +\frac{1}{4r} f_1^{-1} f_5^{+1/2}\left(\frac{f_1}{f_5}\right)'[x^n dx^a]. \end{aligned} \tag{D10}$$

To simplify the notation, we have defined $r^2 = x_m x^m$. Using these spin connections and the background (D4) in the gravitino equation (D3), we get

$$\begin{aligned} &\left[\frac{D}{Dx^M} + \frac{1}{8} f_1^{1/2} f_5^{-1/2}[f_1' f_1^{-7/4} f_5^{+1/4}\Gamma^{\hat{r}\,\hat{0}\,\hat{5}} \right. \\ &\qquad \left. + (f_1 f_5)^{-3/4} f_5' \Gamma^{\hat{\zeta}\hat{\phi}_1\hat{\phi}_2}]\Gamma_M \otimes \sigma_1\right]\epsilon = 0 \\ &\Rightarrow \partial_M \epsilon = 0, \end{aligned} \tag{D11}$$

which is the gravitino equation in the bulk. Note that $\Gamma^{\hat{r}} = \Gamma^{\hat{m}} \frac{x^m}{r}$. Because of cancellation of terms, we conclude that the Killing spinor ϵ is just a constant expressed in terms of vielbeins corresponding to Cartesian coordinates.

We now want to investigate what happens to the bulk gravitino equation (D11) in the near-horizon limit $r \rightarrow 0$. In this limit, the spin connections (D10) become

$$\begin{aligned} &w^{\hat{\mu}\,\hat{n}} \rightarrow (Q_1 Q_5)^{-1/2}[x^n dx^{\mu}], \\ &w^{\hat{m}\,\hat{n}} \rightarrow \frac{1}{r^2}[x^m dx^n - x^n dx^m], \qquad w^{\hat{a}\,\hat{n}} \rightarrow 0. \end{aligned} \tag{D12}$$

Inserting these spin connections and the background (D4) into Eq. (D3), we now find the gravitino equation in the near-horizon limit,

$$\begin{aligned} &\left[\frac{D}{Dx^M} - \frac{1}{4}(Q_1 Q_5)^{-1/4}[\Gamma^{\hat{r}\,\hat{0}\,\hat{5}} + \Gamma^{\hat{\zeta}\hat{\phi}_1\hat{\phi}_2}]\Gamma_M \otimes \sigma_1\right]\epsilon = 0 \\ &\Rightarrow \partial_a \epsilon = 0. \end{aligned} \tag{D13}$$

As a consistency check, the gravitino equation (D13) is indeed equivalent to Mikhailov's equation (E22). For the torus coordinates, there is again a cancellation making the spinor ϵ constant in those directions.

In the near-horizon limit, where we find an $\mathrm{AdS}_3 \times S^3$ structure, it is convenient to move to "polar" coordinates: (r, x^0, x_5) for the AdS_3 and (ζ, ϕ_1, ϕ_2) for the S^3. In this basis, the Killing spinors do have a nontrivial dependence on these six coordinates.

We therefore want to find this dependence by solving the gravitino equation (D13). We proceed as follows. The S^3 and (r, x^0, x_5) parts can be analyzed separately. In fact, the S^3 part is identical to (E6) in Appendix E 3 and can be solved, as detailed in that section. The final solution to that part is given by Eq. (E12).

The (r, x^0, x_5) part is

$$\begin{aligned} &\left[\frac{\partial}{\partial r} \mp \frac{1}{2r}\Gamma^{\hat{0}\,\hat{5}}\right]\epsilon = 0, \qquad \left[\frac{\partial}{\partial x^0} + rD\right]\epsilon = 0, \\ &\left[\frac{\partial}{\partial x_5} \pm rD\right]\epsilon = 0, \end{aligned} \tag{D14}$$

where we defined

$$D = \tfrac{1}{2}(Q_1 Q_5)^{-1/2}(\Gamma^{\hat{0}\,\hat{r}} \pm \Gamma^{\hat{5}\,\hat{r}}). \tag{D15}$$

