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Massive Yang-Mills theory based on the nonlinearly realized gauge group

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We propose a subtraction scheme for a massive Yang-Mills theory realized via a nonlinear representation of the gauge group [here $SU(2)$]. It is based on the subtraction of the poles in $D - 4$ of the amplitudes, in dimensional regularization, after a suitable normalization has been performed. Perturbation theory is in the number of loops, and the procedure is stable under iterative subtraction of the poles. The unphysical Goldstone bosons, the Faddeev-Popov ghosts, and the unphysical mode of the gauge field are expected to cancel out in the unitarity equation. The spontaneous symmetry breaking parameter is not a physical variable. We use the tools already tested in the nonlinear sigma model: hierarchy in the number of Goldstone boson legs and weak-power-counting property (finite number of independent divergent amplitudes at each order). It is intriguing that the model is naturally based on the symmetry $SU(2)_L$ local $\otimes$ $SU(2)_R$ global. By construction the physical amplitudes depend on the mass and on the self-coupling constant of the gauge particle and moreover on the scale parameter of the radiative corrections. The Feynman rules are in the Landau gauge.

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I. INTRODUCTION

With this work we outline a theoretical framework for the explicit evaluation of the Feynman amplitudes of a massive Yang-Mills theory in its perturbative loop expansion. We propose a subtraction scheme for the divergences at $D = 4$ and a robust set of symmetry requirements for the vertex functional in order to guarantee stability under the subtraction procedure, physical unitarity, and predictivity.

Quantization of non-Abelian gauge theories is a subject with a long history in quantum field theory. The perturbative treatment of non-Abelian gauge models was boosted by the observation that the Yang-Mills action [1] can be gauge fixed in such a way to guarantee physical unitarity together with renormalizability by power counting (in the absence of anomalies) [2,3]. The discovery of the nilpotent Becchi-Rouet-Stora-Tyutin (BRST) symmetry [4] then provided a powerful and elegant tool to study algebraically the gauge theories and, in particular, physical unitarity to all orders in the perturbative expansion [5]. The implementation of the BRST symmetry by the Slavnov-Taylor (ST) identity [6] has boosted unexpected progress in quantum field theory (see e.g. [7,8] and references therein).

As is well-known, within this framework a mass term for the non-Abelian gauge field can be accounted for by enlarging the physical spectrum. In fact the mass generation through spontaneous symmetry breaking [9] in the presence of a linearly realized gauge symmetry requires the introduction of (at least) one physical scalar field, known as the Higgs field. Power-counting renormalizability is preserved under this extension [10].

The latter field-theoretic paradigm has led to the extremely successful standard model of particle physics. Still, the question of the origin of spontaneous symmetry breaking remains to be elucidated from the theoretical point of view, and the experimental evidence of the existence of a Higgs particle is still waited for.

This paper is devoted to the analysis of a different approach to the subtraction of the divergences of the massive Yang-Mills theory which relies on the use of a nonlinearly realized gauge group through the introduction of a flat connection. This strategy has been applied in [11–17] to the four-dimensional $SU(2)$ nonlinear sigma model. There the flat connection was coupled to an external vector source transforming as a background gauge field under the local $SU(2)_L$ left symmetry[1] which implements the $SU(2)_L$ invariance of the Haar measure in the path integral.

The present approach can be compared with the infinite mass limit of the Higgs model in the linear case. This has been already done in the case of the sigma model in Ref. [15]. The same conclusions about the absence of a general criterion for an unambiguous removal of the $\log M_H$ parts apply here [16].

In a previous work [17] we found a very powerful technique for integrating the functional equation derived from the invariance of the path-integral Haar measure under local $SU(2)_L$ transformations. Our strategy in building a massive Yang-Mills theory is based on the same technique. We use the gauge field A_μ and the nonlinear sigma model field Ω to construct a *bleached* gauge field a_μ,

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[1] The left symmetry acts on the $SU(2)$ element from the left. In the following a global $SU(2)_R$ symmetry will also be introduced, acting on the group element from the right.

$$a_\mu \equiv \Omega^\dagger A_\mu \Omega + i\Omega^\dagger \partial_\mu \Omega, \qquad (1)$$

which is invariant under $SU(2)_L$ transformations. Notice that each element of the 2×2 matrix is invariant. This opens far too many possibilities than expected for constructing a gauge theory. In order to recover the classical form of the massive Yang-Mills theory, we introduce some more constraints. In particular we impose global $SU(2)_R$ invariance [invariance under local right $SU(2)$ transformation would forbid a mass term]. By this requirement all "right" indices are saturated, and consequently the number of invariants is drastically reduced. This will not be enough. Therefore we will impose other constraints, suggested by our previous works on the nonlinear sigma model. They are aimed at controlling the severe divergences due to the presence of the nonlinear realization of the gauge transformations: weak power counting and hierarchy. The first requirement controls the number of independent divergent amplitudes, while the second guarantees that the amplitudes involving the unphysical Goldstone field (descendant amplitudes) are determined by the amplitudes of the ancestor fields (gauge fields, Faddeev-Popov fields, composite fields associated with nonlinear transformations, etc., i.e. most of the field content present in a power-counting renormalizable gauge theory).

With this set of constraints we get a field theoretical model in the Landau gauge which classically describes a massive non-Abelian gauge field interacting with the Faddeev-Popov (FP) ghosts and nonpolynomially with the unphysical Goldstone bosons. We stress that the model is BRST-, local $SU(2)_L$-, and global $SU(2)_R$-invariant and moreover it satisfies the necessary conditions for the validity of the weak-power-counting theorem. We prove that the resulting equations for the 1-PI generating functional (ST identity, local functional equation, ghost equation, and Landau gauge equation) are valid for the amplitudes constructed in D dimensions by using the Feynman rules for the loop expansion of the model (without any subtraction). Moreover we demonstrate that minimal subtraction for the limit $D = 4$ yields a consistent theory in terms of the parameters of the tree-level effective action plus a mass scale for the radiative corrections. The consistency of the theory relies upon some essential facts: (i) the subtraction of the divergences is achieved by local counterterms; (ii) the number of the independent counterterms is finite at every order of the loop expansion (as a consequence of the hierarchy property and of the validity of the weak-power-counting theorem); (iii) the subtraction procedure does not modify the defining equations; (iv) the validity of the ST identity guarantees the fulfillment of physical unitarity. The last point requires that the Goldstone bosons are unphysical modes together with the FP ghosts and the massless mode present in the Landau gauge description of the vector field.

Moreover it turns out that all the external sources coupled to composite operators, which are necessary in order to perform the subtraction of the divergences, are not physical parameters. In particular K_0, the source coupled to the order parameter field ϕ_0 responsible for the spontaneous breakdown of the gauge symmetry, is not physical. Furthermore the physical amplitudes do not depend on $v \equiv \langle \phi_0 \rangle$.

The proof of physical unitarity (cancellation of unphysical states) has been given in Ref. [18] both in the diagrammatic and in the operatorial formalism, under quite general assumptions which are fulfilled by the subtraction scheme discussed in the present paper. The question of possible violations of the Froissart unitarity bounds [19,20] that may occur at fixed perturbative order and the related issue of resummation of the perturbative series will not be dealt with here.

It is somewhat important to investigate the symmetry properties of the counterterms by cohomological methods. For this purpose we consider the ST equation and the local functional equation at the one-loop level (the linearized ST and local functional equations). The aim is to provide a basis for the counterterms in terms of local invariant solutions of these equations. These solutions are parametrized by representatives of the cohomology of the linearized ST operator on the space spanned by the local solutions of the linearized functional equation [i.e. the variables bleached by a procedure similar to the one used in Eq. (1)].

We have structured the paper according to the logical sequence by which the requirements are imposed on the field theoretical model. In Sec. II we construct the bleached fields according to the nonlinear realization of the gauge group. The presence of unwanted invariants suggests imposing the symmetry under global $SU(2)_R$ transformations. In Sec. III the requirement of weak power counting is imposed. In Sec. IV the ST identity is derived, and it is shown that it is not sufficient to yield the hierarchy. In Sec. V we exploit the invariance of the path-integral measure under local gauge transformations and derive the functional equation which yields both the hierarchy and the subtraction procedure for the $D = 4$ divergences. In Sec. VI we consider the final setup of all the equations (ST identity, local functional equation, ghost equation, Landau gauge equation). In Sec. VII we prove that the unsubtracted vertex functional satisfies all the defining equations in the loop expansion. The structure of the equations suggests the subtraction procedure for the limit $D = 4$. The equations are shown to be stable after the introduction of the counterterms. In Sec. VIII we show that the whole set of identities (ST identity, local functional equation, ghost equation, Landau gauge equation) guarantees the hierarchy and thereby that Goldstone boson amplitudes (descendant) are fixed by the ancestor amplitudes. Section IX contains the implementation of weak power counting to the construction of the tree-level vertex functional $\Gamma^{(0)}$ (massive Yang-Mills theory). In Sec. X we discuss the properties of the local solutions of the linearized equations, and we list a complete set of them compatible with the required dimensions in the one-loop approximation. The conclusions are in Sec. XI. Appendix A gives the Feynman rules, Appendix B proves that ST identity is not enough in order to impose the hierarchy among the ancestor and the descendant amplitudes, Appendix C yields the proof of the weak-power-counting formula, Appendix D lists the linearized ST transforms of the bleached variables, and Appendix E is devoted to the proof of the vacuum expectation value (VEV) independence of the physical amplitudes by using an extended ST identity.

II. NONLINEARLY REALIZED GAUGE SYMMETRIES

The introduction in the Yang-Mills theory of a flat connection gives rise to a peculiar set of invariant variables which can be conveniently described by making use of the technique discussed in [17], which we will briefly summarize here. It turns out that there are many more invariants than in the usual approach based on $SU(2)$ local invariance mediated only by a vector meson. By adding extra fields and, in particular, a flat connection, one gets more terms. The usual field strength term is achieved not only by requiring an invariance under a large group, noticeably a global $SU(2)_R$ beside the local $SU(2)_L$, but also by imposing the weak-power-counting criterion. This last requirement will be dealt with later on.

We will consider a $SU(2)$ gauge group and denote by $A_\mu = A_{a\mu}\frac{\tau_a}{2}$ the gauge connection. τ_a are the Pauli matrices.

The field strength of the gauge field A_μ is defined by

$$G_{\mu\nu}[A] = G_{a\mu\nu}\frac{\tau_a}{2} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \quad (2)$$

The nonlinear sigma model field Ω is an element of the $SU(2)$ group, which is parameterized in terms of the coordinate fields ϕ_a as follows:

$$\Omega = \frac{1}{v}(\phi_0 + i\tau_a\phi_a), \qquad \Omega^\dagger\Omega = 1, \qquad \det\Omega = 1, \qquad \phi_0^2 + \phi_a^2 = v^2, \quad (3)$$

where v is a parameter with dimension equal one. We shall find out that v is not a parameter of the model, because it can be removed by a rescaling of the fields $\vec{\phi}, \phi_0$. The $SU(2)$ flat connection is

$$F_\mu = i\Omega\partial_\mu\Omega^\dagger = F_{a\mu}\frac{\tau_a}{2}, \qquad F_{a\mu} = \frac{2}{v^2}(\phi_0\partial_\mu\phi_a - \partial_\mu\phi_0\phi_a + \epsilon_{abc}\partial_\mu\phi_b\phi_c). \quad (4)$$

The field strength of F_μ vanishes since F_μ is a flat connection:

$$G_{\mu\nu}[F] = 0. \quad (5)$$

Under a local $SU(2)$ left transformation $U_L = \exp(i\alpha_a^L\frac{\tau_a}{2})$, one gets

$$\Omega' = U_L\Omega, \qquad F'_\mu = U_L F_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger, \qquad A'_\mu = U_L A_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger. \quad (6)$$

The nonlinearity of the $SU(2)$ constraint in Eq. (3) implies that the gauge symmetry is nonlinearly realized on the fields ϕ_a, whose infinitesimal transformations are

$$\delta\phi_a = \tfrac{1}{2}\phi_0\alpha_a^L + \tfrac{1}{2}\epsilon_{abc}\phi_b\alpha_c^L, \qquad \phi_0 = \sqrt{v^2 - \phi_a^2}, \qquad \delta\phi_0 = -\tfrac{1}{2}\alpha_a^L\phi_a. \quad (7)$$

Under local $SU(2)_L$ symmetry the combination $A_\mu - F_\mu$ transforms in the adjoint representation of $SU(2)$. Hence one can construct out of $A_\mu - F_\mu$ and Ω a $SU(2)_L$-bleached variable a_μ which is invariant under $SU(2)_L$ local transformations:

$$a_\mu = a_{a\mu}\frac{\tau_a}{2} = \Omega^\dagger(A_\mu - F_\mu)\Omega = \Omega^\dagger A_\mu\Omega - i\partial_\mu\Omega^\dagger\Omega. \quad (8)$$

The $SU(2)_L$ local symmetry is trivialized by the variable a_μ, since any combination of a_μ and its derivatives is $SU(2)_L$ invariant.

One can also consider local $SU(2)_R$ transformations on Ω,

$$\Omega' = \Omega U_R^\dagger, \quad (9)$$

leaving A_μ invariant.

Then one finds that a_μ transforms as a $SU(2)_R$ gauge connection:

$$a'_\mu = U_R a_\mu U_R^\dagger + iU_R\partial_\mu U_R^\dagger. \quad (10)$$

In the presence of a flat connection the interplay of left and right symmetries with renormalizability properties provides very restrictive constraints on the classical action.

In order to discuss this point we start from the Yang-Mills action in the presence of a Stückelberg mass term [18,21]:

$$S = \frac{\Lambda^{(D-4)}}{g^2}\int d^Dx\left(-\frac{1}{4}G_{a\mu\nu}[a]G_a^{\mu\nu}[a] + \frac{M^2}{2}a_{a\mu}^2\right) = \frac{\Lambda^{(D-4)}}{g^2}\int d^Dx\left(-\frac{1}{4}G_{a\mu\nu}[A]G_a^{\mu\nu}[A] + \frac{M^2}{2}(A_{a\mu} - F_{a\mu})^2\right). \quad (11)$$

Λ is a mass scale for continuation in D dimensions.

Notice that the field strength squared of a_μ coincides with the one of the gauge field $A_{a\mu}$ (since a_μ is obtained from A_μ through an operatorial gauge transformation generated by Ω).

S is invariant under local $SU(2)_L$ symmetry (since it only depends on a_μ) and also global $SU(2)_R$ symmetry. It is not invariant under local $SU(2)_R$ symmetry, since the latter forbids the Stückelberg mass term because of the transformation property given in Eq. (10).

Global $SU(2)_R$ symmetry restricts to some extent the number of independent invariants (all right indices are saturated). We find it very intriguing that the symmetry under global $SU(2)_R$ transformations is necessary in order to reproduce a massive Yang-Mills gauge theory. In fact when one uses the present theory for the electroweak model, the $SU(2)_R$ global symmetry plays the role of custodial symmetry [22]. We stress that our approach provides a natural justification of this property.

