

Decay of scalar condensation in quantum field theory

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We consider decay processes of scalar-field condensation in the framework of well-established quantum field theory. We postulate that the quantum state corresponding to the scalar-field condensation is the so-called coherent state with discussing the validity of such a treatment. We show that, by using the unitarity relation of the scattering matrix, the decay rate of the coherent state is systematically calculated. We apply our procedure to derive explicit formulas of decay rates for two cases: (i) we study the case where the scalar condensation decays into a pair of scalar particles and show that our formalism reproduces the results obtained from the parametric-resonance analysis, and (ii) we also calculate the decay rate when the coherent state decays via anomaly.

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I. INTRODUCTION

Scalar-field condensations play very crucial roles in various places in cosmology. Probably the most important example is the inflaton field which is necessary for inflation [1]. In particular, in the slow-roll inflation models [2], energy density of the inflaton condensation provides the energy density to realize a (quasi)de-Sitter universe during inflation. After inflation, the inflaton oscillates around the minimum of the potential and decays into standard-model particles to reheat the universe realizing hot big-bang cosmology. This class of inflation model not only solves flatness and horizon problems in cosmology but also provides density fluctuation consistent with the Wilkinson microwave anisotropy probe data [3]. In addition, it has been pointed out that the density fluctuation may arise from a late-decaying scalar field other than the inflaton, which is called “curvaton” [4]. Another important example is the Affleck-Dine field for baryogenesis [5]. In low-energy supersymmetric models, there exist scalar fields, i.e., scalar partners of quarks, which have baryon number. If some of those fields acquire nonvanishing amplitudes in the early universe, the nonvanishing baryon number may be imprinted into the motion of the scalar-quark condensations due to baryon-number violating operators at an ultrahigh energy scale. Such a scenario is one of the most attractive scenarios to generate large enough baryon asymmetry of the universe.

All of these exotic scalar-field condensations (i.e., inflaton, curvaton, Affleck-Dine field, and so on) oscillate around the minimum of the potential at some stage of the evolution of the universe, and eventually decay into standard-model particles for the cosmological history consistent with observations. Thus, it is important to understand how the scalar-field condensation decays from the viewpoint of the quantum field theory.

The main concern of this paper is to discuss how the oscillating scalar-field condensation decays into other

states. Around the minimum, the potential of the scalar field φ is well estimated by a parabolic one,

$$V = \frac{1}{2}m_\varphi^2\varphi^2. \quad (1.1)$$

Neglecting the effect of the cosmic expansion for simplicity, the solution to the classical equation of motion is given by

$$\varphi = A_\varphi \cos m_\varphi t. \quad (1.2)$$

In this case, the energy density of the condensation is given by $\rho_\varphi = \frac{1}{2}m_\varphi^2 A_\varphi^2$. One should understand how the energy density stored in the oscillation of φ is converted to that of radiation. In the simplest approach, the decay rate of the scalar-field condensation is estimated from the decay rate of a single scalar field (in the vacuum): the energy density of the scalar field in the condensation is approximated to decrease as

$$\frac{d\rho_\varphi}{dt} = -\gamma_\varphi\rho_\varphi, \quad (1.3)$$

where γ_φ is the decay rate of φ in the vacuum. However, it has been also pointed out that, when the scalar field φ is oscillating like Eq. (1.2), wave functions of fields which couple to φ are modified. Consequently, the “decay rate” of the scalar field in the condensation may be significantly different from the one obtained from the field theory in the vacuum. In particular, in some of the cases, an instability band may arise in the wave function of the final-state particles, which results in catastrophic particle production (so-called parametric resonance) [6–9]. Since the decay of scalar-field condensation is very important, it is desirable to have a deep understanding of the decay processes of scalar-field oscillations in the framework of well-established quantum field theory.

In this paper, we consider decay processes of scalar condensations in the framework of the quantum field theory. In our analysis, we neglect the effects of cosmic

expansion as a first step to understand the behavior of the scalar condensation.

We postulate that the scalar condensation corresponds to the so-called coherent state $|\varphi\rangle$ in the quantum field theory; justification of such a treatment will be also discussed. Then, we will show that the decay rate of the coherent state can be systematically calculated by using the unitarity relation of the scattering matrix (S -matrix). Both the coherent state and the S -matrix unitarity have been well-known concepts in the conventional quantum field theory for a long time [10,11]. (In addition, the coherent state was also used in cosmology in the studies of the cosmic perturbation [12], entropy production [13], electroweak baryogenesis [14], and so on.) However, we show that, by combining these two, new insights into the decay of scalar condensation are obtained. Using the fact that the decay rate of the coherent state is proportional to the imaginary part of the so-called T -matrix element $\Im[\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle]$, we give a new formalism with which the decay rate is systematically calculated. We will also apply our formalism to two cases. First, we study the case where the scalar-field condensation is coupled to a real scalar field. We will see that our procedure gives the same decay rate as that obtained by the discussion of parametric resonance. We discuss when Eq. (1.3) is justified and how the instability band for parametric resonance arises in our framework. Second, we consider a scalar condensation which decays into gauge-boson pair via anomaly. We give an explicit expression of the decay rate for such a case.

Organization of this paper is as follows. In Sec. II, we first derive basic formulas which are used in the calculation of the decay rate of the coherent state. In particular, we define the coherent state $|\varphi\rangle$ in the quantum field theory and present important properties of $|\varphi\rangle$. Then, we explain how the decay rate of the state $|\varphi\rangle$ is obtained. In Sec. III, we consider the case where the condensation couples to a real scalar field χ via three-point interaction. Decay of the coherent state via the interaction induced by the anomaly is discussed in Sec. IV. In Sec. V, we summarize our results.

II. BASIC FORMULAS

In this section, we introduce basic formulas used in our analysis. Some of the formulas given in this section are quite well known, and are found in standard textbooks of the quantum field theory [10,11]. However, we give basic formulas for the sake of the following arguments. Then, we discuss how the condensation of scalar field decays via some interaction. Hereafter, the interaction of the scalar field is assumed to be weak enough.

Total decay rate of any state can be related to the imaginary part of the scattering-matrix element due to the S -matrix unitarity [10]. Let us denote the S -matrix as

$$\hat{S} = \hat{\mathbf{1}} + i\hat{\mathcal{T}}, \quad (2.1)$$

where $\hat{\mathbf{1}}$ and $\hat{\mathcal{T}}$ are the unit operator and the so-called T -matrix, respectively. (Here and hereafter, the ‘‘hat’’ is used for operators.) Then, from the unitarity of the S -matrix, the following relation holds:

$$\hat{\mathcal{T}}^\dagger \hat{\mathcal{T}} = i(\hat{\mathcal{T}}^\dagger - \hat{\mathcal{T}}). \quad (2.2)$$

This relation is important for our analysis.

We expect that there exists a quantum state $|\varphi\rangle$ which describes the state with scalar condensation. (Details about $|\varphi\rangle$ will be explained below.) The probability of the state $|\varphi\rangle$ decaying into all the possible final states is related to the imaginary part of the T -matrix element as

$$\text{Prob}(|\varphi\rangle \rightarrow \text{all}) = \sum_f |\langle f|\hat{\mathcal{T}}|\varphi\rangle|^2 = 2\Im[\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle]. \quad (2.3)$$

Now, we consider what the state $|\varphi\rangle$ is. We first quantize the field operator using the free part of the Lagrangian, then treat the interaction terms as perturbations. We denote the free part of the Lagrangian of the real scalar field φ as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_\varphi^2\varphi^2. \quad (2.4)$$

In our analysis, we use the box normalization of the wave functions with the volume L^3 . Then, the field operator is given by

$$\hat{\varphi}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2E_{\mathbf{k}}L^3}} (\hat{a}_{\mathbf{k}}e^{-ikx} + \hat{a}_{\mathbf{k}}^\dagger e^{ikx}), \quad (2.5)$$

where $E_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m_\varphi^2}$. The annihilation and creation operators satisfy the following commutation relations:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}, \quad (2.6)$$

while $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}'}$ (as well as $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}'}^\dagger$) commute.