The split signs correspond to eigenvalues of σ_1, i.e. $\sigma_1 \epsilon = \pm \epsilon$. As a consistency check, it can be verified that these three operators commute. They can be solved analogously to the S^3 part, using the relation

$$\begin{aligned} &\exp\left[\mp \frac{\delta}{2}\Gamma^{\hat{0}\,\hat{5}}\right] r(\Gamma^{\hat{0}} \pm \Gamma^{\hat{5}}) \exp\left[\pm \frac{\delta}{2}\Gamma^{\hat{0}\,\hat{5}}\right] \\ &\quad = re^{-\delta}(\Gamma^{\hat{0}} \pm \Gamma^{\hat{5}}), \end{aligned} \tag{D16}$$

to move factors of the type $\exp[\mp \frac{\delta}{2}\Gamma^{\hat{0}\,\hat{5}}]$ through the D. The solution to (D14) is

$$\begin{aligned} &\epsilon(r, x^0, x_5) = M_1(r, x^0, x_5)\epsilon_0, \\ &M_1(r, x^0, x_5) \equiv \exp[\pm \tfrac{1}{2}\ln(r)\Gamma^{\hat{0}\,\hat{5}}] \\ &\qquad \times \exp[-\tfrac{1}{2}(Q_1 Q_5)^{-1/2}(x^0 \pm x_5)(\Gamma^{\hat{0}\,\hat{r}} \pm \Gamma^{\hat{5}\,\hat{r}})], \end{aligned} \tag{D17}$$

where ϵ_0 is a constant spinor on (r, x^0, x_5). Hence, the full solution to (D13) is obtained by combining the S^3 solution (E12) with the (r, x^0, x_5) solution (D17), i.e.

$$\begin{aligned} &\epsilon(r, x^0, x_5, \zeta, \phi_1, \phi_2) = M_1(r, x^0, x_5) M_2(\zeta, \phi_1, \phi_2)\epsilon_0 \\ &\qquad M_2(\zeta, \phi_1, \phi_2) = \exp[\pm \tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_1\hat{\phi}_2}] \\ &\qquad\qquad \times \exp[+\tfrac{1}{2}(\phi_2 \mp \phi_1)\Gamma^{\hat{\zeta}\hat{\phi}_2}], \end{aligned} \tag{D18}$$

where ϵ_0 is a constant spinor on r, x^0, x_5, S^3, and $M(r, x^0, x_5)$ is already defined in (D17). Note that half of the spinors which satisfy (D6) are still constants in vielbeins corresponding to Cartesian coordinates, as in (D11). The reason is that the projection $\Gamma^{\hat{0}\,\hat{5}} \otimes \sigma_1 \epsilon = +\epsilon$ makes the x^0, x_5 dependence drop out.

3. The D1-D5-P system

In this section, we investigate the D1-D5-P system, i.e. the D1-D5 system with momentum p added. The background is, in fact, almost identical to (D4), only the metric is changed to (50) which can be rewritten as

$$\begin{aligned} ds^2 &= -(Adt + Bdx_5)^2 + C^2 dx_5^2 + \sqrt{f_1 f_5} dx_m dx^m \\ &\quad + \sqrt{\frac{f_1}{f_5}} dx_a dx^a, \end{aligned} \tag{D19}$$

where we defined

$$\begin{aligned}A &= (f_1 f_5)^{-1/4}\sqrt{1-K},\\ B &= (f_1 f_5)^{-1/4}K/\sqrt{1-K},\\ C &= (f_1 f_5)^{-1/4}/\sqrt{1-K},\end{aligned} \tag{D20}$$

where $K = r_p^2/r^2$. As can be verified using (D4) and (D19) in (D2), the dilatino equation remains of the form (D6).

