Domain wall solitons and Hopf algebraic translational symmetries in noncommutative field theories

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Domain wall solitons are the simplest topological objects in field theories. The conventional translational symmetry in a field theory is the generator of a one-parameter family of domain wall solutions, and induces a massless moduli field which propagates along a domain wall. We study similar issues in braided noncommutative field theories possessing Hopf algebraic translational symmetries. As a concrete example, we discuss a domain wall soliton in the scalar ϕ^4 braided noncommutative field theory in Lie-algebraic noncommutative space-time, $[x^i, x^j] = 2i\kappa\epsilon^{ijk}x_k$ $(i, j, k = 1, 2, 3)$, which has a Hopf algebraic translational symmetry. We first discuss the existence of a domain wall soliton in view of Derrick's theorem, and construct explicitly a one-parameter family of solutions in perturbation of the noncommutativity parameter κ. We then find the massless moduli field which propagates on the domain wall soliton. We further extend our analysis to the general Hopf algebraic translational symmetry.

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I. INTRODUCTION

Noncommutative field theories [1–4] are important subjects for studying the Planck scale physics. The most well studied are the noncommutative field theories in Moyal space-time, whose coordinate commutation relation is given by $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with an antisymmetric constant $\theta^{\mu\nu}$. Field theories in Moyal space-time are also known to appear as effective field theories of open string theory with a constant background $B_{\mu\nu}$ field [5,6]. Thus, not only as the simplest field theories in quantum space-time but also as toy models of string theory, various perturbative and nonperturbative aspects such as unitarity [7–9], causality [10], UV-IR mixing [11–13], renormalizability [12], scalar solitons [14–17], instantons [18–23], monopoles [23–25], and other solitonic solutions [26–30] have extensively been analyzed. Recently it has been pointed out that Moyal space-time is invariant under the twisted Poincaré symmetry, which is a kind of Hopf algebraic symmetry [31–33]. There have been various proposals to implement the twisted Poincaré invariance in quantum field theories [34–48]. Gravity in fuzzy space-times has also been discussed in [49–56].

A prominent feature of Hopf algebraic symmetries is the general requirement of nontrivial statistics, which is called braiding, of fields to keep the symmetries at the quantum level. In our previous paper [47], it has been shown that symmetry relations among correlation functions can systematically be derived from Hopf algebraic symmetries in the framework of braided quantum field theories [57], if appropriate braiding of fields can be chosen. This feature is in parallel with the existence of similar relations, such as Ward-Takahashi identities, in field theories possessing conventional symmetries.

The main motivation of this paper is to understand better the physical roles of Hopf algebraic symmetries in another setting. In this paper we study a domain wall soliton in the three-dimensional noncommutative scalar field theory in Lie-algebraic noncommutative space-time $[x^i, x^j] = 2i\kappa\epsilon^{ijk}x_k$ $(i, j, k = 0, 1, 2)$ [58–62]. This noncommutative space-time has also a Hopf algebraic Poincaré symmetry [47,61,62], but the difference from Moyal space-time is that its translational symmetry is Hopf algebraic, while the rotation-boost symmetry is Hopf algebraic in Moyal space-time. Therefore this noncommutative field theory provides an interesting stage for investigating the physical roles of the braiding and the Hopf algebraic translational symmetry on a domain wall soliton, since the conventional translational symmetry in a field theory is the generator of a one-parameter family of domain wall solutions, and induces a massless moduli field which propagates along a domain wall.

This paper is organized as follows. In Sec. II A, we review the three-dimensional noncommutative ϕ^4 theory in the Lie-algebraic noncommutative space-time $[x^i, x^j] = 2i\kappa\epsilon^{ijk}x_k$. In Sec. II B, we apply the criterion of Derrick's theorem [63] to the ϕ^4 theory and conclude that a domain wall solution is possible at least perturbatively in κ. In Sec. II C, we solve the equation of motion to obtain a one-parameter family of the kink solutions in perturbation of the noncommutativity parameter κ. In Sec. II D, we discuss the moduli space. In Sec. II E, we analyze the moduli field, which propagates along the domain wall soliton, and conclude that the moduli field is massless. In Sec. III, we study the general Hopf algebraic translational symmetry. The final section is devoted to the summary and comments.