We postulate that the quantum state describing the scalar-field condensation is the coherent state, which is given by [11]

$$|\varphi\rangle \equiv e^{-|C_\varphi|^2/2} e^{C_\varphi \hat{a}_0^\dagger} |0\rangle, \quad (2.7)$$

where $|0\rangle$ is the vacuum state satisfying $\hat{a}_{\mathbf{k}}|0\rangle = 0$. Notice that the state $|\varphi\rangle$ is properly normalized: $\langle\varphi|\varphi\rangle = 1$. In addition, importantly, the state $|\varphi\rangle$ is an eigenstate of the annihilation operator \hat{a}_0 :

$$\hat{a}_0|\varphi\rangle = C_\varphi|\varphi\rangle. \quad (2.8)$$

We can also see that

$$\varphi(x) \equiv \langle\varphi|\hat{\varphi}(x)|\varphi\rangle = \varphi_-(x) + \varphi_+(x), \quad (2.9)$$

where

$$\varphi_-(x) \equiv \frac{1}{2}A_\varphi e^{-im_\varphi t}, \quad \varphi_+(x) \equiv \frac{1}{2}A_\varphi^* e^{im_\varphi t}, \quad (2.10)$$

with

$$A_\varphi = C_\varphi \sqrt{\frac{2}{m_\varphi L^3}}. \quad (2.11)$$

One can easily see that, for the coherent state $|\varphi\rangle$, the expectation value of the field operator follows the trajectory of the solution to the classical wave equation. Thus,

$$\langle \varphi | T \prod_i \hat{\varphi}(x_i) | \varphi \rangle = \langle \varphi | N \prod_i \hat{\varphi}(x_i) | \varphi \rangle + (\text{all the possible contractions}) = \prod_i \varphi(x_i) + (\text{all the possible contractions}), \quad (2.12)$$

where the symbol T here denotes the time ordering while N is for normal ordering. In addition, in the second equality, we have used Eq. (2.8). Even in more complicated cases, we obtain

$$\langle \varphi | T \prod_i f_i(\hat{\varphi}(x_i)) | \varphi \rangle = \langle \varphi | T \prod_i \sum_n \frac{1}{n!} \left[\frac{d^n f_i}{d\varphi^n} \right]_{\varphi(x_i)} (\hat{\varphi}(x_i) - \varphi(x_i))^n | \varphi \rangle = \prod_i f_i(\varphi(x_i)) + (\text{all the possible contractions}). \quad (2.13)$$

Here, we expand $f_i(\hat{\varphi}(x_i))$ around $\hat{\varphi}(x_i) = \varphi(x_i)$. Then, $\langle \varphi | (\hat{\varphi}(x) - \varphi(x)) | \varphi \rangle = 0$, and Eq. (2.13) is applicable even when the function $f_i(\varphi)$ is singular at $\varphi = 0$.

In the following, we will not consider the processes in which φ is produced due to the decay of the coherent state. (The inclusion of such processes is straightforward.) In such a case, the propagator of φ does not show up in the calculation and the field operator $\hat{\varphi}(x)$ can be simply replaced by the expectation value $\varphi(x)$, in which the contraction terms are irrelevant.

It is also instructive to calculate the expectation values of energy-density operator as

$$\rho_\varphi = L^{-3} \langle \varphi | \sum_{\mathbf{k}} E_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} | \varphi \rangle = \frac{1}{2} m_\varphi^2 |A_\varphi|^2, \quad (2.14)$$

while the expectation value of the number density is also given by

$$n_\varphi = L^{-3} \langle \varphi | \sum_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} | \varphi \rangle = \frac{1}{2} m_\varphi |A_\varphi|^2. \quad (2.15)$$

For the complex scalar field (which we denote as ϕ), similar argument applies. We define the field operator for the complex scalar field as

$$\hat{\phi}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2E_{\mathbf{k}}L^3}} (\hat{a}_{\mathbf{k}} e^{-ikx} + \hat{b}_{\mathbf{k}}^\dagger e^{ikx}), \quad (2.16)$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ ($\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{b}_{\mathbf{k}}^\dagger$) are annihilation (creation) operators. The coherent state for the complex field is given by

$$|\phi\rangle \equiv e^{-(|C_\phi|^2 + |C_{\bar{\phi}}|^2)/2} e^{C_\phi \hat{a}_0^\dagger + C_{\bar{\phi}} \hat{b}_0^\dagger} |0\rangle, \quad (2.17)$$

and

$$\phi(x) \equiv \langle \phi | \hat{\phi}(x) | \phi \rangle = A_\phi e^{-im_\phi t} + A_{\bar{\phi}}^* e^{im_\phi t}, \quad (2.18)$$

where

we expect that the coherent state $|\varphi\rangle$ corresponds to the quantum state where the scalar field is under oscillation.

In calculating T -matrix elements, it is necessary to calculate the expectation values of time-ordered products of field operators. By using the Wick's theorem, such products are calculated as

$$A_\phi = \frac{C_\phi}{\sqrt{2m_\phi L^3}}, \quad A_{\bar{\phi}} = \frac{C_{\bar{\phi}}}{\sqrt{2m_\phi L^3}}. \quad (2.19)$$

The energy density of this state is given by

$$\rho_\phi = 2m_\phi^2 (|A_\phi|^2 + |A_{\bar{\phi}}|^2), \quad (2.20)$$

while we can also calculate the expectation values of the number densities of particle ϕ and antiparticle $\bar{\phi}$ as

$$n_\phi \equiv L^{-3} \langle \phi | \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} | \phi \rangle = 2m_\phi |A_\phi|^2, \quad (2.21)$$

$$n_{\bar{\phi}} \equiv L^{-3} \langle \phi | \sum_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} | \phi \rangle = 2m_\phi |A_{\bar{\phi}}|^2. \quad (2.22)$$

We can see that the number densities of ϕ and its antiparticle are proportional to $|A_\phi|^2$ and $|A_{\bar{\phi}}|^2$, respectively. Thus, when $|A_\phi| > |A_{\bar{\phi}}|$ ($|A_\phi| < |A_{\bar{\phi}}|$), ϕ is more (less) abundant than $\bar{\phi}$ in the condensation. It should be also noted that the function $\phi(x)$ given in Eq. (2.18) gives an elliptical trajectory on the complex ϕ -plane. When $A_\phi = 0$ or $A_{\bar{\phi}} = 0$, the trajectory becomes a circle and, when $|A_\phi| = |A_{\bar{\phi}}|$, the trajectory becomes a straight line.

Before closing this section, we would like to comment that our arguments provide a new procedure of studying the decay of scalar condensation even though the concepts of the S -matrix unitarity and the coherent state have been known for a long time. Importantly, our formalism allows us to systematically calculate the decay rate of the scalar condensation in the framework of quantum field theory, which was not successful before. For instance, as we will discuss in the next section, the band structure of the parametric resonant phenomena, which has been obtained only in the classical treatment, can be understood as a contribution from the quantum corrections with our formula. This fact has a great impact, because our formula makes it possible to evaluate the quantum corrections systemati-

cally, which is impossible in the classical treatment. Furthermore, our formula can be applied to other decays of scalar condensation even in the cases where the classical treatment cannot be used.

III. DECAY INTO SCALAR FIELDS

A. Setup

First, we consider the simplest case where the scalar field φ couples only to the real scalar field χ via the interaction

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}\mu\varphi\chi^2, \quad (3.1)$$

with μ being the coupling constant. With this interaction, the decay rate of single particle in the vacuum is given by

$$\gamma_{\varphi \rightarrow \chi\chi} = \frac{\mu^2}{32\pi m_\varphi} \sqrt{1 - \left(\frac{4m_\chi^2}{m_\varphi^2}\right)}. \quad (3.2)$$

$$\sum_{\mathcal{F}^{(2p)}} i\mathcal{T}^{\mathcal{F}^{(2p)}} = \frac{1}{(2p)!} \left(\frac{i\mu}{2}\right)^{2p} \left\langle 0 \left| T \left[\int d^4x \{\varphi_-(x) + \varphi_+(x)\} \hat{\chi}(x) \hat{\chi}(x) \right]^{2p} \right| 0 \right\rangle, \quad (3.4)$$

where φ_- and φ_+ show up when the field operator $\hat{\varphi}$ is contracted with the creation operator in $|\varphi\rangle$ and the annihilation operator in $\langle\varphi|$, respectively. In order to obtain nonvanishing results, we need to pick up the same number of φ_- and φ_+ .

The right-hand side of Eq. (3.4) contains a contribution from various Feynman diagrams because φ_- and φ_+ can be ordered in many ways. In Fig. 1, we show a typical diagram (after imposing the cut). In our notation, we represent the insertion of φ_- by the white dot \circ while φ_+ by the black dot \bullet ; nonvanishing diagrams have p black and p white dots. Internal lines are the χ -propagator.

