Gravity of magnetic stresses and energy

Giuseppe Bimonte, Enrico Calloni, and Luigi Rosa

Dipartimento di Scienze Fisiche, Università di Napoli Federico II, Complesso Universitario MSA, Via Cintia I-80126 Napoli, Italy
and INFN, Sezione di Napoli, Napoli, Italy

In the framework of designing laboratory tests of relativistic gravity, we investigate the gravitational field produced by the magnetic field of a solenoid. Observing this field might provide a means of testing whether stresses gravitate as predicted by Einstein's theory. A previous study of this problem by Braginsky, Caves, and Thorne predicted that the contribution to the gravitational field resulting from the stresses of the magnetic field and of the solenoid walls would cancel the gravitational field produced by the mass-energy of the magnetic field, resulting in a null magnetically generated gravitational force outside the solenoid. They claim that this null result, once proved experimentally, would demonstrate the stress contribution to gravity. We show that this result is incorrect, as it arises from an incomplete analysis of the stresses, which neglects the axial stresses in the walls. Once the stresses are properly evaluated, we find that the gravitational field outside a long solenoid is in fact independent of Maxwell and material stresses, and it coincides with the Newtonian field produced by the linear mass distribution equivalent to the density of magnetic energy stored in a unit length of the solenoid. We argue that the gravity of Maxwell stress can be directly measured in the vacuum region inside the solenoid, where the Newtonian noise is absent in principle, and the gravity generated by Maxwell stresses is not screened by the negative gravity of magnetic-induced stresses in the solenoid walls.

I. INTRODUCTION

According to general relativity, material and field stresses are sources of gravity because the active gravitational mass density, in the relativistic analogue of Poisson's equation, is proportional to $\rho + T$, where ρ is the density of energy and T is the trace of the stresses [1]. Stress-generated gravity is very important in a number of problems. For example, in astrophysics it affects the maximum mass of neutron stars, but if one intends it, in a broad sense, as the gravity produced by the spatial components of the momentum-energy tensor, it displays its full power in cosmology, where it may well be responsible for the recently discovered accelerated expansion of the Universe [2].

As of today, there exists no direct experimental proof that stresses indeed gravitate, and it is clearly of great interest to investigate the possibility of a *laboratory* experiment to test this prediction of general relativity. Unfortunately, this is very difficult because in ordinary material bodies, of a size that can be handled in a laboratory, the trace of stresses is many orders of magnitude smaller than the energy density associated with the mass density of the body, and therefore its effects are negligible. However, it was realized 30 years ago [3] that a possible way to circumvent this difficulty is by observing the gravity of magnetic fields, which one expects to exist because in general relativity all forms of energy (and stresses) are sources of gravity. Magnetic fields are interesting in this respect because, according to Maxwell theory, the energy density of a magnetic field has the same magnitude as the trace of the Maxwell stress tensor and therefore this type of experiment may provide an excellent tool to probe the

gravity of stresses. With this purpose, the authors of Ref. [3] considered a simple setup, in which the gravity produced by the magnetic field of a long solenoid would be measured by means of a torsion balance, having one of its test masses near the solenoid. Of course, the difficulty of the experiment is due to the fact that magnetically generated gravity is very weak, for experimentally attainable magnetic fields. To get an estimate of the required magnetic fields and balance sensitivity, one may temporarily neglect all stresses and assume, on the basis of the equivalence between mass and energy, that the magnetically generated gravitational field near a long solenoid is the same as that of a cylindrical road, with a linear mass density equal to the magnetic energy (divided by the square of the speed of light c) stored in a unit length of the solenoid. Even for very strong magnetic fields, the effect is very small, if one considers that the mass density equivalent to the energy density of a magnetic field of 10^5 G is as small as 4.4×10^{-13} g cm^{-3}. However, it was argued in [3] that the demands of the experiment could have soon been met, imagining realistic improvements of the technology available in the seventies, in cryogenic low-noise torque balances and superconducting solenoids.

When considering the effect of stresses, one notices that two types of stresses may contribute to the gravitational field of the solenoid: Maxwell stresses of the magnetic field and material stresses that build up in the walls of the solenoid in response to the applied magnetic field. In Ref [3] it was correctly stated that the walls of the solenoid can be considered to be in instantaneous mechanical equilibrium, because in the considered setup the modulation frequency of the magnetic field is extremely low (around

10^{-3} Hz, which represents the typical resonance frequency of a torque balance). The conclusion drawn in [3] was that the inclusion of stresses would lead to a null magnetically generated gravitational force (apart from the Newtonian noise caused by the stress-induced modulation of the mass density of the solenoid walls), because of a purported cancellation occurring between the gravity of stresses and the gravity of magnetic energy.

This result appears suspicious, from the point of view of a well-known paradox, that was pointed out long ago by Tolman [4] in his investigations on the role of stresses as source of gravity. Tolman found the paradox while considering the gravitational field of a static spherical impermeable box filled with a fluid, which undergoes a spherically symmetric transformation that conserves the total energy, but causes a change of pressure, like matter and antimatter annihilating into radiation. One may think that, since the total energy of the system is preserved, the change in pressure determines a change in the active gravitational mass of the box, and a consequent change in the gravitational field outside the box. However, this inference is in contradiction with Birkhoff's theorem, which states that the external gravitational field of a spherically symmetric body is static and therefore it is insensitive to whatever spherically symmetric transformations may occur inside the box. The Tolman paradox was investigated in [5], where the crucial role of the walls that keep the fluid confined was realized. It was shown there that the stresses that build up in the walls in response to the transformation give a negative contribution to the active gravitational mass of the system that just compensates the pressure contribution from the fluid inside, resulting in an overall unchanged total gravitational mass across the transformation, as expected from Birkhoff's theorem. The same problem has been investigated again in a recent paper [6], leading to analogous conclusions. (The key role of the stresses in the walls bounding a relativistic gravitating systems has been discussed by us very recently, in connection with the problem of determining the weight of a Casimir apparatus in a weak gravitational field [7].) The general lesson that one learns from these studies is that the gravitational field *outside* a spherical body is independent of the stresses in its interior, and it is determined solely by the mass-energy content of the body. Since there is no reason to imagine that this is true only for the spherical case, one is led to suspect that the results of [3] may not be correct. This motivated us to reconsider in detail the analysis of [3], and we present here our findings. We realized that the null result found in [3] was determined by a mistaken evaluation of the stresses that build up inside the solenoid walls when the magnetic field is present. In particular, the authors overlooked the axial stresses that arise in response to the axial electrodynamic compression of the solenoid. Besides leading to an incorrect result for the magnetically generated gravitational field outside the solenoid, this error led the authors to

overlook the large Newtonian noise originating from magnetic-induced changes in the length of the solenoid.