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II. NONCOMMUTATIVE ϕ^4 THEORY IN LIE-ALGEBRAIC SPACE-TIME AND THE DOMAIN WALL SOLUTIONS

A. Noncommutative ϕ^4 theory in Lie-algebraic space-time

In this subsection, we review the noncommutative ϕ^4 theory in Lie-algebraic noncommutative space-time whose commutation relation is given by

$$[\hat{x}^i, \hat{x}^j] = 2i\kappa\epsilon^{ijk}\hat{x}_k, \tag{1}$$

where $i, j, k = 0, 1, 2$[1] [58,59], following the constructions of [47,60,61]. Imposing the Jacobi identity and Lorentz invariance, we can determine the commutation relations between the coordinates and momenta as follows [58]:

$$[\hat{P}^i, \hat{x}^j] = -i\eta^{ij}\sqrt{1+\kappa^2\hat{P}^2} + i\kappa\epsilon^{ijk}\hat{P}_k, \tag{2}$$

where we have also imposed $[\hat{P}^i, \hat{P}^j] = 0$. We can identify these operators with the Lie algebra of $ISO(2,2)$ as follows:

$$\hat{x}_i = \kappa(\hat{J}_{-1,i} - \tfrac{1}{2}\epsilon_i^{jk}\hat{J}_{jk}), \tag{3}$$

$$\hat{P}_i = \hat{P}_{\mu=i}, \tag{4}$$

$$1 + \kappa^2\hat{P}^\mu\hat{P}_\mu = 0, \tag{5}$$

where the commutation relations of the Lie algebra of $ISO(2,2)$ are given by

$$[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = -i(\eta_{\mu\rho}\hat{J}_{\nu\sigma} - \eta_{\mu\sigma}\hat{J}_{\nu\rho} - \eta_{\nu\rho}\hat{J}_{\mu\sigma} + \eta_{\nu\sigma}\hat{J}_{\mu\rho}), \tag{6}$$

$$[\hat{J}_{\mu\nu}, \hat{P}_\rho] = -i(\eta_{\mu\rho}\hat{P}_\nu - \eta_{\nu\rho}\hat{P}_\mu), \tag{7}$$

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \tag{8}$$

and the Greek indices run through -1 to 2. From the constraint (5), we can identify the momentum space with the group manifold $SL(2,R)$.

Let $\phi(x)$ be a scalar field in the three-dimensional space-time. Its Fourier transformation is given by

$$\phi(x) = \int dg\tilde{\phi}(g)e^{iP(g)\cdot x}, \tag{9}$$

where $P^i(g)$ are determined by $g = P^{-1} - i\kappa P^i\tilde{\sigma}_i \in SL(2,R)$[2] and $\int dg$ is the Haar measure of $SL(2,R)$. This P^{-1} can take two values,

$$P^{-1} = \pm\sqrt{1+\kappa^2P^iP_i}, \tag{10}$$

for each P_i. This unphysical twofold degeneracy can be deleted by imposing

$$\tilde{\phi}(g) = \tilde{\phi}(-g). \tag{11}$$

The definition of the star product is given by[3]

$$e^{iP(g_1)\cdot x} \star e^{iP(g_2)\cdot x} = e^{iP(g_1g_2)\cdot x}. \tag{12}$$

This determines the coproducts of P^i and P^{-1} via the group product g_1g_2 as

$$\Delta P^i = P^i \otimes P^{-1} + P^{-1} \otimes P^i - \kappa\epsilon^{ijk}P_j \otimes P_k, \tag{13}$$

$$\Delta P^{-1} = P^{-1} \otimes P^{-1} + \kappa^2 P^i \otimes P_i. \tag{14}$$

We consider the ϕ^4 theory in the Lie-algebraic noncommutative space-time. We give the action as follows:

$$S = \int d^3x\Big[-\frac{1}{2}(\partial^i\phi \star \partial_i\phi)(x) + \frac{1}{2}m^2(\phi\star\phi)(x) - \frac{\lambda}{4}(\phi\star\phi\star\phi\star\phi)(x) - \frac{m^4}{4\lambda}\Big], \tag{15}$$

where we have chosen the constant term so that the minima of the potential vanish when $\kappa = 0$.

Carrying out the coordinate integration, one finds a modified energy-momentum conservation: $P^i(g_1g_2\cdots) = P_1 + P_2 + \cdots + \mathcal{O}(\kappa) = 0$ at the classical level. This should be regarded as a consequence of the Hopf algebraic translational symmetry. A naive construction of noncommutative quantum field theory in this space-time leads to disastrous violations of the energy-momentum conservation in the nonplanar diagrams [60]. One can avoid this violation by introducing a nontrivial statistics between scalar fields, which is given by

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[1]The signature of the metric η^{ij} is $(-1,1,1)$, and that of $\eta^{\mu\nu}$ is $(-1,-1,1,1)$.

[2]The definition of $\tilde{\sigma}_i$ is given by

$$\tilde{\sigma}_0 = \sigma_2, \qquad \tilde{\sigma}_1 = i\sigma_3, \qquad \tilde{\sigma}_2 = i\sigma_1,$$

with Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$. We have also changed the normalization of P^{-1} by κ from (5).

[3]In fact, one can produce the commutation relation between the coordinates (1) by differentiating both sides of (12) with respect to $P_1^i \equiv P^i(g_1)$ and $P_2^j \equiv P^j(g_2)$ and then taking the limit $P_1^i, P_2^i \to 0$.

$$\psi(\tilde{\phi}_1(g_1)\tilde{\phi}_2(g_2)) = \tilde{\phi}_2(g_2)\tilde{\phi}_1(g_2^{-1}g_1g_2), \quad (16)$$

where ψ is an exchanging map. This is denoted by braiding. This braiding was first derived from three-dimensional quantum gravity with scalar particles [61]. With this braiding, correlation functions respect the Hopf algebraic symmetry at the full quantum level [47].

