$2n$-dimensional models with topological mass generation

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The 4-dimensional model with topological mass generation that has recently been presented by Dvali, Jackiw, and Pi [G. Dvali, R. Jackiw, and S.-Y. Pi, Phys. Rev. Lett. **96**, 081602 (2006)] is generalized to any even number of dimensions. As in the 4-dimensional model, the $2n$-dimensional model describes a mass-generation phenomenon due to the presence of the chiral anomaly. In addition to this model, new $2n$-dimensional models with topological mass generation are proposed, in which a Stückelberg-type mass term plays a crucial role in the mass generation. The mass generation of a pseudoscalar field such as the η' meson is discussed within this framework.

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I. INTRODUCTION

Recently, Dvali, Jackiw, and Pi have presented a novel 4-dimensional model [1] consisting of well-known topological entities: Chern-Pontryagin density $\mathcal{P}$ and Chern-Simons current $\mathcal{C}^\mu$, $\mathcal{P} = \partial_\mu \mathcal{C}^\mu$. This model can describe the mass-generation phenomenon in a 4-dimensional non-Abelian system without treating details of the underlying dynamics. Dvali *et al.* found the model as a partial, 4-dimensional generalization of the Schwinger model [2] reformulated in terms of the topological entities in 2 dimensions. The reformulated Schwinger model and the 4-dimensional model share the common mass-generation mechanism described in topological terms. Noting this, Dvali *et al.* stated that the present formulation offers a unified topological description of the mass-generation phenomena in seemingly unrelated systems.

In this paper, we first consider a straightforward $2n$-dimensional generalization of the 4-dimensional model and demonstrate that the topological mass generation studied by Dvali *et al.* is present in any even number of dimensions. There, as in the 4-dimensional model, it is verified that the presence of the chiral anomaly is essential for generating mass. Next, we propose a new $2n$-dimensional model with topological mass generation, in which a Stückelberg-type mass term gives rise to mass generation in a gauge invariant manner. In addition, we consider a hybrid of the $2n$-dimensional models mentioned above, in which a mass is caused by both the Stückelberg-type mass term and the presence of the chiral anomaly. The hybrid model is applied, after a few modifications, to the mass generation of a pseudoscalar field such as the η' meson.

In the process of deriving equations of motion in the $2n$-dimensional models, it is necessary to know the variation of the Chern-Simons current in $2n$ dimensions. To find this, we adopt an elegant method developed on $(2n+1)$-dimensional space.

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This paper is organized as follows. Section II introduces the topological entities in $2n$ dimensions. Section III presents a straightforward $2n$-dimensional generalization of the model found by Dvali *et al.* Section IV proposes new $2n$-dimensional models with a Stückelberg-type mass term. Section V contains a summary and discussion. The Appendix is devoted to calculating the variation of the Chern-Simons current in $2n$ dimensions.

II. TOPOLOGICAL ENTITIES

Let A be a (Hermitian) Yang-Mills connection on $2n$-dimensional Minkowski space, M^{2n}, with local coordinates (x^μ). The connection A is assumed to take values in a compact semisimple Lie algebra $\mathfrak{g}$, and hence A can be expanded as $A = gA^a_\mu T_a dx^\mu$. Here, g is a coupling constant with mass dimension $(2-n)$, $\{T_a\}$ are Hermitian basis of $\mathfrak{g}$ satisfying the commutation relations $[T_a, T_b] = if_{ab}{}^c T_c$ and the normalization conditions $\mathrm{Tr}(T_a T_b) = \delta_{ab}$. The curvature 2-form of A is given by

$$F \equiv dA - iA^2 = \tfrac{1}{2} g F^a_{\mu\nu} T_a dx^\mu dx^\nu, \tag{1}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu$. (Throughout this paper, the symbol $\wedge$ of the wedge product is omitted.)