The implementation of the symmetries $SU(2)_L$ local and $SU(2)_R$ global is also of great interest. From Eq. (7) one sees clearly that $SU(2)_L$ global is spontaneously broken since the vacuum expectation value of ϕ_0 is nonzero. The same conclusion is valid for $SU(2)_R$ global. Thus only the symmetry generated by the vector currents $(L + R)$ is unitarily implemented and guarantees a global $SU(2)$ symmetry for the physical amplitudes, while the symmetry generated by the axial currents is spontaneously broken. This is another striking difference from massive Yang-Mills realized in the realm of power-counting renormalizable theories [23], where the $SU(2)$ local symmetry is spontaneously broken in its global sector.

III. WEAK POWER COUNTING I

One should notice that global $SU(2)_R$ symmetry allows for additional independent invariants which are also local $SU(2)_L$ symmetric. For instance we have the following independent Lagrangian terms of dimension: ≤ 4

$$\begin{aligned} &\int d^4x \partial_\mu a_{a\nu} \partial^\mu a_a^\nu, \qquad \int d^4x (\partial a)^2, \\ &\int d^4x a^2, \qquad \int d^4x \epsilon_{abc} \partial_\mu a_{a\nu} a_b^\mu a_c^\nu, \\ &\int d^4x (a^2)^2, \qquad \int d^4x a_{a\mu} a_b^\mu a_{a\nu} a_b^\nu. \end{aligned} \tag{12}$$

Thus the action S in Eq. (11) for $D = 4$ is not the most general Lorentz-invariant functional with couplings of dimension ≥ 0 compatible with local $SU(2)_L$ and global $SU(2)_R$ symmetry.

However, S is uniquely fixed by local $SU(2)_L$ symmetry, global $SU(2)_R$ symmetry, and the requirement of the weak-power-counting property. By this we mean that the number of superficially divergent independent amplitudes is finite at each order in the loop expansion. This property is required to be stable under the procedure of subtraction of the divergences. While the second part of the statement requires some effort, after the subtraction procedure has been given (see Sec. VII), the first part can be easily established, under the assumptions discussed in Sec. IX. The proof of this central result requires extending the main tool developed to deal with the divergences of the nonlinear sigma model, i.e. the hierarchy of the Feynman amplitudes, to the case where the gauge bosons are dynamical (see Appendix C).

The weak-power-counting property limits in a substantial way the number of independent coefficients associated with the monomials in Eq. (12). One observes that each monomial in Eq. (12) is a power series in the Goldstone field $\vec{\phi}$ and moreover it contains in some cases derivatives. The number of derivatives in the Goldstone interaction vertices is critical when one evaluates the superficial degree of divergence of a graph. Appendix A provides some relevant Feynman rules, and Appendix C gives the the superficial degree of divergence of a graph with no external Goldstone lines. In Sec. IX we prove that the number of divergent ancestor amplitudes turns out to be finite only if the monomials of Eq. (12) enter into the combination given by the invariant $(G_a^{\mu\nu})^2$ and the presence of $\vec{\phi}$ is confined in the Stückelberg mass term.

IV. SLAVNOV-TAYLOR IDENTITY I

In order to set up the perturbative framework we use the Landau gauge.

The gauge fixing is performed by BRST techniques. The BRST differential s is obtained in the usual way by promoting the gauge parameters α_a^L to the ghost fields c_a and by introducing the antighosts $\bar{c}_a$ coupled in a BRST doublet to the Nakanishi-Lautrup fields B_a:

$$\begin{aligned} s\phi_a = \tfrac{1}{2}\phi_0 c_a + \tfrac{1}{2}\epsilon_{abc}\phi_b c_c, \qquad sA_{a\mu} = (D_\mu[A]c)_a, \\ s\bar{c}_a = B_a, \qquad sB_a = 0. \end{aligned} \tag{13}$$

In the above equation $D_\mu[A]$ denotes the covariant derivative with respect to (w.r.t.) $A_{a\mu}$:

$$(D_\mu[A])_{ac} = \delta_{ac}\partial_\mu + \epsilon_{abc}A_{b\mu}. \tag{14}$$

The BRST transformation of c_a then follows by nilpotency

$$sc_a = -\tfrac{1}{2}\epsilon_{abc}c_b c_c. \tag{15}$$

The tree-level vertex functional is

$$\begin{aligned} \Gamma^{(0)} &= S + \frac{\Lambda^{(D-4)}}{g^2} s \int d^D x (\bar{c}_a \partial A_a) + \int d^D x (A^*_{a\mu} s A^\mu_a \\ &\quad + \phi^*_a s\phi_a + c^*_a s c_a) \\ &= S + \frac{\Lambda^{(D-4)}}{g^2} \int d^D x (B_a \partial A_a - \bar{c}_a \partial_\mu (D^\mu[A]c)_a) \\ &\quad + \int d^D x (A^*_{a\mu} s A^\mu_a + \phi^*_a s\phi_a + c^*_a s c_a). \end{aligned} \tag{16}$$

In $\Gamma^{(0)}$ we have also included the antifields $A^*_{a\mu}$, ϕ^*_a, and c^*_a coupled to the nonlinear BRST variations of the quantized fields.

We can assign a conserved ghost number by requiring that $A_{a\mu}$, ϕ_a, and B_a have ghost number zero; c_a has ghost number one; $\bar{c}_a$, $A^*_{a\mu}$, and ϕ^*_a have ghost number -1; and finally c^*_a has ghost number -2. With these assignments the vertex functional has zero ghost number.

The propagators derived from $\Gamma^{(0)}$ are collected in Appendix A. From Eq. (A3) one sees that the propagator for ϕ_a goes to infinity like $1/p^2$. Since in S there are interaction vertices with four ϕ's and two derivatives (coming from the square of the flat connection), already at the one-loop level there is an infinite number of divergent amplitudes with an arbitrary number of ϕ legs. This phenomenon is also present in the nonlinear sigma model and has been widely discussed in Refs. [11–17].

In the nonlinear sigma model the way out is to make use of the hierarchy principle [11] for the vertex functional, i.e. to fix the ϕ amplitudes in terms of ancestor amplitudes involving only the insertion of the flat connection and the nonlinear sigma model constraint. This is achieved by making use of the local functional equation expressing the invariance of the path-integral Haar measure under local $SU(2)_L$ transformations. The number of divergent ancestor amplitudes is in turn finite at each order in perturbation theory (weak-power-counting theorem) [12].

In the Stückelberg model the situation is somehow different. The invariance under the BRST symmetry in Eqs. (13) and (15) can be translated into the following ST identity:

$$\mathcal{S}(\Gamma^{(0)}) = \int d^Dx\left(\frac{\delta\Gamma^{(0)}}{\delta A^*_{a\mu}}\frac{\delta\Gamma^{(0)}}{\delta A^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a}\frac{\delta\Gamma^{(0)}}{\delta\phi_a} + \frac{\delta\Gamma^{(0)}}{\delta c^*_a}\frac{\delta\Gamma^{(0)}}{\delta c_a} + B_a\frac{\delta\Gamma^{(0)}}{\delta\bar{c}_a}\right) = 0. \quad (17)$$

This holds provided that the following dependence on the antifields of the tree-level vertex functional $\Gamma^{(0)}$ is imposed:

$$\frac{\delta\Gamma^{(0)}}{\delta A^*_{a\mu}} = (D^\mu[A]c)_a, \qquad \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a} = \frac{1}{2}\phi_0 c_a + \frac{1}{2}\epsilon_{abc}\phi_b c_c,$$
$$\frac{\delta\Gamma^{(0)}}{\delta c^*_a} = -\frac{1}{2}\epsilon_{abc}c_b c_c. \quad (18)$$

The ST identity for the full quantum vertex functional is

$$\mathcal{S}(\Gamma) = 0. \quad (19)$$

In power-counting renormalizable theories the ST identity is the tool to control the symmetry properties of the counterterms and to prove physical unitarity. In the present case it has some limitations; in particular, it does not imply the hierarchy property. In Appendix B an explicit counterexample is fully developed. Here we give a short and simple argument. We want to show that at least one particular amplitude, involving only one $\vec{\phi}$ field, cannot be obtained by using Eq. (19) through the hierarchy mechanism. These one-$\vec{\phi}$ field amplitudes can originate only from the relevant term of the linearized Eq. (19):

$$\int d^Dx \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a(x)}\frac{\delta}{\delta\phi_a(x)}\Gamma^{(n)}. \quad (20)$$

Let us consider a one-loop amplitude given by the integrated monomial

$$\mathcal{A}_1 \equiv \int d^Dy A^*_{a\mu}c_b\partial^\mu\phi_c\epsilon_{abc}. \quad (21)$$

The action of the linearized ST operator in Eq. (20) connects linearly $\vec{\phi}$-dependent amplitudes [such as the example in Eq. (21)] to terms with no $\vec{\phi}$ (hierarchy). We get

$$\int d^Dx \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a(x)}\frac{\delta}{\delta\phi_a(x)}\int d^Dy A^*_{a\mu}c_b\partial^\mu\phi_c\epsilon_{abc}$$
$$= v\int d^Dx A^*_{a\mu}c_b\partial^\mu c_c\epsilon_{abc}\ldots$$
$$= \frac{1}{2}v\int d^Dx A^*_{a\mu}\partial^\mu(c_b c_c)\epsilon_{abc}\ldots, \quad (22)$$

where dots represent terms with higher powers of $\vec{\phi}$, which are irrelevant since we have to put $\vec{\phi}=0$. Similarly the monomial

$$\mathcal{A}_2 \equiv \int d^Dy\partial^\mu A^*_{a\mu}c_b\phi_c\epsilon_{abc} \quad (23)$$

yields

$$\int d^Dx \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a(x)}\frac{\delta}{\delta\phi_a(x)}\int d^Dy\partial^\mu A^*_{a\mu}c_b\phi_c\epsilon_{abc}$$
$$= v\int d^Dx\partial^\mu A^*_{a\mu}c_b c_c\epsilon_{abc}\ldots. \quad (24)$$

Thus there is at least one amplitude that cannot be obtained from the hierarchy procedure since

$$\int d^Dx \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a(x)}\frac{\delta}{\delta\phi_a(x)}(2A_1 + A_2)\bigg|_{\vec{\phi}=0} = 0. \quad (25)$$

Thus the set of ancestor fields (elementary or composite) has to be enlarged in order to fix completely the descendant amplitudes, i.e. those involving one or more $\vec{\phi}$ fields. This will be done by using the functional equation that follows from the invariance of the path-integral measure under local gauge transformations.

V. LOCAL GAUGE TRANSFORMATIONS

In order to overcome the difficulties arising from the absence of a hierarchy in the ST identity, we make use of the local $SU(2)_L$ invariance of the path-integral measure. While the classical action in Eq. (11) is invariant under local gauge transformations in Eq. (6), the gauge-fixing term in Eq. (16) is not. In fact if we extend the gauge transformations (6) to the ghost fields by

$$\delta c_a = \epsilon_{abc}c_b\alpha^L_c, \qquad \delta\bar{c}_a = \epsilon_{abc}\bar{c}_b\alpha^L_c, \quad (26)$$

we get

$$\delta S_{GF} = -\Lambda^{D-4}\int d^D x \partial^\mu \alpha_a^L(x)(sD_\mu[A]\bar{c})_a, \tag{27}$$

where use has been made of the fact that the BRST differential s and the generator of infinitesimal gauge transformation δ are commuting operators

$$[s, \delta] = 0. \tag{28}$$

In order to implement the gauge transformations properties for the 1-PI vertex functional, we have to introduce a new set of external sources coupled to the relevant composite operators. Thus the tree-level 1-PI vertex functional becomes

$$\begin{aligned}\Gamma^{(0)} = S &+ \frac{\Lambda^{D-4}}{g^2} s \int d^D x(\bar{c}_a \partial^\mu A_{a\mu}) \\ &+ \frac{\Lambda^{D-4}}{g^2}\int d^D x(V_a^\mu s(D_\mu[A]\bar{c})_a + \Theta_a^\mu (D_\mu[A]\bar{c})_a) \\ &+ \int d^D x(A^*_{a\mu} sA_a^\mu + \phi_0^* s\phi_0 + \phi_a^* s\phi_a \\ &+ c_a^* sc_a + K_0\phi_0).\end{aligned} \tag{29}$$

This can be recasted in the following form:

$$\begin{aligned}\Gamma^{(0)} = S &+ \frac{\Lambda^{D-4}}{g^2}\int d^D x(B_a(D^\mu[V](A_\mu - V_\mu))_a \\ &- \bar{c}_a(D^\mu[V]D_\mu[A]c)_a) + \frac{\Lambda^{D-4}}{g^2} \\ &\times \int d^D x \Theta_a^\mu (D_\mu[A]\bar{c})_a + \int d^D x(A^*_{a\mu} sA_a^\mu \\ &+ \phi_0^* s\phi_0 + \phi_a^* s\phi_a + c_a^* sc_a + K_0\phi_0).\end{aligned} \tag{30}$$

The gauge-fixing part can be interpreted as the background gauge fixing [24] in the presence of the background connection $V_{a\mu}$.

The tree-level vertex functional in Eq. (30) fulfills a local functional equation which has to be preserved by the quantization procedure (which includes the subtraction of the divergences)

$$\begin{aligned}\mathcal{W}(\Gamma) \equiv \int d^D x \alpha_a^L(x)\Big(&-\partial_\mu \frac{\delta\Gamma}{\delta V_{a\mu}} + \epsilon_{abc}V_{c\mu}\frac{\delta\Gamma}{\delta V_{b\mu}} - \partial_\mu\frac{\delta\Gamma}{\delta A_{a\mu}} + \epsilon_{abc}A_{c\mu}\frac{\delta\Gamma}{\delta A_{b\mu}} + \epsilon_{abc}B_c\frac{\delta\Gamma}{\delta B_b} + \frac{1}{2}K_0\phi_a \\ &+ \frac{1}{2}\frac{\delta\Gamma}{\delta K_0}\frac{\delta\Gamma}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta\Gamma}{\delta\phi_b} + \epsilon_{abc}\bar{c}_c\frac{\delta\Gamma}{\delta\bar{c}_b} + \epsilon_{abc}c_c\frac{\delta\Gamma}{\delta c_b} + \epsilon_{abc}\Theta_{c\mu}\frac{\delta\Gamma}{\delta\Theta_{b\mu}} + \epsilon_{abc}A^*_{c\mu}\frac{\delta\Gamma}{\delta A^*_{b\mu}} \\ &+ \epsilon_{abc}c_c^*\frac{\delta\Gamma}{\delta c_b^*} + \frac{1}{2}\phi_0^*\frac{\delta\Gamma}{\delta\phi_a^*} + \frac{1}{2}\epsilon_{abc}\phi_c^*\frac{\delta\Gamma}{\delta\phi_b^*} - \frac{1}{2}\phi_a^*\frac{\delta\Gamma}{\delta\phi_0^*}\Big) = 0.\end{aligned} \tag{31}$$

The interlacing between the local functional equation in Eq. (31) generated by the gauge transformations and the ST identity will be treated in full detail in the next section.