B. Derrick's theorem in the noncommutative ϕ^4 theory

We consider a domain wall soliton in the noncommutative ϕ^4 theory. At first we consider whether the domain wall solution may exist or not by applying the criterion of Derrick's theorem [63].

Varying the action (15) with respect to $\phi(x)$, we obtain the equation of motion,

$$\partial^2\phi(x) + m^2\phi(x) - \lambda(\phi \star \phi \star \phi)(x) = 0. \quad (17)$$

Since our interest is in a domain wall, we consider only one spatial direction of the coordinates.[4] Let us change the variables P, P^{-1} as follows:[5]

$$P = \frac{1}{\kappa}\sinh(\kappa\theta) \qquad P^{-1} = \cosh(\kappa\theta), \quad (18)$$

where $-\infty < \theta < \infty$. Then the field $\phi(x)$ is given by

$$\phi(x) = \int \frac{d\theta}{2\pi}\tilde{\phi}(\theta)e^{(i/\kappa)\sinh(\kappa\theta)x}. \quad (19)$$

The star product simply becomes

$$e^{(i/\kappa)\sinh(\kappa\theta_1)x} \star e^{(i/\kappa)\sinh(\kappa\theta_2)x} = e^{(i/\kappa)\sinh(\kappa(\theta_1+\theta_2))x}. \quad (20)$$

Here we notice that the nontrivial momentum sum, which comes from the star product, can be described by the usual sum of θ. In fact, from (2), we can find that the commutation relation between $\hat{\theta} = \frac{1}{\kappa}\sinh^{-1}(\kappa\hat{P})$ and $\hat{x}$ becomes

$$[\hat{\theta}, \hat{x}] = -i, \quad (21)$$

and, from (13), the coproduct of $\hat{\theta}$ becomes

$$\Delta\hat{\theta} = \hat{\theta}\otimes 1 + 1 \otimes \hat{\theta}, \quad (22)$$

which is the usual Leibnitz rule.

Using (19) and (20), the equation of motion (17) becomes

$$\int \frac{d\theta}{2\pi}\bigg(-\frac{1}{\kappa^2}\sinh^2(\kappa\theta)\tilde{\phi}(\theta) + m^2\tilde{\phi}(\theta) - \lambda\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\frac{d\theta_3}{2\pi}(2\pi)\delta(\theta-\theta_1-\theta_2-\theta_3) \times \tilde{\phi}(\theta_1)\tilde{\phi}(\theta_2)\tilde{\phi}(\theta_3)\bigg)e^{(i/\kappa)\sinh(\kappa\theta)x} = 0. \quad (23)$$

Thus we find that

$$\bigg(-\frac{1}{\kappa^2}\sinh^2(\kappa\theta) + m^2\bigg)\frac{\tilde{\phi}(\theta)}{2\pi} - \lambda\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\frac{d\theta_3}{2\pi}\delta(\theta-\theta_1-\theta_2-\theta_3) \times \tilde{\phi}(\theta_1)\tilde{\phi}(\theta_2)\tilde{\phi}(\theta_3) = 0. \quad (24)$$

Next we define

$$h(x) = \int\frac{d\theta}{2\pi}\tilde{\phi}(\theta)e^{i\theta x}. \quad (25)$$

Rewriting (24) with $h(x)$, we obtain an equation of motion for $h(x)$:

$$\frac{1}{\kappa^2}\sin^2(\kappa\partial)h(x) + m^2h(x) - \lambda h^3(x) = 0. \quad (26)$$

Now the equation has a familiar local interaction term, but has infinite higher derivative terms. Another very important feature is that, though the star product (20) and hence (17) are not invariant under the simple translation $x \rightarrow x + a$, Eq. (26) has the obvious translational symmetry.

To analyze (26), we may consider an action for $h(x)$, which is given by

$$S_h = \int dx\bigg[-\frac{1}{2\kappa^2}\sin(\kappa\partial)h(x)\sin(\kappa\partial)h(x) + \frac{1}{2}m^2h^2(x) - \frac{\lambda}{4}h^4(x) - \frac{m^4}{4\lambda}\bigg]. \quad (27)$$

Then the problem becomes to find the minimum of the energy $E_h = -S_h$ with an appropriate boundary condition at the infinities $x \rightarrow \pm\infty$, where the field takes the vacuum values $h = \pm m/\sqrt{\lambda}$.

