New infinite-dimensional symmetry groups for heterotic string theory

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Some Riemann-Hilbert (RH) problems are introduced for carrying out symmetry transformations of the 2-dimensional heterotic string theory. A pair of RH transformations are constructed, and they are verified to give an infinite-dimensional symmetry group of the considered theory. This symmetry group has the structure of the semidirect product of the Kac-Moody group $\widehat{O(d, d+n)}$ and Virasoro group. Moreover, the infinitesimal forms of these RH transformations are calculated out, and they are found to give exactly the same results as in my previous paper. These demonstrate that the pair of RH transformations in the current paper provide exponentiations of all the infinitesimal symmetries in my previous paper. The finite forms of symmetry transformations given in the present paper are more important and useful for theoretic studies and new solution generation, etc.

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I. INTRODUCTION

The studies of symmetry structures for the dimensionally reduced low energy effective (super)string theories have attracted much attention in the recent past (e.g. [1–22]) owing to their importance in theoretical and mathematical physics. Such effective string theories describe various interacting matter fields coupled to gravity. The dimensionally reduced heterotic string theory (e.g. [1,2,9,14–16,18,20–22]) is a typical and very important model of this kind. Some analogies between it and the reduced Einstein-Maxwell theory have been noted. However, the mathematical structures of the heterotic string theory are much more complicated. For example, many scalar functions in Einstein gravity correspond, formally, to matrix ones in the string theory; thus the noncommuting property of the matrices gives rise to essential complications for the further study of the latter. Moreover, some important and useful formulas in some studies of the reduced Einstein or Einstein-Maxwell theories (e.g. [23–29]) will have no general analogues in the reduced heterotic string theory, so deeper research and further, extended methods of study are needed.

The present paper is a continuation of my previous paper [22]. In [22], I constructed complex $(2d+n)\times(2d+n)$ matrix H, F potentials and established a pair of Hauser-Ernst (HE)-type linear systems. Based on these linear systems, I explicitly constructed new infinitesimal symmetry transformations of the 2-dimensional heterotic string theory and verified that they constitute an infinite-dimensional Lie algebra, which has the structure of the semidirect product of the Kac-Moody $\widehat{O(d, d+n)}$ and Virasoro algebras. However, for theoretic studies and new solution generation, etc., the more important and useful thing is to find finite symmetry transformations of the considered theory. This is the main aim of the present paper.

In Sec. II, the $(2d+n)\times(2d+n)$ matrix complex F, H potentials and associated pair of HE-type linear systems for the 2-dimensional heterotic string theory [22] are briefly recalled. In Sec. III, I construct a pair of Riemann-Hilbert (RH) transformations relating to the pair of HE-type linear systems and then prove that they are indeed symmetry transformations of the considered heterotic string theory. In Sec. IV, the equivalent integral equation formulations are given out, and the infinite-dimensional group structures of the RH transformations are verified. In Sec. V, infinitesimal forms of the given RH transformations are calculated, which give exactly the same results of my previous paper [22]. These demonstrate that the pair of RH transformations in the present paper provide exponentiations of all the infinitesimal symmetry transformations given in [22]. Finally, Sec. VI provides a summary and discussion.

II. MATRIX COMPLEX F, H POTENTIALS AND HE-TYPE LINEAR SYSTEMS

For later use, here I briefly recall the complex matrix H, F potentials and HE-type linear systems for the 2-dimensional heterotic string theory given in [22].

I start with the action describing the massless sector of heterotic string theory as follows:

$$\mathcal{S} = \int d^{2+d}x\sqrt{|\mathcal{G}|}e^{-\Phi}\Big[\mathcal{R} + \mathcal{G}^{LN}\partial_L\Phi\partial_N\Phi - \tfrac{1}{12}\mathcal{H}_{LNP}\mathcal{H}^{LNP} - \tfrac{1}{4}\mathcal{F}^k_{LN}\mathcal{F}^{kLN}\Big], \quad (2.1)$$

where $\mathcal{R}$ is the Ricci scalar for the metric $\mathcal{G}_{LN}$ $(L, N = 1, 2, \cdots, 2+d)$, Φ is the dilaton field, and

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$$\mathcal{F}^k_{LN} = \partial_L \mathcal{A}^k_N - \partial_N \mathcal{A}^k_L,$$
$$\mathcal{H}_{LNP} = \left(\partial_L \mathcal{B}_{NP} - \tfrac{1}{2}\mathcal{A}^k_L \mathcal{F}^k_{NP}\right) + \text{cyclic.} \tag{2.2}$$

$\mathcal{B}_{NP}$ and $\mathcal{A}^k_L$ $(k = 1, 2, \cdots, n)$ denote the antisymmetric tensor field and $U(1)^n$ gauge fields, respectively. For the heterotic string $d = 8$, $n = 16$, but I keep them arbitrary in the present discussion.

Following Maharana and Schwarz [1] and Sen [2], when dimensionally reducing the above theory from $2 + d$ to 2 dimensions by compactification on a d-dimensional torus and using the fact that the 2-dimensional antisymmetry tensor field and 2-dimensional gauge fields have no dynamics, then (2.1) can be reduced to the following effective action [2,9,18]:

$$S = \int d^2x\sqrt{g}e^{-\phi}\Big[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{8}g^{\mu\nu}\,\mathrm{Tr}(\partial_\mu\mathcal{M}^{-1}\partial_\nu\mathcal{M})\Big], \tag{2.3}$$

where $g^{\mu\nu}$ $(\mu,\ \nu = 1,\ 2)$ denotes the inverse of the 2-dimensional metric $g_{\mu\nu}$ (in this paper I choose the signature of $g_{\mu\nu}$ to be $+\,+$), $g = \det(g_{\mu\nu})$, R is the Ricci scalar for $g_{\mu\nu}$, ϕ is the shifted dilaton field, and the $(2d + n) \times (2d + n)$ matrix $\mathcal{M}$, representing the moduli G, B, and A, is parametrized as

$$\mathcal{M} = \begin{pmatrix} G^{-1} & G^{-1}(B + C) & G^{-1}A \\ (-B + C)G^{-1} & (G - B + C)G^{-1}(G + B + C) & (G - B + C)G^{-1}A \\ A^\top G^{-1} & A^\top G^{-1}(G + B + C) & I_n + A^\top G^{-1}A \end{pmatrix}, \tag{2.4}$$

in which G, B, and A are, respectively, $d \times d$ symmetric, antisymmetric, and $d \times n$ matrix-valued fields coming from the fields of the $(2 + d)$-dimensional heterotic strings, "$\top$" denotes the transposition, $C = \frac{1}{2}AA^\top$ is a $d \times d$ matrix, and I_n denotes the $n \times n$ unit matrix. All of the above fields are assumed to depend only on x^1, x^2. For the present paper I shall equivalently use $M := e^{-\phi}\mathcal{M}$ instead of $\mathcal{M}$. In the conformal gauge $g_{\mu\nu} = e^{2\gamma}\delta_{\mu\nu}$, denoting x^1, x^2 by x, y and $e^{-\phi}$ by ρ for simplicity, the motion equations of the 2-dimensional heterotic string theory can be written as [2,9,18,22]

$$d(\rho^{-1}M\mathcal{L}^* dM) = 0 \tag{2.5}$$

with conditions

$$M^\top = M, \tag{2.6a}$$
$$M\mathcal{L}M = \rho^2\mathcal{L}, \tag{2.6b}$$
$$\mathcal{L} := \begin{pmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}, \tag{2.6c}$$

and $\rho = e^{-\phi}$ is a harmonic function in 2-dimensional $\{x, y\}$.