We remark that the above equation contains a bilinear term, which arises as a consequence of the nonlinearity of the local gauge transformations. This term allows establishing the hierarchy procedure as in the nonlinear sigma model [11,12,17]. The hierarchy tool allows getting all the amplitudes involving at least one ϕ field (descendant amplitudes) from those with no ϕ fields (ancestor amplitudes). The boundary condition for this algorithm is provided by

$$\left.\frac{\delta\Gamma}{\delta K_0(x)}\right|_{\text{All fields and sources}=0} = v \tag{32}$$

[see Eq. (7)]. Since ϕ_0 is not invariant both under left- and right-$SU(2)$ transformations, both are spontaneously broken by the condition (32), and only the $SU(2)_V$ is unitarily implemented. The parameter v that spontaneously breaks the symmetry is not a physical quantity. This can be seen at the tree level in Eqs. (29) and (30) where v can be removed by the change of variables

$$\begin{aligned}&\Gamma^{(0)}[\vec{A}_\mu, \vec{c}, \vec{\bar{c}}, v\vec{\phi}, \vec{B}, \vec{A}^*_\mu, \vec{c}^*, v^{-1}\vec{\phi}^*, v^{-1}\phi_0^*, v^{-1}K_0, v] \\ &\quad = \Gamma^{(0)}[\vec{A}_\mu, \vec{c}, \vec{\bar{c}}, \vec{\phi}, \vec{B}, \vec{A}^*_\mu, \vec{c}^*, \vec{\phi}^*, \phi_0^*, K_0, v]|_{v=1}.\end{aligned} \tag{33}$$

or directly in Eqs. (31) and (32) where v also disappears after one uses in Γ the substitution given in Eq. (33). The rescaling of the field $\vec{\phi}$ has no effects on the physical amplitudes. Also the effect of the rescaling on the external sources ϕ_0^* and K_0 is null for physics. However, this conclusion can be drawn only after the enlargement of the ST transformations to the new variables (Sec. VI) and the discovery that ϕ_0^* and K_0 are not physical variables.

In the sequel we will explicitly use the hierarchy procedure in the one-loop approximation by integrating the linearized form of Eq. (31) as in Ref. [17]. Equation (31) together with the ST identity will be our tool for the symmetric subtraction of the divergences in the perturbative expansion at the point $D = 4$ for dimensionally regularized amplitudes. For that purpose we put in evidence the linearized part of the above equation for future use both in the recursive construction of the counterterms in the loop expansion and in the integration over the variable $\vec{\phi}$.

$$\mathcal{W}_0(\Gamma^{(n)}) = \int d^Dx\alpha_a^L(x)\Big(-\partial_\mu\frac{\delta}{\delta V_{a\mu}} + \epsilon_{abc}V_{c\mu}\frac{\delta}{\delta V_{b\mu}} - \partial_\mu\frac{\delta}{\delta A_{a\mu}} + \epsilon_{abc}A_{c\mu}\frac{\delta}{\delta A_{b\mu}} + \epsilon_{abc}B_c\frac{\delta}{\delta B_b} + \epsilon_{abc}\bar{c}_c\frac{\delta}{\delta \bar{c}_b} + \epsilon_{abc}c_c\frac{\delta}{\delta c_b} + \Big(\frac{1}{2}\delta_{ab}\frac{\delta\Gamma^{(0)}}{\delta K_0} + \frac{1}{2}\epsilon_{abc}\phi_c\Big)\frac{\delta}{\delta\phi_b} + \frac{1}{2}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta K_0} + \epsilon_{abc}\Theta_{c\mu}\frac{\delta}{\delta\Theta_{b\mu}} + \epsilon_{abc}A^*_{c\mu}\frac{\delta}{\delta A^*_{b\mu}} + \epsilon_{abc}c^*_c\frac{\delta}{\delta c^*_b} + \frac{1}{2}\phi_0^*\frac{\delta}{\delta\phi_a^*} + \frac{1}{2}\epsilon_{abc}\phi_c^*\frac{\delta}{\delta\phi_b^*} - \frac{1}{2}\phi_a^*\frac{\delta}{\delta\phi_0^*}\Big)\Gamma^{(n)} = -\frac{1}{2}\sum_{j=1}^{n-1}\int d^Dx\alpha_a^L(x)\frac{\delta\Gamma^{(j)}}{\delta K_0}\frac{\delta\Gamma^{(n-j)}}{\delta\phi_a}. \tag{34}$$

The requirement of the invariance under $\mathcal{W}_{0a}$ corresponds to the invariance under the local transformations

$$\mathcal{W}_0A_{a\mu} = (D_\mu[A]\alpha^L)_a, \qquad \mathcal{W}_0V_{a\mu} = (D_\mu[V]\alpha^L)_a,$$
$$\mathcal{W}_0\phi_a = \frac{1}{2}\phi_0\alpha_a^L + \frac{1}{2}\epsilon_{abc}\phi_b\alpha_c^L,$$
$$\mathcal{W}_0B_a = \epsilon_{abc}B_b\alpha_c^L,$$
$$\mathcal{W}_0\bar{c}_a = \epsilon_{abc}\bar{c}_b\alpha_c^L, \qquad \mathcal{W}_0c_a = \epsilon_{abc}c_b\alpha_c^L,$$
$$\mathcal{W}_0\Theta_{a\mu} = \epsilon_{abc}\Theta_{b\mu}\alpha_c^L,$$
$$\mathcal{W}_0A^*_{a\mu} = \epsilon_{abc}A^*_{b\mu}\alpha_c^L, \qquad \mathcal{W}_0c^*_a = \epsilon_{abc}c^*_b\alpha_c^L,$$
$$\mathcal{W}_0\phi_0^* = -\frac{1}{2}\alpha_a^L\phi_a^*, \qquad \mathcal{W}_0\phi_a^* = \frac{1}{2}\alpha_a^L\phi_0^* + \frac{1}{2}\epsilon_{abc}\phi_b^*\alpha_c^L,$$
$$\mathcal{W}_0K_0 = \frac{1}{2}\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\alpha_a^L, \tag{35}$$

where

$$\mathcal{W}_0 \equiv \int d^Dx\alpha_a^L(x)\mathcal{W}_{0a}(x). \tag{36}$$

The action of $\mathcal{W}_0$ on the fields coincides with the one of the generator of the local gauge transformations. In addition $\mathcal{W}_0$ also acts on the external sources, as displayed in the last four lines of Eq. (35).

The technique discussed in [17] can be used in order to derive a set of bleached variables [in one-to-one correspondence with the original ones appearing in Eq. (35)] which are invariant under $\mathcal{W}_0$.

We first notice that for any $I = I_a\frac{\tau_a}{2}$, transforming in the adjoint representation under the local gauge transformations in Eq. (6)

$$I' = U_LIU_L^\dagger, \tag{37}$$

its bleached counterpart $\tilde{I} = \tilde{I}_a\frac{\tau_a}{2}$ can be obtained by conjugation w.r.t. Ω

$$\tilde{I} = \Omega^\dagger I\Omega. \tag{38}$$

In fact $\tilde{I}$ is invariant under local gauge transformations. In components one finds

$$\tilde{I}_a = R_{ba}I_b, \tag{39}$$

where the matrix R_{ba} is given by

$$R_{ba} \equiv \frac{1}{2}\mathrm{Tr}(\Omega^\dagger\tau_b\Omega\tau_a) = \Big(1 - 2\frac{\vec{\phi}^2}{v_D^2}\Big)\delta_{ba} + 2\frac{\phi_a\phi_b}{v_D^2} + 2\epsilon_{acb}\frac{\phi_0\phi_c}{v_D^2}. \tag{40}$$

This procedure allows to construct the bleached variables

$$\tilde{B}_a, \tilde{\bar{c}}_a, \tilde{c}_a, \tilde{\Theta}_{a\mu}, \tilde{A}^*_{a\mu}, \tilde{c}^*_a. \tag{41}$$

Moreover, since $F_{a\mu}$ transforms as a flat connection under local gauge transformations, the combinations $A_\mu - F_\mu$ and $V_\mu - F_\mu$ both transform in the adjoint representation. The corresponding bleached variables are denoted by $a_{a\mu}$ and $v_{a\mu}$ and are given by

$$a_{a\mu} = R_{ba}(A_{b\mu} - F_{b\mu}), \qquad v_{a\mu} = R_{ba}(V_{b\mu} - F_{b\mu}). \tag{42}$$

Since R_{ba} is invertible, the change of variables leading to the bleached variables in Eqs. (41) and (42) is invertible.

We remark that all the bleached variables in Eqs. (41) and (42) reduce for $\phi = 0$ to their corresponding ancestors. One could also consider the $\mathcal{W}_0$-invariant combination

$$R_{ba}(A_{b\mu} - V_{b\mu}), \tag{43}$$

but this would spoil the correspondence at $\phi = 0$ with a single ancestor variable.

According to Eq. (35) the matrix

$$\Omega^* = \phi_0^* + i\phi_a^*\tau_a \tag{44}$$

transforms as Ω under $\mathcal{W}_0$. In particular the combination

$$\Omega^\dagger\Omega^* = \phi_0\phi_0^* + \phi_a\phi_a^* + i(\phi_a^*\phi_0 - \phi_0^*\phi_a - \epsilon_{abc}\phi_b^*\phi_c)\tau_a$$

is $\mathcal{W}_0$ invariant. This suggests introducing the bleached counterparts of ϕ_0^* and ϕ_a^* as follows:

$$\tilde{\phi}_0^* = \frac{1}{v_D}(\phi_0\phi_0^* + \phi_a\phi_a^*),$$
$$\tilde{\phi}_a^* = \frac{1}{v_D}(\phi_0\phi_a^* - \phi_a\phi_0^* - \epsilon_{abc}\phi_b^*\phi_c). \tag{45}$$

The normalization factor has been chosen in such a way that at $\phi = 0$ $\tilde{\phi}_0^*$ and $\tilde{\phi}_a^*$ reduce to ϕ_0^* and ϕ_a^*, respectively.

Finally it can be proved by the same methods used in [12] that the combination

$$\tilde{K}_0 = \frac{1}{v_D}\left(\frac{v_D^2 K_0}{\phi_0} - \phi_a\frac{\delta}{\delta\phi_a}(\Gamma^{(0)}|_{K_0=0})\right) \tag{46}$$

is $\mathcal{W}_0$ invariant. Again the normalization condition is chosen in such a way that $\tilde{K}_0|_{\phi=0} = K_0$ holds.

The use of the bleached variables will greatly simplify the solution of the local functional equation (34), since in these variables $\mathcal{W}_0$ takes the very simple form

$$\mathcal{W}_0 = \int d^Dx\,\alpha_b^L\zeta_{ab}\frac{\delta}{\delta\phi_a}, \tag{47}$$

where the invertible matrix ζ_{ab} is given by

$$\zeta_{ab} = \tfrac{1}{2}\phi_0\delta_{ab} + \tfrac{1}{2}\epsilon_{acb}\phi_c. \tag{48}$$

VI. SLAVNOV-TAYLOR II

According to the standard algebraic treatment given in [25–27] the background connection $V_{a\mu}$ is paired with the classical ghost $\Theta_{a\mu}$ into a $\mathcal{S}_0$ doublet [28,29]:

$$\mathcal{S}_0 V_{a\mu} = \Theta_{a\mu}, \qquad \mathcal{S}_0\Theta_{a\mu} = 0. \tag{49}$$

This technical device allows guaranteeing that physical observables are not modified by the introduction of the background connection [26,27]. ϕ_0^* and $-K_0$ pair as well into a $\mathcal{S}_0$ doublet:

$$\mathcal{S}_0\phi_0^* = -K_0, \qquad \mathcal{S}_0K_0 = 0. \tag{50}$$

Under the assignments in Eqs. (49) and (50) $\Gamma^{(0)}$ in Eq. (29) is also ST invariant. We remark that, since the source K_0 of the nonlinear constraint in Eq. (3) is the component of a $\mathcal{S}_0$ doublet, it is an unphysical variable (unlike in the nonlinear sigma model). As a consequence the physical amplitudes are not affected by the rescaling performed in Eq. (33), and therefore they do not depend on v.

The ST identity in the presence of the new set of sources is

$$\mathcal{S}(\Gamma) = \int d^Dx\left(\frac{\delta\Gamma}{\delta A^*_{a\mu}}\frac{\delta\Gamma}{\delta A^\mu_a} + \frac{\delta\Gamma}{\delta\phi^*_a}\frac{\delta\Gamma}{\delta\phi_a} + \frac{\delta\Gamma}{\delta c^*_a}\frac{\delta\Gamma}{\delta c_a} + B_a\frac{\delta\Gamma}{\delta\bar{c}_a} + \Theta_{a\mu}\frac{\delta\Gamma}{\delta V_{a\mu}} - K_0\frac{\delta\Gamma}{\delta\phi^*_0}\right) = 0. \tag{51}$$

Γ also obeys the Landau gauge equation

$$\frac{\delta\Gamma}{\delta B_a} = \frac{\Lambda^{D-4}}{g^2}D^\mu[V](A_\mu - V_\mu)_a \tag{52}$$

and the ghost equation

$$\frac{\delta\Gamma}{\delta\bar{c}_a} = \frac{\Lambda^{D-4}}{g^2}\left(-D_\mu[V]\frac{\delta\Gamma}{\delta A^*_\mu} + D_\mu[A]\Theta^\mu\right)_a, \tag{53}$$

which follows as a consequence of the linearity of the gauge-fixing condition. In the background Landau gauge a further identity holds, the antighost equation [30]. However, we will not make use of it in the present construction since it cannot be generalized to different Lorentz-covariant gauges.

The Eqs. (51)–(53) are not independent. By taking the functional derivative of Eq. (51) with respect to B and by using Eq. (52), one obtains the ghost equation (53).

In the perturbative loop expansion we need to recursively use Eq. (51) in order to extract the symmetric counterterms. This leads us to consider the linearized version of the ST identity

$$\mathcal{S}_0(\Gamma^{(n)}) \equiv \int d^Dx\left(\frac{\delta\Gamma^{(0)}}{\delta A^*_{a\mu}}\frac{\delta\Gamma^{(n)}}{\delta A^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta A^\mu_a}\frac{\delta\Gamma^{(n)}}{\delta A^*_{a\mu}} + \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a}\frac{\delta\Gamma^{(n)}}{\delta\phi_a} + \frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta\Gamma^{(n)}}{\delta\phi^*_a} + \frac{\delta\Gamma^{(0)}}{\delta c^*_a}\frac{\delta\Gamma^{(n)}}{\delta c_a} + \frac{\delta\Gamma^{(0)}}{\delta c_a}\frac{\delta\Gamma^{(n)}}{\delta c^*_a} + B_a\frac{\delta\Gamma^{(n)}}{\delta\bar{c}_a} + \Theta_{a\mu}\frac{\delta\Gamma^{(n)}}{\delta V_{a\mu}} - K_0\frac{\delta\Gamma^{(n)}}{\delta\phi^*_0}\right) = -\int d^Dx\sum_{j=1}^{n-1}\left(\frac{\delta\Gamma^{(j)}}{\delta A^*_{a\mu}}\frac{\delta\Gamma^{(n-j)}}{\delta A^\mu_a} + \frac{\delta\Gamma^{(j)}}{\delta\phi^*_a}\frac{\delta\Gamma^{(n-j)}}{\delta\phi_a} + \frac{\delta\Gamma^{(j)}}{\delta c_a}\frac{\delta\Gamma^{(n-j)}}{\delta c^*_a}\right). \tag{54}$$

$\mathcal{S}_0$ is nilpotent.