In this regard, we will consider perturbation in κ. The energy can be expanded in the form

$$E_h = -S_h = \int dx\bigg[\frac{1}{2}\bigg(\sum_{n=1}^{\infty}\kappa^{2n-2}C_n\partial^n h(x)\partial^n h(x)\bigg) + V(h(x))\bigg], \quad (28)$$

where $C_n = 2^{n-1}/(n!(2n-1)!!)$ and $V(h(x)) = -\frac{1}{2}m^2h^2(x) + \frac{\lambda}{4}h^4(x) + \frac{m^4}{4\lambda} \geq 0$. The positivity of all the coefficients C_n will play an essential role in the following discussions.

Let us rescale $x^i \rightarrow x'^i = \mu x^i$ $(0 < \mu < \infty)$ and define $h^{(\mu)}(x) = h(\mu x)$. Derrick's theorem [63] tells us that, if the

[4] In one dimension, there is no nontrivial noncommutativity of coordinates, but the coordinate and the momentum are noncommutative as in (2). Thus a soliton solution is not the same as the commutative case.

[5] When one considers only spatial directions, one can safely take only the positive branch of P^{-1} in (10).

energy for the rescaled field does not have any stationary points with respect to μ, there exist no soliton solutions. In our case, the energy for $h^{(\mu)}(x)$ is given by

$$E_{h^{(\mu)}} = \int dx\left[\frac{1}{2}\left(\sum_{n=1}^{\infty}\kappa^{2n-2}C_n\partial^n h^{(\mu)}(x)\partial^n h^{(\mu)}(x)\right) + V(h^{(\mu)}(x))\right] \tag{29}$$

$$= \int dx'\frac{1}{\mu}\left[\frac{1}{2}\left(\sum_{n=1}^{\infty}\mu^{2n}\kappa^{2n-2}C_n\partial'^n h(x')\partial'^n h^{(\mu)}(x')\right) + V(h^{(\mu)}(x'))\right] \tag{30}$$

$$= \frac{1}{\mu}E_0 + \sum_{n=1}^{\infty}\mu^{2n-1}E_{2n}, \tag{31}$$

where

$$E_0 = \int dxV(h(x)), \qquad E_{2n} = \frac{C_n}{2}\int dx(\partial^n h(x))^2. \tag{32}$$

All the E_0 and E_{2n} are non-negative in general. For an $h(x)$ connecting the distinct vacua, E_0 and at least some of the E_{2n} are positive. Therefore (31) diverges at $\mu \to +0, +\infty$ (or a finite μ_c),[6] and takes a minimum value at a positive finite μ. Thus we conclude that a domain wall solution in this noncommutative field theory is possible.

C. The perturbative solution of $h(x)$

Next we consider the perturbative solution of $h(x)$. We write the perturbation series as $h(x) = h_0(x) + \kappa^2 h_2(x) + \kappa^4 h_4(x) + \cdots$. Inserting this into the equation of motion (26), we obtain for each order of κ^2,

$$\partial^2 h_0(x) + 2h_0(x) - 2h_0^3(x) = 0, \tag{33}$$

$$\partial^2 h_2(x) + 2h_2(x) - 6h_0^2(x)h_2(x) - \tfrac{1}{3}\partial^4 h_0(x) = 0, \tag{34}$$

$$\partial^2 h_4(x) + 2h_4(x) - 6h_0^2(x)h_4(x) - \frac{1}{3}\partial^4 h_2(x) + \frac{2}{45}\partial^6 h_0(x) - 6h_0(x)h_2^2(x) = 0, \tag{35}$$

$$\vdots$$

where we have set $m^2 = 2$, $\lambda = 2$ for simplicity.

Our purpose is to obtain kink solutions whose boundary condition is given by $h(x = \pm\infty) = \pm 1$. Equation (33) is the same as the equation of motion in the commutative case. The general solution of (33) has two integration constants. One is interpreted as the translation of the solution, and the other can be determined by the behavior at $x = -\infty$ or ∞. If one assumes $h_0(x = \pm\infty) \neq \pm 1$, $h_0(x)$ diverges or oscillates at $x = \pm\infty$. For such an $h_0(x)$, the solutions of $h_{2n}(x)$ $(n = 1, 2, \ldots)$ diverge at $x = \pm\infty$, unless $h_{2n}(x = \pm\infty) = 0$. Thus the boundary condition $h(x = \pm\infty) = \pm 1$ cannot be satisfied by the perturbative solution, unless we assume $h_0(x = \pm\infty) = \pm 1$.

For the boundary condition $h_0(x = \pm\infty) = \pm 1$, the solution to Eq. (33) is well known and given by

$$h_0(x) = \tanh(x + a), \tag{36}$$

where $a \in \mathbb{R}$. The arbitrary parameter a results from the translational invariance of Eq. (33).