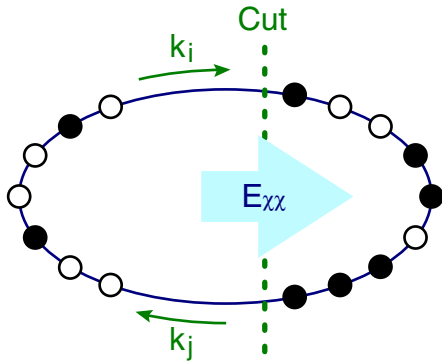


FIG. 1 (color online). An example of the Feynman diagram which gives rise to the imaginary part of $\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle$. The white dot \circ indicates φ_- insertion while the black dot \bullet is for φ_+ insertion. In this example, $p = 8$ and, with the cut shown in the figure, $N_\varphi = 3$. (Notice that other cuts are also possible with this diagram.)

In this section, with the interaction given in Eq. (3.1), we discuss how the coherent state decays.

As discussed in the previous section, the decay rate of the coherent state can be related to the imaginary part of the diagonal element of the T -matrix $\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle$. Importantly, $\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle$ is obtained by calculating loop diagrams in the quantum field theory.

At the one-loop level, in other words, neglecting the fluctuation of φ , $\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle$ is expressed as

$$\langle\varphi|\hat{\mathcal{T}}|\varphi\rangle \equiv \sum_{p=1}^{\infty} \sum_{\mathcal{F}^{(2p)}} \mathcal{T}^{\mathcal{F}^{(2p)}}, \quad (3.3)$$

where the summation over $\mathcal{F}^{(2p)}$ is for all the possible Feynman diagrams with $2p$ external φ . With p being fixed, one can find

The expectation value of $\hat{\mathcal{T}}$ for the given diagram $\mathcal{F}^{(2p)}$ (with $2p$ external φ insertion) is given by

$$i\mathcal{T}^{\mathcal{F}^{(2p)}} = L^3 T S_{\mathcal{F}} \left| \frac{\mu A_\varphi}{2} \right|^{2p} \int \frac{d^4\tilde{k}}{(2\pi)^4} \times \prod_{l=1}^{2p} (k_l^2 - m_\chi^2 + i0^+)^{-2}, \quad (3.5)$$

where $S_{\mathcal{F}}$ is the symmetry factor and

$$L^3 T = \int d^4x. \quad (3.6)$$

In addition, the momentum flowing on the l th propagator is given by

$$k_l \equiv \tilde{k} + \sum_{j=1}^l \varepsilon_j Q_\varphi, \quad (3.7)$$

with

$$Q_\varphi \equiv (m_\varphi, \mathbf{0}), \quad (3.8)$$

and $\varepsilon_j = 1$ and -1 if the J th φ insertion is φ_- and φ_+ , respectively.

The imaginary part of $\mathcal{T}^{\mathcal{F}^{(2p)}}$ is obtained by cutting two propagators (see Fig. 1), which corresponds to the replacements of those two propagators by the δ -functions (with a relevant numerical factor):

$$\Im[\mathcal{T}^{\mathcal{F}^{(2p)}}] = \lim_{\xi \rightarrow m_\chi^2} \sum_{i=1}^{2p-1} \sum_{j=i+1}^{2p} \Im[\mathcal{T}_{i,j}^{\mathcal{F}^{(2p)}}(\xi)], \quad (3.9)$$

where

$$\begin{aligned} \Im[\mathcal{T}_{i,j}^{\mathcal{F}^{(2p)}}(\xi)] &\equiv 2\pi^2 L^3 T S_{\mathcal{F}} \left| \frac{\mu A_{\varphi}}{2} \right|^{2p} \\ &\times \int \frac{d^4 \tilde{k}}{(2\pi)^4} \delta(k_i^2 - \xi_i) \delta(k_j^2 - \xi_j) \\ &\times \prod_{I \neq i,j} (k_I^2 - \xi_I)^{-1}. \end{aligned} \quad (3.10)$$

(For details, see Appendix A.) $\Im[\mathcal{T}_{i,j}^{\mathcal{F}^{(2p)}}]$ is the contribution from the diagram in which the cut is on i th and j th propagators. For $\Im[\mathcal{T}_{i,j}^{\mathcal{F}^{(2p)}}]$, we define the energy flow from one side of the cut to the other, which we denote $E_{\chi\chi}$; with the following non-negative integer,

$$N_{\varphi} = \left| \sum_{I=i+1}^j \varepsilon_I \right|, \quad (3.11)$$

the energy flow is given by

$$E_{\chi\chi} = N_{\varphi} m_{\varphi}. \quad (3.12)$$

Notice that $\Im[\mathcal{T}_{i,j}^{\mathcal{F}^{(2p)}}]$ contributes only to the decay rate of the process in which N_{φ} of φ in the condensation simultaneously annihilates into two χ because we neglect the fluctuations of the φ field. At the perturbative level, such a decay process is kinematically allowed when $E_{\chi\chi} > 2m_{\chi}$.

B. Small-amplitude limit

In this and the next subsections, we concentrate on the case where the amplitude of φ is small. In this case, the leading-order contribution to the $E_{\varphi} = N_{\varphi} m_{\varphi}$ mode is from the diagram with $2N_{\varphi}$ external φ with φ_{-} and φ_{+} being completely separated by the cut. (See Fig. 2.) Concentrating on such a leading-order diagram, the imaginary part of the T -matrix element is given by

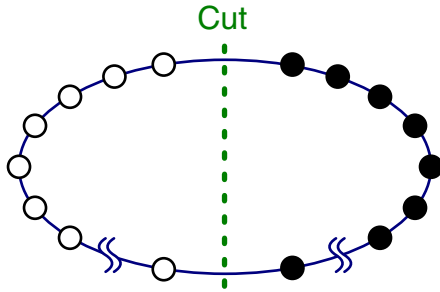


FIG. 2 (color online). Feynman diagram which gives the leading-order contribution to the decay rate in the small-amplitude limit. The white dot \circ indicates φ_{-} insertion while the black dot \bullet is for φ_{+} insertion.

$$\begin{aligned} \Im[\mathcal{T}_{\text{Leading}}^{(N_{\varphi})}] &= L^3 T \pi^2 \left| \frac{\mu A_{\varphi}}{2} \right|^{2N_{\varphi}} \int \frac{d^4 \tilde{k}}{(2\pi)^4} \delta(\tilde{k}^2 - m_{\chi}^2) \\ &\times \delta((\tilde{k} - N_{\varphi} Q_{\varphi})^2 - m_{\chi}^2) \\ &\times \prod_{I=1}^{N_{\varphi}-1} [(\tilde{k} - I Q_{\varphi})^2 - m_{\chi}^2]^{-2}. \end{aligned} \quad (3.13)$$

Constraints from the δ -functions give $\tilde{k} Q_{\varphi} = \frac{1}{2} N_{\varphi} m_{\varphi}^2$. Thus,

$$\prod_{I=1}^{N_{\varphi}-1} [(\tilde{k} - I Q_{\varphi})^2 - m_{\chi}^2]^{-1} \rightarrow m_{\varphi}^{-4(N_{\varphi}-1)} [(N_{\varphi}-1)!]^{-4}, \quad (3.14)$$

and hence

$$\Im[\mathcal{T}_{\text{Leading}}^{(N_{\varphi})}] = L^3 T \frac{\beta_{N_{\varphi}}}{32\pi} \frac{m_{\varphi}^4}{[(N_{\varphi}-1)!]^4} \left| \frac{\mu A_{\varphi}}{2m_{\varphi}^2} \right|^{2N_{\varphi}}, \quad (3.15)$$

where, for $N_{\varphi} m_{\varphi} > 2m_{\chi}$, the velocity $\beta_{N_{\varphi}}$ is given by

$$\beta_{N_{\varphi}} \equiv \sqrt{1 - \frac{4m_{\chi}^4}{N_{\varphi}^2 m_{\varphi}^2}}, \quad (3.16)$$

while $\beta_{N_{\varphi}} = 0$ for $N_{\varphi} m_{\varphi} \leq 2m_{\chi}$.

The decay rate of the coherent state $|\varphi\rangle$ per unit volume is evaluated as

$$(\text{Decay rate per unit volume}) = \frac{\text{Prob}(|\varphi\rangle \rightarrow \text{all})}{L^3 T}, \quad (3.17)$$

and hence is given by

$$\Gamma_{\text{Leading}}^{(N_{\varphi})} = \frac{\beta_{N_{\varphi}}}{16\pi} \frac{m_{\varphi}^4}{[(N_{\varphi}-1)!]^4} \left| \frac{\mu A_{\varphi}}{2m_{\varphi}^2} \right|^{2N_{\varphi}}. \quad (3.18)$$

The above expression is consistent with the result given in the study of the parametric resonance [8]. In addition, the decay rate for the $N_{\varphi} = 1$ mode is related to the decay rate of single particle, which is given in Eq. (3.2), as

$$\Gamma_{\text{Leading}}^{(N_{\varphi}=1)} = n_{\varphi} \gamma_{\varphi \rightarrow \chi\chi}. \quad (3.19)$$

Thus, when the amplitude of φ is small, decay of the coherent state can be treated as the decay of individual particles in the condensation, which justifies the conventional treatment of the decay processes of scalar condensations.