After stresses are properly accounted for, our analysis shows, in a general way, that the total magnetically generated gravitational mass, measured far from the solenoid, is independent of the stresses and is just equal to the total magnetic energy (divided by the square of the speed of light c^2), in accordance with one's intuition and in agreement with earlier studies on the Tolman paradox. We then consider the field near a long solenoid, and we show that the magnetically generated gravitational field is different from zero, and as expected it is equivalent to the Newtonian field generated by a linear mass density that is equal to the instantaneous magnetic energy per unit length (divided by c^2) stored in the solenoid. Since the near field outside the solenoid, like the far field, is independent of the stresses, we conclude that observation of the external field cannot be used to test the gravity of stresses. Moreover, measuring this magnetically generated field will be very hard, because we estimate that magnetic-induced changes in the length of the solenoid produce a Newtonian noise that is many order of magnitudes larger than the magnetically generated gravity. This by no means implies, however, that the gravity of stresses is not observable in this setup, because in the vacuum region *inside* the solenoid the gravity produced by Maxwell stresses is not screened by the negative gravity of the material stresses in the walls, and therefore it contributes to the field as much as the density of magnetic energy. Moreover, it is expected that the Newtonian noise will be much less of a problem, because in the ideal case of a infinitely long and perfectly axially symmetric solenoid, Newtonian noise inside the solenoid is strictly zero.

The paper is organized as follows: in Sec. II we derive, within linearized theory for general relativity, the magnetically generated gravitational pull exerted on a test particle by a solenoid carrying a quasistatic magnetic field. In Sec. III we analyze in detail the contributions from Maxwell and material stresses and we prove that outside the solenoid they cancel each other. Section IV deals with the problem of Newtonian noise, while Sec. V contains a discussion of the results and our conclusions. Finally, in the Appendix we provide explicit formulas for the material stresses that build up within the walls of an idealized solenoid.

II. GRAVITY OF A QUASISTATIC MAGNETIC FIELD

In this section we estimate the pull F_i exerted on a test particle at rest, by the magnetically generated gravitational field of a solenoid $\mathcal{S}$, producing a quasistatic magnetic field B. Since the gravitational fields involved are extremely small, nonlinear effects are negligible and we can safely study the problem using the simple linearized theory for Einstein's general relativity [1]. In this approximation, the

gravitational field $g_{\mu\nu}$ is written as [8]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad (2.1)$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-c^2, 1, 1, 1\}$ is the flat Minkowski metric, and $h_{\mu\nu}$ represents a weak gravitational field. We further split $h_{\mu\nu}$ as

$$h_{\mu\nu} = h_{\mu\nu}|_{B=0} + \gamma_{\mu\nu}, \qquad (2.2)$$

where $h_{\mu\nu}|_{B=0}$ is the field that exists when the solenoid is turned off, while $\gamma_{\mu\nu}$ is the magnetically generated field that is present when the magnetic field B is turned on. The field $h_{\mu\nu}|_{B=0}$ includes the background gravitational existing in the laboratory, together with the small field generated by the walls of the solenoid when no currents flow in it.

To linear order, the pull F_i on a test particle of mass m arising from the magnetically-generated gravitational field is

$$F_i = -m(\Gamma^i_{00} - \Gamma^i_{00}|_{B=0}) = \tfrac{1}{2}m\partial_i\gamma_{00}, \qquad (2.3)$$

where Γ^i_{00} are Christoffel symbols. For a quasistatic magnetic field, linearized theory gives the following equations for $\gamma_{\mu\nu}$:

$$\triangle \bar{\gamma}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \qquad (2.4)$$

In these equations, $\triangle$ denotes the flat space-time Laplacian, and $\bar{\gamma}_{\mu\nu}$ is the field

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\gamma, \qquad (2.5)$$

where $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$. The above equations have to be supplemented by the Lorenz gauge conditions, which for a static field imply

$$\partial_i\bar{\gamma}^i_\mu = 0. \qquad (2.6)$$

It is important to bear in mind that, according to the definition of $\gamma_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$ appearing on the right-hand side (r.h.s.) of Eq. (2.4) represents the sole contribution to the total energy-momentum tensor that arises when the magnetic field is turned on. The solenoid being at rest, and the magnetic field being quasistatic, the nonvanishing components of $T^{\mu\nu}$ read

$$T^{00} = \delta\rho_{\text{walls}} + \mathcal{E}_{\text{mag}}/c^2, \qquad (2.7)$$

$$T^{ij} = T^{ij}_{\text{walls}} + T^{ij}_{\text{mag}}. \qquad (2.8)$$

In the above equations, $\delta\rho_{\text{walls}}$ represents the change in the (classical) mass density of the solenoid walls resulting from possible deformations of the solenoid determined by the magnetic field [9], while T^{ij}_{walls} denotes the extra mechanical stresses that build up within the solenoid walls when the field is turned on. Note that T^{ij}_{walls} does not include the mechanical stresses resulting from the weight of the

solenoid and from the external forces exerted on the solenoid walls by the mounts that hold it. Finally, $\mathcal{E}_{\text{mag}} = B^2/(8\pi)$ denotes the density of magnetic energy, while T^{ij}_{mag} is the Maxwell tensor:

$$T^{ij}_{\text{mag}} = \frac{1}{4\pi}\left(\frac{1}{2}B^2\delta^{ij} - B^iB^j\right). \qquad (2.9)$$

Upon solving Eq. (2.4) it is easy to obtain for the pull F_i the following expression:

$$F_i = -m\partial_i(\delta\Phi_{\text{walls}} + \psi), \qquad (2.10)$$

where

$$\delta\Phi_{\text{walls}} = -G\int d^3\mathbf{y}\,\frac{\delta\rho_{\text{walls}}}{|\mathbf{x}-\mathbf{y}|}, \qquad (2.11)$$

and

$$\psi = -G\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\left(\frac{\mathcal{E}_{\text{mag}} + T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}}{c^2}\right). \qquad (2.12)$$