Consider the Chern-Pontryagin $2n$-form

$$\begin{aligned} P_{2n} &\equiv \mathrm{Tr}\, F^n \\ &= \frac{1}{2^n} g^n h_{a_1\cdots a_n} F^{a_1}_{\mu_1\mu_2} \cdots F^{a_n}_{\mu_{2n-1}\mu_{2n}} \\ &\quad \times dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_{2n-1}} dx^{\mu_{2n}}, \end{aligned} \tag{2}$$

where $h_{a_1\cdots a_n} \equiv \mathrm{Tr}(T_{a_1} \cdots T_{a_n})$. The Bianchi identity $dF = i(AF - FA)$ guarantees $dP_{2n} = 0$. Then, in accordance with Poincaré's lemma, P_{2n} is expressed at least locally as

$$P_{2n} = dC_{2n-1}, \tag{3}$$

with the Chern-Simons $(2n-1)$-form [3]

$$C_{2n-1}(A,F) \equiv n\int_0^1 dt\,\mathrm{Tr}(AF_t^{n-1}), \quad (4)$$

where $F_t \equiv tF - i(t^2-t)A^2$.

We now introduce the Hodge $*$ operator defined by

$$*(dx^{\mu_1}\cdots dx^{\mu_p}) = \frac{1}{(2n-p)!}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_{2n}} \times dx^{\mu_{p+1}}\cdots dx^{\mu_{2n}}. \quad (5)$$

The $*$ operator transforms p-forms into their dual $(2n-p)$-forms. For a p-form $\alpha_p = (p!)^{-1}\alpha_{\mu_1\cdots\mu_p}\times dx^{\mu_1}\cdots dx^{\mu_p}$ on M^{2n}, it is verified that

$$**\alpha_p = (-1)^{p(2n-p)+1}\alpha_p, \quad (6)$$

$$*d*\alpha_p = (-1)^{(p-1)(2n-p)+1}\partial^\mu\alpha_{\mu\mu_1\cdots\mu_{p-1}} \times dx^{\mu_1}\cdots dx^{\mu_{p-1}}. \quad (7)$$

Using (5), the Hodge $*$ operation of P_{2n} is found to be

$$\mathcal{P}_{2n} \equiv *P_{2n} = \frac{1}{2^n}g^n h_{a_1\cdots a_n}\epsilon^{\mu_1\mu_2\cdots\mu_{2n-1}\mu_{2n}}F^{a_1}_{\mu_1\mu_2}\cdots F^{a_n}_{\mu_{2n-1}\mu_{2n}}. \quad (8)$$

The 0-form $\mathcal{P}_{2n}$ is referred to as the Chern-Pontryagin density. Applying the $*$ operator to (3) and using the formulas (6) and (7), we have the dual form of (3):

$$\mathcal{P}_{2n} = \partial_\mu \mathcal{C}^\mu_{2n}, \quad (9)$$

where the $\mathcal{C}^\mu_{2n}$ are the components of the 1-form $\mathcal{C}_{2n} \equiv -*C_{2n-1}$. This 1-form, or simply $\mathcal{C}^\mu_{2n}$, is referred to as the Chern-Simons current. The $\mathcal{P}_{2n}$ and $\mathcal{C}^\mu_{2n}$ are topological entities essential for constructing the $2n$-dimensional models with topological mass generation.

III. MASS GENERATION DUE TO CHIRAL ANOMALY

Now, we show that the mass-generation mechanism studied in Ref. [1] works in any even number of dimensions. The Lagrangian that we adopt, $\mathcal{L}_{2n}$, is a $2n$-dimensional analogue of the Lagrangian for the 4-dimensional model:

$$\mathcal{L}_{2n} = \tfrac{1}{2}\mathcal{P}^2_{2n} + \Lambda^2(\mathcal{C}^\nu_{2n} - \partial_\mu p^{\mu\nu})(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu}). \quad (10)$$

Here, $p^{\mu\nu}$ and $q_{\mu\nu}$ are antisymmetric tensor fields, $\mathcal{J}^5_\nu$ is an axial vector current, and Λ is a constant with mass dimension. (An overall dimensionful constant is omitted.)

