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Gauge-invariant quantities characterizing gauge fields in chromodynamics

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We calculate Lorentz-invariant and gauge-invariant quantities characterizing the product $\sum_a D_R(T^a)F^a_{\mu\nu}$, where $D_R(T^a)$ denotes the matrix for the generator T^a in the representation $R =$ fundamental and adjoint, for color SU(3). We also present analogous results for an SU(2) gauge theory.

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I. INTRODUCTION

Although the properties and interactions of quarks and gluons require for their description a quantum field theory, quantum chromodynamics (QCD), it has proved useful to consider the semiclassical limit of this theory in certain cases. For example, successful models of high-energy particle production and hadronization have made use of a non-Abelian Yang-Mills generalization of the Schwinger mechanism [1,2] in which the chromoelectric field inside a flux tube between an initial quark-antiquark pair is responsible for subsequent nonperturbative production of $q\bar{q}$ pairs and hadronization [3–7]. The Schwinger calculation itself described the nonperturbative production of a charged fermion-antifermion pair by a constant classical electric field, the result of which can also be obtained from the imaginary part of the Euler-Heisenberg effective action [2]. The semiclassical limit of chromoelectric fields has also been used in certain models of relativistic heavy ion collisions [8–11]. For the case of a classical SU(3) gauge field that is constant in space and time and is such that the chromomagnetic field vanishes and all group components of the chromoelectric field point in the same direction (e.g., $\mathbf{E}^a = E^a\hat{z}\forall a$), general formulas for the nonperturbative production of gluon pairs gg and $q\bar{q}$ pairs have recently been given [12,13].

Proceeding from the special case of static, spatially constant classical fields to the general case of spacetime-dependent classical fields, one recalls that Euclidean solutions of classical non-Abelian gauge theories with nontrivial topological index, i.e., instantons, have played an important role in understanding the properties of these theories [14–17]. In particular, analyses of semiclassical effects due to instantons have shown that, in the case of weak $SU(2)_L$, these lead to nonperturbative violation of B and L (conserving $B - L$) [15] and may be significant for baryogenesis at the electroweak phase transition [18], while in the color SU(3) case, these analyses of instanton effects have explained, among other things, the breaking of the global axial vector isoscalar $U(1)_A$ symmetry and hence the fact that the η' meson is not an almost Nambu-Goldstone boson [19]. Classical solutions have also been relevant for classification of Yang-Mills theories (mainly in the SU(2) case) [20–22].

Given this importance of semiclassical color fields, it seems useful to have a set of gauge-invariant quantities that characterize these fields. Accordingly, in this paper, we present such a set. We consider an SU(N) gauge theory, concentrating on the case of color, $N = N_c = 3$, but also giving some results for the simpler case $N = 2$. We calculate certain gauge-invariant and Lorentz-invariant quantities that characterize the product

$$(\mathcal{F}_R)_{\mu\nu} \equiv \sum_a D_R(T^a)F^a_{\mu\nu}, \tag{1.1}$$

where $D_R(T^a)$ denotes the matrix for the generator $T^a \equiv T_a$ of SU(N) in the representation R, a sum over the group index a from 1 to $N^2 - 1$ is understood, and we consider the case of R being the fundamental and adjoint representation. The dimension of the representation R is denoted d_R. We recall that for the fundamental representation, $[D_{\rm fund}(T^a)]_{ij} = (T^a)_{ij}$, $1 \le i, j \le N$ and for the adjoint, $[D_{\rm adj}(T^a)]_{bc} = -ic_{abc}$, $1 \le a, b, c \le N^2 - 1$, where the structure constants c_{abc} of the SU(N) Lie algebra are defined via $[T^a, T^b] = ic_{abc}T^c$ with normalization determined by the standard condition $\text{Tr}(T^aT^b) = (1/2)\delta^{ab}$. We also recall the relation, for SU(N), $\{T_a, T_b\} = (1/N)\delta_{ab}\cdot 1_{N\times N} + d_{abc}T_c$. The field strength tensor is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc_{abc}A^b_\mu A^c_\nu$, where g is the gauge coupling (taken positive without loss of generality).