Of the two terms appearing on the r.h.s. of Eq. (2.10), that involving $\delta\Phi_{\text{walls}}$ just represents a purely classical "Newtonian noise," and we postpone to Sec. V a discussion of its consequences. The interesting term for us is the contribution proportional to ψ, that represents the magnetically generated gravitational field. We see that ψ coincides with the classical gravitational field generated by and effective mass distribution ρ_{eff} equal to

$$\rho_{\text{eff}} = \frac{1}{c^2}(\mathcal{E}_{\text{mag}} + T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}). \qquad (2.13)$$

This is a rather complicated formula, for it involves the trace of the stresses T^{ij}_{walls} in the solenoid walls. It is convenient to define the total "effective gravitational mass" M_{eff} as the integral over all space of ρ_{eff}:

$$M_{\text{eff}} = \int_{\text{all space}} d^3x\rho_{\text{eff}}. \qquad (2.14)$$

We split M_{eff} as

$$M_{\text{eff}} = M_{\text{mag en}} + M_{\text{str}}, \qquad (2.15)$$

where $M_{\text{mag en}}$ is the mass associated with the total magnetic energy E_{mag}

$$M_{\text{mag en}} = \frac{1}{c^2}\int_{\text{all space}} d^3\mathbf{x}\mathcal{E}_{\text{mag}} \equiv \frac{E_{\text{mag}}}{c^2}, \qquad (2.16)$$

while M_{str} is associated with Maxwell and material stresses:

$$M_{\text{str}} \equiv \frac{1}{c^2}\int_{\text{all space}} d^3\mathbf{x}(T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}). \qquad (2.17)$$

Note that both the integrals for $M_{\text{mag en}}$ and M_{str} exist, because at large distances R from the solenoid, the magnetic field falls off like R^{-3} and then $\mathcal{E}_{\text{mag}}$ and T^{ij}_{mag} both decay as R^{-6}. The existence of a contribution to M_{eff}, such

as $M_{\text{mag en}}$, arising from the magnetic energy is not surprising in view of the equivalence between energy and mass, established in the theory of special relativity. On the contrary, the contribution M_{str} from the stresses represents a true general relativistic effect. In the next section it will be proven that M_{str} is always zero, at mechanical equilibrium.

III. CONTRIBUTION FROM STRESSES

To be definite, we imagine that the solenoid $\mathcal{S}$ is hanging by a suitable set of threads, and that apart from the suspension points its surface is free. Now, upon taking the spatial divergence of both sides of Eq. (2.4), and then using the gauge condition Eq. (2.6), we obtain

$$\partial_i(T^{ij}_{\text{walls}} + T^{ij}_{\text{mag}}) = 0. \quad (3.1)$$

Outside the solenoid walls, where $T^{ij}_{\text{walls}} = 0$, the above equations are satisfied as a consequence of the static Maxwell equations in vacuum:

$$\mathbf{\nabla} \cdot \mathbf{B} = \mathbf{0}, \qquad \mathbf{\nabla} \times \mathbf{B} = \mathbf{0}. \quad (3.2)$$

Inside the solenoid walls, instead, Eq. (3.1) expresses the local balance between electrodynamic forces and material stresses, at mechanical equilibrium. At points on the boundary $\partial\mathcal{S}$ of the solenoid walls, Eq. (3.1) must be supplemented by the following boundary condition

$$n_i(T^{ij}_{\text{walls}} + T^{ij}_{\text{mag}})|_{\text{in}} = n_i T^{ij}_{\text{mag}}|_{\text{out}}, \quad (3.3)$$

where n^i is the normal to the surface of the solenoid walls, oriented outwards the solenoid, and the suffixes in (out) denote the values of the fields immediately inside (outside) the solenoid walls. Equation (3.3) expresses the fact that the total electrodynamic self-force on the solenoid is zero, and therefore the threads that support it do not apply any extra force when the magnetic field is turned on. Using Eq. (3.1) and the boundary condition Eq. (3.3), we can now show that M_{str} is always zero. For this purpose, we note that at all points not lying on the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, Eq. (3.1) implies the identity

$$(T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}) = \partial_j[(T^{ij}_{\text{walls}} + T^{ij}_{\text{mag}})x^i]. \quad (3.4)$$

Upon substituting this expression for $T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}$ into the r.h.s. of Eq. (2.17), and then performing the integral of the total divergence by Gauss theorem, we obtain for M_{str} the expression

$$M_{\text{str}} = \int_{\partial\mathcal{S}} d^2\sigma x^j n^i[(T^{ij}_{\text{walls}} + T^{ij}_{\text{mag}})|_{\text{in}} - T^{ij}_{\text{mag}}|_{\text{out}}]$$
$$+ \lim_{R\to\infty} \int_{S_R} d^2\sigma T^{ij}_{\text{mag}} x^j n^i, \quad (3.5)$$

where S_R denotes a two-sphere of radius R centered at any point inside the solenoid. Now, the first integral on the r.h.s. is zero because of the boundary condition Eq. (3.3), and the second vanishes because $T^{ij}_{\text{mag}} n^i x^j$ falls off as R^{-5}.

Therefore, as promised, we obtain

$$M_{\text{str}} = 0. \quad (3.6)$$

The conclusion is that, independently of the shape of the solenoid and of the detailed distribution of the stresses inside its walls, the general conditions of mechanical equilibrium as encoded in Eqs. (3.1) and (3.3) imply that the combined contribution of Maxwell and material stresses to the *total* gravitational mass of the solenoid vanishes. Therefore, the total effective gravitational mass associated with the magnetic field is equal to $M_{\text{mag en}}$:

$$M_{\text{eff}} = M_{\text{mag en}}. \quad (3.7)$$

The gravitational field that is observed far from the solenoid when the magnetic field is turned on is then equal to that of a point charge with mass $M_{\text{mag en}}$, placed at the position of the solenoid.