Under the (infinitesimal) gauge transformation

$$\delta_\omega A^a_\mu = D_\mu\omega^a, \quad (11)$$

the Chern-Pontryagin density $\mathcal{P}_{2n}$ remains invariant, while the Chern-Simons current $\mathcal{C}^\mu_{2n}$ transforms as

$$\delta_\omega \mathcal{C}^\nu_{2n} = \partial_\mu \mathcal{U}^{\mu\nu}_{2n}. \quad (12)$$

Here, $\mathcal{U}^{\mu\nu}_{2n}$ is an antisymmetric tensor that is a polynomial in $(A^a_\mu, F^a_{\mu\nu}, \omega^a)$ and linear in ω^a. (For further details, see the Appendix.) We impose the gauge transformation rule

$$\delta_\omega p^{\mu\nu} = \mathcal{U}^{\mu\nu}_{2n} \quad (13)$$

on $p^{\mu\nu}$ so that the combination $\mathcal{C}^\nu_{2n} - \partial_\mu p^{\mu\nu}$ can be gauge invariant; thereby the gauge invariance of $\mathcal{L}_{2n}$ can be secured. In this sense, $p^{\mu\nu}$ plays the role of the Stückelberg field. By contrast, $q_{\mu\nu}$ is assumed to be gauge invariant, $\delta_\omega q_{\mu\nu} = 0$, by considering the gauge invariance of $\mathcal{J}^5_\nu$. As a result, $\mathcal{L}_{2n}$ remains invariant under the gauge transformation δ_ω. The field $p^{\mu\nu}$ is necessary for the gauge invariance of $\mathcal{L}_{2n}$, while $q_{\mu\nu}$ is necessary to avoid the integrability condition $\partial_\mu\mathcal{J}^5_\nu = \partial_\nu\mathcal{J}^5_\mu$.

As can be seen in the Appendix, the variation of the Chern-Simons current $\mathcal{C}^\mu_{2n}$ is given by [see (A23)]

$$\delta\mathcal{C}^\nu_{2n} = \mathcal{W}^{\mu\nu}_{2n,a}\delta A^a_\mu + \partial_\mu \mathcal{V}^{\mu\nu}_{2n}, \quad (14)$$

where

$$\mathcal{W}^{\mu\nu}_{2n,a} \equiv \frac{n}{2^{n-1}}g^n h_{a_1\cdots a_{n-1}a}\epsilon^{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu} \times F^{a_1}_{\mu_1\mu_2}\cdots F^{a_{n-1}}_{\mu_{2n-3}\mu_{2n-2}}, \quad (15)$$

and $\mathcal{V}^{\mu\nu}_{2n}$ is an antisymmetric tensor that is a polynomial in $(A^a_\mu, F^a_{\mu\nu}, \delta A^a_\mu)$ and linear in δA^a_μ. Using (14), variation of the action $S_{2n} = \int \mathcal{L}_{2n}dx$ with respect to A^a_μ is readily calculated, yielding the equation of motion

$$\{-\partial_\mu\mathcal{P}_{2n} + \Lambda^2(\mathcal{J}^5_\mu - \partial^\rho q_{\rho\mu})\}\mathcal{W}^{\sigma\mu}_{2n,a} - \Lambda^2\partial_\mu(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu})\frac{\delta\mathcal{V}^{\mu\nu}_{2n}}{\delta A^a_\sigma} = 0. \quad (16)$$

Variation of S_{2n} with respect to $p^{\mu\nu}$ and $q_{\mu\nu}$ yields the equations

$$\partial_\mu(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu}) - (\mu \leftrightarrow \nu) = 0, \quad (17)$$

$$\partial^\mu(\mathcal{C}^\nu_{2n} - \partial_\rho p^{\rho\nu}) - (\mu \leftrightarrow \nu) = 0. \quad (18)$$

By virtue of (17), the second line of (16) vanishes. Also, we can strip away $\mathcal{W}^{\sigma\mu}_{2n,a}$ in (16) using the identity $\mathcal{W}^{\sigma\mu}_{2n,a}F^a_{\sigma\nu} = 2\delta^\mu_\nu\mathcal{P}_{2n}$. As a result, provided $\mathcal{P}_{2n} \neq 0$, (16) reduces to

$$-\partial_\mu\mathcal{P}_{2n} + \Lambda^2(\mathcal{J}^5_\mu - \partial^\rho q_{\rho\mu}) = 0. \quad (19)$$

