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Topological discrete algebra, ground-state degeneracy, and quark confinement in QCD

Masatoshi Sato

The Institute for Solid State Physics, The University of Tokyo, Kashiwanoha 5-1-5, Kashiwa-shi, Chiba 277-8581, Japan

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Based on the permutation group formalism, we present a discrete symmetry algebra in QCD. The discrete algebra is hidden symmetry in QCD, which is manifest only on a space-manifold with nontrivial topology. Quark confinement in the presence of dynamical quarks is discussed in terms of the discrete symmetry algebra. It is shown that the quark deconfinement phase has ground-state degeneracy depending on the topology of the space, which gives a gauge-invariant distinction between the confinement and deconfinement phases. We also point out that new quantum numbers relating to the fractional quantum Hall effect exist in the deconfinement phase.

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The purpose of this paper is to present an argument for classification of quantum phases in QCD. Classification of phases in QCD is an old but unsolved problem in quantum field theory. (For recent reviews, see [1,2].) As is well known, behaviors of the Wilson loop [3] (or Polyakov line [4]) and the ’t Hooft loop [5,6] are useful to classify the quark confinement and deconfinement phases in pure Yang Mills theories, but once dynamical quarks are included, they are no longer sufficient to distinguish them. Therefore, the quark confinement has been investigated indirectly by the chiral condensation (for a review, see [7]) or the percolation theory [8]. Here we present a direct argument for the quark confinement in the presence of dynamical quarks. It will be shown below that there exist quantum numbers which distinguish these two phases at zero temperature.

This work is motivated by recent developments of understanding of quantum phases. While many quantum phases and phase transitions can be described by spontaneous symmetry breaking and local order parameters, in recent years it has become increasingly clear that in a wide class of strongly correlated many-body systems, a phase transition driven by a nonthermal parameter may occur at zero temperature which can not be understood by any local order parameter. The characteristic signature of the novel phase is finite ground-state degeneracy depending on the topology of the space, and the underlying order of the novel phase is dubbed as topological order [9,10]. The Laughlin state for the fractional Hall effect is known to have a topological order [11]. At present, many systems including bosonic ones and those at zero magnetic field are identified as possessing topological orders [12–20].

Recently, we have argued that the topological degeneracy in a class of topological orders is due to the emergence of a discrete symmetry [21], which contains three fractional parameters: quasiparticle charge, anyon statistics, and the fractional quantum Hall conductivity. In particular, it is notable that the emergence of collective excitations having fractional quantum numbers with respect to constituent particles in the Hamiltonian is closely related to the existence of the topological order [21–23]. Such charge fractionalization has an interesting similarity to quark deconfinement, which also gives fractional charged excitations. In spite of the essential difference that quarks are not collective excitations but elementary particles, this motivates us to study the quark confinement in terms of topological order.

In the following, we will show that the quark deconfinement phase in QCD indeed has a topological order. Generalizing the argument in [21], we will derive a discrete symmetry algebra in QCD, which we dub topological discrete algebra. The existence of the center of the gauge group is crucial for the derivation. By using the topological discrete algebra, it will be shown that the quark deconfinement phase in QCD has ground-state degeneracy depending on the topology of the system. The topological degeneracy in the thermodynamic limit is a gauge-invariant quantum number that distinguishes the deconfinement phase from the confinement one.

For definiteness we will consider the lattice QCD in the following. The generalization to the continuum one is straightforward. The action of the link variable $U_{n,\mu} \in SU(3)$ is given by

$$S_{\rm G} = \sum_{\rm P} \frac{1}{g^2} \,{\rm tr}(1 - U_{\rm P}), \tag{1}$$

with the plaquette variable $U_{\rm P} = U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}$, and that of the quark ψ_n^f is

$$S_{\rm F} = -\frac{1}{2}\sum_{f}\sum_{n,\mu}(\bar{\psi}_n^f \gamma_\mu U_{n,\mu}\psi_{n+\hat{\mu}}^f - \bar{\psi}_{n+\hat{\mu}}^f \gamma_\mu U_{n,\mu}^{\dagger}\psi_n^f) - \sum_{f,n} m_f \bar{\psi}_n^f \psi_n^f, \tag{2}$$