Next we will solve Eq. (34) for $a = 0$. Let us put

$$h_2(x) = \frac{f(x)}{\cosh^2 x}. \tag{37}$$

Inserting this and (36) for $a = 0$ into (34), we obtain

$$f''(x) - 4\tanh x f'(x) - \frac{8}{3}\left(2\frac{\tanh x}{\cosh^2 x} - \tanh^3 x\right) = 0. \tag{38}$$

Then let us put

$$f'(x) = \cosh^4(x)g(x), \tag{39}$$

and insert this into (38). The equation becomes

$$g'(x) = \frac{8}{3\cosh^4 x}\left(2\frac{\tanh x}{\cosh^2 x} - \tanh^3 x\right). \tag{40}$$

Integrating (40) over x, we obtain

$$g(x) = \frac{2}{3\cosh^4 x} - \frac{4}{3\cosh^6 x} + A_1, \tag{41}$$

where A_1 is an integration constant. Thus the differential equation of $f(x)$ becomes

$$f'(x) = \frac{2}{3} - \frac{4}{3\cosh^2 x} + A_1\cosh^4 x. \tag{42}$$

Integrating this over x and using (37), we obtain

$$h_2(x) = \frac{2x}{3\cosh^2 x} - \frac{4\tanh x}{3\cosh^2 x} + A_1\left(\frac{3x}{8\cosh^2 x} + \frac{3}{8}\tanh x + \frac{1}{4}\cosh^2 x\tanh x\right) + \frac{A_2}{\cosh^2 x}, \tag{43}$$

where A_2 is an integration constant.

Since the term with A_1 is divergent at $x = \pm\infty$, we have to put $A_1 = 0$ from the boundary condition. The A_2 term is allowed but can just be absorbed into the parameter a in (36), because $\tanh(x + \kappa^2 A_2) = \tanh(x) + \kappa^2 A_2/\cosh^2(x) + \cdots$. To systematically kill such redundant integration constants, we impose the oddness condition, $h_{2n}(x) = -h_{2n}(-x)$ for $a = 0$. Then $A_2 = 0$ is also required. Finally, recovering the parameter a, we obtain

$$h_2(x) = \frac{2(x + a)}{3\cosh^2(x + a)} - \frac{4\tanh(x + a)}{3\cosh^2(x + a)}. \tag{44}$$

[6] For example, the convergence radius of the infinite sum is $|\mu| < \mu_c = \pi/4\kappa$ for $h(x) = \tanh(x)$.

In the same way, we can obtain the solution to Eq. (35), which is given by

$$h_4(x) = \frac{134(x+a)}{45\cosh^2(x+a)} - \frac{8(x+a)}{3\cosh^4(x+a)} - \frac{40\tanh(x+a)}{9\cosh^2(x+a)} - \frac{4(x+a)^2\tanh(x+a)}{9\cosh^2(x+a)} + \frac{52\tanh(x+a)}{9\cosh^4(x+a)}. \tag{45}$$

This procedure will be able to be repeated to a required order.

D. The solution of $\phi(x)$ and the moduli space

In the preceding subsection, we have obtained the perturbative solution of $h(x)$. Then we formally know the perturbative soliton solution of $\phi(x)$ through $\tilde{\phi}(\theta)$, which are related to $\phi(x)$ and $h(x)$ by (19) and (25), respectively.

In the following let us discuss the moduli space of the domain wall solution. In $h(x)$, the moduli parameter is just the translation parameter a. This translation corresponds to the phase rotation $\tilde{\phi}(\theta) \to e^{ia\theta}\tilde{\phi}(\theta)$, as can be seen in (25). Therefore the translation on $\phi(x)$ is given by

$$T_a\phi(x) = \int \frac{d\theta}{2\pi}\tilde{\phi}(\theta)e^{i(\theta a + (1/\kappa)\sinh(\kappa\theta)x)} \tag{46}$$

$$= e^{ia\hat{\theta}}\phi(x). \tag{47}$$

This last expression shows that the operator $\hat{\theta}$, which is a nonlinear function of $\hat{P}$, is the generator of the translational moduli. In fact, by using the Leibnitz rule (22) and following the same procedure as a conventional symmetry, one can directly show that, if $\phi(x)$ is a solution to the equation of motion (17), $e^{ia\hat{\theta}}\phi(x)$ is also a solution. The generalization of this fact to the general Hopf algebraic translational symmetry will be discussed in Sec. III.

E. The moduli field from the Hopf algebraic translational symmetry

Another interesting consequence of the conventional translational symmetry in a field theory is the existence of a massless propagating field along a domain wall. This field can be obtained by generalizing the constant moduli parameter a to a field $a(x_\parallel)$ depending on the coordinates along a domain wall. In this subsection, we will study this aspect in our noncommutative field theory.