We also comment here that the decay rate given in Eq. (3.18) is also derived from the tree-level calculation of the $\chi\chi$ pair creation rate in the external oscillating φ field; $\Gamma_{\text{Leading}}^{(N_{\varphi})}$ is equal to the production rate of $\chi\chi$ pair per unit volume with total energy of $E_{\chi\chi} = N_{\varphi} m_{\varphi}$. In general, at the leading order of A_{φ} , the decay rate of the coherent

state is also obtained by calculating the tree-level production rate of the final-state particles treating the scalar condensation as an external field. If we consider higher order contributions, however, such a calculation breaks down; treating the scalar condensation as the external field, denominators of some propagators vanish in certain types of diagrams. Notice that, in Eq. (3.9), such a difficulty does not exist. (See also the following discussion.)

C. Small velocity limit

In the previous subsection, we have calculated leading-order contributions to the decay rates of each mode in the small-amplitude limit. Calculations of the contributions which are higher order in the amplitude are straightforward. In this subsection, we discuss when the small-amplitude expansion breaks down, taking the $N_\varphi = 1$ mode as an example.

If we calculate $O(|A_\varphi|^{2p})$ contributions to the decay rate of such a mode, which are from diagrams with $2p$ external φ insertions, one finds that the imaginary part of the T -matrix element is inversely proportional to the powers of β_1 in the $m_\varphi \rightarrow 2m_\chi$ limit. First, let us derive such a behavior with explicit calculation.

For the $N_\varphi = 1$ mode, the most important Feynman diagrams in the $m_\varphi \rightarrow 2m_\chi$ limit are those in which φ_+ and φ_- insertions are next to each other. (See Fig. 3.) As we will discuss, other types of diagrams with fixed p have less singular behavior when $\beta_1 \rightarrow 0$. We also note here that the diagram in Fig. 3 contributes only to the $N_\varphi = 1$ mode. (The imaginary part of the T -matrix vanishes when $N_\varphi = 0$.)

Taking into account the diagram shown in Fig. 3, the T -matrix element becomes

$$\begin{aligned} i\mathcal{T}_{\beta_1 \rightarrow 0}^{(N_\varphi=1)} &= L^3 T \frac{1}{2p} \left| \frac{\mu A_\varphi}{2} \right|^{2p} \\ &\times \int \frac{d^4 \tilde{k}}{(2\pi)^4} (\tilde{k}^2 - m_\chi^2 + i0^+)^{-p} \\ &\times [(\tilde{k} - Q_\varphi)^2 - m_\chi^2 + i0^+]^{-p}, \end{aligned} \quad (3.20)$$

and the imaginary part of $\mathcal{T}_{\beta_1 \rightarrow 0}^{(N_\varphi=1)}$ is given by

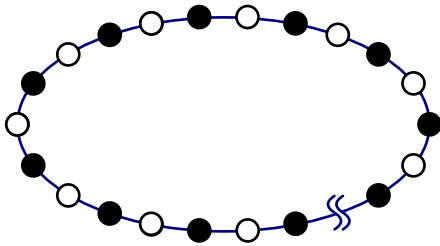


FIG. 3 (color online). Feynman diagrams which give the most singular behavior for the $N_\varphi = 1$ process in the small β_1 limit. The white dot \circ indicates φ_- insertion while the black dot \bullet is for φ_+ insertion.

$$\begin{aligned} \Im[\mathcal{T}_{\beta_1 \rightarrow 0}^{(N_\varphi=1)}] &= -\frac{L^3 T}{32\pi} m_\varphi^4 \beta_1 \left| \frac{\mu A_\varphi}{2} \right|^2 \frac{(4p-7)!!}{p!(p-1)!} \\ &\times \left| \frac{\mu A_\varphi}{2m_\varphi^2 \beta_1^2} \right|^{2(p-1)} + O(\beta_1^{7-4p}), \end{aligned} \quad (3.21)$$

where $(2p-1)!! \equiv \prod_{l=1}^p (2l-1)$ for $p \geq 1$, and $(-3)!! \equiv -1$. (For details, see Appendix B.) Hereafter, we neglect the $O(\beta_1^{7-4p})$ contribution in Eq. (3.21), and the decay rate of the coherent state becomes

$$\begin{aligned} \Gamma_{\beta_1 \rightarrow 0}^{(N_\varphi=1)} &= -\frac{1}{16\pi} m_\varphi^4 \beta_1 \left| \frac{\mu A_\varphi}{2} \right|^2 \sum_{p=1}^{\infty} \frac{(4p-7)!!}{p!(p-1)!} \\ &\times \left| \frac{\mu A_\varphi}{2m_\varphi^2 \beta_1^2} \right|^{2(p-1)}. \end{aligned} \quad (3.22)$$

As discussed in Appendix B, the inverse powers of β_1 stem from the derivative of the function $B(m_\phi^2; \xi_i, \xi_j)$ given in Eq. (A7) with respect to ξ_i or ξ_j . The order of the derivatives is equal to the number of the propagators whose denominators vanish in the on-shell limit $\xi_i \rightarrow m_\chi^2$ ($i = 1 - 2p$). The number of such propagators is maximized for the diagram given in Fig. 3. Thus, we safely neglect other types of diagrams in studying the case of $\beta_1 \rightarrow 0$.

As we mentioned, $\Gamma_{\beta_1 \rightarrow 0}^{(N_\varphi=1)}$ becomes singular when $\beta_1 \rightarrow 0$. In other words, for the $N_\varphi = 1$ mode, the small-amplitude expansion breaks down when $|A_\varphi|$ is comparable to $\mu^{-1} m_\varphi^2 \beta_1^2$. These behaviors are related to the fact that instability bands appear in the solution to the classical wave equation of the scalar field (i.e., χ in our argument) which couples to a oscillating scalar field (i.e., φ). With the interaction given in Eq. (3.1), the wave equation of χ (with the 3-momentum \mathbf{k}) in the oscillating background is given by the Mathieu equation:

$$\frac{d^2 \chi_{\mathbf{k}}}{dt^2} + (\mathbf{k}^2 + m_\chi^2 + \mu |A_\varphi| \cos m_\varphi t) \chi_{\mathbf{k}} = 0. \quad (3.23)$$

Parametrizing the momentum of χ as

$$\mathbf{k}^2 = \frac{1}{4} m_\varphi^2 (\beta_1^2 + \epsilon), \quad (3.24)$$

the lowest instability band in the small-amplitude limit is given by [8]

$$-\theta < \epsilon < \theta, \quad (3.25)$$

where

$$\theta \equiv \frac{2\mu |A_\varphi|}{m_\varphi^2}. \quad (3.26)$$

From the study of the parametric resonance, the momentum of χ produced by the decay of the scalar condensation is in the range given by Eq. (3.25). For the consistency of the calculation, the mass of the initial-state particle φ should be large enough so that \mathbf{k}^2 is positive even for

$\epsilon \sim -\theta$; otherwise, $O(\theta^2)$ contributions may be also important. This argument gives the limitation of the small-amplitude approximation; $\beta_1 \gg \theta^{1/2}$ is required, which results in $|A_\varphi| \gg \mu^{-1} m_\varphi^2 \beta_1^2$.

The above argument is supported by the fact that $\Gamma_{\beta_1 \rightarrow 0}^{(N_\varphi=1)}$ given in Eq. (3.22) is equal to the decay rate obtained by the parametric-resonance analysis. Indeed, $\Gamma_{\beta_1 \rightarrow 0}^{(N_\varphi=1)}$ is also expressed as

$$\begin{aligned} \Gamma_{\beta_1 \rightarrow 0}^{(N_\varphi=1)} &= \frac{1}{2} m_\varphi \int_{|\mathbf{k}_-| < |\mathbf{k}| < |\mathbf{k}_+|} \frac{d^3 \mathbf{k}}{(2\pi)^3} \lambda^{(N_\varphi=1)}(\mathbf{k}) \\ &= \frac{1}{128\pi^2} m_\varphi^4 \int_{-\theta}^{\theta} d\epsilon \sqrt{(\beta_1^2 + \epsilon)(\theta^2 - \epsilon^2)}, \end{aligned} \quad (3.27)$$

where, in the first equality, $|\mathbf{k}_\pm| \equiv \frac{1}{2} m_\varphi \sqrt{\beta_1^2 \pm \theta}$, and $\lambda^{(N_\varphi=1)} = \frac{1}{2} \sqrt{\theta^2 - \epsilon^2}$ is the ‘‘growth-rate factor’’ for the $N_\varphi = 1$ mode obtained in the study of the Mathieu equation [8]. The equivalence of Eqs. (3.22) and (3.27) can be seen by expanding $\sqrt{\beta_1^2 + \epsilon}$ in the integrand of Eq. (3.27) around $\epsilon = 0$ (assuming $\beta_1 > \theta^{1/2}$). Equation (3.27) is nothing but the decay rate of the scalar condensation in the small-amplitude limit derived from the parametric-resonance analysis.