Equation (2.5) implies that we can introduce a $(2d + n) \times (2d + n)$ matrix twist potential $Q(x, y)$ by $dQ = -\rho^{-1}M\mathcal{L}^* dM$, and then from (2.6) and the harmony of $\rho(x, y)$ we can obtain $Q + Q^\top = -2z\mathcal{L}$ with the real field $z = z(x, y)$ introduced by $^*d\rho = dz$. Thus, if we define a complex matrix H potential

$$H := M + iQ \tag{2.7}$$

and denoting $\Omega := i\mathcal{L}$, then the equations about Q and M can be written together as

$$2(z + \rho^*)dH = (H + H^\top)\Omega dH. \tag{2.8}$$

By introducing a complex parameter t and defining

$$A(t) := I - t(H + H^\top)\Omega,\quad (I \text{ is the } (2d + n)\text{-dim unit matrix}), \tag{2.9}$$

$$\Gamma(t) := t\Lambda(t)^{-1}dH, \tag{2.10}$$

$$\Lambda(t) := 1 - 2t(z + \rho^*),$$
$$\Lambda(t)^{-1} = \lambda(t)^{-2}[1 - 2t(z - \rho^*)], \tag{2.11}$$

$$\lambda(t) := [(1 - 2zt)^2 + (2\rho t)^2]^{1/2}, \tag{2.12}$$

then Eq. (2.8) can be rewritten as

$$tdH = A(t)\Gamma(t), \tag{2.13}$$

and the associated HE-type linear system can be established as

$$dF(t) = \Gamma(t)\Omega F(t), \tag{2.14}$$

$$F(0) = I, \tag{2.15a}$$
$$\dot{F}(0) = H\Omega, \tag{2.15b}$$

$$\lambda(t)F(t)^+\Omega F(t) = \Omega, \tag{2.16a}$$
$$F(t)^\top\Omega A(t)F(t) = \Omega. \tag{2.16b}$$

Where $F(t) = F(x, y, t)$ is a $(2d + n) \times (2d + n)$ matrix complex function of x, y, and t, and is holomorphic in a neighborhood of $t = 0$, $\dot{F}(t) := \partial F(t)/\partial t$, $F(t)^+ := F(\bar{t})^\dagger$, "$\dagger$" denotes the Hermitian conjugation, and $\bar{t}$ is the complex conjugation of t. The F potential $F(t)$ is essentially a generating function for the hierarchies of potentials given in Ref. [9].

Besides, by introducing another complex parameter w and defining

$$\tilde{A}(w) := w - (H + H^{\top})\Omega, \tag{2.17}$$

$$\tilde{\Gamma}(w) := \tilde{\Lambda}(w)^{-1} dH, \tag{2.18}$$

$$\begin{aligned} \tilde{\Lambda}(w) &:= w - 2(z + \rho^*), \\ \tilde{\Lambda}(w)^{-1} &= \tilde{\lambda}(w)^{-2}[w - 2(z - \rho^*)], \end{aligned} \tag{2.19}$$

$$\tilde{\lambda}(w) := [(w - 2z)^2 + (2\rho)^2]^{1/2}, \tag{2.20}$$

then Eq. (2.8) can be rewritten as

$$dH = \tilde{A}(w)\tilde{\Gamma}(w), \tag{2.21}$$

and the associated HE-type linear system is

$$d\tilde{F}(w) = \tilde{\Gamma}(w)\Omega\tilde{F}(w), \tag{2.22}$$

$$\tilde{\lambda}(w)\tilde{F}(w)^{+}\Omega\tilde{F}(w) = \Omega, \tag{2.23a}$$

$$\tilde{F}(w)^{\top}\Omega\tilde{A}(w)\tilde{F}(w) = \Omega, \tag{2.23b}$$

where $\tilde{F}(w) = \tilde{F}(x, y, w)$ is another $(2d + n) \times (2d + n)$ matrix complex function of x, y, and w and is analytic around $w = 0$.

III. RIEMANN-HILBERT TRANSFORMATIONS

Let L denote a smooth contour surrounding the origin in the complex plane and be symmetric with respect to the real axis, and L_+ and L_- be the inside and outside (including ∞) of L, respectively. For a complex variable s, if a given complex matrix function $K(s)$ is holomorphic and invertible on L, then there exist a pair of complex matrix functions $X_\pm(s)$ which are (respectively for $\pm$) holomorphic in $L_\pm$, continuous and invertible on $L \cup L_\pm$, such that

$$X_-(s) = X_+(s)K(s), \qquad s \in L. \tag{3.1}$$

We call (3.1) an RH problem. For a fixed kernel $K(s)$, the fundamental solution $X_\pm(s)$ of the RH problem is unique up to a nonsingular constant matrix factor. A suitable boundary condition can cancel this indefiniteness.

By using the above RH problem formulation and solutions $F(t)$, $\tilde{F}(w)$ of linear systems (2.14), (2.15), (2.16), (2.22), and (2.23), we can construct symmetry transformations for the reduced heterotic string theory under consideration. From definitions (2.7), (2.9), (2.10), (2.11), (2.12), (2.17), (2.18), (2.19), and (2.20), we may consistently choose the complex matrix functions $F(t)$ and $\tilde{F}(w)$ as

$$\overline{F(t)} = F(\bar{t}), \qquad \overline{\tilde{F}(w)} = \tilde{F}(\bar{w}) \tag{3.2}$$

in order to ensure the reality of M and Q in the transformed H. We shall take this choice in the following.

A. RH transformation for linear system (2.14), (2.15), and (2.16)

Following the spirit of [28], I introduce a scalar function $v(t)$ which is independent of x, y and holomorphic on $L \cup L_-$ except infinity where it tends to linear divergence such that $v(t)$ is a linear function of t or has singularities in L_+. In addition, $v(t)$ is real when t is real. Furthermore, I introduce two new real functions $\rho' = \rho'(x, y)$, $z' = z'(x, y)$ such that (for fixed x, y) $\lambda(v(t)) = [(1 - 2zv(t))^2 + (2\rho v(t))^2]^{1/2}$ and $\lambda'(t) := [(1 - 2z't)^2 + (2\rho' t)^2]^{1/2}$ have the same zeros in t, and $\dot{v}(t) \neq 0$ at these zeros. Thus we have

$$\frac{1}{2(z' \pm i\rho')} = v^{-1}\left(\frac{1}{2(z \pm i\rho)}\right), \tag{3.3a}$$

$$^*d\rho' = dz'. \tag{3.3b}$$

Equation (3.3a) can be interpreted as a variable transformation under which Eqs. (2.8), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), and (2.16) are transformed. We shall use the notations with prime to denote the transformed functions, e.g. $F(t) \mapsto F'(t)$.