One may contrast the way in which the results of calculations are expressed in terms of gauge-invariant quantities in classical and quantum field theory. In perturbative quantum field theory calculations involving internal gauge boson lines, this entails the cancellation of the associated gauge parameter between different Feynman diagrams contributing to the amplitude for a given process. In a nonperturbative quantum field theory calculation of some gauge-invariant operator $\mathcal{O}$, one actually performs the average over the gauge fields in the path integral, e.g., in the widely used lattice gauge theory formulation,

$$\langle\mathcal{O}\rangle = \frac{\int[\prod_{n,\mu} dU_{n,\mu}d\psi_n d\bar{\psi}_n]\mathcal{O}e^{-S}}{\int[\prod_{n,\mu} dU_{n,\mu}d\psi_n d\bar{\psi}_n]e^{-S}}, \tag{1.2}$$

where S denotes the (Euclidean) action and both the measure and action are gauge invariant. For example, in pure gluodynamics with a Euclidean action $S = -\beta\sum_{\text{plaq.}}(1/N)\text{Re}[\text{Tr}_f(U_{\text{plaq.}})]$ where Tr_f is the trace in the fundamental representation, $U_{\text{plaq.}}$ denotes the product of $U's$ around a plaquette, and $\beta = 2N/g_0^2$, a strong-coupling expansion of a glueball mass would be conveniently expressed in a series in β or, equivalently, as a character expansion. The situation in a (semi)classical gauge theory calculation is different from either of these types of calculations in quantum field theory, since it depends directly on the field strengths. This was already evident from the Schwinger calculation of the production of a fermion-antifermion pair by an electric field $\mathbf{E}$ that is constant in space and time, namely,

$$\frac{dW}{d^4x} = \frac{(qeE)^2}{4\pi^2}\sum_{n=1}^{\infty} n^{-2}e^{-n\pi m^2/(|q|eE)}, \tag{1.3}$$

where q denotes the charge of the fermion. We recall how this is expressed in terms of Lorentz-invariant and gauge-invariant quantities. In this Abelian case the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$ itself is gauge invariant, in contrast to the non-Abelian case. Particle production occurs only if $|\mathbf{E}| > |\mathbf{B}|$, and in this case one can transform to an inertial frame in which the magnetic field is zero, whence the result in Eq. (1.3).

This calculation was generalized recently to the non-Abelian color group $\text{SU}(3)_c$ in the special case in which (i) there is only a chromoelectric field, $\mathbf{E}^a$, i.e., the chromomagnetic field $\mathbf{B}^a = 0$, (ii) $\mathbf{E}^a$ is a constant in space and time, and (iii) all of the group components of $\mathbf{E}^a$ point along the same spatial direction. In this case the production rates for gluon pairs gg and quark-antiquark pairs $q\bar{q}$ [12,13] were calculated. For example, for $q\bar{q}$ it was found that

$$\frac{dW_{q\bar{q}}}{d^4x d^2p_T} = -\frac{1}{4\pi^3}\sum_{j=1}^{3}|g\lambda_{q,j}|\ln[1 - e^{-\pi(p_T^2+m^2)/|g\lambda_{q,j}|}], \tag{1.4}$$

where $\mathbf{p}_T$ denotes the momentum of the quark transverse to the direction of the chromoelectric field $\mathbf{E}^a = E^a\hat{z}$ and where the $\lambda_{q,j}$ depends on two gauge-invariant, Lorentz-invariant quantities

$$C_1 = \sum_a (E^a)^2 \tag{1.5}$$

and

$$C_2 = \Big(\sum_{a,b,c} d_{abc}E^aE^bE^c\Big)^2, \tag{1.6}$$

where the sums of $\text{SU}(3)_c$ group indices a, b, c are from 1 to 8. Integration over $\mathbf{p}_T$ yields

$$\frac{dW_{q\bar{q}}}{d^4x} = \frac{1}{4\pi^2}\sum_{j=1}^{3}(g\lambda_{q,j})^2\sum_{n=1}^{\infty} n^{-2}e^{-n\pi m_q^2/|g\lambda_{q,j}|}. \tag{1.7}$$

As was noted by Schwinger [1], it is necessary to take account of the renormalization of the gauge coupling in the presence of a constant electric field, and the same is true for the non-Abelian case. Thus, strictly speaking, where we write e or g, these refer to running couplings, which run as a function of (invariants of) the respective gauge fields.