The Landau gauge equation (52) yields in the loop expansion at order $n \geq 1$

$$\frac{\delta\Gamma^{(n)}}{\delta B_a} = 0; \tag{55}$$

i.e. the dependence on B_a is only at tree level. Moreover the ghost equation (53) yields at order $n \geq 1$

$$\frac{\delta\Gamma^{(n)}}{\delta\bar{c}_a} = \frac{\Lambda^{D-4}}{g^2}\left(-\partial_\mu\frac{\delta\Gamma^{(n)}}{\delta A^*_{a\mu}} - \epsilon_{abc}V_{b\mu}\frac{\delta\Gamma^{(n)}}{\delta A^*_{c\mu}}\right). \tag{56}$$

The above equation implies that $\Gamma^{(n)}$ depends on $\bar{c}_a$ only through the combination

$$\hat{A}^*_{a\mu} = A^*_{a\mu} + \frac{\Lambda^{D-4}}{g^2}(D_\mu[V]\bar{c})_a. \tag{57}$$

The use of $\hat{A}^*_{a\mu}$ instead of $A^*_{a\mu}$ simplifies the relevant $\mathcal{S}_0$ transforms involving $A^*_{a\mu}$. In fact one finds

$$\mathcal{S}_0\hat{A}^*_{a\mu} = \frac{\delta S}{\delta A_{a\mu}} - \epsilon_{abc}c_b\hat{A}^*_{c\mu},$$
$$\mathcal{S}_0 c^*_a = (D^\mu[A]\hat{A}^*_\mu)_a + \frac{1}{2}\phi^*_0\phi_a - \frac{1}{2}\phi^*_a\phi_0 - \frac{1}{2}\epsilon_{abc}\phi^*_b\phi_c + \epsilon_{abc}c^*_b c_c. \tag{58}$$

With the transformation properties under $\mathcal{S}_0$ given in this section the only field that describes physical states is $\vec{A}_\mu$. The massless mode of $\vec{A}_\mu$ (in the Landau gauge), the Goldstone bosons, and the FP ghosts are unphysical and are expected to give zero contribution in the physical unitarity equation [18]. Moreover also the external sources $A^*_{a\mu}$, c^*_a, ϕ^*_a, ϕ^*_0, and K_0 are unphysical. We stress once again the surprising fact that the external source associated to the order parameter field ϕ_0 is not a physical variable.

The dependence on v of the 1-PI vertex functional can be discussed by means of cohomological tools as shown in Appendix E. This is achieved by introducing an extended ST identity under which also v transforms into an anti-commuting constant ghost θ. This extended ST identity holds for the quantum effective action whose classical approximation $\Gamma^{(0)}_{\rm ext}$ involves an additional θ-dependent part. $\Gamma^{(0)}_{\rm ext}$ reduces for $\theta = 0$ to $\Gamma^{(0)}$ in Eq. (30). The advantage of this procedure is that it allows discussing the dependence on v of the connected Green functions by algebraic methods which are close to those developed in gauge theories in order to discuss the dependence of the connected generating functional on the gauge parameter [31]. One finds that the connected Green functions of BRST-invariant local operators are independent of v. This is a rather remarkable result, since it shows that in the present approach v is an unphysical mass scale. Moreover one can derive an equation allowing control of the dependence of the Green functions involving K_0 in terms of those involving the antifield ϕ^*_0. This reflects the fact that ϕ^*_0 and $-K_0$ form a $\mathcal{S}_0$ doublet [see Eq. (50)]. In this connection we remark that the issue of whether the composite operator ϕ_0, coupled to the external source K_0, is physical or not is a somewhat peculiar problem. By standard cohomological arguments [28] it can be proved that ϕ^*_0 and K_0 do not contribute to the cohomology of the linearized ST operator $\mathcal{S}_0$ (because they form a $\mathcal{S}_0$ doublet [28,29]). Since in the perturbation expansion of gauge theories physical observables can be identified with the cohomology classes of $\mathcal{S}_0$, we conclude that K_0 is unphysical.

VII. PERTURBATIVE SOLUTION IN D DIMENSIONS

It is of paramount importance to establish whether Eqs. (31), (51), and (52) are compatible. For our purpose it would be very satisfactory to prove that the perturbative expansion in the number of loops of the generating functional of the 1-PI functions yields a solution of both equations. This is indeed the case, and the proof of this result is very close to the one already given for the nonlinear sigma model in Ref. [14]. Thus we will not repeat it here. The Feynman rules are taken from the classical action in Eq. (30) in D dimensions. A technical point should be noted regarding the presence of massive tadpoles. In dimensional regularization they are nonzero, unlike in the massless case. Therefore one should keep track of them.

Let us state here only the final formulas. In the present section we perform a rescaling by a factor

$$\Lambda_D \equiv \frac{\Lambda^{(D-4)}}{g^2} \tag{59}$$

of the antifields $A^*_{a\mu}$, c^*_a, and ϕ^*_a and of the external sources ϕ^*_0, K_0:

$$(A^*_{a\mu}, c^*_a, \phi^*_a, \phi^*_0, K_0) \to \Lambda_D(A^*_{a\mu}, c^*_a, \phi^*_a, \phi^*_0, K_0) \tag{60}$$

so that the unperturbed effective action [(29) or (30)] becomes

$$\Gamma^{(0)} = S + \Lambda_D s\int d^Dx(\bar{c}_a\partial^\mu A_{a\mu}) + \Lambda_D\int d^Dx(V^\mu_a s(D_\mu[A]\bar{c})_a + \Theta^\mu_a(D_\mu[A]\bar{c})_a) + \Lambda_D\int d^Dx(A^*_{a\mu}sA^\mu_a + \phi^*_0 s\phi_0 + \phi^*_a s\phi_a + c^*_a s c_a + K_0\phi_0). \tag{61}$$

This rescaling is introduced in order to give D-independent canonical dimensions to all the ancestor fields and sources. It introduces however some Λ_D-dependent factors both in the local gauge functional equation and in the ST identity. We shall account for this change, since it is important for the subtraction procedure. Moreover we denote by

$$\hat{\Gamma} \equiv \Gamma^{(0)} + \sum_{j\geq 1}\hat{\Gamma}^{(j)} \tag{62}$$

the whole set of Feynman rules, including the counterterms. The local functional $\hat{\Gamma}^{(j)}$ collects all the counterterms of order $\hbar^j$.

A. Local gauge equation

In generic D dimensions after the rescaling of Eq. (60) the functional Z, generating the Feynman amplitudes, obeys the equation associated with the local gauge transformations

$$\Big(-\partial_\mu \frac{\delta}{\delta V_{a\mu}} + \epsilon_{abc}V_{c\mu}\frac{\delta}{\delta V_{b\mu}} + \partial_\mu L_a^\mu - \epsilon_{abc}L_{b\mu}\frac{\delta}{\delta L_{c\mu}} - \epsilon_{abc}J_b^B\frac{\delta}{\delta J_c^B} + \epsilon_{abc}\eta_b\frac{\delta}{\delta\eta_c} + \epsilon_{abc}\bar\eta_b\frac{\delta}{\delta\bar\eta_c}$$
$$+ \frac{\Lambda_D}{2}K_0\frac{\delta}{\delta K_a} - \frac{1}{2\Lambda_D}K_a\frac{\delta}{\delta K_0} - \frac{1}{2}\epsilon_{abc}K_b\frac{\delta}{\delta K_c} + \epsilon_{abc}\Theta_{c\mu}\frac{\delta}{\delta\Theta_{b\mu}} + \epsilon_{abc}A^*_{c\mu}\frac{\delta}{\delta A^*_{b\mu}} + \epsilon_{abc}c^*_c\frac{\delta}{\delta c^*_b}$$
$$+ \frac{1}{2}\phi_0^*\frac{\delta}{\delta\phi_a^*} + \frac{1}{2}\epsilon_{abc}\phi_c^*\frac{\delta}{\delta\phi_b^*} - \frac{1}{2}\phi_a^*\frac{\delta}{\delta\phi_0^*}\Big)Z$$
$$= i\Big(-\partial_\mu\frac{\delta\hat\Gamma}{\delta V_{a\mu}} + \epsilon_{abc}V_{c\mu}\frac{\delta\hat\Gamma}{\delta V_{b\mu}} - \partial_\mu\frac{\delta\hat\Gamma}{\delta A_{a\mu}} + \epsilon_{abc}A_{c\mu}\frac{\delta\hat\Gamma}{\delta A_{b\mu}} + \epsilon_{abc}B_c\frac{\delta\hat\Gamma}{\delta B_b} + \epsilon_{abc}\bar c_c\frac{\delta\hat\Gamma}{\delta\bar c_b} + \epsilon_{abc}c_c\frac{\delta\hat\Gamma}{\delta c_b}$$
$$+ \frac{\Lambda_D}{2}K_0\phi_a + \frac{1}{2\Lambda_D}\frac{\delta\hat\Gamma}{\delta K_0}\frac{\delta\hat\Gamma}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta\hat\Gamma}{\delta\phi_b} + \epsilon_{abc}\Theta_{c\mu}\frac{\delta\hat\Gamma}{\delta\Theta_{b\mu}} + \epsilon_{abc}A^*_{c\mu}\frac{\delta\hat\Gamma}{\delta A^*_{b\mu}} + \epsilon_{abc}c^*_c\frac{\delta\hat\Gamma}{\delta c^*_b}$$
$$+ \frac{1}{2}\phi_0^*\frac{\delta\hat\Gamma}{\delta\phi_a^*} + \frac{1}{2}\epsilon_{abc}\phi_c^*\frac{\delta\hat\Gamma}{\delta\phi_b^*} - \frac{1}{2}\phi_a^*\frac{\delta\hat\Gamma}{\delta\phi_0^*}\Big)\cdot Z, \tag{63}$$

where the dot indicates the insertion of the local operators and the field sources are given by

$$L_{a\mu} = -\frac{\delta\Gamma}{\delta A_a^\mu} \qquad K_a = -\frac{\delta\Gamma}{\delta\phi_a} \qquad J_a^B = -\frac{\delta\Gamma}{\delta B_a} \qquad \eta_a = -\frac{\delta\Gamma}{\delta\bar c_a} \qquad \bar\eta_a = \frac{\delta\Gamma}{\delta c_a}. \tag{64}$$

If no counterterms are present, then $\hat\Gamma = \Gamma^{(0)}$. In this case Eq. (63) proves that the unsubtracted amplitudes in D dimensions satisfy the functional equation associated with the local gauge transformations. In fact $\Gamma^{(0)}$ is by construction a solution of Eq. (31), and therefore the right-hand side (R.H.S.) of Eq. (63) is zero. On the other side, if counterterms are introduced, they must obey the identity

$$-\partial_\mu\frac{\delta\hat\Gamma}{\delta V_{a\mu}} + \epsilon_{abc}V_{c\mu}\frac{\delta\hat\Gamma}{\delta V_{b\mu}} - \partial_\mu\frac{\delta\hat\Gamma}{\delta A_{a\mu}} + \epsilon_{abc}A_{c\mu}\frac{\delta\hat\Gamma}{\delta A_{b\mu}} + \epsilon_{abc}B_c\frac{\delta\hat\Gamma}{\delta B_b} + \epsilon_{abc}\bar c_c\frac{\delta\hat\Gamma}{\delta\bar c_b} + \epsilon_{abc}c_c\frac{\delta\hat\Gamma}{\delta c_b} + \frac{\Lambda_D}{2}K_0\phi_a + \frac{1}{2\Lambda_D}\frac{\delta\hat\Gamma}{\delta K_0}$$
$$\times\frac{\delta\hat\Gamma}{\delta\phi_a} + \frac{1}{2}\epsilon_{abc}\phi_c\frac{\delta\hat\Gamma}{\delta\phi_b} + \epsilon_{abc}\Theta_{c\mu}\frac{\delta\hat\Gamma}{\delta\Theta_{b\mu}} + \epsilon_{abc}A^*_{c\mu}\frac{\delta\hat\Gamma}{\delta A^*_{b\mu}} + \epsilon_{abc}c^*_c\frac{\delta\hat\Gamma}{\delta c^*_b} + \frac{1}{2}\phi_0^*\frac{\delta\hat\Gamma}{\delta\phi_a^*} + \frac{1}{2}\epsilon_{abc}\phi_c^*\frac{\delta\hat\Gamma}{\delta\phi_b^*} - \frac{1}{2}\phi_a^*\frac{\delta\hat\Gamma}{\delta\phi_0^*} = 0. \tag{65}$$

B. The subtraction procedure

Equation (65) is the tool used in order to construct the counterterms necessary for the limit $D = 4$. Assume that the subtraction procedure has been performed up to order $n - 1$. Only the pole parts are subtracted by adopting the counterterms structure

$$\hat\Gamma = \Gamma^{(0)} + \Lambda_D\sum_{j\geq 1}\int d^Dx\mathcal{M}^{(j)}, \tag{66}$$

where the local polynomials $\mathcal{M}^{(j)}$ in the fields and sources have no D dependence apart from the poles in $D - 4$. At order n Eq. (34) is then violated since the nth counterterms are not present as they should be according to Eq. (65). The violation is explicitly given by

$$\mathcal{W}_{0a}(\Gamma^{(n)}) + \frac{1}{2\Lambda_D}\sum_{j=1}^{n-1}\frac{\delta\Gamma^{(j)}}{\delta K_0}\frac{\delta\Gamma^{(n-j)}}{\delta\phi_a}$$
$$= \frac{1}{2\Lambda_D}\sum_{j=1}^{n-1}\frac{\delta\hat\Gamma^{(j)}}{\delta K_0}\frac{\delta\hat\Gamma^{(n-j)}}{\delta\phi_a}. \tag{67}$$

According to Eq. (66) the pole part $\mathcal{M}^{(n)}$ has to be extracted from the normalized amplitude

$$\Lambda_D^{-1}\Gamma^{(n)}. \tag{68}$$

By this normalization condition the R.H.S. in Eq. (67) is D independent apart from the poles in $D - 4$ by construction [as stated in Eq. (66)]. Then minimal subtraction on the normalized amplitude (68) removes the breaking terms in the R.H.S. In the left-hand side (L.H.S.) of Eq. (67) $\mathcal{W}_{0a}$ at $n > 0$ contains no Λ_D factor, and therefore the procedure of subtraction [normalization according to Eq. (68) and pure pole subtraction] does not modify the equation.