Motivated by [25,28], for a given solution $F(t)$ of (2.14), (2.15), and (2.16), we select the contour L such that $F(t)$ is holomorphic on $L \cup L_+$ and take the kernel $K(t)$ of (3.1) as

$$K(t) = F'(t)u(t)F(v(t))^{-1}, \tag{3.4}$$

where the complex $(2d + n) \times (2d + n)$ matrix function $u(t)$ (independent of x, y) is holomorphic in $L \cup L_-$ and satisfies

$$u(t)^{\top}\eta u(t) = \eta, \qquad \overline{u(t)} = u(\bar{t}); \tag{3.5}$$

i.e. $u(t) \in O(d, d + n)$ when t is real.

Lemma 1: If $X_\pm(t)$ is a fundamental solution of the RH problem (3.1), (3.4), and (3.5) with boundary condition

$$X_+(0) = I, \tag{3.6}$$

then we can consistently define the following complex functions of t as

$$\begin{aligned} W_1(t) &= X_+^{+}(t)^{-1}\Omega X_+(t)^{-1} \quad \text{on } L \cup L_+ \\ &= (\lambda(v(t))/\lambda'(t))X_-^{+}(t)^{-1}\Omega X_-(t)^{-1} \quad \text{on } L \cup L_-, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} W_2(t) &= X_+^{\top}(t)^{-1}\Omega A'(t)X_+(t)^{-1} \quad \text{on } L \cup L_+ \\ &= X_-^{\top}(t)^{-1}\Omega A(v(t))X_-(t)^{-1} \quad \text{on } L \cup L_-, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} W_3(t) &= dX_+(t)X_+(t)^{-1} + X_+(t)\Gamma'(t)\Omega X_+(t)^{-1} \quad \text{on } L \cup L_+ \\ &= dX_-(t)X_-(t)^{-1} + X_-(t)\Gamma(v(t))\Omega X_-(t)^{-1} \\ &\quad \text{on } L \cup L_-, \end{aligned} \tag{3.9a}$$

$$W_4(t) = \Omega\underline{A}(t)dX_+(t)X_+(t)^{-1} + tX_+^\top(t)^{-1}\Omega dH'\Omega X_+(t)^{-1} \quad \text{on } L\cup L_+$$
$$= \Omega\underline{A}(t)dX_-(t)X_-(t)^{-1} + v(t)X_-^\top(t)^{-1}\Omega dH\Omega X_-(t)^{-1} \quad \text{on } L\cup L_-, \tag{3.10a}$$

and have

$$W_1(t) = \Omega, \tag{3.7b}$$
$$W_2(t) = \Omega\underline{A}(t), \tag{3.8b}$$
$$W_3(t) = \underline{\Gamma}'(t)\Omega, \tag{3.9b}$$
$$W_4(t) = t\Omega d\underline{H}\Omega. \tag{3.10b}$$
$$\underline{A}^+(t)\Omega\underline{A}(t) = \lambda'^2(t)\Omega. \tag{3.11}$$

Where

$$\underline{H} := H' - \dot{X}_+(0)\Omega, \tag{3.12}$$
$$\underline{\Gamma}'(t) := t\Lambda'(t)^{-1}d\underline{H}, \tag{3.13a}$$
$$\underline{A}(t) := I - t(\underline{H} + \underline{H}^\top)\Omega. \tag{3.13b}$$

Proof: From (2.16a), (3.1), (3.4), and (3.5) we obtain

$$X_+^\pm(s)^{-1}\Omega X_+(s)^{-1} = (\lambda(v(s))/\lambda'(s))X_-^\pm(s)^{-1}\Omega X_-(s)^{-1}, \quad s\in L.$$

Noticing the properties of the functions $X_\pm(t)$, $v(t)$, $\lambda(t)$, since $[\lambda(v(t))/\lambda'(t)]$ is nonsingular in $L\cup L_-$, the above equation implies that $W_1(t)$ in (3.7a) is consistently defined and gives an entire function of t. Note that $(\lambda(v(t))/\lambda'(t))X_-^\pm(t)^{-1}\Omega X_-(t)^{-1}$ is regular at $t=\infty$, so $W_1(t)$ is equal to a constant matrix. From the boundary condition (3.6), we get Eq. (3.7b).

To prove (3.8b), we use (2.16b), (3.1), (3.4), and (3.5) to get

$$X_+^\top(s)^{-1}\Omega A'(s)X_+(s)^{-1} = X_-^\top(s)^{-1}\Omega A(v(s))X_-(s)^{-1}, \quad s\in L.$$

This implies that $W_2(t)$ is consistently defined and gives an entire function of t. From the expression of $A(v(t))$ and the property of $v(t)$ at $t=\infty$, we conclude that $W_2(t)$ is linear in t. By using (3.6) we obtain the coefficients in this linear function such that $W_2(t)$ turns to $\Omega\underline{A}(t)$ by definition (3.13b).

The Eq. (3.9b) is proven as follows. From (2.14), (3.1), and (3.4) we have

$$dX_+(s)X_+(s)^{-1} + X_+(s)\Gamma'(s)\Omega X_+(s)^{-1} = dX_-(s)X_-(s)^{-1} + X_-(s)\Gamma(v(s))\Omega X_-(s)^{-1}, \quad s\in L. \tag{3.14}$$

Thus the function $W_3(t)$ defined in (3.9a) is a meromorphic function of t and has simple singularity at the zeros of $\lambda'(t)$ [or, equivalently, $\lambda(v(t))$]. According to the theory of meromorphic function, $W_3(t)$ can be expressed as $U + t\Lambda'(t)^{-1}V$. By using (3.6) and (3.12) we obtain

$$U = 0, \qquad V = d\dot{X}_+(0) + dH'\Omega = d\underline{H}\Omega;$$

thus $W_3(t) = t\Lambda'(t)^{-1}d\underline{H}\Omega$, and this gives (3.9b) by (3.13a).

To prove (3.10b), we note that from (2.10), (2.13), (3.8a), (3.8b), and (3.14) we have

$$\Omega\underline{A}(s)dX_+(s)X_+(s)^{-1} + sX_+^\top(s)^{-1}\Omega dH'\Omega X_+(s)^{-1} = \Omega\underline{A}(s)dX_-(s)X_-(s)^{-1} + v(s)X_-^\top(s)^{-1}\Omega dH\Omega X_-(s)^{-1}, \qquad s\in L.$$

Similar to above, $W_4(t)$ in (3.10a) is a linear function of t, and by using (3.6) we get (3.10b).