where $n = (n_1, n_2, n_3, n_4)$ denotes the site on the lattice, $\hat{\mu}$ the unit vector in the n_μ direction ($\mu = 1, 2, 3, 4$), and f the index of the flavor of the quark. To remove the doublers, we also add the Wilson term $S_{\rm W}$. In addition, any term with a nonthermal parameter that may control quantum phases in QCD can be added if the electric charge is preserved. For example, chemical potential terms for quarks can be included. The partition function $\mathcal{Z}$ is given by

$$\mathcal{Z} = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-(S_{\rm G}+S_{\rm F}+S_{\rm W}+\cdots)}. \tag{3}$$

The topological discrete algebra we will consider is defined only if the topology of the space-manifold is nontrivial, so let us consider the system on a three dimensional torus T^3. (The space-time is four dimensional.) On the torus T^3, the site at $\boldsymbol{n} = (n_1, n_2, n_3)$ is identified with one at $\boldsymbol{n} + \hat{a}N_a\,(a = 1, 2, 3)$, where $\hat{a}$ is the unit vector in the n_a direction, and N_a is an integer [24]. The volume of the torus is $N_1N_2N_3$, and we assume that it is large enough. In the thermodynamic limit, we take $N_1N_2N_3 \to \infty$. In practice, the torus is realized by imposing the periodic boundary conditions in all the spatial directions. The torus T^3 is topologically equivalent to a direct product of three one-dimensional spheres, $T^3 = S^1 \times S^1 \times S^1$, so it has three independent spatial noncontractable loops $C_a(a = 1, 2, 3)$ along the n_a direction. In addition, there are three holes $h_a(a = 1, 2, 3)$ encircled by the noncontractable loops C_a. See Fig. 1.

To define the topological discrete algebra, we introduce an external $U(1)$ electromagnetic gauge field $e^{i\theta_{n,\mu}}$. Consider an adiabatic insertion of the magnetic flux Φ_a through the hole h_a encircled by the noncontractable loop C_a. This process induces the external $U(1)$ electromagnetic gauge field $e^{i\theta_{n,\mu}}$ with $\theta_{n,\mu} = \delta_{\mu,a}\Phi_a/N_a$. Note that the $U(1)$ field strength on T^3 remains to be zero after the flux insertion.

Here one can show that the spectrum of the system is periodic in Φ_a with the period 2π. To see this, let us consider the partition function $\mathcal{Z}(\Phi_a)$ with the inserted flux Φ_a. Since upper quarks (u, c, and t) and lower quarks (d, s, and b) have $2/3$ and $-1/3$ electric charges, respectively, then by the following unitary transformation,

$$\begin{aligned} &\psi_n \to e^{-i2n_a\Phi_a/3N_a}\psi_n, \qquad \bar{\psi}_n \to e^{i2n_a\Phi_a/3N_a}\bar{\psi}_n, \qquad \text{for upper quarks,} \\ &\psi_n \to e^{in_a\Phi_a/3N_a}\psi_n, \qquad \bar{\psi}_n \to e^{-in_a\Phi_a/3N_a}\bar{\psi}_n, \qquad \text{for lower quarks,} \end{aligned} \tag{4}$$

the induced $U(1)$ electromagnetic gauge field is eliminated in the action except on the links between the sites $n = (\boldsymbol{n}, n_4)$ with $n_a = N_a$ and those with $n_a = 1$. The kinetic terms of the quarks on these links acquire the following $U(1)$ phase after the transformation (4),

$$\begin{aligned} &e^{i2\Phi_a/3}\bar{\psi}_n\gamma_a U_{n,a}\psi_{n+\hat{a}}, \qquad \text{for upper quarks,} \\ &e^{-i\Phi_a/3}\bar{\psi}_n\gamma_a U_{n,a}\psi_{n+\hat{a}}, \qquad \text{for lower quarks,} \end{aligned} \tag{5}$$

