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Complete LQG propagator. II. Asymptotic behavior of the vertex

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In a previous article we have shown that there are difficulties in obtaining the correct graviton propagator from the loop-quantum-gravity dynamics defined by the Barrett-Crane vertex amplitude. Here we show that a vertex amplitude that depends nontrivially on the intertwiners can yield the correct propagator. We give an explicit example of asymptotic behavior of a vertex amplitude that gives the correct full graviton propagator in the large distance limit.

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I. INTRODUCTION

A technique for computing n-point functions in a background-independent context has been recently introduced [1,2] and developed [3]. Using this technique, we have found in a previous paper [4] that the definition of the dynamics of loop-quantum-gravity (LQG) by means of the Barrett-Crane (BC) spinfoam vertex [5] *fails* to give the correct tensorial structure of the graviton propagator in the large-distance limit. The natural question is whether this is an intrinsic difficulty of the background-independent loop and spinfoam formalism, or whether it is a specific difficulty of the BC vertex. Here we show that the answer is the latter. We do so by explicitly exhibiting a vertex amplitude W that yields the correct propagator in the large-distance limit. We have no claim that this vertex amplitude is physically correct. In fact, it is a rather artificial object, chosen by simply taking the asymptotic form of the BC vertex and correcting the detail for which the BC vertex fails to work. Thus, W has at best an interest in the asymptotic region. But its existence shows that the background-independent loop and spinfoam formalism *can* yield the full tensorial structure of the perturbative n-point functions.

Furthermore, the properties of W give some indications of the asymptotic that the dynamics can have, if it has to yield the correct low energy limit. The detail of the BC vertex that needs to be corrected turns out to be a *phase* in the intertwiner variables. *A posteriori*, the need for this phase appears pretty obvious on physical grounds, as we shall discuss in detail. This might provide a useful indication for selecting a definition of the dynamics alternative to the one provided by the BC vertex. While the BC vertex is defined by the $SO(4)$ Wigner $10j$ symbol, an alternative vertex given by the square of an $SU(2)$ Wigner $15j$ symbol has been introduced recently [6]. This vertex can be derived also using coherent states techniques and can be extended to the Lorentzian case and to arbitrary values of the Immirzi parameter [7]. It would be very interesting to see whether the asymptotics of this vertex exhibit the phase dependence that we find here to be required for the low energy limit.

In Sec. II we introduce the vertex W and we give a simple explanation of the reason why the additional phase is needed. In the rest of the paper we prove that W yields the correct full tensorial structure of the propagator. In developing this calculation we have stumbled upon an unexpected result that indicates that the state used in [4] is too symmetric. This does not affect the results of [4], but forces us to reconsider the definition of the state. In Sec. III, we discuss this issue in detail and give the appropriate boundary state. In Sec. IV we compute the propagator, and in Sec. V we compare it with the one computed in linearized quantum general relativity.

This paper is not self-contained. It is based on the paper [4], where all relevant definitions are given. For an introduction to the formalism we use, see [2] For a general introduction to background-independent loop-quantum-gravity [8], see [9].

II. VERTEX AND ITS PHASE

Consider the Euclidean graviton propagator $G^{\mu\nu\rho\sigma}(x,y) = \langle 0|h^{\mu\nu}(x)h^{\rho\sigma}(y)|0\rangle$, where $h^{\mu\nu}(x)$ is the difference between the gravitational quantum field and its background value $\delta^{\mu\nu}$, and $|0\rangle$ is the vacuum state, peaked on the flat Euclidean metric $\delta^{\mu\nu}$. Let L be the distance between x and y (in the flat Euclidean metric). Choose a regular 4-simplex with two boundary tetrahedra n and m centered at the points x and y; the indices $i, j, k, l, m, n, \ldots = 1, \ldots, 5$ label the five tetrahedra bounding the 4-simplex. Define $\mathbf{G}^{ij,kl}_{n,m}(L) = G^{\mu\nu\rho\sigma}(x,y)(n_n^{(i)})_\mu (n_n^{(j)})_\nu \times (n_m^{(k)})_\rho (n_m^{(l)})_\sigma$, where $n_m^{(k)}$ (denoted n^{mk} in [4]) is the normal one-form to the triangle bounding the tetrahedra m and k, in the hyperplane defined by the tetrahedron m (with $|n|$ equal to the area of the triangle). Clearly, knowing $\mathbf{G}^{ij,kl}_{n,m}(L)$ is the same as knowing $G^{\mu\nu\rho\sigma}(x,y)$. Following [1,2], $\mathbf{G}^{ij,kl}_{n,m}(L)$ can be computed in a background-independent

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context as the scalar product