Further details about this subtraction procedure are in Appendix D of Ref. [14].

Once again we stress our point of view that, by using the freedom to introduce free parameters describing the general solution $\Delta\hat\Gamma^{(n)}$ of the homogeneous equation

$$\mathcal{W}_0(\Delta\hat\Gamma^{(n)}) = 0, \tag{69}$$

one would destroy the predictivity of the theory, since the theory is not power-counting renormalizable and therefore the new parameters appearing in the quantum corrections cannot be reabsorbed by a redefinition of the constants already present in the classical vertex functional $\Gamma^{(0)}$. Our subtraction prescription is based on a finite number of parameters. Therefore it is predictive and it can be experimentally tested.

C. Comments on the subtraction procedure

Let us look closer into this subtraction procedure by considering the dependence on Λ_D of a generic amplitude. The removal of the divergences requires the insertion of counterterms. Therefore it is important to distinguish the order in the $\hbar$ expansion from the loop number. A counterterm $\mathcal{M}^{(k)}$ in Eq. (66) is of order $\hbar^k$. The vertex functional can be graded according to the $\hbar$ power of the counterterms included in the amplitudes (in D dimensions):

$$\Gamma^{(n)} = \sum_{k=0}^{n} \Gamma^{(n,k)}. \tag{70}$$

$\Gamma^{(n,k)}$ has important properties that are discussed in Ref. [14].

With the Feynman rules given by the $\Gamma^{(0)}$ in Eq. (61), the propagators of the dynamical fields carry a factor Λ_D^{-1}, while every vertex has a factor Λ_D (including the counterterms). Since the number n_L of topological loops for a 1-PI amplitude is given by

$$n_L = I - V + 1, \tag{71}$$

where V is the number of vertices (including the counterterms), then the factor is

$$\Lambda_D^{(1-n_L)}. \tag{72}$$

Therefore $\Gamma^{(n,k)}$ carries an overall factor given by a power of Λ_D, where the exponent is not given by the order in the $\hbar$ expansion but by the number of topological loops

$$\Gamma^{(n)} = \sum_{k=0}^{n} \Lambda_D^{(1-n_L)} \Gamma^{(n,k)} \Big|_{\Lambda_D=1} = \Lambda_D^{(1-n)} \sum_{k=0}^{n} \Lambda_D^{k} \Gamma^{(n,k)} \Big|_{\Lambda_D=1}, \tag{73}$$

where Λ_D can be set to one, by considering it as an independent variable together with Λ and D:

$$g^2 = \frac{\Lambda^{(D-4)}}{\Lambda_D}. \tag{74}$$

The power of $\hbar^n$ of $\Gamma^{(n,k)}$ is given by

$$n = I + \sum_{j\geq 0} V^{(j)}(j-1) + 1 = I - V + 1 + k = n_L + k, \tag{75}$$

where $V^{(j)}$ counts the number of vertices of order $\hbar^j$ and k is the total $\hbar$ power of the counterterms. In particular the tree vertices are of order $\hbar^0$. Also the coupling constant enters in the amplitudes in a powerlike form ($g^{2(n-1)}$), since the subtraction procedure does not alter the dependence on g.

The complex dependence of the vertex functional from Λ_D makes the subtraction procedure nontrivial. In the iterative procedure of subtraction, where the counterterms have been consistently used up to order $n-1$, the 1-PI amplitude $\Gamma_U^{(n)}$ (where the subscript U reminds that the last subtraction at order n has yet to be performed) has a Laurent expansion

$$\Gamma_U^{(n)} = \sum_{j=-M}^{\infty} a_j (D-4)^j. \tag{76}$$

Then the proposed finite part is given by the $(D-4)^0$ coefficient in the Laurent expansion of $\Lambda^{(4-D)}\Gamma_U^{(n)}$. I.e.

$$\sum_{j=0}^{M} \frac{1}{j!} (-\ln(\Lambda))^j a_{-j}. \tag{77}$$

While the counterterms are given by

$$\begin{aligned} \int d^D x \mathcal{M}^{(n)}(x) &= -g^2 \sum_{i=0}^{\infty} \frac{1}{i!} (-\ln(\Lambda))^i (D-4)^i \\ &\quad \times \sum_{j=0}^{M} a_{-j}(D-4)^{-j} \Big|_{\text{Pole Part}} \\ &= -g^2 \sum_{l=1}^{M} \frac{1}{(D-4)^l} \bigg(\sum_{j=l}^{M} \frac{1}{(j-l)!} \\ &\quad \times (-\ln(\Lambda))^{(j-l)} a_{-j} \bigg). \end{aligned} \tag{78}$$

One can easily verify that the finite part for $D = 4$ of

$$\begin{aligned} &\Gamma_U^{(n)} + \Lambda_D \int d^D x \mathcal{M}^{(n)}(x) \Big|_{D=4} \\ &= \Lambda_D \bigg(\frac{1}{\Lambda_D} \Gamma_U^{(n)} + \int d^D x \mathcal{M}^{(n)}(x) \bigg) \Big|_{D=4} \\ &= \frac{1}{g^2} \bigg(\frac{1}{\Lambda_D} \Gamma_U^{(n)} + \int d^D x \mathcal{M}^{(n)}(x) \bigg) \Big|_{D=4} \end{aligned} \tag{79}$$

is indeed the expression given in Eq. (77).

D. Slavnov-Taylor equation

Now we examine the same items for the ST identity (51). The ghost equation (53), being linear in Γ, poses no problems.

As for the functional equation (63) associated with the local gauge transformations, we state the relation between ST identity and the equation for the counterterms

$$\int d^Dx\Big(-\frac{L_{a\mu}}{\Lambda_D}\frac{\delta}{\delta A^*_{a\mu}}-\frac{K_a}{\Lambda_D}\frac{\delta}{\delta\phi^*_a}+\frac{\bar\eta_a}{\Lambda_D}\frac{\delta}{\delta c^*_a}-\eta_a\frac{\delta}{\delta J^B_a}+\Theta_{a\mu}\frac{\delta}{\delta V_{a\mu}}-K_0\frac{\delta}{\delta\phi^*_0}\Big)Z$$
$$=\int d^Dx\Big(\frac{1}{\Lambda_D}\frac{\delta\hat\Gamma}{\delta A^*_{a\mu}}\frac{\delta\hat\Gamma}{\delta A^\mu_a}+\frac{1}{\Lambda_D}\frac{\delta\hat\Gamma}{\delta\phi^*_a}\frac{\delta\hat\Gamma}{\delta\phi_a}+\frac{1}{\Lambda_D}\frac{\delta\hat\Gamma}{\delta c^*_a}\frac{\delta\hat\Gamma}{\delta c_a}+B_a\frac{\delta\Gamma}{\delta\bar c_a}+\Theta_{a\mu}\frac{\delta\hat\Gamma}{\delta V_{a\mu}}-K_0\frac{\delta\hat\Gamma}{\delta\phi^*_0}\Big)\cdot Z. \tag{80}$$

Thus the counterterms in perturbation theory must obey the following equation:

$$\mathcal{S}_0(\hat\Gamma^{(n)})+\frac{1}{\Lambda_D}\sum_{j=1}^{n-1}\Big(\frac{\delta\hat\Gamma^{(j)}}{\delta A^*_{a\mu}}\frac{\delta\hat\Gamma^{(n-j)}}{\delta A^\mu_a}+\frac{\delta\hat\Gamma^{(j)}}{\delta\phi^*_a}\frac{\delta\hat\Gamma^{(n-j)}}{\delta\phi_a}+\frac{\delta\hat\Gamma^{(j)}}{\delta c^*_a}\frac{\delta\hat\Gamma^{(n-j)}}{\delta c_a}\Big)=0. \tag{81}$$

As in Eq. (67) the nonlinear part of Eq. (81) fixes the violation at n loops of Eq. (51) and therefore the implementability of the pure pole subtraction strategy.

E. Subtraction procedure and ST identity

After the subtraction has been performed at $n-1$ order, the nth order correction to the vertex functional obeys the equation

$$\int d^Dx\Big[\frac{1}{\Lambda_D}\Big(\frac{\delta\Gamma^{(0)}}{\delta A^*_{a\mu}}\frac{\delta}{\delta A^\mu_a}+\frac{\delta\Gamma^{(0)}}{\delta A^\mu_a}\frac{\delta}{\delta A^*_{a\mu}}+\frac{\delta\Gamma^{(0)}}{\delta\phi^*_a}\frac{\delta}{\delta\phi_a}+\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta}{\delta\phi^*_a}+\frac{\delta\Gamma^{(0)}}{\delta c^*_a}\frac{\delta}{\delta c_a}+\frac{\delta\Gamma^{(0)}}{\delta c_a}\frac{\delta}{\delta c^*_a}\Big)+B_a\frac{\delta}{\delta\bar c_a}+\Theta_{a\mu}\frac{\delta}{\delta V_{a\mu}}-K_0\frac{\delta}{\delta\phi^*_0}\Big]\Gamma^{(n)}+\frac{1}{\Lambda_D}\int d^Dx\sum_{j=1}^{n-1}\Big(\frac{\delta\Gamma^{(j)}}{\delta A^*_{a\mu}}\frac{\delta\Gamma^{(n-j)}}{\delta A^\mu_a}+\frac{\delta\Gamma^{(j)}}{\delta\phi^*_a}\frac{\delta\Gamma^{(n-j)}}{\delta\phi_a}+\frac{\delta\Gamma^{(j)}}{\delta c_a}\frac{\delta\Gamma^{(n-j)}}{\delta c^*_a}\Big)$$
$$=\frac{1}{\Lambda_D}\sum_{j=1}^{n-1}\Big(\frac{\delta\hat\Gamma^{(j)}}{\delta A^*_{a\mu}}\frac{\delta\hat\Gamma^{(n-j)}}{\delta A^\mu_a}+\frac{\delta\hat\Gamma^{(j)}}{\delta\phi^*_a}\frac{\delta\hat\Gamma^{(n-j)}}{\delta\phi_a}+\frac{\delta\hat\Gamma^{(j)}}{\delta c^*_a}\frac{\delta\hat\Gamma^{(n-j)}}{\delta c_a}\Big). \tag{82}$$

In fact only after the introduction of the counterterm $\hat\Gamma^{(n)}$ is the ST identity expected to be valid by Eq. (80). The missing counterterm can be replaced by the nonlinear part exhibited in Eq. (81), thus yielding the breaking term in the R.H.S. of Eq. (82).

A closer look at the L.H.S. of Eq. (82) shows that the operator acting on $\Gamma^{(n)}$ does not contain Λ_D and that the last nonlinear terms involving $\Gamma^{(j)}(j<n)$ have no pole parts by assumption. We now divide both terms by Λ_D in order to normalize the vertex functional in the L.H.S. according to Eq. (68) and to remove any D dependence in the R.H.S. apart from the pole in $D-4$, in agreement with the normalization of the counterterms in Eq. (66). The subtraction of the poles from $\Lambda_D^{-1}\Gamma^{(n)}$ leaves invariant in form the L.H.S. of the equation, and the breaking terms are removed. I.e. one recovers the ST identity for the subtracted amplitudes at order n.

VIII. WEAK POWER COUNTING II

By making use of the functional equation associated with the local gauge transformations, one can indeed establish a weak-power-counting theorem. The number of independent ancestor amplitudes can be fixed by taking into account the functional identities which are fulfilled by the vertex functional Γ. As we have already discussed, the ST identity (51) is not enough to induce a hierarchy. The Landau gauge equation (52) shows that the dependence on B only enters at tree level (since the R.H.S. of this equation is purely classical).

The ghost equation (53) fixes the dependence on $\bar c_a$. Therefore the field $\bar c_a$ can be neglected in the hierarchy procedure.

The functional equation (31) will in turn fix the dependence on the ϕ's. The ancestor amplitudes can correspondingly be identified with those involving the ancestor variables, i.e. all the fields and sources except the $\vec\phi$ fields.

The weak-power-counting theorem can be stated as follows. The number of independent superficially divergent amplitudes is finite at each order in the loop expansion. These amplitudes involve only the ancestor fields and sources. In particular, given a 1-PI n-loop graph $\mathcal{G}$ with N_A external A legs, N_c external c legs, N_V external V legs, N_Θ external Θ legs, $N_{\phi_0^*}$ external ϕ_0^* legs, N_{K_0} external K_0 legs, $N_{\phi_a^*}$ external ϕ_a^* legs, N_{A^*} external A^* legs, and N_{c^*} external c^* legs, the superficial degree of divergence of $\mathcal{G}$ is bounded by

$$d(\mathcal{G})\le(D-2)n+2-N_A-N_c-N_V-N_{\phi_a^*}-2(N_\Theta+N_{A^*}+N_{\phi_0^*}+N_{c^*}+N_{K_0}). \tag{83}$$

Moreover this property is stable under minimal subtraction in dimensional regularization. A detailed proof of this result is given in Appendix C. From Eq. (83) we see that at each order in the loop expansion there is only a finite number of divergent ancestor amplitudes.

The above result relies on the assumptions discussed in Sec. IX.

Our subtraction scheme is consistent and predictive. It is consistent since the defining equations are stable under the subtraction procedure, and it is predictive because the physical parameters are those of the zero-loop vertex functional plus the scale of the radiative corrections (denoted in the present paper by Λ). Uniqueness of the tree-level vertex functional (see Sec. IX), as dictated by the symmetries and weak power counting, forbids additional terms. Physical unitarity in the Landau gauge has been proved under quite general assumptions [18], and it is based on the ST identity (51). By weak power counting the number of counterterms is finite at each order in the loop expansion [see Eq. (66)]. The nth loop counterterms contain ancestor monomials with dimension bounded by Eq. (83).

We finally notice that from Eq. (83) one can associate a *dimension* (distinct from the canonical dimension) which serves to establish the degree of divergence of a graph. This allows establishing a grading in the local solutions of the homogeneous equations for the counterterms [Eq. (87)]. This technique will be used in Sec. X for the construction of a basis for the counterterms in the one-loop approximation.