where n satisfies $n_a = N_a$. (Note that the site $n + \hat{a}$ with $n_a = N_a$ is identified with the site n with $n_a = 1$.) If Φ_a is 2π, these $U(1)$ phases coincide with an element of the center of $SU(3)$, $e^{-2\pi i/3}$. So by changing the integral variable in (5) as $U_{n,a} \to e^{2\pi i/3}U_{n,a}$, one finds

$$\mathcal{Z}(2\pi) = \mathcal{Z}(0). \tag{6}$$

Therefore, the spectrum of the system is periodic in Φ_a with the period 2π. The adiabatic insertion of the unit flux, $\Phi_a = 2\pi$, defines a kind of symmetry of the system, and we denote it by U_a.

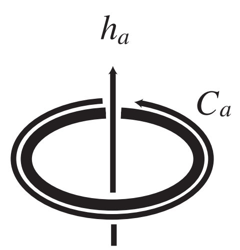

FIG. 1. The noncontractable loop C_a and the hole h_a on S^1. Each S^1 in $T^3 = S^1 \times S^1 \times S^1$ has the noncontractable loop and the hole.

Let us first consider the quark deconfinement phase. In the quark deconfinement phase, the physical states are classified by the representation of the permutation group for quarks. For N identical quarks on T^3, the permutation group consists of $\sigma_i\,(i = 1, \cdots, N-1)$, which exchanges the ith and $(i+1)$th quarks, and $\tau_i^a\,(a = 1, 2, 3, i = 1, \cdots, N)$, which represents moving the ith quark along the noncontractable loop C_a in the n_a direction. The permutation group on T^3 is given by

$$\begin{aligned} \sigma_k^2 &= 1, \qquad 1 \le k \le N-1, \\ (\sigma_k\sigma_{k+1})^3 &= 1, \qquad 1 \le k \le N-2, \\ \sigma_k\sigma_l &= \sigma_l\sigma_k, \qquad 1 \le k \le N-3, \qquad |l-k| \ge 2, \\ \tau_{i+1}^a &= \sigma_i\tau_i^a\sigma_i, \qquad 1 \le i \le N-1, \qquad a = 1, 2, 3, \\ \tau_1^a\sigma_j &= \sigma_j\tau_1^a, \qquad 2 \le j \le N, \qquad a = 1, 2, 3, \\ \tau_i^a\tau_j^b &= \tau_j^b\tau_i^a, \qquad i, j = 1, \cdots, N, \qquad a, b = 1, 2, 3. \end{aligned} \tag{7}$$

These generators have nontrivial commutation relations with $U_a\,(a = 1, 2, 3)$ since quarks have fractional charges. If we apply τ_i^a after the flux insertion U_a, the induced gauge field $e^{i\theta_{n,\mu}}$ will give rise to an Aharanov-Bohm phase $e^{-i2\pi/3}$. (Both upper and lower quarks acquire the same $U(1)$ phase.) Therefore, we obtain

$$\tau_i^a U_a = e^{-2\pi i/3}U_a\tau_i^a, \tag{8}$$

where $a = 1$, 2, 3 and $i = 1, \cdots, N$. On the other hand, because τ_i^b ($b \neq a$) and σ_i do not encircle the inserted flux by U_a, we have

$$\tau_i^b U_a = U_a \tau_i^b, \qquad \sigma_i U_a = U_a \sigma_i \tag{9}$$

with $a \neq b$.

Now from the commutation relations (8) and (9), it is easy to verify that $U_a U_b U_a^{-1} U_b^{-1}$ ($a \neq b$) commutes with all the permutation group generators. So by Schur's lemma, for any irreducible representation of the permutation group, $U_a U_b U_a^{-1} U_b^{-1}$ is a (unimodular) c-number. Therefore, we have the following commutation relation,

$$U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a, \qquad \lambda_{a,b} = -\lambda_{b,a}. \tag{10}$$

Here it can be shown that $\lambda_{a,b}$ is given by

$$\lambda_{a,b} = \frac{k_{a,b}}{3}, \tag{11}$$

with an integer $k_{a,b}$. This is because that U_a^3 also commutes with all the permutation group generators, which implies $U_a^3 U_b = U_b U_a^3$ for any irreducible representation of the permutation group. Comparing this to (10), we obtain (11).