Equation (3.27) [and Eq. (3.18)] also shows the fact that, at least at the small-amplitude limit, the results from the parametric-resonance analysis is obtained in our procedure where the quantum state describing the scalar condensation is postulated to be the coherent state. The equivalence of two approaches is also expected from the fact that the basic equations governing the behavior of the parametric resonance is derived in our framework. In particular, we can calculate the density matrix of the final-state particle χ in the quantum field theory. We can see that the Mathieu equation shows up in the calculation and that the resultant density matrix is the same as the one obtained in the study of parametric resonance. These subjects will be discussed in the next subsection.

D. Calculation of the density matrix

In the classical treatment of the parametric resonant system, it is well known that the Mathieu equation appears as the equation of motion for the χ field. Thus, the equation is also expected to be obtained in the quantum-field-theory treatment. In this subsection, we show the derivations of the Mathieu equation and the density matrix of χ explicitly.

Since we are interested in the case where the scalar field φ initially forms the scalar condensation oscillating around the minimum of its potential, we describe the initial state (which is taken at $t = 0$ in this subsection) as

$$|i\rangle = |\varphi\rangle \otimes |0\rangle_\chi, \quad (3.28)$$

where the first and second kets represent the states for φ

and χ , respectively. In addition, $|0\rangle_\chi$ is the vacuum of the χ field. (In the following, we omit the subscript χ .)

For our argument, it is convenient to use the density matrix of the total system in the Schroedinger picture. The density matrix at the time T is simply given by

$$\hat{\rho}_{\text{tot}}(T) = e^{-i\hat{H}T} |i\rangle\langle i| e^{i\hat{H}T}, \quad (3.29)$$

with \hat{H} being the Hamiltonian of the total system. We consider the properties of $\hat{\rho}_{\text{tot}}$ in the coordinate basis:

$$|q \otimes X\rangle \equiv |q\rangle \otimes |X\rangle, \quad (3.30)$$

where $|q\rangle$ and $|X\rangle$ are eigenstates of the field operators \hat{q} and \hat{X} , respectively:

$$\hat{q}(t, \mathbf{x})|q\rangle = q(\mathbf{x})|q\rangle, \quad \hat{X}(t, \mathbf{x})|X\rangle = X(\mathbf{x})|X\rangle. \quad (3.31)$$

Then, the density matrix in the coordinate basis, $\rho_{\text{tot}}[q, X; q', X'] \equiv \langle q \otimes X | \hat{\rho}_{\text{tot}}(T) | q' \otimes X' \rangle$, is given by

$$\begin{aligned} \rho_{\text{tot}}[q, X; q', X'] &= \int \mathcal{D}q_i \int \mathcal{D}q'_i \int \mathcal{D}X_i \int \mathcal{D}X'_i \langle q_i | \varphi \rangle \\ &\quad \times \langle X_i | 0 \rangle \langle \varphi | q'_i \rangle \langle 0 | X'_i \rangle K[q, X; q_i, X_i] \\ &\quad \times K^*[q', X'; q'_i, X'_i]. \end{aligned} \quad (3.32)$$

The kernel is represented in the path integral form as

$$K[q, X; q_i, X_i] = \int_{\chi(0, \mathbf{x})=X_i(\mathbf{x})}^{\chi(T, \mathbf{x})=X(\mathbf{x})} \mathcal{D}\chi \int_{\varphi(0, \mathbf{x})=q_i(\mathbf{x})}^{\varphi(T, \mathbf{x})=q(\mathbf{x})} \mathcal{D}\varphi e^{iS_{\text{tot}}}, \quad (3.33)$$

where S_{tot} is the total action. For the explicit form of the kernel, see [15].

In order to study the behavior of the χ field, we derive the reduced density matrix of χ by tracing out q and q' variables:

$$\rho_{\text{red}}[X; X'] \equiv \int \mathcal{D}q \rho_{\text{tot}}[q, X; q, X']. \quad (3.34)$$

For this purpose, we define

$$\tilde{q}_{\mathbf{k}} \equiv L^{-3/2} \int d^3 \mathbf{x} q(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}, \quad (3.35)$$

$$\tilde{X}_{\mathbf{k}} \equiv L^{-3/2} \int d^3 \mathbf{x} X(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}. \quad (3.36)$$

Then, with the use of the properties of the coherent state, we obtain the following relations:

$$\begin{aligned} \langle q | \varphi \rangle &= \exp \left[-\frac{1}{2} m_\varphi (\tilde{q}_0 - L^{3/2} A_\varphi)^2 \right] \\ &\quad \times \prod_{\mathbf{k} \neq 0} \exp \left[-\frac{1}{2} E_{\mathbf{k}} |\tilde{q}_{\mathbf{k}}|^2 \right], \end{aligned} \quad (3.37)$$

$$\langle X | 0 \rangle = \prod_{\mathbf{k}} \exp \left[-\frac{1}{2} \omega_{\mathbf{k}} |\tilde{X}_{\mathbf{k}}|^2 \right], \quad (3.38)$$

where, in this subsection, we consider the case where A_φ is

real. (A_φ can be taken to be real with a relevant shift of the time variable.) In addition, $\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m_\chi^2$. In Eqs. (3.37) and (3.38), we omit unimportant numerical constants. Using Eq. (3.37), the reduced density matrix becomes

$$\rho_{\text{red}}[X; X'] = \int \mathcal{D}X_i \mathcal{D}X'_i \langle X_i | 0 \rangle \langle 0 | X'_i \rangle \int_{\chi(0, \mathbf{x})=X_i}^{\chi(T, \mathbf{x})=X} \mathcal{D}\chi \times \int_{\chi'(0, \mathbf{x})=X'_i}^{\chi'(T, \mathbf{x})=X'} \mathcal{D}\chi' e^{i\bar{S}[\chi] - i\bar{S}[\chi'] + \mathcal{C}[\chi, \chi']}, \quad (3.39)$$

where

$$\bar{S}[\chi] = \int_0^T dt \int d^3\mathbf{x} \left[\frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} m_\chi^2 \chi^2 - \frac{1}{2} \mu A_\varphi \chi^2 \cos m_\varphi t \right], \quad (3.40)$$

while \mathcal{C} gives the collision terms:

$$\mathcal{C}[\chi, \chi'] = -\frac{\mu^2}{4} \int_0^T dt \int_0^t dt' \int d^3\mathbf{x} \int d^3\mathbf{x}' \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \times \cos \mathbf{k}(\mathbf{x} - \mathbf{x}') [\chi^2(x) - \chi'^2(x)] \times [e^{-iE_{\mathbf{k}}(t-t')} \chi^2(x') - e^{iE_{\mathbf{k}}(t-t')} \chi'^2(x')]. \quad (3.41)$$

It can be seen that the periodic perturbation term, $\chi^2 \cos m_\varphi t$, appears in the reduced density matrix. Collision terms, which are proportional to χ^4 , χ'^4 , and $\chi^2 \chi'^2$, also appear after the integration. Since no approximation was made to derive the density matrix, the above formula can be used at any amplitude of the φ field, A_φ . Furthermore, we can derive the kinetic equation by calculating the correlation function of the χ field on the reduced density matrix, which allows us to describe the nonlinear dynamics of the χ system, as pointed out in [16].