We go back to the three-dimensional case. For simplicity, we set $\kappa = 1$. We change the variable $P^i(g)$ as follows:

$$P^i = \sinh(\sqrt{k^2})\frac{k^i}{\sqrt{k^2}}. \tag{48}$$

This k^i is the three-dimensional analog of θ in the previous subsections. The field $\phi(x)$ can be rewritten as

$$\begin{aligned}\phi(x) &= \int \frac{d^3P}{(2\pi)^3\sqrt{1+P^2}}\tilde{\phi}(P)e^{iP\cdot x} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{\sinh^2(\sqrt{k^2})}{k^2}\tilde{\phi}(k)e^{i\sinh(\sqrt{k^2})k^i/\sqrt{k^2}x_i} \\ &\equiv \int \frac{d^3k}{(2\pi)^3}\tilde{\varphi}(k)e^{i\sinh(\sqrt{k^2})k^i/\sqrt{k^2}x_i}.\end{aligned} \tag{49}$$

Let us define

$$h(\hat{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{\varphi}(k)e^{ik\cdot\hat{x}} \tag{50}$$

as in (25). Then it can be shown that the action (15) is equivalent to the following action [58]:

$$S = \langle 0|\bigg(-\frac{1}{2}h(\hat{x})[\hat{P}^i,[\hat{P}_i,h(\hat{x})]] + \frac{1}{2}m^2h(\hat{x})^2 - \frac{\lambda}{4}h(\hat{x})^4\bigg)|0\rangle, \tag{51}$$

where $|0\rangle$ denotes the momentum zero eigenstate $\hat{P}^i|0\rangle = 0$, and

$$[\hat{P}^i,\hat{x}^j] = -i\eta^{ij}\sqrt{1+\hat{P}^2} + i\epsilon^{ijk}\hat{P}_k, \tag{52}$$

$$[\hat{P}^i,\hat{P}^j] = 0. \tag{53}$$

From the commutation relation, the following relation is satisfied [58]:

$$\hat{P}^i e^{ik\cdot\hat{x}}|0\rangle = \sinh(\sqrt{k^2})\frac{k^i}{\sqrt{k^2}}e^{ik\cdot\hat{x}}|0\rangle = P^i e^{ik\cdot\hat{x}}|0\rangle. \tag{54}$$

Thus $e^{ik\cdot\hat{x}}|0\rangle$ is the eigenstate of $\hat{P}^i$ with an eigenvalue P^i. In the following discussions, we use the notation $|P^i\rangle \equiv e^{ik\cdot\hat{x}}|0\rangle$.

The equation of motion from (51) is

$$(-[\hat{P}^2,h(\hat{x})] + m^2h(\hat{x}) - \lambda h(\hat{x})^3)|0\rangle = 0. \tag{55}$$

As has been discussed in the preceding subsections, there exists a one-parameter family of domain wall solutions $h^a_{\rm sol}(\hat{x})$ to (55), where a is the translational parameter. One may expand the solution with respect to a as $h^a_{\rm sol}(\hat{x}) = h_{\rm sol}(\hat{x}^1) + af(\hat{x}^1) + \cdots$, where we have chosen $\hat{x}^1$ as the spatial direction perpendicular to the domain wall.[7] Then, putting this expansion into (55) and taking the first order of a, $f(\hat{x}^1)$ is shown to satisfy

$$(-[\hat{P}^2,f(\hat{x}^1)] + m^2f(\hat{x}^1) - 3\lambda(h_{\rm sol}(\hat{x}^1))^2f(\hat{x}^1))|0\rangle = 0. \tag{56}$$

To study the property of the moduli field, we will replace a to $a(\hat{x}_0,\hat{x}_2)$. In doing so, the braiding property (16) plays essential roles. For general $h_1(\hat{x})$, $h_2(\hat{x})$, we have the fol-

[7] The following discussions do not depend on the value of a where the expansion with respect to a is carried out.

lowing commuting property:

$$\begin{aligned} h_1(\hat{x})h_2(\hat{x}) &= \int dg_1 \int dg_2 \tilde{\phi}_1(g_1)\tilde{\phi}_2(g_2) e^{ik(g_1)\cdot\hat{x}} e^{ik(g_2)\cdot\hat{x}} \\ &= \int dg_1 \int dg_2 \tilde{\phi}_2(g_2)\tilde{\phi}_1(g_2^{-1}g_1g_2) \\ &\quad \times e^{ik(g_1)\cdot\hat{x}} e^{ik(g_2)\cdot\hat{x}} \\ &= \int dg_1 \int dg_2 \tilde{\phi}_2(g_2)\tilde{\phi}_1(g_1) e^{ik(g_2g_1g_2^{-1})\cdot\hat{x}} \\ &\quad \times e^{ik(g_2)\cdot\hat{x}} \\ &= \int dg_1 \int dg_2 \tilde{\phi}_2(g_2)\tilde{\phi}_1(g_1) e^{ik(g_2g_1)\cdot\hat{x}} \\ &= h_2(\hat{x})h_1(\hat{x}), \end{aligned} \tag{57}$$

where we have used the invariance of the Haar measure. Inserting $h(\hat{x}) = h_{\rm sol}(\hat{x}^1) + a(\hat{x_0}, \hat{x_2})f(\hat{x}^1)$ into the equation of motion (55) and taking the first order of $a(\hat{x_0}, \hat{x_2})$, we obtain

$$\begin{aligned} &(-[\hat{P}^2, a(\hat{x_0}, \hat{x_2})f(\hat{x}^1)] + m^2 a(\hat{x_0}, \hat{x_2})f(\hat{x}^1) \\ &\quad - 3\lambda a(\hat{x_0}, \hat{x_2})(h_{\rm sol}(\hat{x}^1))^2 f(\hat{x}^1))|0\rangle = 0. \end{aligned} \tag{58}$$