We also notice that $\lambda_{a,b}$ is zero if the time-reversal symmetry is preserved. For time-reversal invariant systems, we have the time-reversal transformation T with $T U_a T^{-1} = c_a U_a^\dagger$ (c_a is a constant.) Applying T to (10) and using the anti-Hermiticity of T, we find

$$U_a^\dagger U_b^\dagger = e^{-2\pi i \lambda_{a,b}} U_b^\dagger U_a^\dagger. \tag{12}$$

From (10) and (12), it is found that $\lambda_{a,b} = 0$ in time-reversal invariant systems.

The constants $\lambda_{a,b}$ are new quantum numbers in the deconfinement phase. Later, we will show that they are closely related to the fractional quantum Hall effects.

Since quarks are fermions, then the exchange operator σ_i in (7) satisfies $\sigma_i = -1$ [25]. In this case, the permutation group representation is uniquely determined as

$$\tau_i^a = T_a \tag{13}$$

with matrices T_a satisfying

$$T_a T_b = T_b T_a. \tag{14}$$

Therefore, the commutation relations (8)–(10) now reduce to

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a, \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a, \tag{15}$$

with $a, b = 1$, 2,3. The topological discrete algebra in the quark deconfinement phase consists of the flux insertion operator U_a and the quark winding operator T_a with the commutation relations (14) and (15).

Now we show our main claim in this paper: *If there is a mass gap to excitations above the ground state, then the quark deconfinement phase has at least* 3^3 *-fold ground-state degeneracy on* T^3.

To show this, consider the following process. First, create a pair of quark and antiquark out of a vacuum, then move the quark by $T_a (\equiv \tau_i^a)$. After the quark returns to the original position, we pair annihilate the quark and the antiquark. Suppose that there is a mass gap to excitations above the vacuum space and these operations do not close the mass gap, then these processes define the operation of T_a from a vacuum to a vacuum. Since T_a ($a = 1, 2, 3$) commutes with each other, we take the basis of the vacuum space which diagonalizes T_1, T_2, and T_3 simultaneously,

$$T_a |\boldsymbol{\eta}\rangle = e^{i\eta_a} |\boldsymbol{\eta}\rangle, \qquad \boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3). \tag{16}$$

Then by applying U_a ($a = 1, 2, 3$) to this and using (15), we have

$$\begin{aligned} T_1(U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle) &= e^{i(\eta_1 - 2\pi r/3)} U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle, \\ T_2(U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle) &= e^{i(\eta_2 - 2\pi s/3)} U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle, \\ T_3(U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle) &= e^{i(\eta_3 - 2\pi t/3)} U_1^r U_2^s U_3^t |\boldsymbol{\eta}\rangle, \end{aligned} \tag{17}$$

where r, s, and t are integers. Because theses new states have 3^3 distinct sets of eigenvalues of T_a's, we find that the ground state (vacuum) in the quark deconfinement phase has at least 3^3-fold degeneracy on T^3.

Note that the ground-state degeneracy obtained in the quark deconfinement phase depends on the topology of the space-manifold. This is easily seen by considering the system in a 3-dimensional box with the free boundary conditions, which is homotopic to a 3-dimensional ball B^3. Because no noncontractable loop exists on this space-manifold, the operator τ_i^a does not exist and the permutation group consists of only the exchange operator σ_i. Moreover, the operator U_a ($a = 1, 2, 3$) can not be defined since there is no hole in B^3. So on B^3 no topological discrete algebra is derived and no ground-state degeneracy is obtained from this algebra. In general, if the space-manifold on which the system is defined has l independent spatial noncontractable loops, then the minimal ground-state degeneracy in the deconfinement phase is 3^l.