As for (3.11), we first note that from (2.16) we have $A^+(t)\Omega A(t) = \lambda^2(t)\Omega$. Then from (3.7a), (3.7b), (3.8a), and (3.8b) it follows that

$$\underline{A}^+(t)\Omega\underline{A}(t) = \begin{cases} X_+^\pm(t)^{-1}A'^+(t)\Omega X_+^\star(t)^{-1}\Omega^{-1}X_+^\top(t)^{-1}\Omega A'(t)X_+(t)^{-1} & \text{on } L\cup L_+ \\ X_-^\pm(t)^{-1}A^+(v(t))\Omega X_-^\star(t)^{-1}\Omega^{-1}X_+^\top(t)^{-1}\Omega A(v(t))X_+(t)^{-1} & \text{on } L\cup L_- \end{cases}$$
$$= \begin{cases} X_+^\pm(t)^{-1}A'^+(t)\Omega A'(t)X_+(t)^{-1} & \text{on } L\cup L_+ \\ (\lambda'(t)/\lambda(v(t)))X_-^\pm(t)^{-1}A^+(v(t))\Omega A(v(t))X_-(t)^{-1} & \text{on } L\cup L_+ \end{cases}$$
$$= \lambda'^2(t)\Omega.$$

Theorem 1: Let $X_\pm(t) = X_\pm(x,y,t)$ be a fundamental solution of the RH problem (3.1), (3.4), (3.5), and (3.6); then the complex matrix function given by

$$\underline{F}(t) = X_+(t)F'(t) \quad \text{in } L\cup L_+$$
$$= X_-(t)F(v(t))u^{-1}(t) \quad \text{in } L\cup L_- \tag{3.15}$$

is holomorphic on $L\cup L_+$ and satisfies

$$d\underline{F}(t) = \underline{\Gamma}'(t)\Omega\underline{F}(t), \tag{3.16a}$$
$$\underline{A}(t)d\underline{F}(t) = td\underline{H}\Omega\underline{F}(t), \tag{3.16b}$$
$$\underline{F}(0) = I, \tag{3.17a}$$
$$\dot{F}(0) = \underline{H}\Omega, \tag{3.17b}$$
$$\lambda'(t)\underline{F}(t)^+\Omega\underline{F}(t) = \Omega, \tag{3.17c}$$
$$\underline{F}(t)^\top\Omega\underline{A}(t)\underline{F}(t) = \Omega. \tag{3.17d}$$

Proof: Equation (3.16a) is derived from (2.14), (3.9a), (3.9b), and (3.15); (3.16b) is deduced from (2.13), (2.14), (3.8a), (3.8b), (3.10a), (3.10b), and (3.15); (3.17a) follows simply from (2.15a), (3.6), and (3.15); (3.17b) follows from

Eqs. (2.15), (3.6), and (3.15) and definition (3.12); (3.17c) follows from (2.16a), (3.7a), (3.7b), and (3.15); (3.17d) is derived from (2.16b), (3.8a), (3.8b), and (3.15).

Theorem 2: The new complex function $\underline{H}$ given by (3.12) is an H potential of the considered heterotic string theory. Explicitly, $\underline{H}$ satisfies

$$2(z' + \rho'^*)d\underline{H} = (\underline{H} + \underline{H}^\top)\Omega d\underline{H}, \tag{3.18a}$$

$$\underline{H} - \underline{H}^\dagger = -2\Omega z', \tag{3.18b}$$

$$\underline{M} := \mathrm{Re}(\underline{H}), \qquad \underline{M}^\top = \underline{M}, \tag{3.18c}$$

$$\underline{M}\Omega\underline{M} = \rho'^2\Omega. \tag{3.18d}$$

Proof: To prove (3.18a), note that Eqs. (3.13a), (3.16a), and (3.16b) imply $\Lambda'(t)d\underline{F}(t) = \underline{A}(t)d\underline{F}(t)$. Thus from (2.11) and (3.13b) we have

$$2(z' + \rho'^*)d\underline{F}(t) = (\underline{H} + \underline{H}^\top)\Omega d\underline{F}(t).$$

Taking the t derivative of the above equation and then setting $t = 0$, we obtain (3.18a) by using (3.17b). Equation (3.18b) is derived by taking the t derivative of Eq. (3.17c) and then setting $t = 0$ and noting (3.17a) and (3.17b). Equation (3.18c) is a trivial implication of (3.18b). To prove (3.18d), note that (3.18b) and (3.18c) imply $\underline{A}(t) = (1 - 2tz')I - 2t\underline{M}\Omega$. Thus from (3.11) we have $(1 - 2tz')^2\Omega - 4t^2\Omega\underline{M}\Omega\underline{M}\Omega = \lambda'(t)^2\Omega$; this gives (3.18d).

B. RH transformation for linear system (2.22) and (2.23)

Here we need another scalar function $\tilde{v}(w)$, which has the same properties as $v(t)$ but the variable t is replaced by w, and according to the properties of $\tilde{v}(w)$ we may write

$$\tilde{v}(w)|_{w\to\infty} = aw, \qquad a > 0 \text{ (real number)}. \tag{3.19}$$

Relating to $\tilde{v}(w)$ we introduce two real functions $\rho'' = \rho''(x, y)$, $z'' = z''(x, y)$ such that (for fixed x, y) $\tilde{\lambda}(\tilde{v}(w)) = [(\tilde{v}(w) - 2z)^2 + (2\rho)^2]^{1/2}$ and $\tilde{\lambda}''(t) := [(w - 2z'')^2 + (2\rho'')^2]^{1/2}$ have the same zeros in w, and $\dot{\tilde{v}}(w) \neq 0$ at these zeros. Thus we have

$$2(z'' \pm i\rho'') = \tilde{v}^{-1}(2(z \pm i\rho)), \tag{3.20a}$$

$$^*d\rho'' = dz''. \tag{3.20b}$$

Equation (3.20a) can be interpreted as another variable transformation; the corresponding transformed functions will be denoted with double prime "″," e.g. $\tilde{F}(w) \mapsto \tilde{F}''(w)$.

Consider an RH problem relating to (2.22) and (2.23) as follows. We use $\tilde{L}$ and $\tilde{L}_\pm$ in the w plane. For a given solution $\tilde{F}(w)$ of (2.22) and (2.23), we select the contour $\tilde{L}$ such that $\tilde{F}(w)$ is holomorphic on $\tilde{L} \cup \tilde{L}_+$ and take the kernel of (3.1) as

$$\tilde{K}(w) = a^{-1/2}\tilde{F}''(w)\tilde{u}(w)\tilde{F}(\tilde{v}(w))^{-1}, \tag{3.21}$$

where the positive real number a is the same as in (3.19), and $\tilde{u}(w)$ has the same properties as $u(t)$ except that t is replaced by w.

By virtue of the RH problem (3.1) with kernel (3.21), we can obtain another RH transformation for the heterotic string theory. First, we have the following.

Lemma 2: Let $\tilde{X}_\pm(w)$ be a fundamental solution of the RH problem (3.1) and (3.21) with boundary condition

$$\tilde{X}_-(\infty) = I; \tag{3.22}$$

then we can consistently define the following functions of w as

$$\begin{aligned}\tilde{W}_1(w) &= \tilde{X}_+^\top(w)^{-1}\Omega\tilde{X}_+(w)^{-1} \quad \text{on } \tilde{L} \cup \tilde{L}_+ \\ &= a^{-1}(\tilde{\lambda}(\tilde{v}(w))/\tilde{\lambda}''(w))\tilde{X}_-^\top(w)^{-1}\Omega\tilde{X}_-(w)^{-1} \\ &\quad \text{on } \tilde{L} \cup \tilde{L}_-,\end{aligned} \tag{3.23a}$$