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \langle W|(E^{(i)}_n \cdot E^{(j)}_n - n^{(i)}_n \cdot n^{(j)}_n) \times (E^{(k)}_m \cdot E^{(l)}_m - n^{(k)}_m \cdot n^{(l)}_m)|\Psi_{\mathbf{q}}\rangle, \quad (1)$$

for an appropriate $\mathbf{q}$. Here $\langle W|$ is the boundary functional, which can be intuitively understood as the path integral of the Einstein-Hilbert action on a finite spacetime region $\mathcal{R}$, with given boundary configuration. The operator $E^{(i)}_n$ (denoted $E^{(ni)}_n$ in [4]) is the triad operator at the point n, contracted with $n^{(i)}_n$. $|\Psi_{\mathbf{q}}\rangle$ is a state on the boundary of $\mathcal{R}$, picked on a given classical boundary (extrinsic and extrinsic) geometry $\mathbf{q}$.

Fixing such a boundary geometry is equivalent to fixing a background metric g in the interior, where g is the solution of the Einstein equations with boundary data $\mathbf{q}$. The existence of such a background metric is part of the definition of the propagator, which is a measure of fluctuations around a given background. Criticisms to the approach of [1,2] have been raised on the grounds that a propagator makes no sense in a background-independent context because it is a quantity that depends on a background geometry. These criticisms neglect the fact that the information about the background, over which the propagator is defined, is coded into the boundary state via $\mathbf{q}$.

We are interested in the value of (1) to first order in the group field theory expansion parameter λ, and in the limit in which the boundary surface (whose size is determined by $\mathbf{q}$) is large. On the physical interpretation of this limit, see [6]. To first order, the leading contribution to W has support only on spin networks with a 4-simplex graph. If $\mathbf{j} = (j_{nm})$ and $\mathbf{i} = (i_n)$ are, respectively, the ten spins and the five intertwiners that color this graph, then in this approximation (1) reads

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})(E^{(i)}_n \cdot E^{(j)}_n - n^{(i)}_n \cdot n^{(j)}_n) \times (E^{(k)}_m \cdot E^{(l)}_m - n^{(k)}_m \cdot n^{(l)}_m)\Psi(\mathbf{j},\mathbf{i}). \quad (2)$$

To this order, W is just determined by the amplitude of a single vertex. In [1,2,4], (a suitable adjustment of) the BC vertex was chosen for W. The propagator depends only on the asymptotic behavior of the vertex. This has the structure [10]

$$W_{\mathrm{BC}}(\mathbf{j}) \sim e^{(i/2)(\delta\mathbf{j}G\delta\mathbf{j})}e^{i\Phi\cdot\delta\mathbf{j}} + e^{-(i/2)(\delta\mathbf{j}G\delta\mathbf{j})}e^{-i\Phi\cdot\delta\mathbf{j}}, \quad (3)$$

where G is the 10×10 matrix given by the second derivatives of the $4d$ Regge action around the symmetric state, $\delta\mathbf{j}$ is the difference between the ten spins $\mathbf{j}$ and their background value j_0, and Φ is a $10d$ vector with all equal components, which were shown in [1,2] to precisely match those determined by the background extrinsic curvature. The diagonal components of the propagator determined by (1) turn out to be correct at first [1] and second [2] order, but the nondiagonal components fail to do so [4].

Here we make a different choice for W. We choose a vertex W with an asymptotic form that includes a Gaussian intertwiner-intertwiner and spin-intertwiner dependence, and—most crucially—a phase dependence on the intertwiner variables. To write this, introduce a $15d$ vector $\delta\mathbf{I} = (\delta\mathbf{j}, \delta\mathbf{i})$, where $\delta\mathbf{i}$ is the difference between the five intertwiners $\mathbf{i}$ and their background value i_0, and write $\delta I_\alpha = (\delta j_{nm}, \delta i_n) = (j_{nm} - j_0, i_n - i_0)$, where $\alpha = (nm, n)$. Then we assume the following form for W (which we only use below in the asymptotic limit):

$$W(\mathbf{j},\mathbf{i}) = e^{(i/2)(\delta\mathbf{I}G\delta\mathbf{I})}e^{i\phi\cdot\delta\mathbf{I}} + e^{-(i/2)(\delta\mathbf{I}G\delta\mathbf{I})}e^{-i\phi\cdot\delta\mathbf{I}} = w(\mathbf{j},\mathbf{i}) + \overline{w(\mathbf{j},\mathbf{i})}. \quad (4)$$

Here G is now a 15×15 real matrix, for which we only assume that it respects the symmetries of the problem and that it scales as $1/j_0$. The quantity $\phi = (\phi_{nm}, \phi_n)$ is now a $15d$ vector: its 10 spin components ϕ_{nm} just reproduce the spin phase dependence of (3); while its five intertwiner components are equal and we fix them to have value

$$\phi_n = \frac{\pi}{2}. \quad (5)$$

This phase dependence is the crucial detail that makes the calculation work.