Then, from (56), we obtain

$$[\hat{P}^2, a(\hat{x_0}, \hat{x_2})]f(\hat{x}^1)|0\rangle = 0. \tag{59}$$

After the Fourier transformation, we find

$$\int_{P_1}\int_{P_2} \tilde{a}(P_1)\tilde{f}(P_2)(P(g_1g_2)^2 - P_2^2)|P(g_1g_2)\rangle = 0, \tag{60}$$

where $P_1^i = (P_1^0, 0, P_1^2)$ and $P_2^i = (0, P_2, 0)$. From the formula of the coproduct of P^2, which is given by

$$\begin{aligned} \Delta(P^2) &= P^2 \otimes 1 + 1 \otimes P^2 + P^2 \otimes P^2 \\ &\quad + 2\sqrt{1+P^2}P^i \otimes \sqrt{1+P^2}P_i + P^iP^j \otimes P_iP_j, \end{aligned} \tag{61}$$

Eq. (60) becomes

$$\int_{P_1}\int_{P_2} P_1^2 \tilde{a}(P_1)(1+P_2^2)\tilde{f}(P_2)|P(g_1g_2)\rangle = 0. \tag{62}$$

Operating $\langle x|$ from the left, we find

$$\begin{aligned} &\int_{P_1}\int_{P_2} P_1^2 \tilde{a}(P_1)(1+P_2^2)\tilde{f}(P_2) e^{iP(g_1g_2)\cdot x} \\ &= \int_{P_1}\int_{P_2} P_1^2 \tilde{a}(P_1)(1+P_2^2)\tilde{f}(P_2) e^{iP_1\cdot x} \star e^{iP_2\cdot x} \\ &= -\partial^2 a(x_0, x_2)(1-\partial^2)f(x_1) = 0. \end{aligned} \tag{63}$$

Thus, since $(1-\partial^2)f(x_1)$ does not vanish constantly, we obtain

$$\partial^2 a(x_0, x_2) = 0. \tag{64}$$

Thus we conclude that the moduli field is massless.

The preceding discussions in the operator formalism can be repeated with the star product. Putting the expansion $\phi_{\rm sol}^a(x) = \phi_{\rm sol}(x^1) + ag(x^1) + \cdots$ into the equation of motion (17), one obtains

$$\partial^2 g(x^1) + m^2 g(x^1) - 3\lambda\phi_{\rm sol}(x^1) \star \phi_{\rm sol}(x^1) \star g(x^1) = 0. \tag{65}$$

Next we define the moduli field $a(x^0, x^2)$, and consider $\phi(x) = \phi_{\rm sol}(x^1) + a(x^0, x^2) \star g(x^1)$. Putting this into the equation of motion and taking the first order of $a(x^0, x^2)$, we obtain

$$\begin{aligned} &\partial^2(a(x^0, x^2) \star g(x^1)) + m^2(a(x^0, x^2) \star g(x^1)) \\ &\quad - 3\lambda\phi_{\rm sol}(x^1) \star \phi_{\rm sol}(x^1) \star a(x^0, x^2) \star g(x^1) = 0, \end{aligned} \tag{66}$$

where we have used the property similar to (16) for the star product. The first term of (66) can easily be computed by using the coproduct of P^2. Using (61) and (10), $\partial^2(a \star g)$ becomes

$$\begin{aligned} \partial^2(a(x^0, x^2) \star g(x^1)) &= a(x^0, x^2) \star \partial^2 g(x^1) + \partial^2 a(x^0, x^2) \\ &\quad \star g(x^1) - \partial^2 a(x^0, x^2) \star \partial^2 g(x^1). \end{aligned} \tag{67}$$

Thus (66) becomes

$$\begin{aligned} 0 &= a(x^0, x^2) \star (\partial^2 g(x^1) + m^2 g(x^1) - 3\lambda g(x^1) \star \phi_{\rm sol}(x) \\ &\quad \star \phi_{\rm sol}(x)) + \partial^2 a(x^0, x^2) \star g(x^1) \\ &\quad - \partial^2 a(x^0, x^2) \star \partial^2 g(x^1) \\ &= (g(x^1) - \partial^2 g(x^1)) \star \partial^2 a(x^0, x^2), \end{aligned} \tag{68}$$

where we have used (65). Thus we obtain the same conclusion as above.

III. THE GENERAL HOPF ALGEBRAIC TRANSLATIONAL SYMMETRY

In the preceding section, the discussions are restricted to the specific noncommutative field theory. However, it is interesting to know what holds for the general Hopf algebraic translational symmetry. In this section, we will show that the results in the preceding section are the general consequence of a Hopf algebraic translational symmetry.