When the amplitude A_φ is small enough, we can treat the collision terms as perturbations. Then, the reduced density matrix is, at the leading-order calculation, written by using the wave functional of the χ field:

$$\rho_{\text{red}}[X; X'] = \Psi[T, X] \times \Psi^*[T; X'], \quad (3.42)$$

where

$$\Psi[T, X] = \int \mathcal{D}X_i \langle X_i | 0 \rangle \int_{\chi(0, \mathbf{x})=X_i}^{\chi(T, \mathbf{x})=X} \mathcal{D}\chi e^{i\bar{S}[\chi]}. \quad (3.43)$$

The wave functional is written as a product of an infinite set of wave functions of harmonic oscillators as

$$\Psi[T, X] = \prod_{\mathbf{k}} \int_{-\infty}^{\infty} d\tilde{X}_{i, \mathbf{k}} \exp \left[-\frac{1}{2} \omega_{\mathbf{k}} |\tilde{X}_{i, \mathbf{k}}|^2 \right] \times \int_{\tilde{\chi}_{\mathbf{k}}(0)=\tilde{X}_{i, \mathbf{k}}}^{\tilde{\chi}_{\mathbf{k}}(T)=\tilde{X}_{i, \mathbf{k}}} \mathcal{D}\tilde{\chi}_{\mathbf{k}} e^{i\bar{S}_{\mathbf{k}}[\tilde{\chi}_{\mathbf{k}}]}, \quad (3.44)$$

where

$$\bar{S}_{\mathbf{k}}[\tilde{\chi}_{\mathbf{k}}] = \int_0^T dt \left[\frac{1}{2} |\dot{\tilde{\chi}}_{\mathbf{k}}|^2 - \frac{1}{2} (\mathbf{k}^2 + m_\chi^2 + \mu A_\varphi \cos m_\varphi t) |\tilde{\chi}_{\mathbf{k}}|^2 \right], \quad (3.45)$$

with the ‘‘dot’’ being the derivative with respect to time. $\Psi[T, X]$ satisfies the boundary condition $\Psi[0, X] = \prod_{\mathbf{k}} e^{-\omega_{\mathbf{k}} |\tilde{X}_{\mathbf{k}}|^2/2}$ (up to normalization), and its evolution is governed by the wave equation derived from the action given in Eq. (3.45). Thus, $\Psi[T, X]$ is nothing but the wave functional obtained in [8]:

$$\Psi[T, X] = \prod_{\mathbf{k}} \frac{1}{\sqrt{u_{\mathbf{k}}(T)}} \exp \left[\frac{i}{2} \frac{\dot{u}_{\mathbf{k}}(T)}{u_{\mathbf{k}}(T)} |\tilde{X}_{\mathbf{k}}|^2 \right], \quad (3.46)$$

where $u_{\mathbf{k}}(t)$ is the solution to the Mathieu equation:

$$\frac{d^2 u_{\mathbf{k}}}{dt^2} + (\omega_{\mathbf{k}}^2 + \mu A_\varphi \cos m_\varphi t) u_{\mathbf{k}} = 0, \quad (3.47)$$

with the conditions $u_{\mathbf{k}}(0) = \sqrt{\pi/\omega_{\mathbf{k}}}$ and $\dot{u}_{\mathbf{k}}(0) = i\sqrt{\omega_{\mathbf{k}}\pi}$. This fact supports that the scalar condensation is well described by the coherent state in the quantum field theory.

Before closing this section, we emphasize that the quantity $\mathfrak{S}[\langle \varphi | \mathcal{T} | \varphi \rangle]$ is relatively easily calculated with a wide variety of interactions and final states. Thus, for some applications, our procedure is more powerful than the approach using the Mathieu equation.

IV. DECAY VIA ANOMALY

Next, let us consider the case where a complex scalar field ϕ may decay via chiral and conformal anomalies. As the fundamental theory, we expect that there exists chiral fermions which have gauge quantum numbers and that the complex scalar field couples to the chiral fermions through a Yukawa interaction. To make our discussion definite, we consider $SU(N_c)$ gauge interaction; chiral fermions Q_L and Q_R^c are in fundamental and antifundamental representations of $SU(N_c)$, respectively, while ϕ is singlet.

The complex scalar field couples to chiral fermions Q_L and Q_R^c via the Yukawa interaction

$$\mathcal{L}_{\text{Yukawa}} = -y(\phi Q_L Q_R^c + \text{H.c.}). \quad (4.1)$$

It should be noted that, in this model, there exists anomalous $U(1)$ symmetry, which we call $U(1)_A$; charges of ϕ , Q_L , and Q_R^c are 1, $-\frac{1}{2}$, and $-\frac{1}{2}$, respectively.

When the amplitude of ϕ is large, fermions Q_L and Q_R^c acquire Dirac mass. Thus, when $m_\phi \ll y|\phi|$, effective mass of the fermions is much larger than the mass of ϕ . In this case, the decay process $\phi \rightarrow Q_L Q_R^c$ is expected to be kinematically forbidden.

When $m_\phi \ll y|\phi|$, it is rather convenient to consider the low-energy effective field theory by integrating out the fermions. The relevant (light) fields in the low-energy effective field theory are ϕ and gauge fields as far as the effective mass of the fermions are much larger than m_ϕ . In the following, we concentrate on such a case; thus, we assume that the inequality $m_\phi \ll y|\phi(x)|$ always holds at any point of the trajectory of ϕ .

We first consider the effects of the operator induced by the chiral anomaly:

$$\mathcal{L}_{\text{eff}} = -i\lambda_1(\ln\phi - \ln\phi^\dagger)F^{\mu\nu}\tilde{F}_{\mu\nu}, \quad (4.2)$$

where $F_{\mu\nu}$ is the field-strength tensor and

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \quad (4.3)$$

In Eq. (4.2) and hereafter, summation over the adjoint gauge index is implicit. In addition,

$$\lambda_1 \equiv \frac{g^2}{32\pi^2}T_R, \quad (4.4)$$

with g being the gauge coupling constant of $SU(N_c)$, and $T_R = \frac{1}{2}$.

Now, let us discuss the decay of the coherent state given in Eq. (2.17). At the leading order in λ_1 , which is of $\mathcal{O}(\lambda_1^2)$ in the calculation of $\langle\phi|\hat{\mathcal{T}}|\phi\rangle$, we obtain

$$\begin{aligned} \langle\phi|i\hat{\mathcal{T}}|\phi\rangle &= -\frac{1}{2}\lambda_1^2 \int d^4x d^4x' \langle 0|T\hat{F}_{\mu\nu}(x)\hat{\tilde{F}}^{\mu\nu}(x)\hat{F}_{\mu'\nu'}(x') \\ &\quad \times \hat{\tilde{F}}^{\mu'\nu'}(x')|0\rangle[\ln\phi(x) - \ln\phi^\dagger(x)] \\ &\quad \times [\ln\phi(x') - \ln\phi^\dagger(x')]. \end{aligned} \quad (4.5)$$

In the following, we consider the case that $A_\phi \geq A_{\bar{\phi}}$. Then, we expand $\ln\phi(x)$ as

$$\ln\phi(x) = \ln(A_\phi e^{-iQ_\phi x}) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{A_{\bar{\phi}}^*}{A_\phi}\right)^n e^{2niQ_\phi x}, \quad (4.6)$$

where $Q_\phi = (m_\phi, \mathbf{0})$. At the lowest order in λ_1 , the decay rate of the coherent state can be obtained by calculating the two-point functions of several types of operators with relevant momentum injection. For the local operator $\hat{\mathcal{O}}(x)$, let us define

$$I_{\mathcal{O}}(Q) \equiv -i \int d^4x_1 d^4x_2 \langle 0|T\hat{\mathcal{O}}(x_1)\hat{\mathcal{O}}(x_2)|0\rangle e^{iQ(x_1-x_2)}. \quad (4.7)$$

Then, by using the fact that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is expressed as a total derivative,

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= \partial_\mu K^\mu \\ &= \frac{1}{2}\partial_\mu[\epsilon^{\mu\nu\rho\sigma}A_\nu(\partial_\rho A_\sigma) + (\text{gauge field})^3], \end{aligned} \quad (4.8)$$

Eq. (4.5) becomes

$$\begin{aligned} \langle\phi|\hat{\mathcal{T}}|\phi\rangle &= -2\lambda_1^2 I_{Q_\phi^\mu K_\mu}(0) - \lambda_1^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left|\frac{A_{\bar{\phi}}}{A_\phi}\right|^{2n} \\ &\quad \times I_{F_{\mu\nu}\tilde{F}^{\mu\nu}}(2nQ_\phi). \end{aligned} \quad (4.9)$$

It is notable that $I_{Q_\phi^\mu K_\mu}(0)$ has no imaginary part because there is no momentum injection into the internal gauge-boson lines from the $Q_\phi^\mu K_\mu$ -vertex. Thus, the coherent state does not decay if $A_\phi = 0$ or $A_{\bar{\phi}} = 0$. (This statement holds even after taking into account the higher order terms in λ_1 .) This fact can be understood by the conservation of the $U(1)_A$ charge. With a fixed value of the total energy of the system, $U(1)_A$ charge is maximized when $A_\phi = 0$ or $A_{\bar{\phi}} = 0$. Thus, if $U(1)_A$ charge is conserved, the decay of ϕ in the condensation into the gauge bosons is forbidden. Of course, the interaction given in Eq. (4.2) breaks $U(1)_A$ symmetry because \mathcal{L}_{eff} is not invariant under the $U(1)_A$ transformation. This is due to the fact that \mathcal{L}_{eff} is induced by the chiral anomaly. However, we can add new fermions, which we call Q'_L and Q'_R , to have conserved $U(1)$ symmetry. Indeed, with Q'_L and Q'_R , which are in fundamental and antifundamental representation of $SU(N_c)$, respectively, we can define nonanomalous $U(1)_A$ symmetry by assigning charge $+\frac{1}{2}$ to both of Q'_L and Q'_R . (Notice that Q'_L and Q'_R do not have to couple to ϕ .) In this case, conservation of the $U(1)_A$ charge is obvious and the decay of ϕ into the gauge bosons is completely forbidden. Thus, the $U(1)_A$ charge stored in the scalar condensation cannot be released by the interaction given in Eq. (4.2). This fact may have some relevance in the study of the decay of scalar condensations in various cosmological scenarios, in particular, in the Affleck-Dine scenario [5]. In the absence of Q'_L and Q'_R , instanton effects may generate new interactions which explicitly breaks $U(1)_A$ symmetry. In such a case, decay of the coherent state occurs via such new interactions.