Let us illustrate up front the reason that this additional phase cures the problems that appeared with the BC vertex. The boundary state must have an intertwiner dependence, in order to have the correct semiclassical value of the mean values of the angles between the faces of the boundary tetrahedra. The mean value of an intertwiner variable i_n—namely of the virtual link of the intertwiner in a given pairing—must have a certain value i_0. For this, it is sufficient, say, that the state be a Gaussian around i_0. However, in quantum geometry the different angles of a tertrahedron do not commute [11]. Therefore a state with a behavior like $\exp\{-(i_n - i_0)^2\}$ will be peaked on the virtual spin i_n in one pairing, but *it will not be peaked in the virtual spin in a different pairing*. Therefore, the other angles of the tetrahedron will not be peaked on the correct semiclassical value. We can of course write a Gaussian which is peaked on a variable as well as on another, noncommuting, variable. For instance, a standard Schrödinger wave packet $\psi(x) = \exp\{-\frac{(x-x_0)^2}{2}\sigma + ip_0x\}$ is peaked on position as well as momentum. But in order to do so, we must have a phase dependence on the x. Similarly, the boundary state needs a phase dependence on the intertwiner variable i_n, in order to be peaked on all angles. As shown in [12], the correct value for this is $\exp\{i\frac{\pi}{2}i_n\}$. Now, the general mechanism through which the dynamical kernel reproduces the semiclassical dynamics in quantum mechanics is the cancellation of the phases between the propagation kernel and the boundary state. If this does not happen, the rapidly oscillating phases suppress the amplitude. For instance, in the nonrelativistic quantum mechanics of a free particle, the propagation kernel $K(x, y)$ in a time t has a phase dependence from small fluctuations $\delta x = x - x_0$ and $\delta y = y - y_0$ of the form

$$K(x_0 + \delta x, y_0 + \delta y) = \langle x_0 + \delta x| e^{-(i/\hbar)(p^2/2m)t} | y_0 + \delta y\rangle \sim C e^{-ip_0\delta x} e^{ip_0 \delta y}, \quad (6)$$

where $p_0 = m(y_0 - x_0)/t$. This phase precisely cancels the phase of an initial and final wave packet ψ_i and ψ_f centered on x_0 and y_0, if these have the correct momentum. That is,

$$\langle \psi_f | e^{-(i/\hbar)Ht} | \psi_i\rangle = \int dx \int dy e^{-((x-x_0)^2/2\sigma)-(i/\hbar)p_f x} K(x,y) \times e^{-((y-y_0)^2/2\sigma)+(i/\hbar)p_i y} \quad (7)$$

is suppressed by the oscillating phases unless $p_i = p_f = p_0$. This is the standard mechanism through which quantum theory reproduces the (semi-)classical behavior. In quantum gravity, it is reasonable to expect the same to happen if we have to recover the Einstein equations in the semiclassical limit. That is, the propagation kernel W must have a phase dependence that matches the one in a semiclassical boundary state. This is precisely the role of the phase $\exp\{i\frac{\pi}{2} i_n\}$ that we have included in (4).

In the rest of the paper we show that a vertex amplitude that has the phase dependence as above can reproduce the tensorial structure of the graviton propagator. First, however, we must improve the definition of the vertex given above, and correct a problem with the definition of the state in [4].

III. BOUNDARY STATE AND SYMMETRY

Following [1,2], we consider a boundary state defined as a Gaussian wave packet, centered on the values determined by the background geometry $\mathbf{q}$. Here

$$\Phi_{\mathbf{q}}(\mathbf{j}, \mathbf{i}) = C e^{-(1/2j_0)(\delta \mathbf{I} A \delta \mathbf{I}) + i\phi\cdot\delta\mathbf{I}}. \quad (8)$$

Where A is a 15×15 matrix and the normalization factor C is determined by $\langle W|\Phi_{\mathbf{q}}\rangle = 1$. The spin phase coefficients are fixed by the background extrinsic geometry [1]. The intertwiner phase coefficients are fixed by the requirement that the state remain peaked after a change of pairing to the value i_0 [4,12].