To obtain the gravitino equation, we again use vielbeins (D9) and corresponding spin connections (D10). However, in the t and x_5 directions, we instead use

$$e^{\hat{t}} = Adt + Bdx_5, \qquad e^{\hat{x}} = Cdx_5. \tag{D21}$$

The relevant new spin connections are

$$\begin{aligned}\omega^{\hat{x}\hat{m}} &= \frac{x^m}{r}(f_1 f_5)^{-1/4}[-\mathcal{A} e^{\hat{t}} + C'/C e^{\hat{x}}]\\ &= \frac{x^m}{r}(f_1 f_5)^{-1/4}[-\mathcal{A}(Adt + Bdx_5) + C' dx_5],\\ \omega^{\hat{x}\hat{t}} &= (f_1 f_5)^{-1/4}\mathcal{A}\frac{x^m}{r}e^{\hat{m}} = \mathcal{A}dr,\\ \omega^{\hat{t}\hat{m}} &= \frac{x^m}{r}(f_1 f_5)^{-1/4}[A'/A e^{\hat{t}} + \mathcal{A} e^{\hat{x}}]\\ &= \frac{x^m}{r}(f_1 f_5)^{-1/4}[A'/A(Adt + Bdx_5) + \mathcal{A} C dx_5],\end{aligned} \tag{D22}$$

where we defined

$$\mathcal{A} = \frac{1}{2}\frac{AB' - A'B}{AC} = \frac{K'}{1-K}. \tag{D23}$$

Using (D4), (D10), (D19), and (D22), we can now obtain the $M = x_5$ component of the gravitino equation (D3), which is $\hat{D}_5\epsilon = 0$, where

$$\begin{aligned}\hat{D}_5 &= \partial_5 - \frac{1}{8}\frac{(f_1 f_5)^{-1/2}}{\sqrt{1-K}}\bigg[\bigg[\frac{(f_1 f_5)'}{f_1 f_5} - 2K'\bigg]\Gamma^{\hat{x}\hat{r}}\\ &\quad + \bigg[2K' - K\frac{(f_1 f_5)'}{f_1 f_5}\bigg]\Gamma^{\hat{t}\hat{r}}\\ &\quad - \frac{(f_1 f_5)'}{f_1 f_5}(\Gamma^{\hat{r}\hat{t}} - K\Gamma^{\hat{r}\hat{x}})\otimes\sigma_1\bigg]\\ &= \partial_5 - \frac{1}{8}\frac{(f_1 f_5)^{-1/2}}{\sqrt{1-K}}\bigg[\frac{(f_1 f_5)'}{f_1 f_5}\Gamma^{\hat{x}\hat{r}}(\mathbb{1} - \Gamma^{\hat{0}\hat{5}}\otimes\sigma_1)\\ &\quad - K\frac{(f_1 f_5)'}{f_1 f_5}\Gamma^{\hat{t}\hat{r}}(\mathbb{1} - \Gamma^{\hat{0}\hat{5}}\otimes\sigma_1) - 2K'\Gamma^{\hat{x}\hat{r}}(\mathbb{1} + \Gamma^{\hat{0}\hat{5}})\bigg].\end{aligned} \tag{D24}$$

Hence, using the additional constraint

$$\Gamma^{\hat{0}\hat{5}}\epsilon = -\epsilon \tag{D25}$$

in addition to the ones obtained in Eq. (D6), we find that

$$\partial_5\epsilon = 0.$$

Similarly, using (D6) and (D25), the $M = t$ component of the gravitino equation becomes

$$\partial_0\epsilon = 0.$$

However, the $M = m$ ($m = 1, 2, 3, 4$) component leads to

$$\bigg[\partial_r - \frac{1}{4}\frac{K'}{(1-K)}\Gamma^{\hat{0}\hat{5}}\bigg]\epsilon = \bigg[\partial_r + \frac{1}{4}\frac{K'}{(1-K)}\bigg]\epsilon = 0, \tag{D26}$$

using (D25), so the spinor must have a nontrivial dependence on r,

$$\epsilon(r) = (1-K)^{-1/4}\epsilon_0. \tag{D27}$$

For the remaining components, the gravitino equation just reduces to $\partial_M\epsilon = 0$. Hence, the full Killing spinor is just given by (D27), with ϵ_0 a constant with respect to the vielbeins (D9) (except the x^0 and x_5 directions) and (D21) (except the t and x directions).