Obviously, Eq. (3.6) does not imply that the magnetically generated stresses produce no gravity at all, because it only states that Maxwell and material stresses cancel each other *on average*, namely, after integrating over all space. While this is sufficient to conclude that stresses do not contribute to the far field, it still remains a possibility that stresses produce significant gravitational effects in the vicinity of the solenoid, because the near field depends on the detailed *spatial* distribution of the stresses. The study of the near field is clearly much more complicated in general, because it requires a detailed determination of the mechanical stresses inside the walls of the solenoid. The study of the stresses that arises in a solenoid generating a strong magnetic field has received much attention in the literature over the years, in view of its great practical importance (see, for example, Ref. [10] and references therein), and in general it is a difficult problem that involves making a definite model for the constitutive equations characterizing the material, and it usually requires numerical tools. We shall not discuss this difficult problem here, and we are content with a few simple considerations that can be drawn on the basis of general mechanical equations, without any consideration of specific constitutive equations. To simplify the problem, we consider below a very long cylindrical solenoid and we discuss separately the gravitational field outside and inside the solenoid.

A. External near field

We consider, as in Ref. [3], a very long cylindrical solenoid, constituted by a (nonmagnetic) pipe with inner and outer radii R_1 and R_2, respectively, and length $L \gg R_2$. We suppose for simplicity that the electric current producing the magnetic field flows along the inner surface of the pipe, in the positive azimuthal direction, and that it has a uniform surface density j. We let $\{x, y, z\}$ represent a Cartesian coordinate system whose z axis coincides with the solenoid axis, and whose origin lies at the center of the

solenoid, and we let $r = \sqrt{x^2 + y^2}$ be the distance from the solenoid axis.

Axial symmetry obviously implies that the effective mass density ρ_{eff} in Eq. (2.13) is a function only of r and z. As a first step, we show that ρ_{eff} is significantly different from zero only inside the solenoid, i.e., for $r \leq R_2$ and $|z| \leq L/2$. This is obvious for the contribution to ρ_{eff} arising from the material stresses, because T^{ij}_{walls} vanish outside the solenoid walls. Then, upon noting that

$$T^{ii}_{\text{mag}} = \mathcal{E}_{\text{mag}}, \tag{3.8}$$

as can be seen by taking the trace of the Maxwell stresses in Eq. (2.9), we see that the contribution to ρ_{eff} arising from the magnetic field is equal to twice $\mathcal{E}_{\text{mag}}/c^2$. We can estimate the integral of $\mathcal{E}_{\text{mag}}$ outside the solenoid as follows: the external magnetic field coincides with the field of a cylindrical magnet having length L and radius R_1, carrying a uniform magnetization $m = j/c$ along the positive z direction. The field of such a magnet coincides with the sum of the fields $\mathbf{B}_1$ and $\mathbf{B}_2$ produced by the opposite surface distributions of magnetic charges on the opposite caps of the magnet (at $z = \pm L/2$), with uniform surface densities $\sigma_m = \pm j/c$. The total energy $E^{\text{ext}}_{\text{mag}}$ of the external field can then be estimated to be

$$E^{\text{ext}}_{\text{mag}} = \frac{1}{8\pi} \int_{\text{out}} d^3x (B_1^2 + B_2^2) + \frac{1}{4\pi} \int_{\text{out}} d^3x \mathbf{B}_1 \cdot \mathbf{B}_2. \tag{3.9}$$

The first integral on the r.h.s. of the above equation represents the sum of the magnetic energies of two isolated pole distributions at $z = \pm L/2$. Therefore, it is independent of the solenoid length L, and on dimensional grounds one expects it to be of the form

$$\frac{1}{8\pi} \int_{\text{out}} d^3x (B_1^2 + B_2^2) = \frac{B_{\text{in}}^2}{8\pi} 2AR_1^3, \tag{3.10}$$

where $B_{\text{in}} = 4\pi j/c$ is the magnetic field inside the solenoid and A is some dimensionless numerical constant. As for the second integral on the r.h.s of Eq. (3.9), it represents the interaction energy among the two poles of the magnet, and it can be approximated as the interaction energy of two opposite pointlike magnetic charges of magnitude $q_m = \pi R_1^2 j/c$ at distance L:

$$\frac{1}{4\pi} \int_{\text{out}} d^3x \mathbf{B}_1 \cdot \mathbf{B}_2 \simeq -\frac{q_m^2}{L} = -\frac{2\pi^2 j^2 R_1^4}{c^2 L} = -\frac{B_{\text{in}}^2 R_1^4}{8L}. \tag{3.11}$$

Adding up Eqs. (3.10) and (3.11), we obtain for $E^{\text{ext}}_{\text{mag}}$ the expression

$$E^{\text{ext}}_{\text{mag}} \simeq \left(2A - \frac{\pi R_1}{L}\right) \frac{B_{\text{in}}^2}{8\pi} R_1^3. \tag{3.12}$$

On the other hand, the internal magnetic energy $E^{\text{int}}_{\text{mag}}$ can be estimated to be

$$E^{\text{int}}_{\text{mag}} = \frac{B_{\text{in}}^2}{8\pi} \times \pi R_1^2 L, \tag{3.13}$$

and therefore we obtain for the ratio of $E^{\text{ext}}_{\text{mag}}/E^{\text{int}}_{\text{mag}}$ the estimate

$$\frac{E^{\text{ext}}_{\text{mag}}}{E^{\text{int}}_{\text{mag}}} = \frac{2A}{\pi} \frac{R_1}{L} - \left(\frac{R_1}{L}\right)^2, \tag{3.14}$$

which shows that $E^{\text{ext}}_{\text{mag}}$ becomes negligible with respect to $E^{\text{int}}_{\text{mag}}$ for $R_1/L \ll 1$.