$$\begin{aligned}\tilde{W}_2(w) &= \tilde{X}_+^\top(w)^{-1}\Omega\tilde{A}''(w)\tilde{X}_+(w)^{-1} \quad \text{on } \tilde{L} \cup \tilde{L}_+ \\ &= a^{-1}\tilde{X}_-^\top(w)^{-1}\Omega\tilde{A}(\tilde{v}(w))\tilde{X}_-(w)^{-1} \quad \text{on } \tilde{L} \cup \tilde{L}_-,\end{aligned} \tag{3.24a}$$

$$\begin{aligned}\tilde{W}_3(w) &= d\tilde{X}_+(w)\tilde{X}_+(w)^{-1} + \tilde{X}_+(w)\tilde{\Gamma}''(w)\Omega\tilde{X}_+(w)^{-1} \\ &\quad \text{on } \tilde{L} \cup \tilde{L}_+ \\ &= d\tilde{X}_-(w)\tilde{X}_-(w)^{-1} + \tilde{X}_-(w)\tilde{\Gamma}(\tilde{v}(w))\Omega\tilde{X}_-(w)^{-1} \\ &\quad \text{on } \tilde{L} \cup \tilde{L}_-,\end{aligned} \tag{3.25a}$$

$$\begin{aligned}\tilde{W}_4(w) &= \Omega\tilde{A}(w)d\tilde{X}_+(w)\tilde{X}_+(w)^{-1} \\ &\quad + \tilde{X}_+^\top(w)^{-1}\Omega dH''\Omega\tilde{X}_+(w)^{-1} \quad \text{on } \tilde{L} \cup \tilde{L}_+ \\ &= \Omega\tilde{A}(w)d\tilde{X}_-(w)\tilde{X}_-(w)^{-1} \\ &\quad + a^{-1}\tilde{X}_-^\top(w)^{-1}\Omega dH\Omega\tilde{X}_-(w)^{-1} \quad \text{on } \tilde{L} \cup \tilde{L}_-,\end{aligned} \tag{3.26a}$$

and have

$$\tilde{W}_1(w) = \Omega, \tag{3.23b}$$

$$\tilde{W}_2(w) = \Omega\tilde{A}(w), \tag{3.24b}$$

$$\tilde{W}_3(w) = \tilde{\Gamma}''(w)\Omega, \tag{3.25b}$$

$$\tilde{W}_4(w) = \Omega d\tilde{H}\Omega. \tag{3.26b}$$

$$\tilde{A}^+(w)\Omega\tilde{A}(w) = \tilde{\lambda}''^2(w)\Omega. \tag{3.27}$$

Where

$$\tilde{H} := a^{-1}H'' - \partial_\tau\tilde{X}_-(w)|_{\tau=0}\Omega, \qquad \tau := w^{-1}, \tag{3.28}$$

$$\tilde{\Gamma}''(w) := \tilde{\Lambda}''(w)^{-1} d\tilde{H}, \tag{3.29a}$$

$$\tilde{A}(w) := w - (\tilde{H} + \tilde{H}^{\top})\Omega. \tag{3.29b}$$

Proof: The proof is similar to that of Lemma 1. However, here the boundary conditions at $w = \infty$ such as (3.19) and (3.22) are used.

Theorem 3: The complex matrix function given by

$$\begin{aligned}\tilde{F}(w) &= a^{-1/2}\tilde{X}_{+}(w)\tilde{F}''(w) \quad \text{in } \tilde{L} \cup \tilde{L}_{+} \\ &= \tilde{X}_{-}(w)\tilde{F}(\tilde{v}(w))\tilde{u}(w)^{-1} \quad \text{in } \tilde{L} \cup \tilde{L}_{-}\end{aligned} \tag{3.30}$$

is holomorphic on $L \cup L_{+}$ and satisfies

$$d\tilde{F}(w) = \tilde{\Gamma}''(w)\Omega\tilde{F}(w), \tag{3.31a}$$

$$\tilde{A}(w)d\tilde{F}(w) = d\tilde{H}\Omega\tilde{F}(w), \tag{3.31b}$$

$$\tilde{\lambda}''(w)\tilde{F}(w)^{+}\Omega\tilde{F}(w) = \Omega, \tag{3.32a}$$

$$\tilde{F}(w)^{\top}\Omega\tilde{A}(w)\tilde{F}(w) = \Omega. \tag{3.32b}$$

Proof: The proof is similar to that of Theorem 1.

Theorem 4: The new complex function $\tilde{H}$ given by (3.28) is an H potential of the considered heterotic string theory. Explicitly, $\tilde{H}$ satisfies

$$2(z'' + \rho''^{*})d\tilde{H} = (\tilde{H} + \tilde{H}^{\top})\Omega d\tilde{H}, \tag{3.33a}$$

$$\tilde{H} - \tilde{H}^{\dagger} = -2\Omega z'', \tag{3.33b}$$

$$\tilde{M} := \mathrm{Re}(\tilde{H}), \qquad \tilde{M}^{\top} = \tilde{M}, \tag{3.33c}$$

$$\tilde{M}\Omega\tilde{M} = \rho''^{2}\Omega. \tag{3.33d}$$

Proof: First we note that Eqs. (3.29a) and (3.31a) imply

$$\tilde{\Lambda}''(w)d\tilde{F}(w)\tilde{F}(w)^{-1} = d\tilde{H}\Omega. \tag{3.34}$$

To prove (3.33a), use (3.31a) and (3.31b) to get $\tilde{A}(w)d\tilde{F}(w) = \tilde{\Lambda}''(w)d\tilde{F}(w)$. Multiplying this equation from the left by $\tilde{\Lambda}''(w)$ and from the right by $\tilde{F}(w)^{-1}$, then we obtain (3.33a) by using (3.34) and the definitions of $\tilde{A}(w)$ and $\tilde{\Lambda}''(w)$.

To prove (3.33b), note that from (2.19), (2.20), and (3.32a) we have

$$2\Omega dz'' = [\tilde{\Lambda}''(w)d\tilde{F}(w)\tilde{F}(w)^{-1}\Omega]^{+} - \tilde{\Lambda}''(w)d\tilde{F}(w)\tilde{F}(w)^{-1}\Omega.$$

This, by using (3.34), is followed by $d\tilde{H} - d\tilde{H}^{\dagger} = -2\Omega dz''$ and then gives (3.33b) by selecting some suitable integral constant.

The proof of (3.33c) and (3.33d) is similar to that of (3.18c) and (3.18d).

IV. EQUIVALENT INTEGRAL EQUATIONS AND GROUP PROPERTIES OF THE RH TRANSFORMATIONS

First, noting the analytic property of $X_{-}(t)$ in (3.15) on $L \cup L_{-}$ (including ∞), we have

$$\frac{1}{2\pi i}\int_{L}\frac{X_{-}(s)}{s(s-t)}ds = 0, \qquad t \in L_{+}. \tag{4.1}$$

Substituting Eq. (3.15) into (4.1), it follows that

$$\frac{1}{2\pi i}\int_{L}\frac{\underline{F}(s)u(s)F(v(s))^{-1}}{s(s-t)}ds = 0, \qquad t \in L_{+}, \tag{4.2}$$

subject to the condition $\underline{F}(0) = I$.