IX. UNIQUENESS OF THE TREE-LEVEL VERTEX FUNCTIONAL

We are now in a position to prove the uniqueness of the tree-level vertex functional in Eq. (29). The dependence on the antifields is fixed by the boundary conditions in Eq. (18). The dependence on B_a and on the antighost field $\bar{c}_a$ is determined by Eqs. (52) and (53), respectively. The local $SU(2)_L$ symmetry is implemented through Eq. (31). Then the ST identity in Eq. (51) fixes the dependence on $V_{a\mu}$ and K_0, as well as on the ghosts c_a. However, by requiring global $SU(2)_R$ invariance there is still the freedom to add any global $SU(2)_R$ invariant constructed out of the bleached variable $a_{a\mu}$. This residual freedom is indeed limited by the weak-power-counting theorem. For that purpose we first notice that only invariants up to dimension four in the $A_{a\mu}$ variables are allowed by the UV behavior of the $A_{a\mu}$ propagator. Such an argument is shared also by power-counting renormalizable theories. This limits the possible interaction terms to the set of invariants in Eq. (12). Then the central idea of the argument is that only $(G_{a\mu\nu}[a])^2$ is independent of the fields $\vec{\phi}$ and ϕ_0, in a way already shown in Eq. (11). If any dependence on the fields $\vec{\phi}$ and ϕ_0 in the dimension four a_μ monomials in Eq. (12) remains, then we get infinitely many divergent graphs for the ancestor amplitudes already at one-loop level (violation of the weak-power-counting rule).

First we notice that only the combination

$$\int d^D x(\partial_\mu a_{a\nu}\partial^\mu a_a^\nu - (\partial a_a)^2) \tag{84}$$

is allowed by the requirement of the absence of negative metric modes in the ϕ_a sector. In fact if we expand $a_{a\mu}$ in powers of ϕ_a according to Eq. (8) after setting the gauge field $A_{a\mu}$ to zero, we find at the lowest order

$$\int d^D x\partial_\mu a_{a\nu}\partial^\mu a_a^\nu \sim \int d^D x(\partial a_a)^2 \sim \frac{4}{v^2}\int d^D x\phi_a\Box^2\phi_a. \tag{85}$$

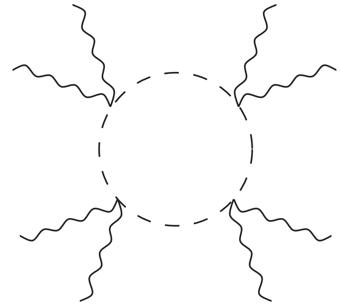

FIG. 1. A weak-power-counting violating graph.

We now notice that each of the invariants in Eq. (12), with the exclusion of the mass term $\int d^4 x a^2$, contains vertices with two A's, two ϕ's, and two derivatives. These vertices destroy the weak-power-counting bound in Eq. (83) since they give rise already at one-loop level to divergent graphs involving an arbitrary number of external A legs (see Fig. 1) with superficial degree of divergence $d(\mathcal{G}) = 4$.

By requiring that these interaction vertices vanish, one finds that only the following combination is allowed (up to an overall constant):

$$\int d^D x G_{a\mu\nu}[a]G_a^{\mu\nu}[a]. \tag{86}$$

This is a rather remarkable result. The tree-level vertex functional in Eq. (29) (which embodies the Yang-Mills action with a Stückelberg mass term) is uniquely determined by symmetry requirements and the weak-power-counting property. In particular, the symmetry content of the model allows for the anomalous trilinear and quadrilinear couplings in Eq. (12), but the latter are excluded on the basis of the weak-power-counting criterion.

X. ONE LOOP

As is well-known there is no consistent theory of pure massive Yang-Mills in the framework of power-counting renormalizable field theory (i.e. physical unitarity is violated). The present formulation aims to overcome the limitation of power-counting renormalizability and yet to provide a consistent physical theory. This can be tested already at one loop. In particular one can verify that the conditions for the validity of physical unitarity are met and moreover that the divergences can be consistently organized in counterterms which preserve the defining equations (symmetric subtraction). Some one-loop calculations will be published elsewhere. Here we provide a theoretical analysis of the counterterms by means of the local solutions of the linearized equations (65) and (81), which [together with the Landau gauge equation (52)] at one loop take the form

$$\mathcal{W}_0(\hat{\Gamma}^{(1)}) = 0 \qquad \mathcal{S}_0(\hat{\Gamma}^{(1)}) = 0 \qquad \frac{\delta\hat{\Gamma}^{(1)}}{\delta B_a} = 0. \tag{87}$$

We want to provide a basis for the local solutions of Eq. (87). The FP ghost number has to be zero, and moreover the dimensions of the monomial must match those of the pole part of the Feynman amplitudes. The analysis of this problem is made easy by the fact that

$$[\mathcal{W}_0, \mathcal{S}_0] = 0. \tag{88}$$

This can be proved by the following steps: (i) The commutator is either zero or of first order in the functional derivatives; thus Eq. (88) needs to be checked only on the fields and sources. (ii) On fields Eq. (88) reduces to Eq. (28). (iii) In order to test Eq. (88) on the sources we use the identities

$$\mathcal{W}_0(\Gamma^{(0)}) = -\frac{1}{2}\int d^Dx\alpha_a^L\Big(K_0\phi_a - \phi_0\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\Big)$$
$$\mathcal{S}_0(\Gamma^{(0)}) = -\int d^Dx\Big(B_a\frac{\delta}{\delta\bar{c}_a} + \Theta_{a\mu}\frac{\delta}{\delta V_{a\mu}} - K_0\frac{\delta}{\delta\phi_0^*}\Big)\Gamma^{(0)}. \tag{89}$$

Eq. (88) is used in order to construct the solutions of Eq. (87). The solutions of

$$\mathcal{W}_0(\mathcal{M}) = 0 \tag{90}$$

are constructed by using bleached fields. Then Eq. (88) says that the ST transform of $\mathcal{M}$ also satisfies Eq. (90):

$$\mathcal{W}_0(\mathcal{S}_0(\mathcal{M})) = \mathcal{S}_0(\mathcal{W}_0(\mathcal{M})) = 0. \tag{91}$$

In Appendix D we explicitly realize Eq. (91) by showing that the ST transform by $\mathcal{S}_0$ of bleached fields and sources remains bleached. The results are used here to perform the ST transforms on monomials that yield solutions of Eq. (87) of dimensions equal or less than four. The output of this calculation provides a basis for the counterterms at the one-loop level. The invariants can be divided into two classes: one depends only on the bleached gauge field a_μ, and the other is given by all possible $\mathcal{S}_0$-exact local functionals of dimension less than or equal to four. The first class is given by

$$\begin{aligned} &\mathcal{I}_1 = \int d^Dx\,\mathrm{Tr}\partial_\mu a_\nu\partial^\mu a^\nu, \quad \mathcal{I}_2 = \int d^Dx\,\mathrm{Tr}(\partial a)^2,\\ &\mathcal{I}_3 = i\int d^Dx\,\mathrm{Tr}(\partial_\mu a_\nu[a^\mu, a^\nu]),\\ &\mathcal{I}_4 = \int d^Dx\,\mathrm{Tr}(a^2)\mathrm{Tr}(a^2),\\ &\mathcal{I}_5 = \int d^Dx\,\mathrm{Tr}(a_\mu a_\nu)\mathrm{Tr}(a^\mu a^\nu), \quad \mathcal{I}_6 = \int d^Dx\,\mathrm{Tr}(a^2). \end{aligned} \tag{92}$$

By explicit calculation one shows that

$$\begin{aligned} &\mathcal{I}_1 - \mathcal{I}_2 - 3\mathcal{I}_3 + 2\mathcal{I}_4 - 2\mathcal{I}_5 - M^2\mathcal{I}_6\\ &= -\int d^Dx\,\mathrm{Tr}[a^\mu(D^\rho G_{\rho\mu}[a] + M^2 a_\mu)]\\ &= -\frac{g^2}{\Lambda^{D-4}}\mathcal{S}_0\,\mathrm{Tr}\int d^Dx\tilde{\bar{A}}^*_\mu a^\mu; \end{aligned} \tag{93}$$

i.e. the invariants $\mathcal{I}_1 - \mathcal{I}_6$ are linearly independent, but $\mathcal{S}_0$ dependent (cohomologically dependent). Moreover it should be remembered that $\mathcal{I}_1 - \mathcal{I}_6$ can be linearly combined to reproduce $\int d^Dx(G_{a\mu\nu})^2$, which is independent from $\vec{\phi}$. Thus one of the invariants out of $\mathcal{I}_1 - \mathcal{I}_6$ can be discarded in favor of the squared field strength. This has some advantages if one projects to amplitudes involving the Goldstone $\vec{\phi}$ field.

The second class contains the $\mathcal{S}_0$-exact local functionals

$$\begin{aligned} &\mathcal{I}_7 = \mathcal{S}_0\int d^Dx\,\mathrm{Tr}(\tilde{\bar{A}}^*_\mu v^\mu) = \frac{\Lambda^{D-4}}{g^2}\int d^Dx\,\mathrm{Tr}[v^\mu(D^\rho G_{\rho\mu}[a] + M^2 a_\mu)] - \int d^Dx\,\mathrm{Tr}(\tilde{\bar{A}}^*_\mu\tilde{\Theta}^\mu) + \int d^Dx\,\mathrm{Tr}\tilde{\bar{A}}^*_\mu(D^\mu[v]\tilde{c}),\\ &\mathcal{I}_8 = \mathcal{S}_0\Big[\int d^Dx\,\mathrm{Tr}(\tilde{\Omega}^*\mathcal{S}_0\,\mathrm{Tr}(\tilde{\Omega}^*))\Big] = \int d^Dx[(\mathrm{Tr}(\tilde{K}))^2 - (\mathrm{Tr}(\tilde{c}\tilde{\Omega}^*))^2 + 2i\,\mathrm{Tr}(\tilde{K})\,\mathrm{Tr}(\tilde{c}\tilde{\Omega}^*)]\\ &\mathcal{I}_9 = \mathcal{S}_0\int d^Dx\,\mathrm{Tr}(\tilde{\Omega}^*)\,\mathrm{Tr}(a^2) = -i\int d^Dx\,\mathrm{Tr}(\tilde{c}\tilde{\Omega}^*)\,\mathrm{Tr}(a^2) - \int d^Dx\,\mathrm{Tr}(\tilde{K})\,\mathrm{Tr}(a^2),\\ &\mathcal{I}_{10} = \mathcal{S}_0\int d^Dx\,\mathrm{Tr}(\tilde{c}^*\tilde{c}) = \int d^Dx\Big(\mathrm{Tr}((D^\mu[a]\tilde{\bar{A}}^*_\mu)\tilde{c}) - \frac{i}{4}\,\mathrm{Tr}((\tilde{\Omega}^*)^\dagger\tilde{c}) + \frac{i}{2}\,\mathrm{Tr}(\tilde{c}^*\{\tilde{c},\tilde{c}\})\Big),\\ &\mathcal{I}_{11} = \mathcal{S}_0\int d^Dx\,\mathrm{Tr}(\tilde{\Omega}^*) = -i\int d^Dx\,\mathrm{Tr}(\tilde{c}\tilde{\Omega}^*) - \int d^Dx\,\mathrm{Tr}(\tilde{K}). \end{aligned} \tag{94}$$

The last invariant, $\mathcal{I}_{11}$, although of lower dimensions, has been included for a possible use in gauges different from Landau's.

At the one-loop level the weak-power-counting criterion fixes the upper bound for the dimensions of the local invariants. On the basis of this argument we have omitted invariants like

$$\mathcal{S}_0 \int d^D x \mathrm{Tr}(\tilde{\Omega}^* \tilde{K}) = \int d^D x \mathrm{Tr}([-i\tilde{c}\tilde{\Omega}^* - \tilde{K}]\tilde{K} + i\tilde{\Omega}^* \tilde{c}\tilde{K}) = -\int d^D x \mathrm{Tr}(i\tilde{c}\{\tilde{\Omega}^*, \tilde{K}\} + \tilde{K}^2), \quad (95)$$

since it has terms of dimension five according to the counting of Eq. (83).

XI. CONCLUSIONS

A consistent theory of massive Yang-Mills can be formulated in spite of the fact that the starting set of Feynman rules corresponds to a power-counting nonrenormalizable theory. Consistency is based on the existence of a subtraction scheme for the divergences which does not alter the set of defining equations. Physical unitarity, locality of the counterterms, a finite number of subtractions at each order of the loop expansion (more correctly: expansion in $\hbar$), and a finite number of physical parameters are essential properties of the procedure of subtraction. The symmetry of the model is the gauge group $SU(2)$ left (local) $\otimes$ $SU(2)$ right (global). Moreover BRST invariance is enforced in order to guarantee physical unitarity. The managing of the divergences is based on techniques already tested in the nonlinear sigma model: hierarchy, weak power counting, and dimensional subtraction on properly normalized 1-PI amplitudes. The spontaneous breakdown of the global axial symmetry is via a vacuum expectation value which has no physical significance. The global vector symmetry remains unitarily implemented.

APPENDIX A: FEYNMAN RULES

In order to fix the Feynman rules we find it convenient to use the tree-level effective action (61) instead of the original form in Eq. (30). By this choice both the local functional equation (31) and the ST identity (16) acquire an explicit dependence on $\Lambda_D = \Lambda^{D-4}/g^2$ (as discussed in Sec. VII).

The advantage resides in the fact that with the rescaled effective action (61) the dependence of the 1-PI amplitudes on Λ_D can be easily traced: any n_L-loop amplitude contains $\Lambda_D^{1-n_L}$ as a factor [see Eq. (73)]. Then one can discard any dependence from Λ_D in the intermediate steps and recover it at the end of the calculations. In particular when one evaluates the counterterms, the prescription (68) requires that at any loop order the amplitudes must be normalized by the prefactor Λ^{4-D}, before the subtraction of the poles [see Eq. (77)]. On the other side, if physical matrix elements are required, the normalization of the asymptotic states has to be taken into account. Thus at the tree-level approximation, one gets for physical S-matrix elements

$$S_{A_1 \ldots A_{N_A}} = g^{N_A} W^C_{A_1 \ldots A_{N_A}}, \quad (A1)$$

where $W^C_{A_1 \ldots A_{N_A}}$ denotes the connected amputated Green function with physical polarizations inserted on the gauge boson legs $A_1, \ldots, A_{N_A}$.