Taking the divergence of (19) and considering antisymmetry of $q_{\rho\mu}$, we have

$$\partial^2\mathcal{P}_{2n} - \Lambda^2\partial^\mu\mathcal{J}^5_\mu = 0. \quad (20)$$

Now, we expect that the axial vector current possesses an anomalous divergence:

$$\partial^\mu\mathcal{J}^5_\mu = -N\mathcal{P}_{2n}, \quad (21)$$

where N is a dimensionless positive constant. Then, (20) becomes

$$\partial^2 \mathcal{P}_{2n} + N\Lambda^2 \mathcal{P}_{2n} = 0. \tag{22}$$

This shows that the pseudoscalar $\mathcal{P}_{2n}$ has acquired the mass $\sqrt{N}\Lambda$. It should be stressed that the mass $\sqrt{N}\Lambda$ is generated owing to the presence of the chiral anomaly. The topological mass generation studied by Dvali *et al.* [1] is thus valid in any even number of dimensions.

IV. OTHER MODELS

Until now, we have merely considered a $2n$-dimensional generalization of the 4-dimensional model given in Ref. [1]. In this section, we propose new $2n$-dimensional models with topological mass generation.

A. A Stückelberg-type model

With the topological entities $\mathcal{P}_{2n}$ and $\mathcal{C}^{\mu}_{2n}$ and the antisymmetric tensor field $p^{\mu\nu}$, we first propose a model governed by the Lagrangian

$$\tilde{\mathcal{L}}_{2n} = \tfrac{1}{2}\mathcal{P}^2_{2n} - \tfrac{1}{2}m^2(\mathcal{C}^{\nu}_{2n} - \partial_\mu p^{\mu\nu})(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}), \tag{23}$$

where m is a constant with mass dimension. Obviously, $\tilde{\mathcal{L}}_{2n}$ is invariant under the gauge transformation δ_ω.

Variation of the action $\tilde{S}_{2n} = \int \tilde{\mathcal{L}}_{2n} dx$ with respect to A^a_μ gives, with the help of (14), the equation of motion

$$\{-\partial_\mu \mathcal{P}_{2n} - m^2(\mathcal{C}_{2n,\mu} - \partial^\rho p_{\rho\mu})\}\mathcal{W}^{\sigma\mu}_{2n,a} + m^2\partial_\mu(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu})\frac{\delta \mathcal{V}^{\mu\nu}_{2n}}{\delta A^a_\sigma} = 0. \tag{24}$$

Variation of $\tilde{S}_{2n}$ with respect to $p^{\mu\nu}$ yields the equation

$$\partial_\mu(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}) - (\mu \leftrightarrow \nu) = 0. \tag{25}$$

By virtue of (25), the second line of (24) vanishes. Also, we can strip away $\mathcal{W}^{\sigma\mu}_{2n,a}$ in (24) in the same manner as what we used under (18). Consequently, provided $\mathcal{P}_{2n} \neq 0$, (24) reduces to

$$-\partial_\mu \mathcal{P}_{2n} - m^2(\mathcal{C}_{2n,\mu} - \partial^\rho p_{\rho\mu}) = 0. \tag{26}$$

Taking the divergence of (26), and noting (9) and antisymmetry of $p_{\rho\mu}$, we have

$$\partial^2 \mathcal{P}_{2n} + m^2 \mathcal{P}_{2n} = 0. \tag{27}$$

This shows that the pseudoscalar $\mathcal{P}_{2n}$ has the mass m, which is immediately caused by the second term on the right-hand side of (23). Because this term provides a mass in a gauge invariant manner, it can be called the Stückelberg-type mass term of $\mathcal{P}_{2n}$. Accordingly, we refer to the present model as the Stückelberg-type model. The mass-generation mechanism in this model is obviously different from that in the model presented in Sec. III.