As for RH transformation (3.30), by condition (3.22) we have

$$\frac{1}{2\pi i}\int_{\tilde{L}}\frac{\tilde{X}_{-}(s)}{(s-w)}ds = I, \qquad w \in \tilde{L}_{+}. \tag{4.3}$$

Now from (3.30) we obtain

$$\frac{1}{2\pi i}\int_{\tilde{L}}\frac{\underline{\tilde{F}}(s)\tilde{u}(s)\tilde{F}(\tilde{v}(s))^{-1}}{(s-w)}ds = I, \qquad w \in \tilde{L}_{+}. \tag{4.4}$$

In order to show the group structure of the above RH transformations explicitly, from the properties of $v(t)$, we introduce $\xi(t) := v^{-1}(t)$ on the contour L and define the action of (u, ξ) on any function $\Psi(t)$ as

$$(u, \xi)\Psi(t) := u(t)\Psi(\xi^{-1}(t)) = u(t)\Psi(v(t)). \tag{4.5}$$

Then the integral Eq. (4.2) can be rewritten as

$$\frac{1}{2\pi i}\int_{L}\frac{\underline{F}(s)(u, \xi)F(s)^{-1}}{s(s-t)}ds = 0, \qquad t \in L_{+}. \tag{4.6}$$

If we carry out the RH transformation 2 times successively and denote

$$(u, \xi)\colon F(t) \to \underline{F}(t), \qquad (u_1, \xi_1)\colon \underline{F}(t) \to \underline{\underline{F}}(t), \tag{4.7}$$

then from (3.15) [or equivalently (4.6)] we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{L}\frac{\underline{\underline{F}}(s)(u_1, \xi_1)(u, \xi)F(s)^{-1}}{s(s-t)}ds \\ &= \frac{1}{2\pi i}\int_{L}\frac{\underline{\underline{F}}(s)[u_1(s)u(v_1(s))]F(v(v_1(s)))^{-1}}{s(s-t)}ds \\ &= \frac{1}{2\pi i}\int_{L}\frac{\underline{\underline{F}}(s)(u_1\gamma_{\xi_1}(u), \xi_1\xi)F(s)^{-1}}{s(s-t)}ds = 0, \\ &t \in L_{+},\end{aligned}$$

where we have used the homomorphism $\gamma\colon \{\xi\} \to \mathrm{Aut}\{u\}$ defined by

$$\gamma\colon \xi \to \gamma_{\xi}, \qquad \gamma_{\xi}\colon u(t) \to \gamma_{\xi}(u)(t) = u(\xi^{-1}(t)). \tag{4.8}$$

Thus, we have an RH transformation $(u_2, \xi_2)\colon F(t) \to \underline{\underline{F}}(t)$ such that

$$(u_2, \xi_2) = (u_1, \xi_1)(u, \xi) = (u_1 \gamma_{\xi_1}(u), \xi_1 \xi). \quad (4.9)$$

Similarly, by introducing $\tilde{\xi}(w) := \tilde{v}^{-1}(w)$ on the contour $\tilde{L}$ and defining

$$(\tilde{u}, \tilde{\xi})\tilde{\Psi}(w) := \tilde{u}(w)\tilde{\Psi}(\tilde{\xi}^{-1}(w)) = \tilde{u}(w)\tilde{\Psi}(\tilde{v}(w)), \quad (4.10)$$

we obtain an RH transformation $(\tilde{u}_2, \tilde{\xi}_2)$: $\tilde{F}(w) \to \underline{\underline{F}}(w)$ such that

$$(\tilde{u}_2, \tilde{\xi}_2) = (\tilde{u}_1, \tilde{\xi}_1)(\tilde{u}, \tilde{\xi}) = (\tilde{u}_1 \tilde{\gamma}_{\tilde{\xi}_1}(\tilde{u}), \tilde{\xi}_1 \tilde{\xi}). \quad (4.11)$$

According to the loop group theory [30], the composition laws (4.9) and (4.11) show that the RH transformations (3.15) [or (4.2)] and (3.30) [or (4.4)], joined together, provide a representation of the semidirect product of the affine Kac-Moody group $\widehat{O(d, d+n)}$ and Virasoro group. As special cases, when $v(t) = t$, $\tilde{v}(w) = w$, we obtain a representation of the Kac-Moody group $\widehat{O(d, d+n)}$. When $u(t) = I$, $\tilde{u}(w) = I$, we get a representation of the Virasoro group. These results demonstrate that the heterotic string theory under consideration possesses rich symmetry structures.

V. INFINITESIMAL RH TRANSFORMATIONS

In order to find the relationship between the results in the present paper and that in [22], we discuss the infinitesimal forms of the above RH transformations.

Setting $v(t) = t$ and considering the following infinitesimal transformation

$$u(t) = I + \delta u(t), \qquad \underline{F}(t) = F(t) + \delta F(t),$$

then by (2.15a), Eq. (4.2) becomes

$$\delta F(t)F(t)^{-1} = -\frac{t}{2\pi i}\int_L \frac{F(s)\delta u(s)F(s)^{-1}}{s(s-t)}ds, \qquad t \in L_+. \quad (5.1)$$

Noticing the properties of $u(t)$, without loss of generality, we can select $\delta u(s) = \delta_\alpha^{(k)} u(s) = T_\alpha s^{-k}$ $(k \geq 0)$. Where $T_\alpha = T_a \alpha^a$, T_a are generators of $o(d, d+n)$ [the Lie algebra of $O(d, d+n)$], α^a are infinitesimal real constants. Substituting these into (5.1), we have

$$\delta_\alpha^{(k)} F(t)F(t)^{-1} = -\frac{t}{2\pi i}\int_L \frac{s^{-k}F(s)T_\alpha F(s)^{-1}}{s(s-t)}ds, \quad t \in L_+;$$

then the parameterized transformation $\delta_\alpha(t')F(t) = \sum_{k=0}^\infty t'^k \delta_\alpha^{(k)} F(t)$ $(t' \in L_+)$ is given by

$$\begin{aligned}\delta_\alpha(t')F(t)F(t)^{-1} &= -\frac{t}{2\pi i}\int_L \sum_{k=0}^\infty \frac{t'^k F(s)T_\alpha F(s)^{-1}}{s^{k+1}(s-t)}ds \\ &= -\frac{t}{2\pi i}\int_L \frac{F(s)T_\alpha F(s)^{-1}}{(s-t')(s-t)}ds \\ &= \frac{t}{t-t'}[F(t')T_\alpha F(t')^{-1} - F(t)T_\alpha F(t)^{-1}].\end{aligned} \quad (5.2)$$

Similarly, taking $\tilde{v}(w) = w$ and considering the infinitesimal transformation $\tilde{u}(w) = I + \tilde{\delta}\,\tilde{u}(w)$, $\underline{\tilde{F}}(w) = \tilde{F}(w) + \tilde{\delta}\,\tilde{F}(w)$, then (4.4) becomes

$$\tilde{\delta}\,\tilde{F}(w)\tilde{F}(w)^{-1} = -\frac{1}{2\pi i}\int_{\tilde{L}} \frac{\tilde{F}(s)\tilde{\delta}\,\tilde{u}(s)\tilde{F}(s)^{-1}}{(s-w)}ds, \quad w \in \tilde{L}_+. \quad (5.3)$$