Consider now a point P in the vicinity of the solenoid, but far from its ends. The above estimation of the external magnetic stresses and energy shows that the magnetically generated gravitational field at P is determined by the stresses and the magnetic energy that are present inside the solenoid and within its material walls. Since far from the solenoid's ends the magnetic field and the material stresses are approximately independent of the z coordinate, we see from Eq. (2.12) that the field ψ at P coincides with the classical field of an infinite cylindrical rod, with a uniform linear mass density σ_{eff} equal to

$$\sigma_{\text{eff}} = \frac{1}{c^2} \int_0^{R_2} dr 2\pi r (\mathcal{E}_{\text{mag}} + T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}). \tag{3.15}$$

Now, we can split σ_{eff} analogously to what we did with M_{eff} in Eq. (2.15):

$$\sigma_{\text{eff}} = \sigma_{\text{mag en}} + \sigma_{\text{str}}, \tag{3.16}$$

where

$$\sigma_{\text{mag en}} = \frac{1}{c^2} \int_0^{R_2} dr 2\pi r \mathcal{E}_{\text{mag}} \equiv \frac{\tilde{\mathcal{E}}_{\text{mag}}}{c^2}, \tag{3.17}$$

with $\tilde{\mathcal{E}}_{\text{mag}}$ the magnetic energy per unit length of the solenoid, and

$$\sigma_{\text{str}} = \frac{1}{c^2} \int_0^{R_2} dr 2\pi r (T^{ii}_{\text{walls}} + T^{ii}_{\text{mag}}). \tag{3.18}$$

We can easily see that σ_{str} vanishes. Indeed, neglecting the contributions to M_{str} from the external magnetic field, which we have seen to be small, as well as the contribution from the small region near the solenoid's ends, we can then express M_{str} as

$$M_{\text{str}} = L\sigma_{\text{str}}. \tag{3.19}$$

Since according to Eq. (3.6) M_{str} is zero, it follows at once

$$\sigma_{\text{str}} = 0. \tag{3.20}$$

We conclude that also near the solenoid the magnetically generated gravitational field ψ is independent of the stresses, and it simply coincides with the field generated by a cylindrical distribution of mass, having a linear density that is equal to the instantaneous magnetic energy stored in the solenoid (divided by c^2) per unit length:

$$\sigma_{\rm eff} = \frac{\tilde{\mathcal{E}}_{\rm mag}}{c^2}. \tag{3.21}$$

These results are in sharp contrast with the findings of Ref. [3], where it was concluded that the contribution from Maxwell and material stresses is *different from zero*, and of such a magnitude as to cancel the gravitational field produced by the mass energy of the magnetic field, resulting in a null magnetically generated gravitational field ψ outside the solenoid. A detailed analysis of the sketchy computations in [3] shows that this incorrect conclusion arose from an incomplete evaluation of the material stresses that build up inside the solenoid walls, as the authors only considered the effect of the radial electrodynamic forces pushing to increase the radius of the solenoid, but they overlooked the existence of an axial force tending to compress the solenoid [11]. When the contribution from the axial stresses is accounted for, our result Eq. (3.21) is recovered. As a further check of the fundamental Eq. (3.20), in the Appendix we provide the explicit formulas for the material stresses that build up within the walls of an idealized solenoid.

B. Internal field

We consider now the gravitational field in the vacuum region in the *interior* of the solenoid, i.e., for $r < R_1$. Since $T^{ij}_{\rm walls}$ is zero for $r < R_1$, the field ψ coincides with the classical potential generated by a linear mass density $\sigma^{\rm int}_{\rm eff}(r)$:

$$\sigma^{\rm int}_{\rm eff}(r) = \frac{1}{c^2} \int_0^r dr' 2\pi r' (\mathcal{E}_{\rm mag} + T^{ii}_{\rm mag}). \tag{3.22}$$

Differently from the external region, in the interior of the solenoid Maxwell stresses are not screened by material stresses, and therefore they do contribute to the *internal* gravitational field. Upon recalling that $T^{ii}_{\rm mag} = \mathcal{E}_{\rm mag}$, see Eq. (3.8), we see that Maxwell stresses contribute to the internal field as much as magnetic energy, and then we can rewrite Eq. (3.22) as

$$\sigma^{\rm int}_{\rm eff}(r) = \frac{1}{c^2} \int_0^r dr' 2\pi r' 2\mathcal{E}_{\rm mag} \equiv \frac{2\tilde{\mathcal{E}}^{\rm int}_{\rm mag}(r)}{c^2}. \tag{3.23}$$

It is interesting to consider a solenoid with thin walls. Since in such a case the magnetic energy contained in the region of space occupied by the solenoid walls is negligible, we have

$$\tilde{\mathcal{E}}^{\rm int}_{\rm mag}(R_1) \simeq \tilde{\mathcal{E}}_{\rm mag}, \tag{3.24}$$

and therefore Eq. (3.23) implies that a test mass placed immediately inside the solenoid would feel an oscillating pull towards the solenoid's axis that is *twice* as strong as the pull observed just outside the solenoid:

$$F(R_1) = 2F(R_2). \tag{3.25}$$

This result arises because, for $r < R_1$, the gravity originating from Maxwell stresses [the second term inside the brackets in Eq. (3.22)] is not screened by the negative gravity of the magnetic-induced stresses in the walls of the solenoid.

IV. NEWTONIAN NOISE

Producing strong magnetic fields and designing sensitive torque balances may not be enough to ensure that one would be able to actually observe the magnetically generated gravitational field ψ. For that to be possible, one has to make sure that the Newtonian noise $\delta\Phi_{\rm walls}$ is not exceedingly large compared to ψ. The order-of-magnitude estimate presented below shows that there is little prospect of measuring ψ outside the solenoid, for we estimate that outside the solenoid $\delta\Phi_{\rm walls}$ is about 9 orders of magnitude larger that ψ. At the end, we shall briefly comment on the chances of measuring ψ *inside* the solenoid, where the Newtonian noise is expected to be much smaller.

As we pointed out in the previous sections, $\delta\Phi_{\rm walls}$ comes about because electrodynamic forces deform the solenoid walls, resulting in a change of shape and density of the walls. An accurate determination of $\delta\Phi_{\rm walls}$ requires a detailed model for the solenoid, and is beyond the scope of this paper. We shall be content with simple considerations based on order-of-magnitude estimates.

We consider separately radial electrodynamic forces, which tend to increase the radius of the solenoid, and axial electrodynamic forces, which tend to make the solenoid shorter. Radial forces were the only source of Newtonian noise that was considered in [3] because, as observed earlier, the authors did not take account of the axial compression of the solenoid. In principle, radial deformations are innocuous because, for a perfectly cylindrical solenoid, a symmetric radial deformation does not alter the axial mass density of the solenoid, and therefore it produces no Newtonian noise. Real solenoids of course are not perfectly symmetrical, and therefore one expects that slightly asymmetrical radial deformations will actually produce some noise. A possible remedy for this problem was pointed out in Ref. [3], and consists in averaging over azimuthal inhomogeneities in the radial deformation, by setting the solenoid in rotation around its axis, with an angular frequency much larger than the modulation frequency of the magnetic field.