B. A hybrid model

Next, we propose a hybrid of the previous two models. The Lagrangian that we adopt to define the hybrid is

$$\hat{\mathcal{L}}_{2n} = \tfrac{1}{2}\mathcal{P}^2_{2n} - \tfrac{1}{2}m^2(\mathcal{C}^{\nu}_{2n} - \partial_\mu p^{\mu\nu})(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}) + \Lambda^2(\mathcal{C}^{\nu}_{2n} - \partial_\mu p^{\mu\nu})(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu}). \tag{28}$$

This certainly inherits characteristics of the Lagrangians (10) and (23). Variation of the action $\hat{S}_{2n} = \int \hat{\mathcal{L}}_{2n} dx$ with respect to A^a_μ gives the equation of motion

$$\{-\partial_\mu \mathcal{P}_{2n} - m^2(\mathcal{C}_{2n,\mu} - \partial^\rho p_{\rho\mu}) + \Lambda^2(\mathcal{J}^5_\mu - \partial^\rho q_{\rho\mu})\}\mathcal{W}^{\sigma\mu}_{2n,a} + \{m^2\partial_\mu(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}) - \Lambda^2\partial_\mu(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu})\} \times \frac{\delta \mathcal{V}^{\mu\nu}_{2n}}{\delta A^a_\sigma} = 0. \tag{29}$$

Variation of $\hat{S}_{2n}$ with respect to $p^{\mu\nu}$ and $q_{\mu\nu}$ yields the equations

$$m^2\partial_\mu(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}) - \Lambda^2\partial_\mu(\mathcal{J}^5_\nu - \partial^\rho q_{\rho\nu}) - (\mu \leftrightarrow \nu) = 0, \tag{30}$$

$$\partial^\mu(\mathcal{C}^{\nu}_{2n} - \partial_\rho p^{\rho\nu}) - (\mu \leftrightarrow \nu) = 0. \tag{31}$$

Combining (30) and (31) leads to (17). In the same procedure as what was taken to derive (20) and (27) from (16) and (24), respectively, we obtain, from (29) and (30),

$$\partial^2 \mathcal{P}_{2n} + m^2 \mathcal{P}_{2n} - \Lambda^2 \partial^\mu \mathcal{J}^5_\mu = 0. \tag{32}$$

When the chiral anomaly is presented, (21) holds and (32) becomes

$$\partial^2 \mathcal{P}_{2n} + (m^2 + N\Lambda^2)\mathcal{P}_{2n} = 0. \tag{33}$$

This demonstrates that the pseudoscalar $\mathcal{P}_{2n}$ has the mass $\hat{m} \equiv \sqrt{m^2 + N\Lambda^2}$. Obviously, the mass $\hat{m}$ is caused by both the Stückelberg-type mass term and the presence of the chiral anomaly. The hybrid model can be reduced to either of the previous models depending on choices of the mass parameters m and Λ.

V. SUMMARY AND DISCUSSION

The topological mass generation studied by Dvali *et al.* is valid in any even number of dimensions with no essential changes. That is, the $2n$-dimensional Chern-Pontryagin density $\mathcal{P}_{2n}$ acquires a mass owing to the presence of the chiral anomaly. Here, just as in the 4-dimensional model, the presence of the chiral anomaly is assumed without specifying its dynamical origin. To bring the $2n$-dimensional model close to a complete one, it will be necessary to investigate the underlying dynamics that leads to the mass generation due to the chiral anomaly.

By incorporating the Stückelberg-type mass term into the Lagrangian (10), the $2n$-dimensional model is extended to the hybrid model governed by the Lagrangian (28). The hybrid model becomes the Stückelberg-type model in the absence of the chiral anomaly. Now we concentrate our discussion on the hybrid model, because it involves the other two models. In the case $n = 1$, the hybrid model reduces to the 2-dimensional massive Yang-Mills theory with a vector current.

In the case $n \geq 2$, the Lagrangian (28) consists of higher dimensional terms such as $\mathcal{P}_{2n}^2$. For this reason, (28) cannot be regarded as a fundamental Lagrangian; (28) should be viewed as an effective Lagrangian that is derived from a fundamental gauge theory. The hybrid model in the case $n \geq 2$ will be applied to a phenomenological description of mass-generation phenomena expected in the fundamental theory. In this connection, now we propose an application of the hybrid model to the mass generation of a pseudoscalar field.