Using the relation

$$\Im[I_{F_{\mu\nu}\tilde{F}^{\mu\nu}}(Q)] = -\frac{N_c^2 - 1}{4\pi}(Q^2)^2 L^3 T, \quad (4.10)$$

the decay rate is given by

$$\begin{aligned} \Gamma_{F\bar{F}} &= \frac{\text{Prob}(|\phi\rangle \rightarrow \text{all})}{L^3 T} \\ &= \frac{8}{\pi}(N_c^2 - 1)\lambda_1^2 m_\phi^4 \sum_{n=1}^{\infty} n^2 \left|\frac{A_{\bar{\phi}}}{A_\phi}\right|^{2n}. \end{aligned} \quad (4.11)$$

Since the decay rate vanishes if $A_\phi = 0$ or $A_{\bar{\phi}} = 0$, the decay of the coherent state in this case should be understood as an annihilation between ϕ and $\bar{\phi}$ in the condensation; the same number of ϕ and $\bar{\phi}$ annihilates into the gauge-boson pair.

In the study of the decay of coherent state, the energy-loss rate is also important. Using the fact that the imaginary part of $I_{F_{\mu\nu}\tilde{F}^{\mu\nu}}(2nQ_\phi)$ is from the decay process into two

gauge bosons with the total energy of $2nm_\phi$, the energy-loss rate can be calculated. So far, we have considered the case where $A_\phi \geq A_{\bar{\phi}}$. However, the decay rate for the case of $A_\phi \leq A_{\bar{\phi}}$ is derived by interchanging $A_\phi \leftrightarrow A_{\bar{\phi}}$ in the result. Thus, we obtain

$$\left[\frac{d\rho_\phi}{dt} \right]_{F\bar{F}} = -\frac{16}{\pi} (N_c^2 - 1) \lambda_1^2 m_\phi^5 \sum_{n=1}^{\infty} n^3 \left[\frac{\min(n_\phi, n_{\bar{\phi}})}{\max(n_\phi, n_{\bar{\phi}})} \right]^n, \quad (4.12)$$

where we have used the fact that the number densities of ϕ and $\bar{\phi}$ are proportional to $|A_\phi|^2$ and $|A_{\bar{\phi}}|^2$, respectively. [See Eqs. (2.21) and (2.22).] One may simplify the above energy-loss rate by using

$$\sum_{n=1}^{\infty} n^3 r^n = \frac{r(1+4r+r^2)}{(1-r)^4}. \quad (4.13)$$

Here, we emphasize that the results given in Eqs. (4.11) and (4.12) can be used for any value of the amplitude (as far as the effective mass of Q_L and Q_R^c are much larger than m_ϕ). When $n_\phi \gg n_{\bar{\phi}}$ or $n_\phi \ll n_{\bar{\phi}}$, which corresponds to the case where the classical motion of the scalar condensation is almost circular, the energy-loss rate is well approximated by the leading term in Eq. (4.12). On the contrary, in the limit of $n_{\bar{\phi}} \rightarrow n_\phi$, higher order terms become important and the energy-loss rate is enhanced. In this case, however, one should note that, at some point of the classical trajectory, $|\phi|$ approaches to the origin. Then, the effective mass of the fermions Q_L and Q_R^c may become so small that the effective field theory, which is obtained by integrating out these fermions, may break down. We also note that, with the ratio $n_\phi/n_{\bar{\phi}}$ being fixed, the decay and energy-loss rates are independent of the amplitude of the scalar condensation.

Before closing this section, we also present the result for the case where the scalar field ϕ couples to the gauge field as

$$\mathcal{L}_{\text{eff}} = \lambda_R (\ln \phi + \ln \phi^\dagger) F^{\mu\nu} F_{\mu\nu}. \quad (4.14)$$

This type of interaction is also generated by integrating out particles which acquire masses from the condensation of ϕ (like Q_L and Q_R^c). At the leading order in λ_R , the energy-loss rate is given by

$$\left[\frac{d\rho_\phi}{dt} \right]_{FF} = -\frac{16}{\pi} (N_c^2 - 1) \lambda_R^2 m_\phi^5 \sum_{n=1}^{\infty} n^3 \left[\frac{\min(n_\phi, n_{\bar{\phi}})}{\max(n_\phi, n_{\bar{\phi}})} \right]^n. \quad (4.15)$$

V. SUMMARY

In this paper, we have discussed the decay processes of the scalar condensation. We postulated that the quantum state corresponding to the scalar oscillation is the so-called

coherent state in the quantum field theory. Then, by using the S -matrix unitarity, we have developed the method to calculate the decay rate of the coherent state. We believe that our procedure can be applied to a large class of models which may contain various types of interactions.

Then, in order to demonstrate how the decay rate is calculated, we considered two examples. First, we studied the case where the scalar field φ couples to another scalar field χ via three-point interaction. Using the small-amplitude approximation, we have calculated the decay rate for the process where N_φ ($N_\varphi = 1, 2, 3, \dots$) of φ in the condensation simultaneously annihilate into a pair of χ . For the case of $N_\varphi = 1$, we have seen that the result is the same as that in the conventional approach where the decay rate of the scalar condensation is estimated by the product of the decay rate of single φ in the vacuum and the number density of φ . We have also pointed out that the small-amplitude approximation breaks down when the amplitude becomes close to $\mu^{-1} m_\varphi^2 \beta_1^2$, where μ is the coupling constant and β_1 is the velocity of χ in the $N_\varphi = 1$ mode. Such a behavior is also expected from the discussion based on the parametric resonance. Indeed, our procedure reproduced the decay rate of the scalar condensation calculated from the parametric-resonance analysis.

The second example was the case where the complex scalar-field decays into gauge bosons via the interaction induced by the chiral anomaly. We have considered the case where the scalar potential has $U(1)_A$ symmetry to rotate $\phi \rightarrow e^{i\alpha} \phi$ at the classical level, which is broken by the effect of the chiral anomaly. In this case, we could calculate the decay rate without using the small-amplitude approximation. We have seen that the decay process is forbidden unless both the particle ϕ and its antiparticle $\bar{\phi}$ exist in the condensation, and that the ‘‘decay’’ of the coherent state is due to the annihilation between them. Thus, $U(1)_A$ charge stored in the condensation cannot be released by the effective interaction induced by the chiral anomaly.

Finally, we comment on the effects of the cosmic expansion, which were neglected in our analysis. With the cosmic expansion, scalar field should be quantized in non-flat background in which the wave function of the scalar field is modified. In such a case, creation and annihilation operators mix each other due to the change of the vacuum state [17]. In addition, the Hubble friction term also induces the damp of the amplitude of the scalar field φ . In general, in the expanding universe, these effects should be taken into account and the decay rates obtained in our analysis are expected to be modified. However, if the expansion rate of the universe H is small enough compared to the mass of the scalar field m_φ , wave function of φ is well approximated by that obtained in the flat background (i.e., $\sim e^{\mp ikx}$) for the time scale $\sim m_\varphi^{-1}$. Thus, we expect that our procedure provides a good approximation of the decay rate when $H \ll m_\varphi$; effects of the Hubble friction

term can be taken into account as an adiabatic decrease of the amplitude.