II. GENERALITIES ON QUANTITIES CHARACTERIZING $(\mathcal{F}_R)_{\mu\nu}$

We now proceed to analyze the general case where both a non-Abelian electric and magnetic field are present and where neither is a constant in space or time. Under a (local) SU(N) gauge transformation generated by the unitary matrix U,

$$(\mathcal{F}_R)_{\mu\nu} \to D_R(U)(\mathcal{F}_R)_{\mu\nu}D_R(U)^{-1}, \tag{2.1}$$

where $D_R(U^{-1}) = [D_R(U)]^{-1}$. $(\mathcal{F}_R)_{\mu\nu}$ is a matrix (of dimension $d_R \times d_R$) in group space. Given that $(\mathcal{F}_R)_{\mu\nu}$ transforms as in Eq. (2.1), it follows that the characteristic polynomial equation for $(\mathcal{F}_R)_{\mu\nu}$ is invariant under a gauge transformation, and hence so are its roots, the eigenvalues.

Since we will carry out various matrix manipulations with the field strength tensor, it will be convenient to use the pseudo-Euclidean metric, in which there is no distinction between covariant and contravariant indices. In this case, with the ordering of the indices given by $x^\mu = (\mathbf{x}, it)$, the field strength tensor takes the form

$$F^a_{\mu\nu} = \begin{bmatrix} 0 & B_3^a & -B_2^a & -iE_1^a \\ -B_3^a & 0 & B_1^a & -iE_2^a \\ B_2^a & -B_1^a & 0 & -iE_3^a \\ iE_1^a & iE_2^a & iE_3^a & 0 \end{bmatrix} \tag{2.2}$$

for $a = 1, \ldots, N^2 - 1$. The dual field strength tensor is then $\tilde{F}^a_{\mu\nu} = (i/2)\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, where $\epsilon_{\mu\nu\rho\sigma}$ is totally antisymmetric, with $\epsilon_{1234} = 1$. For each group index a, $F^a_{\mu\nu}$ changes via a similarity transformation under a (homogeneous) Lorentz transformation $\mathcal{U}$, viz.,

$$F^a \to \mathcal{U}F^a\mathcal{U}^{-1} \tag{2.3}$$

in a notation suppressing explicit Lorentz indices. Therefore, the characteristic polynomial equation for this matrix, and its roots, are Lorentz invariant. For an individual a, these eigenvalues are not gauge invariant, but they will be useful at intermediate steps in our calculation of the gauge-invariant quantities characterizing $(\mathcal{F}_R)_{\mu\nu}$. The eigenvalues are determined from the characteristic polynomial equation

$$\det(F^a - \lambda \cdot 1) = 0, \tag{2.4}$$

where here 1 is the 4×4 identity matrix. We use the following relations, which hold individually for each group index a:

$$\mathrm{Tr}_{\mathrm{Lor.}}[(F^a)^2] = F^a_{\mu\nu}F^a_{\nu\mu} = 2(|\mathbf{E}^a|^2 - |\mathbf{B}^a|^2) \tag{2.5}$$

and

$$\mathrm{Tr}_{\mathrm{Lor.}}[(F^a)^4] \equiv F^a_{\mu\nu}F^a_{\nu\rho}F^a_{\rho\sigma}F^a_{\sigma\mu} = 2(|\mathbf{E}^a|^2 - |\mathbf{B}^a|^2)^2 + 4(\mathbf{E}^a\cdot\mathbf{B}^a)^2. \tag{2.6}$$