4. Lunin-Mathur geometries

In this section, we write down and solve the dilatino Killing spinor equation for the Lunin-Mathur geometries. This serves to verify that these geometries do preserve the same supersymmetries as the naive D1-D5 geometry.

Using the metric in (52), we may first define a vielbein in terms of the following orthonormal 1-forms:

$$\begin{aligned}e^{\hat{t}} &= \left(\frac{H}{1+K}\right)^{1/4}(dt - A_{\hat{n}}dx^{\hat{n}}),\\ e^{\hat{5}} &= \left(\frac{H}{1+K}\right)^{1/4}(dx_5 + B_{\hat{n}}dx^{\hat{n}}),\\ e^{\hat{m}} &= \left(\frac{H}{1+K}\right)^{-1/4}dx^{\hat{m}},\\ e^{\hat{a}} &= \{H(1+K)\}^{1/4}dx^{\hat{a}}.\end{aligned} \tag{D28}$$

The field strength may now be computed in terms of the RR 2-forms given in (52), and we find

$$\begin{aligned}G^{(3)} &= dC^{(2)}\\ &= \left(\frac{H}{1+K}\right)^{3/4}\epsilon^{\hat{n}}_{\hat{m}\hat{l}\hat{p}}\partial_n H^{-1}e^{\hat{m}}\wedge e^{\hat{l}}\wedge e^{\hat{p}}\\ &\quad + \left(\frac{1+K}{H}\right)^{1/4}\partial_n\frac{1}{(1+K)}e^{\hat{t}}\wedge e^{\hat{5}}\wedge e^{\hat{n}}\\ &\quad + \frac{1}{1+K}(-\partial_m B_n e^{\hat{5}} - \partial_m A_n e^{\hat{t}})\wedge e^{\hat{n}}\wedge e^{\hat{m}}.\end{aligned} \tag{D29}$$

Notice that under Poincare duality

$$\begin{aligned} G^{(7)} &= *_{10}G^{(3)} \\ &= \bigg[\bigg(\frac{H}{1+K}\bigg)^{3/4}\epsilon^{\hat n}_{\hat m\hat l\hat p}\partial_n K e^{\hat m}\wedge e^{\hat l}\wedge e^{\hat p} \\ &\quad + \bigg(\frac{1+K}{H}\bigg)^{1/4}\partial_n H e^{\hat t}\wedge e^{\hat 5}\wedge e^{\hat n} + H(-\partial_m B_n e^{\hat 5} \\ &\quad - \partial_m A_n e^{\hat t})\wedge e^{\hat n}\wedge e^{\hat m}\bigg]\wedge dx^6\wedge dx^7\wedge dx^8\wedge dx^9. \end{aligned} \tag{D30}$$

This justifies the result (86) for $C'^{(2)}$ (recall by definition $G^{(7)} = dC'^{(2)}\wedge dx^6\wedge\ldots\wedge dx^9$) which we claimed was obtained by interchanging $H \leftrightarrow \frac{1}{1+K}$ in $C^{(2)}$.

Now, substituting the result (D29) into the dilatino equation (D2), we find that it becomes

$$\begin{aligned} &H^{1/4}(1+K)^{-5/4}(\partial_m K)[\mathbb{1} - \Gamma^{\hat 0\hat 5}\otimes\sigma_1]\epsilon \\ &\quad + H^{5/4}(1+K)^{-1/4}[-(\partial_m H^{-1})\Gamma^{\hat m} \\ &\quad - (*_4 dH^{-1})_{mnp}\Gamma^{\hat m\hat n\hat p}\otimes\sigma_1]\epsilon \\ &\quad + H^{3/4}(1+K)^{-3/4}\Gamma^{\hat 0}[\Gamma^{\hat m\hat n}(\partial_m B_n) \\ &\qquad - \Gamma^{\hat m\hat n\hat 0\hat 5}(\partial_m A_n)\otimes\sigma_1]\epsilon = 0, \end{aligned} \tag{D31}$$

which is satisfied when, in addition to the properties of the metric (52), we use the projections

$$\Gamma^{\hat 6\hat 7\hat 8\hat 9}\epsilon = \epsilon, \qquad \Gamma^{\hat 0\hat 5}\otimes\sigma_1\epsilon = \epsilon. \tag{D32}$$

This confirms the result we quoted in Section II.