Let us now consider the quark confinement phase. In contrast to the quark deconfinement phase, the topological discrete algebra on T^3 becomes trivial and no topological degeneracy is required in the quark confinement phase as follows. In the quark confinement phase, the permutation group for hadrons, not for quarks, classifies the physical states. The permutation group for hadrons on T^3 is also defined by (7) if σ_i and τ_i^a are interpreted as those for hadrons, however, all the generators of the permutation group for hadrons commute with the flux insertion operators U_a ($a = 1, 2.3$),

$$\tau_i^a U_b = U_b \tau_i^a, \qquad \sigma_i U_a = U_a \sigma_i, \qquad (a, b = 1, 2, 3). \tag{18}$$

This is because any hadron has an integer electric charge, so the movement τ_i^a of a hadron around the inserted flux

$\Phi_a = 2\pi$ gives only the trivial Aharanov-Bohm phase. Then from (18) and the Schur's lemma, it is found that the flux insertion operator U_a reduces to a unimodular constant for any irreducible representation of the permutation group for hadrons. In addition, since the representation of the permutation group for a hadron is fermion or boson, τ_i^a for a hadron is again uniquely determined as

$$\tau_i^a = \tilde{T}_a \tag{19}$$

with mutually commuting matrices $\tilde{T}_a$ $(a = 1, 2, 3)$,

$$\tilde{T}_a \tilde{T}_b = \tilde{T}_b \tilde{T}_a. \tag{20}$$

Because all the elements of the topological discrete algebra, $\tilde{T}_a$ and U_a, commute with each other, no additional ground-state degeneracy is required on T^3 in this phase.

Our results here indicate that the deconfinement phase in QCD is topologically ordered, and it is distinguished clearly from the quark confinement phase in the concept of topological order.

In the static limit of QCD, the minimal topological degeneracy obtained above is reproduced by the following conventional argument using the Wilson loop. In this limit, all quarks are infinitely heavy and decoupled from the dynamics. So the system is effectively described by the pure $SU(3)$ gauge theory. The pure $SU(3)$ gauge theory is invariant under the transformations,

$$U_{n,a} \to e^{2\pi i m/3} U_{n,a}, \qquad (m = 1, 2, 3), \qquad n_a \text{ fixed}, \tag{21}$$

which rotate all spacelike links in the n_a direction at a fixed n_a by an element of the center of $SU(3)$. On T^3, we can introduce the Wilson loop $W(C_a)$ along the noncontractable loop C_a,

$$W(C_a) = \mathrm{tr} \prod_{n \in C_a} U_{n,a}, \tag{22}$$

and it is transformed by this center symmetry (21) as

$$W(C_a) \to e^{2\pi i m/3} W(C_a). \tag{23}$$

So the expectation value $\langle W(C_a)\rangle$ is a gauge-invariant order parameter for the center symmetry. In the quark confinement phase, from the area law, it follows that in the temporal gauge

$$\langle W(C_a, \tau) W^\dagger(C_a, \tau')\rangle \sim e^{-\sigma N_a |\tau - \tau'|}, \tag{24}$$

with a positive constant σ, and the imaginary times $\tau = it$ and $\tau' = it'$. Thus using the cluster property

$$\langle W(C_a, \tau) W^\dagger(C_a, \tau')\rangle \overset{|\tau-\tau'|\to\infty}{\to} |\langle W(C_a)\rangle|^2, \tag{25}$$

we have

$$\langle W(C_a)\rangle = 0. \tag{26}$$

So the center symmetry is not broken, and no additional ground-state degeneracy is required on T^3 in the quark confinement phase. On the other hand, in the quark deconfinement phase, it is possible that $\langle W(C_a)\rangle \neq 0$ since it obeys the perimeter law, and the center symmetry can be spontaneously broken. If $\langle W(C_a)\rangle \neq 0$ for all C_a's $(a = 1, 2, 3)$, we have 3^3 different set of $\langle W(C_a)\rangle$'s, which are related to each other by the center symmetry. Thus the ground-state degeneracy is 3^3-fold, and it coincides with the minimal ground-state degeneracy on T^3 obtained from the topological discrete algebra.