Selecting $\tilde{\delta}\,\tilde{u}(s) = \tilde{\delta}_\alpha^{(k)}\tilde{u}(s) = T_\alpha s^{-k}$ $(k \geq 1)$ and denoting the corresponding $\tilde{\delta}\,\tilde{F}(w)$ by $\tilde{\delta}_\alpha^{(k)}\tilde{F}(w)$, then from (5.3), the parameterized infinitesimal transformation $\tilde{\delta}_\alpha(w')\tilde{F}(w) = \sum_{k=1}^\infty w'^k \tilde{\delta}_\alpha^{(k)}\tilde{F}(w)$ $(w' \in \tilde{L}_+)$ is given by

$$\tilde{\delta}_\alpha(w')\tilde{F}(w)\tilde{F}(w)^{-1} = \frac{w'}{w-w'}[\tilde{F}(w')T_\alpha \tilde{F}(w')^{-1} - \tilde{F}(w)T_\alpha \tilde{F}(w)^{-1}]. \quad (5.4)$$

The RH transformations (4.2) and (4.4), in fact, contain more symmetries of the heterotic string theory. To show this, we need "cross" infinitesimal variations $\tilde{\delta}F(t)$ and $\delta\tilde{F}(w)$ brought about by $u(t) = I + \tilde{\delta}u(t)$ and $\tilde{u}(w) = I + \delta\tilde{u}(w)$, respectively. Considering the relation between t and w in (2.14) and (2.22), we can select $\tilde{\delta}_\alpha^{(k)}u(t) = T_\alpha t^k$ $(k \geq 1)$ and $\delta_\alpha^{(k)}\tilde{u}(w) = T_\alpha w^k$ $(k \geq 0)$. Correspondingly, Eqs. (5.1) and (5.3) give, respectively,

$$\tilde{\delta}_\alpha^{(k)} F(t)F(t)^{-1} = -\frac{t}{2\pi i}\int_L \frac{s^k F(s)T_\alpha F(s)^{-1}}{s(s-t)}ds;$$

$$\delta_\alpha^{(k)}\tilde{F}(w)\tilde{F}(w)^{-1} = -\frac{1}{2\pi i}\int_{\tilde{L}} \frac{s^k \tilde{F}(s)T_\alpha \tilde{F}(s)^{-1}}{(s-w)}ds.$$

To obtain explicit expressions of the corresponding parameterized cross infinitesimal transformations $\tilde{\delta}_\alpha(w)F(t) = \sum_{k=1}^\infty w^k \tilde{\delta}_\alpha^{(k)}F(t)$ and $\delta_\alpha(t)\tilde{F}(w) = \sum_{k=0}^\infty t^k \delta_\alpha^{(k)}\tilde{F}(w)$, we note that since $F(t)$, $\tilde{F}(w)$ have different analytic properties, from (2.14), (2.15), (2.16), (2.22), and (2.23) we can set

$$\tilde{F}(w)|_{w=1/t} = t^{1/2}F(t), \qquad F(t)|_{t=1/w} = w^{1/2}\tilde{F}(w). \quad (5.5)$$

Thus we have

$$\tilde{\delta}_{\alpha}(w)F(t)F(t)^{-1} = -\frac{tw}{2\pi i}\int_L \frac{F(s)T_\alpha F(s)^{-1}}{(1-sw)(s-t)}ds = \frac{tw}{1-tw}[\tilde{F}(w)T_\alpha\tilde{F}(w)^{-1} - F(t)T_\alpha F(t)^{-1}]; \tag{5.6}$$

$$\delta_\alpha(t)\tilde{F}(w)\tilde{F}(w)^{-1} = -\frac{1}{2\pi i}\int_{\tilde{L}}\frac{\tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}}{(1-ts)(s-w)}ds = \frac{1}{1-tw}[F(t)T_\alpha F(t)^{-1} - \tilde{F}(w)T_\alpha\tilde{F}(w)^{-1}]. \tag{5.7}$$

Next we consider the cases $u(t) = I$ and $\tilde{u}(w) = I$ of the transformations (4.2) and (4.4). We first calculate infinitesimal transformations brought about by

$$v(t) = t + \Delta(t), \tag{5.8a}$$

$$\tilde{v}(w) = w + \tilde{\Delta}(w), \tag{5.8b}$$

where $\Delta(t)$ and $\tilde{\Delta}(w)$ are infinitesimal functions of t and w, respectively. For (5.8a), we have

$$\underline{F}(t) = F(t) + \Delta F(t),$$
$$F(v(t))^{-1} = F(t)^{-1} + \partial_t[F(t)^{-1}]\Delta(t). \tag{5.9}$$

Substituting $u(t) = I$ and (5.9) into (4.2), we obtain

$$\Delta F(t)F(t)^{-1} = \frac{t}{2\pi i}\int_L \frac{\dot{F}(s)F(s)^{-1}}{s(s-t)}\Delta(s)ds, \qquad t\in L_+. \tag{5.10}$$

Noticing the properties of $v(t)$, without loss of generality, we select $\Delta(s) = \Delta_\sigma^{(k)}(s) = \sigma s^{1-k}$ $(k\geq 0)$ and denote the corresponding $\Delta F(t)$ by $\Delta_\sigma^{(k)}F(t)$, where σ is an infinitesimal real constant; then from (5.10), the parameterized transformation $\Delta_\sigma(t')F(t) = \sum_{k=0}^\infty t'^k\Delta_\sigma^{(k)}F(t)$ $(t'\in L_+)$ is given by

$$\Delta_\sigma(t')F(t)F(t)^{-1} = \frac{\sigma t}{2\pi i}\int_L\frac{s\dot{F}(s)F(s)^{-1}}{(s-t')(s-t)}ds = \frac{\sigma t}{t-t'}[t\dot{F}(t)F(t)^{-1} - t'\dot{F}(t')F(t')^{-1}]. \tag{5.11}$$

Similarly, for (4.4) and (5.8b), we obtain

$$\tilde{\Delta}\tilde{F}(w)\tilde{F}(w)^{-1} = \frac{1}{2\pi i}\int_{\tilde{L}}\frac{\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}}{(s-w)}\tilde{\Delta}(s)ds, \qquad w\in\tilde{L}_+. \tag{5.12}$$

Selecting $\tilde{\Delta}(s) = \tilde{\Delta}_\epsilon^{(k)}(s) = \epsilon s^{1-k}$ $(k\geq 1)$ and denoting the corresponding $\tilde{\Delta}\tilde{F}(w)$ by $\tilde{\Delta}_\epsilon^{(k)}\tilde{F}(w)$ (ϵ is an infinitesimal real constant), then from (5.12), the parameterized transformation $\tilde{\Delta}_\epsilon(w')\tilde{F}(w) = \sum_{k=1}^\infty w'^k\tilde{\Delta}_\epsilon^{(k)}\tilde{F}(w)$ $(w'\in\tilde{L}_+)$ is given by

$$\tilde{\Delta}_\epsilon(w')\tilde{F}(w)\tilde{F}(w)^{-1} = \frac{\epsilon w'}{w-w'}[w\dot{\tilde{F}}(w)\tilde{F}(w)^{-1} - w'\dot{\tilde{F}}(w')\tilde{F}(w')^{-1}]. \tag{5.13}$$