APPENDIX E: KILLING SPINOR EQUATIONS IN GLOBAL ADS

In this appendix, we analyze the Killing spinor equations in global AdS. In Appendixes E 1 and E 2, we write down the dilatino and gravitino equations, respectively. We proceed to solving the gravitino equation in Appendix E 3. Finally, we compare the results to those of Mikhailov in Appendix E 4.

1. The dilatino equation

In this section, we will find the dilatino equation and the corresponding projection conditions in global AdS. The background we are working with is defined by the following metric, dilaton, and the 3-form RR field strength:

$$\begin{aligned} ds^2 &= (Q_1Q_5)^{1/2}[-\cosh^2(\rho)dt^2 + \sinh^2(\rho)d\theta^2 + d\rho^2] \\ &\quad + (Q_1Q_5)^{1/2}[d\zeta^2 + \cos^2(\zeta)d\phi_1^2 + \sin^2(\zeta)d\phi_2^2] \\ &\quad + ds^2_{T^4} \\ e^{2\phi} &= \frac{Q_1}{Q_5}, \\ G^{(3)} &= +Q_5\sinh\rho dt\wedge d\theta\wedge d\rho + Q_5\sin(2\zeta)d\zeta\wedge d\phi_1 \\ &\quad \wedge d\phi_2. \end{aligned} \tag{E1}$$

Using this in Eq. (D2), we find the dilatino equation

$$[\Gamma^{\hat\rho\hat t\hat\theta} + \Gamma^{\hat\zeta\hat\phi_1\hat\phi_2}]\otimes\sigma_1\epsilon = 0 \Rightarrow \Gamma^{\hat 6\hat 7\hat 8\hat 9}\epsilon = \epsilon. \tag{E2}$$

Note that it implies the usual torus projection.

2. The gravitino equation

In this section, we want to find the gravitino equation in the global AdS background (E1). We begin by defining the vielbeins

$$\begin{aligned} e^{\hat t} &= (Q_1Q_5)^{+1/4}\cosh(\rho)dt, \\ e^{\hat\theta} &= (Q_1Q_5)^{+1/4}\sinh(\rho)d\theta, \\ e^{\hat\rho} &= (Q_1Q_5)^{+1/4}d\rho, \\ e^{\hat\zeta} &= (Q_1Q_5)^{+1/4}d\zeta, \\ e^{\hat\phi_1} &= (Q_1Q_5)^{+1/4}\cos(\zeta)d\phi_1, \\ e^{\hat\phi_2} &= (Q_1Q_5)^{+1/4}\sin(\zeta)d\phi_2. \end{aligned} \tag{E3}$$

Thus, the corresponding nonvanishing spin connections are

$$\begin{aligned} w^{\hat t\hat\rho} &= \sinh(\rho)dt, & w^{\hat\theta\hat\rho} &= \cosh(\rho)d\theta, \\ w^{\hat\phi_1\hat\zeta} &= -\sin(\zeta)d\phi_1, & w^{\hat\phi_2\hat\zeta} &= \cos(\zeta)d\phi_2. \end{aligned} \tag{E4}$$

Using the background (E1) and the spin connections (E4) in Eq. (D3), we find the gravitino equation

$$\bigg[\frac{D}{Dx^M} - \frac{1}{4}(Q_1Q_5)^{-1/4}[\Gamma^{\hat\rho\hat t\hat\theta} + \Gamma^{\hat\zeta\hat\phi_1\hat\phi_2}]\Gamma_M\otimes\sigma_1\bigg]\epsilon = 0. \tag{E5}$$

As a consistency check, the gravitino equation (E5) is indeed equivalent to Mikhailov's equation (E22). We show how to solve the gravitino equation (E5) in Appendix E 3.