The quadratic part in the quantized fields of $\Gamma^{(0)}$ (where Λ_D has been discarded) is

$$\int d^D x\Big(-\frac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu})^2 + \frac{M^2}{2}\Big(A_{a\mu} - \frac{2}{v}\partial_\mu \phi_a\Big)^2 + B_a \partial A_a - \bar{c}_a \Box c_a\Big). \quad (A2)$$

It is straightforward to get the propagators

$$\begin{aligned} \Delta_{A_{a\mu}A_{b\nu}} &= \frac{-i}{p^2 - M^2}\Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\delta_{ab}, \\ \Delta_{\phi_a\phi_b} &= \frac{i}{4}\frac{v^2}{M^2}\frac{1}{p^2}\delta_{ab}, \qquad \Delta_{B_a A_{a\mu}} = \frac{p_\mu}{p^2}\delta_{ab}, \\ \Delta_{B_a\phi_b} &= -i\frac{v}{2p^2}, \qquad \Delta_{c_a\bar{c}_b} = \frac{i}{p^2}\delta_{ab}, \\ \Delta_{B_a B_b} &= 0, \qquad \Delta_{A_{a\mu}\phi_b} = 0. \end{aligned} \quad (A3)$$

APPENDIX B: ABSENCE OF THE HIERARCHY BASED ON SLAVNOV-TAYLOR IDENTITY

At one-loop order the ST identity in Eq. (19) reads

$$\begin{aligned} \mathcal{S}_0(\Gamma^{(1)}) = \int d^D x\Big(&\frac{\delta\Gamma^{(0)}}{\delta A^*_{a\mu}}\frac{\delta\Gamma^{(1)}}{\delta A^\mu_a} + \frac{\delta\Gamma^{(0)}}{\delta A_{a\mu}}\frac{\delta\Gamma^{(1)}}{\delta A^*_{a\mu}} \\ &+ \frac{\delta\Gamma^{(0)}}{\delta\phi^*_a}\frac{\delta\Gamma^{(1)}}{\delta\phi_a} + \frac{\delta\Gamma^{(0)}}{\delta\phi_a}\frac{\delta\Gamma^{(1)}}{\delta\phi^*_a} + \frac{\delta\Gamma^{(0)}}{\delta c^*_a}\frac{\delta\Gamma^{(1)}}{\delta c_a} \\ &+ \frac{\delta\Gamma^{(0)}}{\delta c_a}\frac{\delta\Gamma^{(1)}}{\delta c^*_a} + B_a\frac{\delta\Gamma^{(1)}}{\delta\bar{c}_a}\Big) = 0. \end{aligned} \quad (B1)$$

In order to show that Eq. (B1) does not uniquely fix the dependence on the ϕ's once the amplitudes involving all the remaining variables are known (absence of the hierarchy), we will construct two different solutions, $\mathcal{I}$ and $\mathcal{I}'$, of Eq. (B1) which coincide at $\phi_a = 0$.

For that purpose we notice that

$$\begin{aligned} \mathcal{I} &= \mathcal{S}_0\Big(\int d^D x(A^*_{a\mu} + \partial_\mu\bar{c}_a)A^\mu_a\Big) \\ &= \int d^D x\Big(A_{a\mu}\frac{\delta S}{\delta A_{a\mu}} - (A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu c_a\Big) \end{aligned} \quad (B2)$$

is $\mathcal{S}_0$ invariant due to the nilpotency of $\mathcal{S}_0$. Nilpotency holds as a consequence of the tree-level ST identity in Eq. (17).

At $\phi = 0$ $\mathcal{I}$ reduces to

$$\begin{aligned} \mathcal{I}_{\phi=0} = \int\Big[&\frac{\Lambda^{D-4}}{g^2}(-\partial_\mu A_{\nu a}\partial^\mu A^\nu_a + (\partial A_a)^2 \\ &- 3\epsilon_{abc}\partial_\mu A_{\nu a}A^\mu_b A^\nu_c - (A^2)^2 + A_{a\mu}A^\mu_b A_{a\nu}A^\nu_b \\ &+ M^2 A^2_{\mu a}) - (A^*_{\mu a} + \partial_\mu\bar{c}_a)\partial^\mu c_a\Big]. \end{aligned} \quad (B3)$$

We now set

$$\mathcal{S}_0\phi_a = \Omega_{ab}c_b \equiv (\tfrac{1}{2}\phi_0\delta_{ab} + \tfrac{1}{2}\epsilon_{acb}\phi_c)c_b. \tag{B4}$$

The matrix Ω_{ab} is invertible due to the nonlinear constraint in Eq. (3). Let us now replace in the first two lines of Eq. (B3) ∂_μ with the covariant derivative w.r.t. $F_{a\mu}$. We substitute $A_{a\mu}$ with the combination $I_{a\mu} = A_{a\mu} - F_{a\mu}$. Moreover we make separately $\mathcal{S}_0$ invariant the last line of Eq. (B3) as follows:

$$\begin{aligned}\mathcal{I}' = \int\Big[&\frac{\Lambda^{D-4}}{g^2}(-(D[F]_\mu I_\nu)_a(D[F]^\mu I^\nu)_a + (D[F]I)_a^2\\ &- 3\epsilon_{abc}(D_\mu[F]I_\nu)_a I_b^\mu I_c^\nu - (I^2)^2 + I_{a\mu}I_b^\mu I_{a\nu}I_b^\nu\\ &+ M^2I^2) + \mathcal{S}_0[(A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu(\Omega^{-1}_{ap}\phi_p)]\Big].\end{aligned} \tag{B5}$$

By construction $\mathcal{I}'$ is also $\mathcal{S}_0$ invariant. Moreover at $\phi = 0$ $\mathcal{I}$ and $\mathcal{I}'$ coincide, as can be easily checked by noticing that

$$\begin{aligned}&\mathcal{S}_0((A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu(\Omega^{-1}_{ap}\phi_p))\\ &\quad= \mathcal{S}_0(A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu(\Omega^{-1}_{ap}\phi_p)\\ &\qquad- (A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu(\mathcal{S}_0(\Omega^{-1}_{ap})\phi_p)\\ &\qquad- (A^*_{a\mu} + \partial_\mu\bar{c}_a)\partial^\mu c_a.\end{aligned} \tag{B6}$$

However, $\mathcal{I}$ and $\mathcal{I}'$ differ in their ϕ-dependent terms. Let us consider for instance the sector $A^*c\phi$. By using integration by parts, a basis of monomials involving just one derivative is given by $\epsilon_{abc}\partial A^*_a c_b\phi_c$, $\epsilon_{abc}A^*_{a\mu}\partial^\mu c_b\phi_c$. We project on the latter monomial. The only term contributing to this monomial in $\mathcal{I}'$ is

$$\frac{2}{v}A^*_{a\mu}\epsilon_{abc}\partial^\mu c_b\phi_c. \tag{B7}$$

On the other hand, there is no similar contribution in $\mathcal{I}$ (since in $\mathcal{I}$ $A^*_{a\mu}$ does not couple to the ϕ's).

This means that we have found two different $\mathcal{S}_0$ invariants with the same ancestor amplitudes. This gives an explicit counterexample showing that the ST identity is not sufficient in order to implement the hierarchy principle.

APPENDIX C: PROOF OF THE WEAK-POWER-COUNTING FORMULA

In this appendix we prove the power-counting formula in Eq. (83).

Let $\mathcal{G}$ be an arbitrary n-loop 1-PI ancestor graph with I internal lines, V vertices, and a given set $\{N_A, N_c, N_V, N_\Omega, N_{\phi_0^*}, N_{K_0}, N_{\phi_a^*}, N_{A^*}, N_{c^*}\}$ of external legs.

By Eq. (A3) all propagators but those involving the field B behave as p^{-2} as p goes to infinity, while those involving B behave as p^{-1}. Let us denote by $\hat{I}$ the number of internal lines associated with propagators which do not involve B, and by I_B the number of internal lines with propagators involving B. One has

$$I = \hat{I} + I_B. \tag{C1}$$

According to the Feynman rules generated by the tree-level vertex functional in Eq. (29), the superficial degree of divergence of $\mathcal{G}$ is

$$\begin{aligned}d(\mathcal{G}) = nD - 2\hat{I} - I_B + V_{AAA} + \sum_k V_{A\phi^k}\\ + 2\sum_k V_{\phi^k} + V_{\bar{c}cA} + V_{\bar{c}cV}.\end{aligned} \tag{C2}$$

In the above equation we have denoted by V_{AAA} the number of vertices in $\mathcal{G}$ with three A fields, with $V_{A\phi^k}$ the number of vertices with one A and k ϕ's, and so on. By using Eq. (C1) we can rewrite Eq. (C2) as

$$\begin{aligned}d(\mathcal{G}) = nD - 2I + I_B + V_{AAA} + \sum_k V_{A\phi^k}\\ + 2\sum_k V_{\phi^k} + V_{\bar{c}cA} + V_{\bar{c}cV}.\end{aligned} \tag{C3}$$

Moreover, since B only enters into the trilinear vertex $\Gamma^{(0)}_{B_aV_{b\mu}A_{c\nu}}$, the number of BVA vertices must coincide with the number of propagators involving B:

$$I_B = V_{BVA}. \tag{C4}$$

The total number of vertices V is given by

$$\begin{aligned}V = {}& V_{AAA} + V_{AAAA} + \sum_k V_{A\phi^k} + \sum_k V_{\phi^k} + V_{BVA} + V_{\bar{c}cA}\\ &+ V_{\bar{c}cV} + V_{\bar{c}cVA} + V_{\bar{c}A\Omega} + V_{\phi_0^*\phi c} + \sum_k V_{\phi_a^*\phi^k c}\\ &+ V_{A^*Ac} + V_{c^*cc} + \sum_k V_{K_0\phi^k}.\end{aligned} \tag{C5}$$

Euler's formula yields

$$I = n + V - 1. \tag{C6}$$

By using Eqs. (C4)–(C6) into Eq. (C2) one gets

$$
\begin{aligned}
d(\mathcal{G}) &= (D-2)n + 2 + I_B - V_{AAA} - \sum_k V_{A\phi^k} - V_{\bar{c}cA} - V_{\bar{c}cV} \\
&\quad - 2\Big[V_{AAAA} + V_{BVA} + V_{\bar{c}cVA} + V_{\bar{c}A\Omega} + V_{\phi_0^*\phi c} + \sum_k V_{\phi_a^*\phi^k c} + V_{A^*Ac} + V_{c^*cc} + \sum_k V_{K_0\phi^k}\Big] \\
&= (D-2)n + 2 - V_{AAA} - \sum_k V_{A\phi^k} - V_{\bar{c}cA} - V_{\bar{c}cV} - V_{BVA} \\
&\quad - 2\Big[V_{AAAA} + V_{\bar{c}cVA} + V_{\bar{c}A\Omega} + V_{\phi_0^*\phi c} + \sum_k V_{\phi_a^*\phi^k c} + V_{A^*Ac} + V_{c^*cc} + \sum_k V_{K_0\phi^k}\Big]. \qquad \text{(C7)}
\end{aligned}
$$

Clearly one has

$$
\begin{aligned}
&V_{\bar{c}A\Omega} = N_\Omega, \qquad V_{\phi_0^*\phi c} = N_{\phi_0^*}, \qquad V_{A^*Ac} = N_{A^*}, \\
&V_{c^*cc} = N_{c^*}, \qquad \sum_k V_{\phi_a^*\phi^k c} = N_{\phi_a^*}, \qquad \text{(C8)} \\
&\sum_k V_{K_0\phi^k} = N_{K_0}, \qquad V_{\bar{c}cV} + V_{BVA} + V_{\bar{c}cVA} = N_V.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&V_{AAA} + \sum_k V_{A\phi^k} + 2V_{AAAA} + V_{\bar{c}cA} \\
&\quad + V_{\bar{c}cVA} + \sum_k V_{\phi_a^*\phi^k c} \geq N_A + N_c. \qquad \text{(C9)}
\end{aligned}
$$

In fact the quadrilinear vertex V_{AAAA} can give one or two external A lines.

By using Eqs. (C8) and (C9) into Eq. (C7) we obtain in a straightforward way the following bound:

$$
\begin{aligned}
d(\mathcal{G}) &\leq (D-2)n + 2 - N_A - N_c - N_V - N_{\phi_a^*} \\
&\quad - 2(N_\Omega + N_{A^*} + N_{\phi_0^*} + N_{c^*} + N_{K_0}). \qquad \text{(C10)}
\end{aligned}
$$

This establishes the validity of the weak-power-counting formula.

APPENDIX D: $\mathcal{S}_0$ TRANSFORMS OF THE BLEACHED VARIABLES

In this appendix we derive the $\mathcal{S}_0$ transforms of the bleached variables. For that purpose it is useful to work in matrix notation.

The $\mathcal{S}_0$ transform of Ω in Eq. (3) is

$$\mathcal{S}_0\Omega = ic\Omega, \qquad \text{(D1)}$$

where

$$c = c_a\frac{\tau_a}{2}. \qquad \text{(D2)}$$

Moreover

$$\mathcal{S}_0 c = \frac{i}{2}\{c, c\}. \qquad \text{(D3)}$$

It follows by direct computation that the bleached partner of c

$$\tilde{c} = \Omega^\dagger c \Omega \qquad \text{(D4)}$$

transforms as follows under $\mathcal{S}_0$:

$$\mathcal{S}_0\tilde{c} = -\frac{i}{2}\{\tilde{c}, \tilde{c}\}. \qquad \text{(D5)}$$

a_μ in Eq. (8) is $\mathcal{S}_0$ invariant. On the other hand the $\mathcal{S}_0$ transform of

$$v_\mu = v_{a\mu}\frac{\tau_a}{2} = \Omega^\dagger(V_\mu - F_\mu)\Omega \qquad \text{(D6)}$$

yields

$$\mathcal{S}_0 v_\mu = \Omega^\dagger(\Theta_\mu - D_\mu[V]c)\Omega = \tilde{\Theta}_\mu - D_\mu[v]\tilde{c}, \qquad \text{(D7)}$$

and

$$\mathcal{S}_0\tilde{\Theta}_\mu = -i\{\tilde{c}, \tilde{\Theta}_\mu\}. \qquad \text{(D8)}$$

We now move to the study of the antifield-dependent sector.