Now we will address the physical meaning of $\lambda_{a,b}$. For this purpose, consider the degenerate ground states $\phi_K = |\mathbf{\Phi}\rangle_K$ $(K = 1, \cdots, d)$ with inserted fluxes $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$. Since U_a inserts the unit flux 2π adiabatically, they satisfy

$$U_a |\mathbf{\Phi}\rangle_K = e^{i\gamma_a(\mathbf{\Phi})} |\mathbf{\Phi} + \hat{a}2\pi\rangle_K, \tag{27}$$

where $\gamma_a(\mathbf{\Phi})$ is the quantum holonomy given by

$$\gamma_a(\mathbf{\Phi}) = i \int_{\Phi_a}^{\Phi_a + 2\pi} d\Phi_a \left\langle \phi_K \left| \frac{\partial}{\partial \Phi_a} \right| \phi_K \right\rangle. \tag{28}$$

In general the degenerate ground states are related to each other by the operators U_a and T_a, or some other symmetry, so $\gamma_a(\mathbf{\Phi})$ is independent of K. From $U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$, it follows

$$\begin{aligned} &\gamma_a(\mathbf{\Phi} + \hat{b}2\pi) + \gamma_b(\mathbf{\Phi}) - \gamma_b(\mathbf{\Phi} + \hat{a}2\pi) - \gamma_a(\mathbf{\Phi}) \\ &\quad = 2\pi\lambda_{a,b} + 2\pi M, \end{aligned} \tag{29}$$

where M is an integer. Then by the Stokes's theorem, the Hall conductance σ_{ab} $(a \neq b)$ [26]

$$\begin{aligned} \sigma_{ab} = &-\frac{e^2}{hd} \sum_{K=1}^{d} \int_0^{2\pi} \int_0^{2\pi} \frac{d\Phi_a d\Phi_b}{2\pi i} \left[\left\langle \frac{\partial \phi_K}{\partial \Phi_a} \middle| \frac{\partial \phi_K}{\partial \Phi_b} \right\rangle \right. \\ &\left. - (\Phi_a \leftrightarrow \Phi_b) \right] \end{aligned} \tag{30}$$

is rewritten as

$$\begin{aligned} \sigma_{ab} &= -\frac{e^2}{hd} \sum_{K=1}^{d} \int_0^{2\pi} \int_0^{2\pi} \frac{d\Phi_a d\Phi_b}{2\pi i} \left[\frac{\partial}{\partial \Phi_a} \left\langle \phi_K \left| \frac{\partial}{\partial \Phi_b} \right| \phi_K \right\rangle \right. \\ &\quad \left. - (\Phi_a \leftrightarrow \Phi_b) \right] \\ &= \frac{e^2}{hd} \sum_{K=1}^{d} \frac{1}{2\pi} \left[\gamma_b(\hat{a}2\pi) - \gamma_b(\mathbf{0}) - \gamma_a(\hat{b}2\pi) + \gamma_a(\mathbf{0}) \right] \\ &= -\frac{e^2}{h}(\lambda_{a,b} + M). \end{aligned} \tag{31}$$

This indicates clearly that a fractional $\lambda_{a,b}$ implies the fractional quantum Hall effect.

So far, we have considered the electromagnetic gauge field in order to derive the topological discrete algebra. This is not unique. If the baryon number is preserved, we can use another $U(1)$ gauge field obtained by gauging the baryon number. This gauge field is fictitious but useful. In particular, the topological discrete algebra is easily generalized to $SU(\mathcal{N}_c)$ QCD by using the fictitious $U(1)$ gauge field as follows. In $SU(\mathcal{N}_c)$ QCD, a baryon consists of $\mathcal{N}_c$ quarks, so we assign the baryon number $\mathcal{Q}_B = 1/\mathcal{N}_c$ to each quark. Then, consider an adiabatic insertion of the fictitious $U(1)$ flux by 2π through the hole h_a of T^3. By using the center of $SU(\mathcal{N}_c)$, it is shown again that the spectrum of the system is periodic in the flux insertion. Since quarks have a fractional baryon number, a nontrivial Aharanov-Bohm phase arises when a quark goes around the flux. Therefore, a similar analysis to the above leads to the following nontrivial discrete algebra on T^3 in the quark deconfinement phase,