As we shall now see, the real trouble comes from the axial compression of the solenoid. To estimate the Newtonian noise introduced by this compression, we consider a cylindrical long solenoid of length L, whose walls have a cross sectional area $A_{\rm walls}$. We assume for simplicity that the axial compression $T^{zz}_{\rm walls}$ is uniform throughout the section of the walls, and that it does not exceed the elastic limit of the material. If we let $F_{\rm ax}$ the total axial compression

$$F_{\text{ax}} = \int_{R_1}^{R_2} dr 2\pi r T^{zz}_{\text{walls}}, \tag{4.1}$$

from Hook's law we estimate that the length of solenoid will suffer a fractional change of magnitude:

$$\frac{\delta L}{L} = -\frac{1}{E} \times \frac{F_{\text{ax}}}{A_{\text{walls}}}, \tag{4.2}$$

where E is the Young modulus for the material of walls. In the Appendix we show that, sufficiently far from the end points, the axial compression F_{ax} has magnitude

$$F_{\text{ax}} = \tilde{\mathcal{E}}_{\text{mag}}. \tag{4.3}$$

Using this formula in the r.h.s. of Eq. (4.2), we obtain an estimate of the relative change in the solenoid length:

$$\frac{\delta L}{L} = -\frac{1}{E} \times \frac{\tilde{\mathcal{E}}_{\text{mag}}}{A_{\text{walls}}}. \tag{4.4}$$

Consider now the total mass σ_{walls} per unit length of the solenoid. Obviously, under a change δL in the solenoid length, σ_{walls} changes by the amount

$$\delta\sigma_{\text{walls}} = -\frac{\delta L}{L}\sigma_{\text{walls}}. \tag{4.5}$$

Then, from Eq. (4.4) we obtain

$$\delta\sigma_{\text{walls}} = \frac{\tilde{\mathcal{E}}_{\text{mag}}}{E} \times \frac{\sigma_{\text{walls}}}{A_{\text{walls}}} = \frac{\tilde{\mathcal{E}}_{\text{mag}}}{E}\rho_{\text{walls}}, \tag{4.6}$$

where $\rho_{\text{walls}} = \sigma_{\text{walls}}/A_{\text{walls}}$ is the mass density of the material for the walls. Having estimated the change $\delta\sigma_{\text{walls}}$ in the linear mass density of the solenoid, we can easily obtain an estimate for the ratio $\psi/\delta\Phi_{\text{walls}}$ among the magnetically generated field ψ and the Newtonian noise. Since the former is proportional to σ_{eff} and the latter to $\delta\sigma_{\text{walls}}$, we find

$$\frac{\psi}{\delta\Phi_{\text{walls}}} = \frac{\sigma_{\text{eff}}}{\delta\sigma_{\text{walls}}} = \frac{\tilde{\mathcal{E}}_{\text{mag}}}{c^2} \times \frac{E}{\tilde{\mathcal{E}}_{\text{mag}}\rho_{\text{walls}}} = \frac{E}{c^2\rho_{\text{walls}}}, \tag{4.7}$$

where in the second passage we used Eq. (3.21). It should be noted that the result is independent of the strength of the magnetic field. In the case of stainless steel, which has $E = 2 \times 10^{11}$ N/m^2 and $\rho = 8$ g/cm^3, we obtain

$$\frac{\psi}{\delta\Phi_{\text{walls}}} = 2.8 \times 10^{-10}, \tag{4.8}$$

and we see that the Newtonian noise is over 9 orders of magnitude larger than the magnetically generated field.

This elementary analysis shows that it will be extremely difficult to observe the oscillating field ψ outside the solenoid. However, the Newtonian noise should be much less of a problem *inside* the solenoid, which we showed to be the interesting region for the purpose of testing the gravity of stresses. This is because, in the ideal case of an infinitely long and perfectly axially symmetric solenoid, the Newtonian noise inside the solenoid is strictly zero.

V. CONCLUSIONS

According to general relativity, stresses act as a source of gravity on the same footing as energy. While stress-generated gravity is normally negligible, it is thought to play an important role in astrophysics, where it contributes to determining the maximum mass of neutron stars, and it is perhaps determinant in cosmology, where "negative" pressure-generated gravity may be the cause of the recently discovered accelerated expansion of the Universe. The importance of these problems makes it highly desirable to design a laboratory test, still lacking as we write, to verify if stresses actually gravitate as predicted by general relativity or not. A test of this sort was proposed long ago in [3], and it involved measuring the gravitational pull on a test mass placed outside a long solenoid, carrying a slowly alternating current. The conclusion was that in general relativity the oscillating magnetic field inside the solenoid produces a null gravitational force on the test mass, because the attractive gravity generated by the energy and stresses of the magnetic field was found to cancel against the negative gravity generated by the material stresses that build up inside the solenoid walls. In this paper we demonstrated that this result is incorrect, as it hinges on a mistaken analysis of the material stresses, in which the electrodynamic axial compression of the solenoid was overlooked. After amending this mistake, we found that the contribution to the external gravitational field from Maxwell stresses and material stresses within the walls cancel each other, and therefore the resulting gravitational field is determined solely by the linear density of magnetic energy stored inside the solenoid. Thus observation of the external field cannot be used to test the gravity of stresses. Moreover, observing this field is extremely unlikely because of the enormous Newtonian noise that results from small changes in the length of the solenoid caused by the axial electrodynamic compression.

The interesting region for testing the gravity of stresses is the one inside the solenoid, because there the gravity of Maxwell stresses is not screened by the gravity of material stresses existing in the solenoid walls, and therefore they contribute as much as the magnetic energy in generating gravity. In the internal region the Newtonian noise should also be much less of a problem, because in the ideal case of a long solenoid, with perfect axial symmetry, Newtonian noise is zero. The major experimental difficulty that we foresee, apart from control of the residual noise resulting from asymmetries of the solenoid, is to find means of accurately measuring the gravitational field in the presence of strong magnetic fields.