For that purpose we first evaluate the $\mathcal{S}_0$ variation of

$$\hat{A}^*_\mu = \hat{A}^*_{a\mu}\frac{\tau_a}{2} \qquad \text{(D9)}$$

and get according to Eq. (58)

$$
\begin{aligned}
\mathcal{S}_0\hat{A}^*_\mu &= \frac{\delta S}{\delta A_{a\mu}}\frac{\tau_a}{2} - \epsilon_{abc}c_b\hat{A}^*_{c\mu}\frac{\tau_a}{2} \\
&= \frac{\Lambda^{D-4}}{g^2}\Big[D^\rho G_{a\rho\mu}\frac{\tau_a}{2} + M^2(A_{a\mu} - F_{a\mu})\frac{\tau_a}{2}\Big] \\
&\quad + i\{c, \hat{A}^*_\mu\}. \qquad \text{(D10)}
\end{aligned}
$$

We need to express the R.H.S. of the above equation in terms of bleached variables. The bleached counterpart of $\hat{A}^*_\mu$ is

$$\tilde{\hat{A}}{}^*{}_\mu = \Omega^\dagger\hat{A}^*_\mu\Omega. \qquad \text{(D11)}$$

The transition from A_μ to the bleached gauge field a_μ is achieved by means of a $SU(2)_L$ gauge transformation of parameters Ω:

$$A_\mu = \Omega a_\mu \Omega^\dagger + i\Omega\partial_\mu\Omega^\dagger. \qquad \text{(D12)}$$

Since the terms between square brackets in Eq. (D10) transform in the adjoint representation under $SU(2)_L$ gauge transformations, we get, by taking into account Eqs. (D4) and (D11),

$$\mathcal{S}_0 \hat{A}^*_\mu = \frac{\Lambda^{D-4}}{g^2}\Omega[D^\rho G_{\rho\mu}[a] + M^2 a_\mu]\Omega^\dagger + i\Omega\{\tilde{c}, \tilde{\hat{A}}^*_\mu\}\Omega^\dagger \tag{D13}$$

and finally

$$\mathcal{S}_0 \tilde{\hat{A}}^*_\mu = \frac{\Lambda^{D-4}}{g^2}[D^\rho G_{\rho\mu}[a] + M^2 a_\mu]. \tag{D14}$$

The matrix Ω^* in Eq. (44) has the following $\mathcal{S}_0$ transform

$$\mathcal{S}_0\Omega^* = -K_0 + i\frac{\delta\Gamma^{(0)}}{\delta\phi_a}\tau_a = -(K_0 + iK_a\tau_a) \equiv -K, \tag{D15}$$

where we have introduced the notation

$$K_a \equiv -\frac{\delta\Gamma^{(0)}}{\delta\phi_a}. \tag{D16}$$

Under local left multiplication K transforms as Ω [12]. The bleached counterpart of Ω^* is

$$\tilde{\Omega}^* = \Omega^\dagger\Omega^*. \tag{D17}$$

Its $\mathcal{S}_0$ transform gives

$$\mathcal{S}_0\tilde{\Omega}^* = -i\Omega^\dagger c\Omega^* - \Omega^\dagger K = -i\tilde{c}\tilde{\Omega}^* - \tilde{K} \qquad \mathcal{S}_0\tilde{K} = -i\tilde{c}\,\tilde{K}. \tag{D18}$$

Finally we consider the $\mathcal{S}_0$ variation of

$$c^* = c^*_a\frac{\tau_a}{2}. \tag{D19}$$

It is convenient to rewrite the couplings between the antifields (ϕ^*_0, ϕ^*_a) and the BRST variations $(s\phi_0, s\phi_a)$ in Eq. (29) in the following way:

$$\int d^Dx(\phi^*_0 s\phi_0 + \phi^*_a s\phi_a) = \int d^Dx\frac{1}{2}\,\mathrm{Tr}[(\Omega^*)^\dagger s\Omega] = \int d^Dx\frac{1}{2}\,\mathrm{Tr}\Big[(\Omega^*)^\dagger ic_a\frac{\tau_a}{2}\Omega\Big]. \tag{D20}$$

One finds

$$\begin{aligned}\mathcal{S}_0 c^* &= \frac{\delta\Gamma^{(0)}}{\delta c_a}\frac{\tau_a}{2} \\ &= D^\mu[A]\tilde{\hat{A}}^*_\mu - \frac{i}{2}\,\mathrm{Tr}\Big[(\Omega^*)^\dagger\frac{\tau_a}{2}\Omega\Big]\frac{\tau_a}{2} - i[c^*, c] \\ &= \Omega(D^\mu[a]\tilde{\hat{A}}^*_\mu)\Omega^\dagger - \frac{i}{2}\,\mathrm{Tr}\Big[(\tilde{\Omega}^*)^\dagger\Omega^\dagger\frac{\tau_a}{2}\Omega\Big]\frac{\tau_a}{2} \\ &\quad - i\Omega[\tilde{c}^*, \tilde{c}]\Omega^\dagger.\end{aligned} \tag{D21}$$

Then we consider the $\mathcal{S}_0$ variation of

$$\tilde{c}^* = \Omega^\dagger c^*\Omega, \tag{D22}$$

and we get

$$\mathcal{S}_0\tilde{c}^* = (D^\mu[a]\tilde{\hat{A}}^*_\mu) - \frac{i}{2}\,\mathrm{Tr}\Big[(\tilde{\Omega}^*)^\dagger\Omega^\dagger\frac{\tau_a}{2}\Omega\Big]\Omega^\dagger\frac{\tau_a}{2}\Omega. \tag{D23}$$

Since the matrices $\mathcal{T}_a = \Omega^\dagger\frac{\tau_a}{2}\Omega$ are unitarily equivalent to the Pauli matrices, the bleached matrix $(\tilde{\Omega}^*)^\dagger$ can be decomposed as follows:

$$(\tilde{\Omega}^*)^\dagger = \tfrac{1}{2}\mathrm{Tr}[(\tilde{\Omega}^*)^\dagger]\mathbf{1} + 2\,\mathrm{Tr}[(\tilde{\Omega}^*)^\dagger\mathcal{T}_a]\mathcal{T}_a. \tag{D24}$$

And thus finally the R.H.S. of Eq. (D23) can be rewritten as

$$\mathcal{S}_0\tilde{c}^* = (D^\mu[a]\tilde{\hat{A}}^*_\mu) - \frac{i}{4}(\tilde{\Omega}^*)^\dagger + \frac{i}{8}\,\mathrm{Tr}[(\tilde{\Omega}^*)^\dagger]\mathbf{1}. \tag{D25}$$

The results of this appendix are quite remarkable. The $\mathcal{S}_0$transforms of bleached variables are bleached. The $\mathcal{S}_0$ transform of the bleached antifield $\tilde{\hat{A}}^*_\mu$ is the equation of motion of the original Stückelberg action S in the bleached gauge field a_μ [see Eq. (D14)].

APPENDIX E: DEPENDENCE ON v

In this appendix we derive an extended ST identity allowing control of the dependence of the Green functions on v through cohomological methods. For that purpose we allow v to transform under ST differential $\mathcal{S}_0$ according to

$$\mathcal{S}_0 v = \theta, \qquad \mathcal{S}_0\theta = 0, \qquad \theta^2 = 0. \tag{E1}$$

The ST identity (51) is then modified to

$$\mathcal{S}(\Gamma) = \int d^Dx\Big(\frac{\delta\Gamma}{\delta A^*_{a\mu}}\frac{\delta\Gamma}{\delta A^\mu_a} + \frac{\delta\Gamma}{\delta\phi^*_a}\frac{\delta\Gamma}{\delta\phi_a} + \frac{\delta\Gamma}{\delta c^*_a}\frac{\delta\Gamma}{\delta c_a} + B_a\frac{\delta\Gamma}{\delta\bar{c}_a} + \Theta_{a\mu}\frac{\delta\Gamma}{\delta V_{a\mu}} - K_0\frac{\delta\Gamma}{\delta\phi^*_0}\Big) + \theta\frac{\partial\Gamma}{\partial v} = 0. \tag{E2}$$

The effective action at the tree-level $\Gamma^{(0)}$ [(29) and (30)] is a solution of the above equation only after adding an extra term dependent on v and θ:

$$\begin{aligned}\Gamma^{(0)}_{\rm ext} = S &+ \frac{\Lambda^{D-4}}{g^2}\int d^Dx(B_a(D^\mu[V](A_\mu - V_\mu))_a \\ &- \bar{c}_a(D^\mu[V]D_\mu[A]c)_a) + \frac{\Lambda^{D-4}}{g^2}\int d^Dx\Theta^\mu_a(D_\mu[A]\bar{c})_a \\ &+ \int d^Dx\Big(A^*_{a\mu}sA^\mu_a + \phi^*_0 s\phi_0 + \phi^*_a s\phi_a \\ &+ c^*_a sc_a + K_0\phi_0 + \phi^*_0\frac{\theta}{v}\phi_0 + \phi^*_a\frac{\theta}{v}\phi_a\Big).\end{aligned} \tag{E3}$$

Now we can discuss the dependence of the physical amplitudes from the parameter v. For this purpose it is convenient to introduce the connected generating functional W [we use the same notations as in Eq. (64)]

$$W = \Gamma + \int d^D x (L_{a\mu} A_a^\mu + K_a \phi_a + J_a^B B_a + \eta_a \bar{c}_a + \bar{\eta}_a c_a). \tag{E4}$$

The ST identity for W reads

$$\mathcal{S}(W) = \int d^D x \Big(-L_{a\mu} \frac{\delta W}{\delta A^*_{a\mu}} - K_a \frac{\delta W}{\delta \phi_a^*} - \bar{\eta}_a \frac{\delta W}{\delta c_a^*} - \frac{\delta W}{\delta J_a^B} \eta_a + \Theta_{a\mu} \frac{\delta W}{\delta V_{a\mu}} - K_0 \frac{\delta W}{\delta \phi_0^*} \Big) + \theta \frac{\partial W}{\partial v} = 0. \tag{E5}$$

This equation can be used in order to study the dependence of the Green functions on v. In particular let $\beta_{i_1}(x), \ldots, \beta_{i_n}(x_n)$ denote a set of additional external sources coupled to BRST-invariant local operators $\mathcal{O}_{i_1}(x_1), \ldots, \mathcal{O}_{i_n}(x_n)$. By differentiating Eq. (E5) w.r.t. θ and $\beta(x_{i_1}), \ldots, \beta(x_{i_n})$ and by setting all sources (collectively denoted by ζ) to zero, one gets

$$\frac{\partial}{\partial v} \frac{\delta^n W}{\delta \beta_{i_1}(x_1) \ldots \delta \beta_{i_n}(x_n)} \Big|_{\zeta=0} = 0; \tag{E6}$$

i.e. the Green functions of the operators $\mathcal{O}_i(x_i)$ are v independent. Moreover by differentiating Eq. (E5) w.r.t. θ and K_0 we get

$$\frac{\partial}{\partial v} \frac{\delta W}{\delta K_0(x)} \Big|_{\zeta=0} = \frac{\partial}{\partial \theta} \frac{\delta W}{\delta \phi_0^*(x)} \Big|_{\zeta=0}. \tag{E7}$$

This equation is a consequence of the fact that ϕ_0^* and $-K_0$ form a $\mathcal{S}_0$ doublet [see Eq. (50)]. We remark that a device technically similar to the one adopted here (pairing of v, θ into a $\mathcal{S}_0$ doublet) has been used in the context of gauge theories in order to discuss the dependence on the gauge parameter. However, we stress an important difference: in the present case the dependence on v is not confined to the BRST-exact sector of the tree-level vertex functional, since it also enters through the combination $\frac{\phi_a}{v}$ in the Stückelberg mass term and in the term $K_0 \phi_0$ of (E3). Therefore v cannot be identified *tout court* with a kind of gauge parameter.

[1] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[2] L. D. Faddeev and V. N. Popov, Phys. Lett. **25B**, 29 (1967).
[3] G. 't Hooft, Nucl. Phys. **B33**, 173 (1971).
[4] C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. **42**, 127 (1975); Ann. Phys. (N.Y.) **98**, 287 (1976); I. V. Tyutin, Lebedev Institute Report No. N39, 1975.
[5] G. Curci and R. Ferrari, Nuovo Cimento Soc. Ital. Fis. **35A**, 273 (1976); T. Kugo and I. Ojima, Phys. Lett. **73B**, 459 (1978).
[6] A. A. Slavnov, Teor. Mat. Fiz. **10**, 153 (1972) [Theor. Math. Phys. **10**, 99 (1972)]; J. C. Taylor, Nucl. Phys. **B33**, 436 (1971).
[7] J. Zinn-Justin, in *Trends in Elementary Particle Theory*, Lecture Notes in Physics Vol. 37 (Springer-Verlag, Berlin, 1975), p. 2; S. D. Joglekar and B. W. Lee, Ann. Phys. (N.Y.) **97**, 160 (1976); I. A. Batalin and G. A. Vilkovisky, Phys. Lett. **102B**, 27 (1981); **120B**, 166 (1983); Phys. Rev. D **28**, 2567 (1983); **30**, 508(E) (1984); Nucl. Phys. **B234**, 106 (1984); J. Math. Phys. (N.Y.) **26**, 172 (1985). For further references see O. Piguet and S. P. Sorella, *Algebraic Renormalization* Lecture Notes in Physics, Vol. M28, (Springer-Verlag, Berlin, 1995); and J. Gomis, J. Paris, and S. Samuel, Phys. Rep. **259**, 1 (1995).
[8] H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D **12**, 482 (1975); **12**, 3159 (1975); **12**, 467 (1975).
[9] P. W. Higgs, Phys. Lett. **12**, 132 (1964); Phys. Rev. Lett. **13**, 508 (1964); Phys. Rev. **145**, 1156 (1966); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Lett. **13**, 585 (1964); T. W. B. Kibble, Phys. Rev. **155**, 1554 (1967).
[10] B. W. Lee and J. Zinn-Justin, Phys. Rev. D **5**, 3121 (1972); **5**, 3137 (1972); **8**, 4654(E) (1973); G. 't Hooft, Nucl. Phys. **B35**, 167 (1971); G. 't Hooft and M. J. G. Veltman, Nucl. Phys. **B44**, 189 (1972); Nucl. Phys. **B50**, 318 (1972).
[11] R. Ferrari, J. High Energy Phys. 08 (2005) 048.
[12] R. Ferrari and A. Quadri, Int. J. Theor. Phys. **45**, 2497 (2006).
[13] R. Ferrari and A. Quadri, J. High Energy Phys. 01 (2006) 003.
[14] D. Bettinelli, R. Ferrari, and A. Quadri, arXiv:hep-th/0701197 [Int. J. Mod. Phys. A. (to be published)].
[15] D. Bettinelli, R. Ferrari, and A. Quadri, Int. J. Theor. Phys. **46**, 2560 (2007).
[16] S. Dittmaier and C. Grosse-Knetter, Phys. Rev. D **52**, 7276 (1995).
[17] D. Bettinelli, R. Ferrari, and A. Quadri, J. High Energy Phys. 03 (2007) 065.
[18] R. Ferrari and A. Quadri, J. High Energy Phys. 11 (2004) 019.
[19] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[20] J. Horejsi, *Introduction to Electroweak Unification: Standard Model from Tree Unitarity* (World Scientific, Singapore, 1994).
[21] E. C. G. Stückelberg, Helv. Phys. Acta **11**, 299 (1938); H. Ruegg and M. Ruiz-Altaba, Int. J. Mod. Phys. A **19**, 3265 (2004).
[22] R. Ferrari, M. Picariello, and A. Quadri, Phys. Lett. B **611**, 215 (2005).
[23] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **28**, 1494 (1972).
[24] L. F. Abbott, Nucl. Phys. **B185**, 189 (1981); L. F. Abbott, M. T. Grisaru, and R. K. Schaefer, Nucl. Phys. **B229**, 372 (1983); A. Denner, G. Weiglein, and S. Dittmaier, Nucl.

Phys. **B440**, 95 (1995).
[25] P. A. Grassi, Nucl. Phys. **B462**, 524 (1996); **B537**, 527 (1999).
[26] C. Becchi and R. Collina, Nucl. Phys. **B562**, 412 (1999).
[27] R. Ferrari, M. Picariello, and A. Quadri, Ann. Phys. (N.Y.) **294**, 165 (2001).
[28] G. Barnich, F. Brandt, and M. Henneaux, Phys. Rep. **338**, 439 (2000).
[29] A. Quadri, J. High Energy Phys. 05 (2002) 051.
[30] P. A. Grassi, T. Hurth, and A. Quadri, Phys. Rev. D **70**, 105014 (2004).
[31] O. Piguet and K. Sibold, Nucl. Phys. **B253**, 517 (1985).