3. Solving the gravitino equation

In this section, we show how to solve the gravitino equation (E5). We proceed as follows. In fact, the S^3 and AdS_3 parts split, and can be analyzed separately. We begin with the S^3 part, which is

$$\bigg[\frac{\partial}{\partial\zeta} \mp \frac{1}{2}\Gamma^{\hat\phi_1\hat\phi_2}\bigg]\epsilon = 0, \qquad \bigg[\frac{\partial}{\partial\phi_1} + A\bigg]\epsilon = 0, \qquad \bigg[\frac{\partial}{\partial\phi_2} \mp A\bigg]\epsilon = 0, \tag{E6}$$

where we defined

$$A \equiv \tfrac{1}{2}\sin(\zeta)\Gamma^{\hat\zeta\hat\phi_1} \pm \tfrac{1}{2}\cos(\zeta)\Gamma^{\hat\zeta\hat\phi_2}. \tag{E7}$$

As a consistency check, it can be verified that these three operators commute. The split signs correspond to eigenvalues of σ_1, i.e. $\sigma_1\epsilon = \pm\epsilon$. The first equation of (E6) implies that

$$\epsilon(\zeta, \phi_1, \phi_2) = \exp[\pm\tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_1\hat{\phi}_2}]\Psi(\phi_1, \phi_2). \tag{E8}$$

The second equation of (E6) then implies that

$$\Psi(\phi_1, \phi_2) = \exp[\mp\tfrac{1}{2}\phi_1\Gamma^{\hat{\zeta}\hat{\phi}_2}]\chi(\phi_2), \tag{E9}$$

where we used the relation

$$\exp[\pm\tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_2\hat{\phi}_1}](\cos(\zeta)\Gamma^{\hat{\zeta}\hat{\phi}_2} \pm \sin(\zeta)\Gamma^{\hat{\zeta}\hat{\phi}_1}) \times \exp[\mp\tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_2\hat{\phi}_1}] = \Gamma^{\hat{\zeta}\hat{\phi}_2} \tag{E10}$$

to move the factor $\exp[\pm\frac{1}{2}\zeta\Gamma^{\hat{\phi}_1\hat{\phi}_2}]$ through the A. The third equation of (E6) similarly implies that

$$\chi(\phi_2) = \exp[+\tfrac{1}{2}\phi_2\Gamma^{\hat{\zeta}\hat{\phi}_2}\epsilon_0], \tag{E11}$$

where ϵ_0 is a constant spinor on S^3. So the solution to (E6) is

$$\epsilon(\zeta, \phi_1, \phi_2) = \exp[\pm\tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_1\hat{\phi}_2}]\exp[+\tfrac{1}{2}(\phi_2 \mp \phi_1)\Gamma^{\hat{\zeta}\hat{\phi}_2}]\epsilon_0. \tag{E12}$$

We now proceed to the AdS_3 part, which is

$$\left[\frac{\partial}{\partial\rho} \mp \frac{1}{2}\Gamma^{\hat{t}\hat{\theta}}\right]\tilde{\epsilon} = 0, \qquad \left[\frac{\partial}{\partial t} + B\right]\tilde{\epsilon} = 0, \qquad \left[\frac{\partial}{\partial\theta} \mp B\right]\tilde{\epsilon} = 0, \tag{E13}$$

where we defined

$$B \equiv \tfrac{1}{2}\sinh(\rho)\Gamma^{\hat{\rho}\hat{t}} \pm \tfrac{1}{2}\cosh(\rho)\Gamma^{\hat{\rho}\hat{\theta}}. \tag{E14}$$