APPENDIX

In this appendix we provide the explicit formulas for the stresses that build up within the walls of the idealized solenoid considered in Sec. III A, consisting of a cylindrical pipe carrying a uniform azimuthal current concentrated on its inner face. The expressions presented below provide an explicit verification of the important formulas, Eqs. (3.20) and (4.3).

We consider first the effect of the radial magnetic pressure P on the inner face of the pipe. Far from the solenoid's ends, P is uniform and its magnitude is equal to the radial component of the Maxwell stress tensor T_{mag}^{rr} inside the solenoid:

$$P = \frac{B_{\mathrm{in}}^2}{8\pi}. \tag{A1}$$

This radial pressure determines transverse stresses $T_{\mathrm{walls}}^{rr}(r)$ and $T_{\mathrm{walls}}^{\phi\phi}(r)$ in the pipe's walls, whose expressions are well known [12] and read

$$
\begin{aligned}
T_{\mathrm{walls}}^{rr} &= P \frac{R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} - 1\right), \\
T_{\mathrm{walls}}^{\phi\phi} &= -P \frac{R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} + 1\right).
\end{aligned}
\tag{A2}
$$

Besides these transverse stresses, the magnetic field determines also axial stresses $T_{\mathrm{walls}}^{zz}(r)$ inside the walls. We derive below the average value F_{ax} of T_{walls}^{zz}, as given in Eq. (4.3). In view of the key role played by the axial compression F_{ax}, and in order to explain its physical origin, we provide two different derivations of Eq. (4.3). The first derivation is based on the general equilibrium conditions Eq. (3.1). Indeed, using Eq. (3.1), one can prove the following identity holding far from the solenoid's ends:

$$\int_0^{R_2} dr 2\pi r \sum_{j=x,y}(T_{\mathrm{walls}}^{jj} + T_{\mathrm{mag}}^{jj}) = 0. \tag{A3}$$

To obtain it, we observe that far from the end points, stresses are independent of z, and therefore Eq. (3.1) reduces to

$$\sum_{k=x,y}\partial_k(T_{\mathrm{walls}}^{jk} + T_{\mathrm{mag}}^{jk}) = 0. \tag{A4}$$

Therefore, we have the identity

$$\sum_{j=x,y}(T_{\mathrm{sol}}^{jj} + T_{\mathrm{mag}}^{jj}) = \sum_{j,k=x,y}\partial_k[(T_{\mathrm{sol}}^{jk} + T_{\mathrm{mag}}^{jk})x^j]. \tag{A5}$$

Upon integrating both sides of the above equation on a cross section Σ of the solenoid, we obtain

$$\int_0^{R_2} dr 2\pi r \sum_{j=x,y}(T_{\mathrm{walls}}^{jj} + T_{\mathrm{mag}}^{jj})$$

$$= R_2 \int_0^{2\pi} d\theta \sum_{j,k=x,y}(T_{\mathrm{walls}}^{jk} + T_{\mathrm{mag}}^{jk})x^j n^k|_{r=R_2}. \tag{A6}$$

The integral on the r.h.s. vanishes, because $T_{\mathrm{walls}}^{jk}n^k|_{r=R_2}$ is zero in view of Eq. (3.3), while $T_{\mathrm{mag}}^{jk}n^k|_{r=R_2}$ vanishes because the magnetic field is negligible outside a long solenoid [see the discussion of the external field following Eq. (3.7)]. Therefore, the left-hand side of Eq. (A6) is zero and this proves Eq. (A3). Indeed, it is easy to verify that Eq. (A3) is satisfied by the explicit expressions for the transverse material stresses given in Eqs. (A2), together with the Maxwell stresses Eq. (2.9).

By using Eq. (A3), we can now easily obtain F_{ax}. To do this, we recall the identity

$$\int_0^{R_2} dr 2\pi r(T_{\mathrm{walls}}^{ii} + T_{\mathrm{mag}}^{ii}) = 0, \tag{A7}$$

which is a direct consequence of Eq. (3.20). Upon subtracting Eq. (A3) from Eq. (A7), we then obtain

$$\int_0^{R_2} dr 2\pi r(T_{\mathrm{walls}}^{zz} + T_{\mathrm{mag}}^{zz}) = 0.$$

It follows from the above equation that

$$F_{\mathrm{ax}} = \int_{R_1}^{R_2} dr 2\pi r T_{\mathrm{walls}}^{zz} = -\int_0^{R_2} dr 2\pi r T_{\mathrm{mag}}^{zz}. \tag{A8}$$

After substituting into the r.h.s. of the above formula the following expression of T_{mag}^{zz} inside the solenoid,

$$T_{\mathrm{mag}}^{zz} = -\frac{B_{\mathrm{in}}^2}{8\pi} = -\mathcal{E}_{\mathrm{mag}}, \tag{A9}$$

we immediately obtain Eq. (4.3).

In order to clarify the physical origin of the axial force F_{ax}, it is useful to provide a more direct derivation of Eq. (4.3). For this purpose, we consider the cylindrical sheet Σ of radius R_1 and height L that contains all the current flowing in the solenoid, and we imagine splitting it in two parts Σ_1 and Σ_2 consisting, respectively, of the points of Σ that lie above and below a plane of equation $z = \bar{z}$. If we imagine Σ_1 and Σ_2 as consisting of a large number of closed circular current loops, it is clear by Ampere's law that an attractive axial force $\mathbf{F}^{(\mathrm{Amp})}(\bar{z})$ arises between Σ_1 and Σ_2, and we show below that for $\bar{z}$ far from the end points $\mathbf{F}^{(\mathrm{Amp})}(\bar{z})$ has a constant magnitude equal to $\tilde{\mathcal{E}}_{\mathrm{mag}}$.