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APPENDIX A: CALCULATION OF THE IMAGINARY PART

Although the technique which will be explained here is well known, in this Appendix, we show how Eq. (3.9) is derived for the sake of some of the readers. For this purpose, we calculate the imaginary part of the following quantity:

$$I_{\mathcal{F}}(Q_{\varphi}) \equiv -i \int \frac{d^4 \tilde{k}}{(2\pi)^4} \prod_{I=1}^{2p} (k_I^2 - m_{\chi}^2 + i0^+)^2. \quad (\text{A1})$$

$$\begin{aligned} I_{\mathcal{F}} &= -\pi \lim_{\xi \rightarrow m_{\chi}^2} \sum_{i=1}^{2p} \int \frac{d^3 \tilde{\mathbf{k}}}{(2\pi)^4} \left[\frac{1}{2\tilde{k}_0} \prod_{I \neq i} (k_I^2 - \xi_I + i0^+)^{-1} \right]_{\tilde{k}_0 = -m_{\varphi} \sum_{J=1}^I \varepsilon_J + \sqrt{\mathbf{k}_I^2 + \xi_I}} \\ &= -\pi \lim_{\xi \rightarrow m_{\chi}^2} \sum_{i=1}^{2p} \int \frac{d^4 \tilde{k}}{(2\pi)^4} \delta(k_i^2 - \xi_i) \prod_{I \neq i} (k_I^2 - \xi_I + i0^+)^{-1}, \end{aligned} \quad (\text{A4})$$

where, in the second equality, the \tilde{k}_0 -integration is performed in the region where $k_{i0} \geq 0$. Using the relation $(x + i0^+)^{-1} = P(x^{-1}) - i\pi\delta(x)$ (where ‘‘P’’ is for the principal value), we obtain

$$\begin{aligned} \Im[I_{\mathcal{F}}] &= 2\pi^2 \lim_{\xi \rightarrow m_{\chi}^2} \sum_{i=1}^{2p-1} \sum_{j=i+1}^{2p} \int \frac{d^4 \tilde{k}}{(2\pi)^4} \delta(k_i^2 - \xi_i) \\ &\quad \times \delta(k_j^2 - \xi_j) \prod_{I \neq i,j} (k_I^2 - \xi_I)^{-1}. \end{aligned} \quad (\text{A5})$$

Substituting the above expression into Eq. (3.5), we obtain Eq. (3.9).

With the quantity $N_{\varphi} = |\sum_{I=i+1}^j \varepsilon_I|$, k_i and k_j are related as $k_i = k_j + N_{\varphi} Q_{\varphi}$ (or $k_i = k_j - N_{\varphi} Q_{\varphi}$). Thus, if $N_{\varphi} = 0$, $k_i = k_j$ and the imaginary part vanishes. Notice also that the constraints from the δ -functions can be solved; by shifting the integration variable \tilde{k} , k_i and k_j can be taken to be \tilde{k} and $\tilde{k} - N_{\varphi} Q_{\varphi}$, respectively. Then, the constraints from the δ -functions become $\tilde{k} Q_{\varphi} = \frac{1}{2} N_{\varphi} m_{\varphi}^2$. Consequently, the product $\prod_{I \neq i,j} (k_I^2 - \xi_I)^{-1}$ becomes \tilde{k} -independent. The remaining part is proportional to the two-body phase space for the process where the parent particle with mass $N_{\varphi} m_{\varphi}$ decays into two daughter particles with masses $\xi_i^{1/2}$ and $\xi_j^{1/2}$:

Here,

$$k_I \equiv \tilde{k} + \sum_{J=1}^I \varepsilon_J Q_{\varphi}, \quad (\text{A2})$$

where $\varepsilon_I = \pm 1$ (with $\varepsilon_1 + \dots + \varepsilon_{2p} = 0$), and $Q_{\varphi} = (m_{\varphi}, \mathbf{0})$.

In order to calculate the imaginary part of $I_{\mathcal{F}}$, it is convenient to rewrite $I_{\mathcal{F}}$ as

$$I_{\mathcal{F}} = -i \lim_{\xi \rightarrow m_{\chi}^2} \int \frac{d^4 \tilde{k}}{(2\pi)^4} \prod_{I=1}^{2p} (k_I^2 - \xi_I + i0^+)^{-1}, \quad (\text{A3})$$

where the limit $\xi \rightarrow m_{\chi}^2$ indicates that $\xi_I \rightarrow m_{\chi}^2$ ($I = 1 - 2p$); before taking the limit, ξ_I are all set to be different.

The integrand of Eq. (A3) has poles at $\tilde{k}_0 = -m_{\varphi} \sum_{J=1}^I \varepsilon_J \pm \sqrt{\mathbf{k}_I^2 + \xi_I}$ ($I = 1 - 2p$) and, after \tilde{k}_0 -integration, $I_{\mathcal{F}}$ becomes

$$\int \frac{d^4 \tilde{k}}{(2\pi)^4} \delta(k_i^2 - \xi_i) \delta(k_j^2 - \xi_j) = \frac{1}{32\pi^3} B(N_{\varphi}^2 m_{\varphi}^2; \xi_i, \xi_j), \quad (\text{A6})$$

where, for $\sqrt{Q^2} > \xi_i^{1/2} + \xi_j^{1/2}$,

$$B(Q^2; \xi_i, \xi_j) \equiv \frac{1}{Q^2} \sqrt{(Q^2)^2 - 2(\xi_i + \xi_j)Q^2 + (\xi_i - \xi_j)^2}, \quad (\text{A7})$$

while $B(Q^2; \xi_i, \xi_j) = 0$ for $\sqrt{Q^2} \leq \xi_i^{1/2} + \xi_j^{1/2}$. Notice that the function B is related to $\beta_{N_{\varphi}}$ given in Eq. (3.16) as

$$\beta_{N_{\varphi}} = B(N_{\varphi}^2 m_{\varphi}^2; m_{\chi}^2, m_{\chi}^2). \quad (\text{A8})$$

APPENDIX B: DERIVATION OF EQ. (3.21)

In this Appendix, we calculate the imaginary part of the following integral:

$$\begin{aligned} I_{\text{Fig. 3}} &= -i \int \frac{d^4 \tilde{k}}{(2\pi)^4} (\tilde{k}^2 - m_{\chi}^2 + i0^+)^{-p} \\ &\quad \times [(\tilde{k} - Q_{\varphi})^2 - m_{\chi}^2 + i0^+]^{-p}, \end{aligned} \quad (\text{B1})$$

to derive Eq. (3.21) from Eq. (3.20). Using the procedure given in Appendix A, we express $I_{\text{Fig. 3}}$ as

$$I_{\text{Fig. 3}} = -i \lim_{\xi^{(l)} \rightarrow m_x^2} \int \frac{d^4 \tilde{k}}{(2\pi)^4} \prod_{l=1}^p (\tilde{k}^2 - \xi_l + i0^+)^{-1} \\ \times \prod_{j=1}^p [(\tilde{k} - Q_\varphi)^2 - \xi_j' + i0^+]^{-1}. \quad (\text{B2})$$

The imaginary part of this quantity is obtained with

$$\prod_{l=1}^p (\tilde{k}^2 - \xi_l + i0^+)^{-1} \rightarrow -i\pi \sum_{i=1}^p \prod_{l \neq i} (\xi_i - \xi_l)^{-1} \delta(\tilde{k}^2 - \xi_i), \quad (\text{B3})$$

and with the similar replacement of the second product. Then, the imaginary part of $I_{\text{Fig. 3}}$ becomes

$$\Im[I_{\text{Fig. 3}}] = \frac{1}{16\pi} \lim_{\xi^{(l)} \rightarrow m_x^2} \sum_{i=1}^p \sum_{j=1}^p \prod_{l \neq i} (\xi_i - \xi_l)^{-1} \\ \times \prod_{j \neq i} (\xi_j' - \xi_j')^{-1} B(m_\varphi^2; \xi_i, \xi_j'). \quad (\text{B4})$$

We can use the relation

$$\lim_{x_1 \rightarrow x} \cdots \lim_{x_p \rightarrow x} \sum_{i=1}^p f(x_i) \prod_{j \neq i} (x_i - x_j)^{-1} \\ = \frac{1}{(p-1)!} \frac{d^{p-1}}{dx^{p-1}} f(x), \quad (\text{B5})$$

to obtain

$$\Im[I_{\text{Fig. 3}}] = \frac{1}{16\pi} \frac{1}{[(p-1)!]^2} \left[\frac{\partial^{(p-1)}}{\partial \xi^{(p-1)}} \right. \\ \left. \times \frac{\partial^{(p-1)}}{\partial \xi'^{(p-1)}} B(m_\varphi^2; \xi, \xi') \right]_{\xi = \xi' = m_x^2}. \quad (\text{B6})$$

Taking the $O(\beta_1^{1-4(p-1)})$ term from the above expression, which is the most singular one when $\beta_1 \rightarrow 0$, Eq. (3.21) is derived.

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