As in Ref. [1], we consider the axial vector current of the form

$$\mathcal{J}_\mu^5 = \sqrt{N}\Lambda^{-1}\partial_\mu \eta_0, \tag{34}$$

where η_0 is a pseudoscalar field. Adding an η_0 kinetic term to (28), and removing $q_{\rho\nu}$ and a total derivative, we have the Lagrangian

$$\hat{\mathcal{L}}'_{2n} = \tfrac{1}{2}\mathcal{P}_{2n}^2 - \tfrac{1}{2}m^2(\mathcal{C}_{2n}^\nu - \partial_\mu p^{\mu\nu})(\mathcal{C}_{2n,\nu} - \partial^\rho p_{\rho\nu}) - \sqrt{N}\Lambda\mathcal{P}_{2n}\eta_0 + \tfrac{1}{2}\partial_\mu\eta_0\partial^\mu\eta_0. \tag{35}$$

This is gauge invariant and leads to the field equations

$$-\partial_\mu\mathcal{P}_{2n} - m^2(\mathcal{C}_{2n,\mu} - \partial^\rho p_{\rho\mu}) + \sqrt{N}\Lambda\partial_\mu\eta_0 = 0, \tag{36}$$

$$\partial^2\eta_0 + \sqrt{N}\Lambda\mathcal{P}_{2n} = 0, \tag{37}$$

and (31). Because the divergence of (34) reproduces (21) with the help of (37), the chiral anomaly is considered in the Lagrangian (35). Taking the divergence of (36) and using (9) and (37) yield (33). Hence, as before, $\mathcal{P}_{2n}$ acquires the mass $\hat{m}$. Using (37), (33) can be written in terms of η_0:

$$(\partial^2 + \hat{m}^2)\partial^2\eta_0 = 0. \tag{38}$$

This equation implies that η_0 possesses both the massless and massive modes. Because the massive mode is recognized to be physical, it follows that η_0 can behave as a pseudoscalar field with the mass $\hat{m}$ [4]. In this way, a mass of the field η_0 is generated.

The Lagrangian (35) in 4 dimensions, $\hat{\mathcal{L}}'_4$, is very similar to what Di Vecchia used for solving the U(1) problem in a simple model [5]. The similarity can be seen by identifying $\hat{m}$ and m with the masses of the singlet and nonsinglet pseudoscalar-mesons, respectively. (The η' mass is evaluated by taking into account the mixing between the singlet meson η_0 and a nonsinglet meson.) A remarkable difference between $\hat{\mathcal{L}}'_4$ and Di Vecchia's Lagrangian, $\mathcal{L}_{\rm D}$, is that whereas $\mathcal{L}_{\rm D}$ contains the mass term $\mathcal{M} \equiv -\frac{1}{2}m^2\eta_0^2$, $\hat{\mathcal{L}}'_4$ does not contain it. Instead of $\mathcal{M}$, $\hat{\mathcal{L}}'_4$ contains the Stückelberg-type mass term to provide the mass m. Unlike $\mathcal{M}$, the Stückelberg-type mass term does not break the symmetry under a constant shift of η_0. In spite of such a difference, the hybrid model should have a close connection with the effective Lagrangian approach to the U(1) problem [5,6].

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APPENDIX: VARIATION OF THE CHERN-SIMONS CURRENT

In this Appendix, we calculate the variation of the Chern-Simons current $\mathcal{C}_{2n}^\mu$. For this purpose, we adopt a geometric method developed on the product space $M^{2n} \times \mathbb{R}$, a direct product of $2n$-dimensional Minkowski space M^{2n} and 1-dimensional real space $\mathbb{R}$. The exterior derivative in $M^{2n} \times \mathbb{R}$ takes the form

$$\boldsymbol{d} = d + \delta_y = \frac{\partial}{\partial x^\mu}dx^\mu + \frac{\partial}{\partial y}dy, \tag{A1}$$

where y denotes the coordinate of $\mathbb{R}$. We now consider the following Yang-Mills connection defined on $M^{2n} \times \mathbb{R}$:

$$\boldsymbol{A} = A + \Omega = gA_\mu^a T_a dx^\mu + g\omega^a T_a dy, \tag{A2}$$

where A is a 1-form that, at $y = 0$, agrees with the connection A that is already present in M^{2n}. The components (A_μ^a, ω^a) of $\boldsymbol{A}$ are understood to be functions of (x^μ, y). The curvature 2-form of $\boldsymbol{A}$ is defined in the manner same as (1):

$$\boldsymbol{F} \equiv \boldsymbol{dA} - i\boldsymbol{A}^2. \tag{A3}$$

Substituting (A1) and (A2) into (A3) and noting the nilpotency $dydy = 0$, we have

$$\boldsymbol{F} = F + \Xi, \tag{A4}$$

with $\Xi \equiv \delta_y A + D\Omega$. Here, $D\Omega$ is the exterior covariant derivative of Ω: $D\Omega \equiv d\Omega - i(A\Omega + \Omega A)$. Obviously, Ξ can be expressed as $\Xi = g\xi_\mu^a T_a dy dx^\mu$, with ξ_μ^a being functions of (x^μ, y). Now we write the definition of Ξ as

$$\delta_y A = -D\Omega + \Xi. \tag{A5}$$

This expression can be read as a transformation rule of A. In fact, the right-hand side is understood as the sum of the (infinitesimal) gauge transformation with a parameter Ω and the shift transformation with a parameter Ξ. For the sake of convenience, we decompose (A5) into the sum of the two transformation rules:

$$\delta_\Omega \boldsymbol{A} = -D\Omega, \tag{A6}$$

$$\delta_\Xi \boldsymbol{A} = \Xi, \tag{A7}$$

in such a way that $\delta_y \boldsymbol{A} = \delta_\Omega A + \delta_\Xi A$. Accordingly, the exterior derivative $\boldsymbol{d}$ is expressed as

$$\boldsymbol{d} = d + \delta_\Omega + \delta_\Xi. \tag{A8}$$

The transformation rules (A6) and (A7) can be written in terms of the component fields as

$$\delta_\omega A^a_\mu = D_\mu \omega^a, \tag{A9}$$

$$\delta_\xi A^a_\mu = \xi^a_\mu, \tag{A10}$$

with $D_\mu \omega^a \equiv \partial_\mu \omega^a + g f_{bc}{}^a A_\mu{}^b \omega^c$. Here, δ_ω and δ_ξ are defined by $\delta_\Omega = \delta_\omega dy$ and $\delta_\Xi = \delta_\xi dy$, respectively.

Replacing (A, F) in formula (3) by $(\boldsymbol{A}, \boldsymbol{F})$, we have an analogue of (3) valid in $M^{2n} \times \mathbb{R}$:

$$\mathrm{Tr}\,\boldsymbol{F}^n = \boldsymbol{d}\boldsymbol{C}_{2n-1}, \tag{A11}$$

where $\boldsymbol{C}_{2n-1} \equiv C_{2n-1}(\boldsymbol{A}, \boldsymbol{F})$. The $(2n-1)$-form $\boldsymbol{C}_{2n-1}$ can be expanded in powers of dy; by virtue of the nilpotency $dydy = 0$, the expansion has only a finite number of expansion terms:

$$\begin{aligned}\boldsymbol{C}_{2n-1} &= C_{2n-1}(A + \Omega, F + \Xi) \\ &= C_{2n-1}(A, F) + U_{2n-1}(A, F, \Omega) \\ &\quad + V_{2n-1}(A, F, \Xi).\end{aligned} \tag{A12}$$

Here, U_{2n-1} is first order in Ω and includes no Ξ, while V_{2n-1} is first order in Ξ and includes no Ω. Concrete forms for U_{2n-1} and V_{2n-1} can be found from (4) and (A12). Applying $\boldsymbol{d}$ to (A12) gives

$$\begin{aligned}\boldsymbol{d}\boldsymbol{C}_{2n-1} &= dC_{2n-1} + dU_{2n-1} + dV_{2n-1} + \delta_\Omega C_{2n-1} \\ &\quad + \delta_\Xi C_{2n-1}.\end{aligned} \tag{A13}$$