As before, we also need cross infinitesimal transformations $\tilde{\Delta}F(t)$ and $\Delta\tilde{F}(w)$ brought about by variations $v(t) = t + \tilde{\Delta}(t)$, $\tilde{v}(w) = w + \Delta(w)$. In these cases, the RH transformations (4.2) and (4.4) give

$$\tilde{\Delta}F(t)F(t)^{-1} = \frac{t}{2\pi i}\int_L\frac{\dot{F}(s)F(s)^{-1}}{s(s-t)}\tilde{\Delta}(s)ds; \tag{5.14}$$

$$\Delta\tilde{F}(w)\tilde{F}(w)^{-1} = \frac{1}{2\pi i}\int_{\tilde{L}}\frac{\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}}{(s-w)}\Delta(s)ds. \tag{5.15}$$

We select $\tilde{\Delta}(t) = \tilde{\Delta}_\epsilon^{(k)}(t) = -\epsilon t^{1+k}$ $(k\geq 1)$, $\Delta(w) = \Delta_\sigma^{(k)}(w) = -\sigma w^{1+k}$ $(k\geq 0)$ and denote the corresponding transformations by $\tilde{\Delta}_\epsilon^{(k)}F(t)$, $\Delta_\sigma^{(k)}\tilde{F}(w)$, respectively. In addition, from relations in (5.5), we have

$$[\dot{\tilde{F}}(w)\tilde{F}(w)^{-1}]|_{w=1/t} = -t\left[t\dot{F}(t)F(t)^{-1} + \frac{1}{2}\right];$$
$$[\dot{F}(t)F(t)^{-1}]|_{t=1/w} = -w\left[w\dot{\tilde{F}}(w)\tilde{F}(w)^{-1} + \frac{1}{2}\right]. \tag{5.16}$$

Thus the parameterized transformations $\tilde{\Delta}_\epsilon(w)F(t) = \sum_{k=1}^\infty w^k\tilde{\Delta}_\epsilon^{(k)}F(t)$ and $\Delta_\sigma(t)\tilde{F}(w) = \sum_{k=0}^\infty t^k\Delta_\sigma^{(k)}\tilde{F}(w)$ are given, respectively, by

$$\tilde{\Delta}_\epsilon(w)F(t)F(t)^{-1} = -\frac{\epsilon tw}{2\pi i}\int_L\frac{s\dot{F}(s)F(s)^{-1}}{(1-sw)(s-t)}ds = \frac{\epsilon tw}{tw-1}\left[t\dot{F}(t)F(t)^{-1} + w\dot{\tilde{F}}(w)\tilde{F}(w)^{-1} + \frac{1}{2}\right]; \tag{5.17}$$

$$\Delta_\sigma(t)\tilde{F}(w)\tilde{F}(w)^{-1} = -\frac{\sigma}{2\pi i}\int_{\tilde{L}}\frac{s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}}{(1-st)(s-w)}ds = \frac{\sigma}{tw-1}\left[w\dot{\tilde{F}}(w)\tilde{F}(w)^{-1} + t\dot{F}(t)F(t)^{-1} + \frac{1}{2}\right]. \tag{5.18}$$

Finally, we give the corresponding infinitesimal transformations of ρ, z. Writing

$$\eta_\pm := z\pm i\rho, \qquad \eta'_\pm := z'\pm i\rho',$$
$$\eta''_\pm := z''\pm i\rho'', \qquad t_\pm := (2\eta_\pm)^{-1},$$
$$w_\pm := 2\eta_\pm, \tag{5.19}$$

then from (3.3a) and (5.8a), we have

$$\Delta_\sigma^{(k)}\eta_\pm = (\eta'_\pm - \eta_\pm)^{(k)}_{(\sigma)} = \frac{1}{2t_\pm^2}\Delta_\sigma^{(k)}(t_\pm) = \sigma\eta_\pm(2\eta_\pm)^k, \quad k \geq 0,$$
$$\Delta_\sigma(t)\eta_\pm = \sum_{k=0}^\infty t^k\Delta_\sigma^{(k)}\eta_\pm = \sigma\eta_\pm\sum_{k=0}^\infty t^k(2\eta_\pm)^k = \sigma\frac{\eta_\pm}{1-2t\eta_\pm};$$

therefore

$$\Delta_\sigma(t)z = \frac{\sigma}{\lambda(t)^2}[z(1-2tz) - 2t\rho^2], \quad \Delta_\sigma(t)\rho = \frac{\sigma}{\lambda(t)^2}\rho. \tag{5.20}$$

Similarly, from (3.20a), (5.8b), and (5.19), we obtain

$$\tilde{\Delta}_\epsilon^{(k)}\eta_\pm = (\eta''_\pm - \eta_\pm)^{(k)}_{(\epsilon)} = -\frac{1}{2}\tilde{\Delta}_\epsilon^{(k)}(w_\pm) = -\frac{\epsilon}{2}w_\pm^{1-k} = -\epsilon\eta_\pm(2\eta_\pm)^{-k}, \qquad k \geq 1,$$
$$\tilde{\Delta}_\epsilon(w)\eta_\pm = \sum_{k=1}^\infty w^k\tilde{\Delta}_\epsilon^{(k)}\eta_\pm = -\epsilon\eta_\pm\sum_{k=1}^\infty w^k(2\eta_\pm)^{-k} = \epsilon w\frac{\eta_\pm}{w-2\eta_\pm};$$

thus

$$\tilde{\Delta}_\epsilon(w)z = \frac{\epsilon w}{\tilde{\lambda}(w)^2}[z(w-2z) - 2\rho^2], \quad \tilde{\Delta}_\epsilon(w)\rho = \frac{\epsilon w^2}{\tilde{\lambda}(w)^2}\rho. \tag{5.21}$$

Equations (5.2), (5.4), (5.6), (5.7), (5.11), (5.13), (5.17), and (5.18) give exactly the same infinitesimal transformations of $F(t)$, $\tilde{F}(w)$ and the associated H potentials [from Eq. (2.15)] as constructed in [22], while (5.20) and (5.21) give the same infinitesimal transformations of ρ, z as in [22]. These results demonstrate that the pair of RH transformations in this paper provide exponentiations of all the infinitesimal symmetry transformations given in [22].

VI. SUMMARY AND DISCUSSION

The symmetry structures of the dimensionally reduced heterotic string theory are studied further. We construct a pair of RH transformations (3.15) and (3.30) [or equivalently (4.2) and (4.4)] relating to the pair of HE-type linear systems (2.14), (2.15), (2.16), (2.22), and (2.23). These RH transformations generate new F and H potentials from old ones and give an infinite-dimensional symmetry transformation group of the considered heterotic string theory. This symmetry group is verified to have the structure of the semidirect product of the complete Kac-Moody group $\widehat{O(d, d+n)}$ and Virasoro group. [However, the transformation (3.15)—or equivalently (4.2)—gives the "positive half" symmetry subgroup only.] Moreover, we find that the infinitesimal forms of these RH transformations give exactly the same as the infinitesimal symmetry transformations in [22]; these show that the RH transformations in the present paper provide us with exponentiations of all infinitesimal symmetries in [22]. Of course, the RH transformations constructed here give out symmetry transformations in finite form, which are more important and useful for theoretic studies and new solution generation, etc.

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