(Note also that $\mathrm{Tr}_{\mathrm{Lor.}}(F^a\tilde{F}^a) = F_{\mu\nu}\tilde{F}_{\nu\mu} = -4\mathbf{E}^a\cdot\mathbf{B}^a$.) With these inputs, the characteristic polynomial equation takes the form, for each a,

$$(\lambda^a)^4 - (|\mathbf{E}^a|^2 - |\mathbf{B}^a|^2)(\lambda^a)^2 - (\mathbf{E}^a\cdot\mathbf{B}^a)^2 = 0. \tag{2.7}$$

The solutions are

$$\lambda^a_1 = -\lambda^a_3 = \sqrt{x^a_1}, \tag{2.8}$$

$$\lambda^a_2 = -\lambda^a_4 = \sqrt{x^a_2}, \tag{2.9}$$

where

$$x^a_{1,2} = \tfrac{1}{2}[|\mathbf{E}^a|^2 - |\mathbf{B}^a|^2 \pm [(|\mathbf{E}^a|^2 - |\mathbf{B}^a|^2)^2 + 4(\mathbf{E}^a\cdot\mathbf{B}^a)^2]^{1/2}] = \tfrac{1}{4}[F^a_{\mu\nu}F^a_{\nu\mu} \pm [(F^a_{\mu\nu}F^a_{\nu\mu})^2 + (F^a_{\mu\nu}\tilde{F}^a_{\nu\mu})^2]^{1/2}]. \tag{2.10}$$

Although a parity or time reversal transformation flips the sign of $F^a_{\mu\nu}\tilde{F}^a_{\nu\mu}$, it leaves the $x^a_{1,2}$ invariant since they depend on $F^a_{\mu\nu}\tilde{F}^a_{\nu\mu}$ only via its square.

For the general SU(N) case we define

$$C_{k1} = \sum_a (\lambda^a_k)^2 \tag{2.11}$$

and

$$C_{k2} = \Big[\sum_{a,b,c} d_{abc}\lambda^a_k\lambda^b_k\lambda^c_k\Big]^2, \tag{2.12}$$

where the sums over the SU(N) group indices run over $a, b, c = 1, \ldots, N^2 - 1$, and

$$r_k \equiv \frac{3C_{k2}}{(C_{k1})^3}. \tag{2.13}$$

These quantities will be used below.

III. INVARIANTS FOR $(\mathcal{F}_R)_{\mu\nu}$: GENERAL METHOD FOR SU(N)

We next use these Lorentz-invariant eigenvalues λ^a_k to calculate the gauge-invariant quantities characterizing $(\mathcal{F}_R)_{\mu\nu}$. For a $d_R \times d_R$ dimensional matrix A in group space we denote $\mathrm{Tr}_R(A) \equiv \sum_{i=1}^{d_R} A_{ii}$. Taking the trace over group indices and Lorentz indices, we have

$$\mathrm{Tr}_{\mathrm{R}}[\mathrm{Tr}_{\mathrm{Lor.}}h(\mathcal{F}_R)] = \sum_{k=1}^4 \mathrm{Tr}_{\mathrm{R}}\{h(T^a\lambda^a_k)\} = \sum_{k=1}^4 \mathrm{Tr}_{\mathrm{R}}(h(V_k)) = \sum_{k=1}^4\sum_{\ell=1}^{d_R} h(\Lambda_{k\ell}), \tag{3.1}$$

where $\Lambda_{k\ell}$ for $\ell = 1, \ldots, d_R$ are the eigenvalues of

$$(V_k)_{ij} \equiv \sum_a [D_R(T^a)]_{ij}\lambda^a_k, \qquad 1 \le i, j \le d_R \tag{3.2}$$

for each k. To find the gauge-invariant and Lorentz-invariant quantities characterizing $(\mathcal{F}_R)_{\mu\nu}$, one can evaluate the group traces of the set of matrices $V_k, V_k^2, \ldots, V_k^{d_R}$.