Also, the following expansion is valid with (A4):

$$\mathrm{Tr}\,\boldsymbol{F}^n = \mathrm{Tr}\,F^n + n\,\mathrm{Tr}(F^{n-1}\Xi). \tag{A14}$$

Substituting (A13) and (A14) into (A11) and decomposing the resultant with respect to Ω and Ξ, we have

$$\mathrm{Tr}\,F^n = dC_{2n-1}, \tag{A15}$$

$$\delta_\Omega C_{2n-1} = -dU_{2n-1}, \tag{A16}$$

$$\delta_\Xi C_{2n-1} = n\,\mathrm{Tr}(F^{n-1}\Xi) - dV_{2n-1}. \tag{A17}$$

Equation (A15) is identical to (3), (A16) is the (infinitesimal) gauge transformation of C_{2n-1}, and (A17) is the shift transformation of C_{2n-1}. In this way, the transformation rules of C_{2n-1} have together been derived.

We can write (A16) and (A17) as

$$\delta_\omega C_{2n-1} = dU_{2n-2}, \tag{A18}$$

$$\delta_\xi C_{2n-1} = n\,\mathrm{Tr}(F^{n-1}\xi) + dV_{2n-2}, \tag{A19}$$

with $\xi \equiv g\xi^a_\mu T_a dx^\mu$. Here, U_{2n-2} and V_{2n-2} are $(2n-2)$-forms defined by $U_{2n-1} = U_{2n-2}dy$ and $V_{2n-1} = V_{2n-2}dy$, respectively. We hereafter treat (A18) and (A19) as transformation rules in M^{2n} by setting $y = 0$. Applying the $*$ operator to (A18) and (A19) and using the formulas (6) and (7) lead to the dual forms:

$$\delta_\omega \mathcal{C}^\nu_{2n} = \partial_\mu \mathcal{U}^{\mu\nu}_{2n}, \tag{A20}$$

$$\delta_\xi \mathcal{C}^\nu_{2n} = \mathcal{W}^{\mu\nu}_{2n,a}\xi^a_\mu + \partial_\mu \mathcal{V}^{\mu\nu}_{2n}, \tag{A21}$$

where

$$\begin{aligned}\mathcal{W}^{\mu\nu}_{2n,a} &\equiv \frac{n}{2^{n-1}} g^n h_{a_1\cdots a_{n-1}a} \epsilon^{\mu_1\mu_2\cdots\mu_{2n-3}\mu_{2n-2}\mu\nu} \\ &\quad \times F^{a_1}_{\mu_1\mu_2}\cdots F^{a_{n-1}}_{\mu_{2n-3}\mu_{2n-2}},\end{aligned} \tag{A22}$$

the $\mathcal{U}^{\mu\nu}_{2n}$ are the components of the 2-form $\mathcal{U}_{2n} \equiv -*U_{2n-2}$, and the $\mathcal{V}^{\mu\nu}_{2n}$ are the components of the 2-form $\mathcal{V}_{2n} \equiv -*V_{2n-2}$. Obviously, $\mathcal{U}^{\mu\nu}_{2n}$, $\mathcal{V}^{\mu\nu}_{2n}$, and $\mathcal{W}^{\mu\nu}_{2n,a}$ are antisymmetric tensors.

Because ξ^a_μ are arbitrary functions of x^μ, the shift transformation (A10) can be identified with the variation of A^a_μ. Replacing ξ^a_μ by the variation δA^a_μ, we express (A21) in the form of the variation of $\mathcal{C}^\nu_{2n}$:

$$\delta\mathcal{C}^\nu_{2n} = \mathcal{W}^{\mu\nu}_{2n,a}\delta A^a_\mu + \partial_\mu \mathcal{V}^{\mu\nu}_{2n}, \tag{A23}$$

where $\mathcal{V}^{\mu\nu}_{2n}$ here is linear in δA^a_μ. Thus, the variation of the Chern-Simons current has been obtained using a geometric method.

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