Indeed, axial symmetry implies that $\mathbf{F}^{(\mathrm{Amp})}(\bar{z})$ is along the z axis, and we let $F_z^{(\mathrm{el})}(\bar{z})$ its z component. Now, Ampere's law gives the following expression for the elementary force $dF_z^{(\mathrm{Amp})}(z_1, z_2)$ between two infinitesimal

circular current loops within Σ_1 and Σ_2:

$$dF_z^{(Amp)}(z_1, z_2) = -\frac{dj_1 dj_2}{c^2} \oint \oint (\vec{dl}_1 \cdot \vec{dl}_2) \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3},$$
(A10)

where $dj_i = jdz_i$, and $\vec{dl}_i$ are line elements tangential to the surface elements, and parallel to the surface current density $\vec{j}$. Using cylindrical coordinates, the above integral can be rewritten as

$$dF_z^{(Amp)} = -\frac{2\pi j^2 R_1^2}{c^2} dz_1 dz_2 \int_0^{2\pi} d\theta \frac{z\cos\theta}{\sqrt{z^2 + 2R_1^2(1 - \cos\theta)}},$$
(A11)

with $z = z_1 - z_2$. Since the integrand is positive, we see that the two rings attract each other, as expected. Upon integrating over z_1 and z_2 we then obtain for $F_z^{(Amp)}(\bar{z})$ the expression

$$F_z^{(Amp)}(\bar{z}) = -\frac{2\pi^2 R_1^2 j^2}{c^2} I(\bar{z}),$$
(A12)

where $I(\bar{z})$ is the integral

$$I(\bar{z}) = \frac{1}{\pi} \int_{\bar{z}}^{L/2} dz_1 \int_{-L/2}^{\bar{z}} dz_2 \int_0^{2\pi} d\theta$$
$$\times \frac{z\cos\theta}{\sqrt{z^2 + 2R_1^2(1 - \cos\theta)}}.$$
(A13)

The integrals over z_1 and z_2 in $I(\bar{z})$ can be done by the change of variables $(z_1, z_2) \rightarrow (w, z)$, where $w = (z_1 + z_2)/2$. The result is

$$I(\bar{z}) = \frac{1}{\pi} \int_0^{2\pi} d\theta \left\{ \log\left[1 - \frac{2\bar{z}}{L} \right.\right.$$
$$+ \sqrt{\left(1 - \frac{2\bar{z}}{L}\right)^2 + 8\frac{R_1^2}{L^2}(1 - \cos\theta)} \right]$$
$$+ \log\left[1 + \frac{2\bar{z}}{L} + \sqrt{\left(1 + \frac{2\bar{z}}{L}\right)^2 + 8\frac{R_1^2}{L^2}(1 - \cos\theta)} \right]$$
$$- \log\left[1 + \sqrt{1 + 2\frac{R_1^2}{L^2}(1 - \cos\theta)} \right]$$
$$\left. - \frac{1}{2}\log(1 - \cos\theta) \right\} \cos\theta,$$
(A14)

where we omitted a few terms that are zero upon integrat-

ing over θ. The positive quantity $I(\bar{z})$ reaches its maximum value at the center of solenoid (for $\bar{z} = 0$), and monotonically decreases towards zero when $\bar{z}$ approaches the end points at $\pm L/2$. For a long solenoid, $R_1/L \ll 1$, and far from the end points, $(L/2 - |z|)/R_1 \gg 1$, $I(\bar{z})$ becomes independent of $\bar{z}$ and its limiting value for an infinitely long solenoid can be obtained by observing that for $R_1/L \rightarrow 0$ the first three terms between the curly brackets of the above integral become independent of θ and therefore, after multiplication by $\cos\theta$, they integrate to zero, leaving us with

$$\lim_{R_1/L \rightarrow 0} I(\bar{z}) = -\int_0^{2\pi} \frac{d\theta}{2\pi} \cos\theta \log(1 - \cos\theta) = 1.$$
(A15)

Upon inserting this value into Eq. (A12), we see, as expected, that in the limit of a long solenoid, and for $\bar{z}$ far from the end points, the current sheets Σ_1 and Σ_2 attract each other with a force of magnitude

$$\lim_{R_1/L \rightarrow 0} F^{(Amp)}(\bar{z}) = \frac{2\pi^2 R_1^2 j^2}{c^2}.$$
(A16)

This formula can be conveniently expressed in terms of the magnetic energy density by noticing that inside a long solenoid the strength of the magnetic field B_{in} is related to the surface current density as

$$B_{in} = \frac{4\pi j}{c}.$$
(A17)

Using this formula, we can rewrite Eq. (A16) as

$$\lim_{R_1/L \rightarrow 0} F^{(Amp)}(\bar{z}) = \pi R_1^2 \times \frac{B_{in}^2}{8\pi} = \tilde{\mathcal{E}}_{mag}.$$
(A18)

Obviously, since the current sheet Σ is anchored to the inner face of the solenoid, the electrodynamic force $F_z^{(Amp)}(\bar{z})$ compresses the pipe and, at mechanical equilibrium, it is balanced by the axial stresses T_{walls}^{zz} inside the pipe walls:

$$F_{ax}(\bar{z}) = F^{(Amp)}(\bar{z}).$$
(A19)

It should be noted that according to this formula the compression $F_{ax}(\bar{z})$ vanishes at the ends of the pipe, and it increases as one moves towards the middle of the pipe. For a very long pipe, Eq. (A18) implies that far from the ends F_{ax} approaches the constant value

$$F_{ax} = \tilde{\mathcal{E}}_{mag},$$
(A20)

which reproduces again Eq. (4.3).

[1] C. Misner, K. P. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2] J. Glanz, Science **282**, 2156 (1998).

[3] V. B. Braginsky, C. M. Caves, and K. S. Thorne, Phys. Rev. D **15**, 2047 (1977).

[4] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon, Oxford, 1934).

[5] C. W. Misner and P. Putnam, Phys. Rev. **116**, 1045 (1959).

[6] J. Ehlers, I. Ozsvath, E. L. Schucking, and Y. Shang, Phys. Rev. D **72**, 124003 (2005).

[7] G. Bimonte, E. Calloni, G. Esposito, and L. Rosa, Phys. Rev. D **76**, 025008 (2007); **74**, 085011 (2006); **75**, 049904(E) (2007); **75**, 089901(E) (2007).

[8] In what follows, Greek letters denote space-time indices, while Latin letters denote space indices.

[9] We neglect here the tiny relativistic correction to the mass density of the walls resulting from the change of the wall's internal energy, as determined by their deformation.

[10] D. Melville and P. G. Mattocks, J. Phys. D **5**, 1745 (1972).

[11] P. L. Kapitza, Proc. R. Soc. A **115**, 658 (1927).

[12] W. C. Young and R. G. Budynas, *Roark's Formulas for Stress and Strain* (McGraw-Hill, New York, 2002).