IV. INVARIANTS FOR $(\mathcal{F}_I)_{\mu\nu}$ FOR SU(2)

We label the representations of SU(2) by an isospin, T, taking on integral or half-integral values. The single diagonal generator has the form $T^3 = \mathrm{diag}(-I, -I+1, \ldots, I-1, I)$, so that the components of a representation are $|I, I_3\rangle$ satisfying $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle$ and $T_3|I, I_3\rangle = I_3|I, I_3\rangle$. Applying our procedure, we find that for this theory, $\sum_a D_I(T^a)F^a_{\mu\nu}$ is characterized by the invariants

$$\Lambda_{k\ell} = I_\ell C_{k1}, \tag{4.1}$$

where $I_\ell = I_3$, with C_{k1} evaluated for $N = 2$ in Eq. (2.11). $\Lambda_{k\ell}$ does not depend on C_{k2} since $d_{abc} = 0$ for SU(2).

V. INVARIANTS FOR $(\mathcal{F}_f)_{\mu\nu}$ IN SU(3)$_c$

We first consider a function h that has a Taylor series expansion in powers of $((\mathcal{F})_R)_{\mu\nu}$; we then apply this for the case where R is the fundamental (f) representation of SU(3)$_c$, for which we need the traces over group indices of V_k, V_k^2, and V_k^3. We find, for each k,

$$\sum_{\ell=1}^3 \Lambda_{k\ell} = 0, \tag{5.1}$$

$$\sum_{\ell=1}^3 (\Lambda_{k\ell})^2 = \frac{1}{2}\sum_a (\lambda^a_k)^2, \tag{5.2}$$

$$\sum_{\ell=1}^3 (\Lambda_{k\ell})^3 = \frac{1}{4}\sum_{a,b,c} d_{abc}\lambda^a_k\lambda^b_k\lambda^c_k. \tag{5.3}$$

For a particular k, the solution of the above equations is as follows:

$$\Lambda_{k\ell} = \sqrt{\frac{C_{k1}}{3}}\cos\Big(\theta_k + \frac{2(\ell-1)\pi}{3}\Big), \qquad \ell = 1, 2, 3, \tag{5.4}$$

where θ_k is given by

$$\cos^2(3\theta_k) = r_k \qquad (5.5)$$

and here C_{k1} and C_{k2} are given by Eqs. (2.11) and (2.12) evaluated for $N = 3$, and r_k by Eq. (2.13). (Note that $0 \le 3C_{k2}/C_{k1}^3 \le 1$.) Because of the symmetries $\lambda_1^a = -\lambda_3^a$ and $\lambda_2^a = -\lambda_4^a$, there are thus 4 independent invariants here, which can be taken to be C_{k1} and C_{k2} for $k = 1, 2$. For the case of identically zero chromomagnetic field, $\mathbf{B}^a = 0 \forall a$, $C_{21} = C_{22} = 0$, while C_{11} and C_{12} reduce to the quantities denoted C_1 and C_2 in Eqs. (1.5) and (1.6).

VI. INVARIANTS FOR $(\mathcal{F}_{\mathrm{adj}})_{\mu\nu}$

We next calculate the invariants for $(\mathcal{F}_{\mathrm{adj}})_{\mu\nu}$ using the relation $[D_{\mathrm{adj}}(T^a)]_{bc} = -ic_{abc}$. Again, we focus on the case of color, $N = 3$, setting $c_{abc} = f_{abc}$, and first evaluate the determinant

$$\begin{aligned}\mathrm{Det}[f^{abc}\lambda_k^a - \Lambda\delta_{bc}] &= \Lambda^2[\Lambda^6 + \mathcal{A}_k\Lambda^4 + \mathcal{B}_k\Lambda^2 + \mathcal{C}_k] \\ &= \Lambda^2\Pi_{\ell=1}^3(\Lambda - i\Lambda_{k\ell})(\Lambda + i\Lambda_{k\ell}).\end{aligned} \qquad (6.1)$$