We first assume that, in considering domain wall solutions, only one direction of momentum is relevant. Then the (associative) coproduct of the momentum may be written as

$$\Delta(\hat{P}) = \sum_i a_i(\hat{P}) \otimes b_i(\hat{P}). \tag{69}$$

This defines the associative sum of two momenta $\oplus$.

Let us consider a small momentum P_ε. One may consider its n sum,

$$\overbrace{P_n \equiv P_\varepsilon \oplus P_\varepsilon \oplus \cdots \oplus P_\varepsilon^n}^{n}. \tag{70}$$

For such P_n, let us define

$$\theta(P_n) = nP_\varepsilon. \tag{71}$$

Then $\theta(P)$ can be shown to define an additive quantity for $\oplus$ as

$$\theta(P_n \oplus P_m) = \theta(P_{n+m}) \tag{72}$$

$$= (n+m)P_\varepsilon \tag{73}$$

$$= \theta(P_n) + \theta(P_m), \tag{74}$$

where we have used the associativity of $\oplus$. This shows the usual Leibnitz rule for the coproduct of $\hat{\theta}$,

$$\Delta(\hat{\theta}) = \hat{\theta} \otimes 1 + 1 \otimes \hat{\theta}. \tag{75}$$

The above discussions may be generalized to negative n's, and further to a continuous momentum by considering the limit $P_\varepsilon \to 0$.

In the actual computation, it is convenient to consider a differential equation for $\theta(P)$ as

$$\frac{d\theta(P)}{dP} = \lim_{P_\varepsilon \to 0} \frac{\theta(P_\varepsilon \oplus P) - \theta(P)}{P_\varepsilon \oplus P - P} = \lim_{P_\varepsilon \to 0} \frac{P_\varepsilon}{P_\varepsilon \oplus P - P}. \tag{76}$$

The last limit can be computed from a given coproduct of momentum.[8] The initial condition should be taken as $\theta(0) = 0$.

For the noncommutative field theory in the preceding section, the coproduct of momentum is given by (13), and the differential equation (76) becomes

$$\frac{d\theta(P)}{dP} = \lim_{P_\varepsilon \to 0} \frac{P_\varepsilon}{\sqrt{1+\kappa^2 P^2}P_\varepsilon + \sqrt{1+\kappa^2 P_\varepsilon^2}P - P} = \frac{1}{\sqrt{1+\kappa^2 P^2}}. \tag{77}$$

With the initial condition $\theta(0) = 0$, the solution is actually given by $P = \frac{1}{\kappa}\sinh(\kappa\theta)$, which agrees with (18).

As explained in Sec. II E, the usual Leibnitz rule (75) for $\hat{\theta}$ implies that $e^{ia\hat{\theta}}\phi(x)$ forms a one-parameter family of domain wall solutions, provided that $\phi(x)$ is such a solution. It would be physically reasonable to assume that there exists at least one domain wall solution which connects distinct vacua with the same energy, if a theory has multiple vacua and is physically sensible. Therefore a noncommutative field theory possessing a Hopf algebraic translational symmetry will have a one-parameter family of domain wall solutions, if it has multiple vacua with the same energy. The associated moduli field will also have a vanishing mass, since the zero mode of the moduli field is the parameter itself, and its potential should be flat in this direction.

[8]For the limit to have a finite value, $0 \oplus P = P$ is necessary. This is mathematically obtained from the axiom $(\mathrm{id} \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{id})\Delta = 1$ with $\epsilon(P) = 0$, where ϵ is the counit map.

IV. SUMMARY AND COMMENTS

We have studied the domain wall soliton and its moduli field in the braided ϕ^4 noncommutative field theory in the three-dimensional Lie-algebraic noncommutative spacetime $[x^i, x^j] = 2i\kappa\epsilon^{ijk}x_k$. This noncommutative spacetime is known to have a Hopf algebraic translational symmetry, and provides an interesting stage for investigating the physical roles of a Hopf algebraic translational symmetry on domain walls. We have found that there exists a one-parameter family of the solutions, and the mass of the moduli field propagating along the domain wall vanishes. We have also argued that the results should also hold in the general noncommutative field theory with a Hopf algebraic translational symmetry. This conclusion agrees with what can be obtained from the conventional translational symmetry of the usual field theory. Therefore our results show another evidence for the physical importance of Hopf algebraic symmetries as much as the standard Lie-algebraic symmetries.

Two comments are in order. First, we have used the braiding property when we analyze the equation of motion for the moduli field. Therefore the nontrivial statistics of both the domain wall and the moduli field seem to play significant roles in their dynamics. Second, in our discussions, the operator $\hat{\theta}$, which has a Lie-algebraic coproduct, plays essential roles. The general derivation of $\hat{\theta}$ in the preceding section is based on that there is only one relevant direction. Therefore, if one considers higher dimensional topological objects such as instantons, it is not at all clear whether we will obtain results in parallel with the usual field theories.

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