$$\begin{aligned} &T_aT_b = T_bT_a, \qquad T_a\mathcal{U}_b = e^{(2\pi i/\mathcal{N}_c)\delta_{a,b}}\mathcal{U}_bT_a,\\ &\mathcal{U}_a\mathcal{U}_b = e^{2\pi i k_{a,b}/l_{a,b}}\mathcal{U}_b\mathcal{U}_a, \qquad (a, b = 1, 2, 3)\end{aligned} \tag{32}$$

where $\mathcal{U}_a$ denotes the flux insertion operator for the fictitious $U(1)$ gauge field, T_a is the quark winding operator defined in the same way as (13), $k_{a,b}$ and $l_{a,b}$ are coprime integers, and $l_{a,b}$ is a divisor of $\mathcal{N}_c$. Using these relations, we find that the minimal ground-state degeneracy on T^3 in the quark deconfinement phase is $\mathcal{N}_c^3$. For $\mathcal{N}_c = 3$, it reproduces the result obtained by using the electromagnetic gauge field. On the other hand, in the quark confinement phase, the topological discrete algebra is trivial and no topological degeneracy arises on T^3 because the baryon number of any hadron is an integer.

The topological discrete algebra constructed above has a similarity to the 't Hooft algebra [5,6]. For example our relations

$$T_aU_b = e^{-(2\pi i/3)\delta_{a,b}}U_bT_a \tag{33}$$

in (15) and

$$T_a\mathcal{U}_b = e^{(2\pi i/\mathcal{N}_c)\delta_{a,b}}\mathcal{U}_bT_a \tag{34}$$

in (32) correspond to the following relation given by the 't Hooft,

$$W(C)B(C') = B(C')W(C)e^{2\pi i n/\mathcal{N}_c}, \tag{35}$$

where C and C' denote closed curves in 3-dimensional space, n the number of times the curve C' winds around C in a certain direction, and $B(C')$ the 't Hooft loop along C'. However, there exist essential distinctions between them. First of all, the 't Hooft algebra is defined when dynamical quarks are absent, but our topological discrete algebra is defined in the presence of dynamical quarks. Second, the topological discrete algebra in the quark deconfinement phase is different from that in the quark confinement phase, but the 't Hooft algebra is the same in both phases. Third, new quantum numbers $\lambda_{a,b}$, which are missing in the 't Hooft algebra, exist in the topological discrete algebra.

Throughout this paper, we have assumed that the system has a finite gap. While color charges are screened in the presence of the gluon mass gap, our result indicates that the quark confinement is not synonymous with the color screening. In addition, the topological discrete algebra (14) and (15) [or (32)] itself is valid even in gapless systems. The concept of ground-state degeneracy becomes subtle in gapless systems, but the degeneracy could be identified by examining the finite scaling carefully.

Finally, we would like to mention possible generalizations of this work. First, the present argument can be generalized to other gauge groups if there are the centers in the gauge groups. It is also applicable even when the system contains bosonic matter fields such as the Higgs fields since it is irrespective of fermionic nature of the matter field.

Another interesting issue is generalization of the present consideration to the finite temperature case. We have noticed that the present argument is based on the notion of ground-state degeneracy, so its meaning is not clear at finite temperature. The thermo field dynamics formalism [27,28], which introduces "temperature dependent vacuum" in finite temperature systems, might be useful for this purpose.

To conclude, we have argued phases in QCD by using a discrete symmetry algebra which is manifest only on a space with nontrivial topology. The topological degeneracy of the ground state, which indicates the presence of a topological order, is derived and it is found that even in the presence of dynamical quarks it is a good quantum number distinguishing the quark confinement phase from the deconfinement one.

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[24] In this paper, the greek index μ has the range 1, 2, 3, 4 and the latin a and b have the range 1, 2, 3. The italic n denotes a four dimensional vector $n = (n_1, n_2, n_3, n_4)$, and the bold $\boldsymbol{n}$ denotes a three dimensional vector $\boldsymbol{n} = (n_1, n_2, n_3)$.
[25] We notice that the statistical property of the matter field are not relevant to the following argument. Even if $\sigma_i = 1$, the permutation group representation is simplified as (13) with (14).
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