Since $f^{abc}\lambda_k^c\lambda_k^a = 0$, it follows that λ_k^a is an eigenvector of the matrix $(V_k)^{ab} = f^{abc}\lambda_k^c$ with zero eigenvalue. Since for $N = 3$, $V_k^{ab} = f^{abc}\lambda_k^c$ is an even-dimensional real antisymmetric matrix, its eigenvalues (i) are comprised of opposite-sign pairs, and (ii) are pure imaginary (so the eigenvalues of $-if^{abc}\lambda_k^c$, which are the $\Lambda_{k\ell}$, are real), whence

$$\begin{aligned}[D_{\mathrm{adj}}(T^a)_{bc}\lambda_k^c]_{\mathrm{eigenvalues}} = (&\Lambda_{k1}, \Lambda_{k2}, \Lambda_{k3}, 0, -\Lambda_{k1}, \\ &-\Lambda_{k2}, -\Lambda_{k3}, 0).\end{aligned} \qquad (6.2)$$

The coefficients $\mathcal{A}_{k,n}$ of Λ^n in Eq. (6.1) are $\mathcal{A}_{k,8} = 1$ and

$$\mathcal{A}_k = \tfrac{3}{2}C_{1k}, \qquad (6.3)$$

$$\mathcal{B}_k = \tfrac{9}{16}C_{1k}^2, \qquad (6.4)$$

and

$$\mathcal{C}_k = \frac{C_{1k}^3}{16}(1 - r_k). \qquad (6.5)$$

From Eq. (6.1) we find

$$\sum_{\ell=1}^3 \Lambda_{k\ell}^2 = \mathcal{A}_k, \qquad (6.6)$$

$$\Lambda_{k1}^2\Lambda_{k2}^2 + \Lambda_{k2}^2\Lambda_{k3}^2 + \Lambda_{k3}^2\Lambda_{k1}^2 = \mathcal{B}_k, \qquad (6.7)$$

and

$$\Lambda_{k1}^2\Lambda_{k2}^2\Lambda_{k3}^2 = \mathcal{C}_k. \qquad (6.8)$$

We define

$$\cos(3\phi_k) = 2r_k - 1. \qquad (6.9)$$

The solution of these three equations is

$$\Lambda_{k\ell} = \left[\frac{C_{k1}}{2}\left\{\cos\left(\phi_k + \frac{2(\ell - 1)\pi}{3}\right)\right\}\right]^{1/2} \qquad (6.10)$$

for $\ell = 1, 2, 3$. Again, owing to the symmetries (2.8) and (2.9), these eigenvalues depend on four functionally independent invariants, which may be taken to be C_{k1} and C_{k2} for $k = 1, 2$. We note that an equivalent set of solutions for the $\Lambda_{k\ell}$ is

$$\left[\frac{C_{k1}}{2}(1 - \cos\theta_k')\right]^{1/2}; \qquad \left[\frac{C_{k1}}{2}\left[1 + \cos\left(\frac{\pi}{3} \pm \theta_k'\right)\right]\right]^{1/2}, \qquad (6.11)$$

where [23]

$$\cos(3\theta_k') = -1 + 2r_k. \qquad (6.12)$$

VII. SOME FURTHER REMARKS

We add here some further remarks. First, since the invariants for $(\mathcal{F}_f)_{\mu\nu}$ and $(\mathcal{F}_{\mathrm{adj}})_{\mu\nu}$ depend on the same set of four independent invariants, they clearly can be related to each other. We note parenthetically that the Schwinger mechanism for pair production in an electric field was generalized to an expression for pair production in an oscillatory electric field in Refs. [24,25] (see also [26]). Expressions for dW/d^4x were given for the case of a general time-dependent electric field and also for the $SU(3)_c$ case with general time-dependent $\mathbf{E}^a = E^a(t)\hat{z}$ in Ref. [27]. Our set of invariants may be used for the still more general problem of nonperturbative particle production for spacetime-dependent classical gauge fields.

VIII. DISCUSSION AND CONCLUSION

In this paper we have given a general method for calculating the gauge-invariant and Lorentz-invariant quantities characterizing the products $\sum_a D_R(T^a)F_{\mu\nu}^a$ for an $SU(N)$ gauge group. We have applied our method to compute these quantities for all representations of SU(2) and for the fundamental and adjoint representations of SU(3). Our results apply for the most general case of spacetime-dependent gauge fields and can provide a convenient set of quantities in terms of which to express calculations for classical chromodynamics.

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