Again, we can verify that these three operators commute. We now make the change of variables defined by

$$\rho = i\zeta, \qquad t = i\phi_1, \qquad \theta = \phi_2, \tag{E15}$$

which turn (E13) into

$$\left[\frac{\partial}{\partial\zeta} \pm \frac{1}{2}\Gamma^{\hat{\phi}_1\hat{\phi}_2}\right]\tilde{\epsilon} = 0, \qquad \left[\frac{\partial}{\partial\phi_1} + C\right]\tilde{\epsilon} = 0, \qquad \left[i\frac{\partial}{\partial\phi_2} \mp C\right]\tilde{\epsilon} = 0, \tag{E16}$$

where we defined

$$C \equiv \tfrac{1}{2}\sin(\zeta)\Gamma^{\hat{\zeta}\hat{\phi}_1} \mp \tfrac{1}{2}\cos(\zeta)\Gamma^{\hat{\zeta}\hat{\phi}_2}. \tag{E17}$$

This system of equations can be analyzed analogously to the S^3 case, and we find the solution

$$\tilde{\epsilon}(\rho, t, \theta) = \exp[\pm\tfrac{1}{2}\rho\Gamma^{\hat{t}\hat{\theta}}]\exp[+\tfrac{1}{2}(\theta \mp t)\Gamma^{\hat{\rho}\hat{\theta}}]\tilde{\epsilon}_0, \tag{E18}$$

where $\tilde{\epsilon}_0$ is a constant spinor on AdS_3. Thus, the full solution to (E5) is

$$\begin{aligned}\epsilon(\zeta, \phi_1, \phi_2, \rho, t, \theta) &= \epsilon(\zeta, \phi_1, \phi_2)\tilde{\epsilon}(\rho, t, \theta) \\ &= \exp[\pm\tfrac{1}{2}\zeta\Gamma^{\hat{\phi}_1\hat{\phi}_2}] \\ &\quad \times \exp[+\tfrac{1}{2}(\phi_2 \mp \phi_1)\Gamma^{\hat{\zeta}\hat{\phi}_2}] \\ &\quad \times \exp[\pm\tfrac{1}{2}\rho\Gamma^{\hat{t}\hat{\theta}}] \\ &\quad \times \exp[+\tfrac{1}{2}(\theta \mp t)\Gamma^{\hat{\rho}\hat{\theta}}]\epsilon_0,\end{aligned} \tag{E19}$$

where ϵ_0 is a constant spinor on $AdS_3 \times S^3$.

4. Comparison to Mikhailov

In this section, we compare our gravitino equations to those of Mikhailov [26]. In particular, Mikhailov writes down the general form of the equation a spinor in global $AdS_3 \times S^3$ must satisfy as

$$\left(\frac{D}{Dx^p} - \frac{1}{2}\frac{\partial f}{\partial R}\Gamma_p\Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\right)\epsilon_{++} = 0, \tag{E20}$$

where $p = 0, 1, 2$ and $0, 1, 2$ are the three coordinates on either AdS_3 or S^3. The spinor ϵ_{++} is defined by requiring that $\sigma_1\epsilon_{++} = +\epsilon_{++}$. Equation (E20) presupposes an embedding in a higher dimensional space. In more detail, we can embed $M = S^3$ or AdS_3 in $N = \mathbb{R}^4$ or $\mathbb{R}^{2,2}$ as

$$ds_N^2 = dR^2 + e^{2f(R)}ds_M^2 = dR^2 + R^2 d\Omega_M^2, \Rightarrow 2f(R) = \log(R^2) \Rightarrow \frac{\partial f}{\partial R} = \frac{1}{R}, \tag{E21}$$

where R is a radial coordinate in N. For us, $R^2 = (Q_1Q_5)^{1/2}$, which means that Eq. (E20) becomes

$$\left(\frac{D}{Dx^p} - \frac{1}{2}(Q_1Q_5)^{-1/4}\Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\Gamma_p\right)\epsilon_{++} = 0. \tag{E22}$$

Mikhailov's equation (E22) is indeed equivalent to (D13) and (E5).

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