At each node n we have three possible pairings, which we denote as x_n, y_n, and z_n. For instance, at the node 5, let $x_5 = \{(12)(34)\}$, $y_5 = \{(13)(24)\}$, $z_5 = \{(14)(23)\}$, and denote $i_{x_5} = i_{\{(12)(34)\}}$ the intertwiner in the pairing x_5, and so on. The vertex (4) and the state (8) are written in terms of the intertwiner variable i_n, which is the virtual link of the node *n in one chosen pairing*. Because of this, the definition of these states depends on the pairing chosen. It follows that the vertex and the state do not have the full symmetry of the 4-simplex. The corresponding propagator turns out not to be invariant under $SO(4)$, as it should in the Euclidean theory. In [4], a simple strategy was adopted in order to overcome this difficulty: sum over the three pairings at each of the five nodes. The state was defined as

$$|\Psi_{\mathbf{q}}\rangle = \sum_{m_n} \sum_{\mathbf{j}, \mathbf{i}_{m_n}} \Phi_{\mathbf{q}}(\mathbf{j}, \mathbf{i}_{m_n}) |\mathbf{j}, \mathbf{i}_{m_n}\rangle, \quad (9)$$

where $m_n = x, y, z$ for each node n. This sum implements the full symmetry of the 4-simplex. Summing over the three bases removes the basis dependence.

In developing the calculations presented in this paper, at first we adopted this same strategy. To our surprise, nothing worked, and something quite strange happened: the dependence on the intertwiner variables i_n mysteriously canceled out in all components of the propagator!

The solution of the puzzle was to realize that to sum over the three basis with a correlation matrix A does implement the symmetry of the 4-simplex, but not just this symmetry. It implements a larger symmetry that has the effect of canceling the intertwiner dependence. Geometrically, this additional symmetry can be viewed as an *independent* rotation of each of the five tetrahedra forming the boundary of the 4-simplex.

To understand what happens, consider for instance the correlation $\langle j_{12} i_{x_5}\rangle$ between the spin j_{12}, which is the quantum number of the area of a triangle, and the intertwiner i_{x_5}, which is the quantum number of the angle θ_{12} between the faces 2 and 3 of the tetrahedron 5. More precisely, i_{x_5} is the eigenvalue of the quantity $A_2^2 + A_3^2 + A_2 A_3 \cos(\theta_{12})$, where A_i is the area of the face i of the tetrahedron 5. Now, if the state is summed over pairings, then it does not distinguish pairings; hence,

$$\langle j_{12} i_{x_5}\rangle = \tfrac{1}{3}(\langle j_{12} i_{x_5}\rangle + \langle j_{12} i_{y_5}\rangle + \langle j_{12} i_{z_5}\rangle). \quad (10)$$

That is,

$$\langle j_{12} i_{x_5}\rangle = \tfrac{1}{3}\langle j_{12}(3A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_1 A_2 \cos(\theta_{12}) + A_1 A_3 \cos(\theta_{13}) + A_1 A_4 \cos(\theta_{14}))\rangle. \quad (11)$$

But let n_i, $i = 1, \ldots, 4$ be the normal to the face i of the tetrahedron 5, with length $|n_i| = A_i$. The closure relation reads

$$\sum_{i=1,4} n_i = 0. \quad (12)$$

Taking the scalar product with n_1 gives

$$A_1^2 + A_1 A_2 \cos(\theta_{12}) + A_1 A_3 \cos(\theta_{13}) + A_1 A_4 \cos(\theta_{14}) = 0. \quad (13)$$

It follows from this equation and (11) that

$$\langle j_{12} i_{x_5}\rangle = \tfrac{1}{3}\langle j_{12}(2A_1^2 + A_2^2 + A_3^2 + A_4^2)\rangle = \tfrac{1}{3}(2\langle j_{12} j_{15}\rangle + \langle j_{12} j_{25}\rangle + \langle j_{12} j_{35}\rangle + \langle j_{12} j_{45}\rangle). \quad (14)$$

That is, the spin-intertwiner correlations are just functions of the spin-spin correlations for a state with this symmetry.

The intertwiner dependence drops out. This means that the propagator is completely unaffected from the correlations involving the intertwiners. It then turns out that the sole spin-spin correlations in the state are not sufficient to give the full tensorial structure of the propagator.

The solution of the difficulty is just to choose a boundary state and a kernel W that do not have the extra symmetry. The simplest possibility is to choose an arbitrary pairing, and then to symmetrize *only* under the symmetries of the 4-simplex. These are generated by the 5! permutations σ of the five vertices of the 4-simplex. A permutation $\sigma : \{1, 2, 3, 4, 5\} \to \{\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\}$ acts naturally on the boundary states

$$\sigma|j_{nm}, i_{x_n}\rangle = |j_{\sigma(n)\sigma(m)}, i_{\sigma(x_n)}\rangle, \tag{15}$$

where the action $\sigma(x_n)$ of the permutation on a node is defined by

$$\sigma(\{(ab)(cd)_n\}) = \{(\sigma(a)\sigma(b))(\sigma(c)\sigma(d))_{\sigma(n)}\} \tag{16}$$

and can therefore change the original pairing at the node.

We therefore define the boundary state by replacing (9) with

$$|\Psi_{\mathbf{q}}\rangle = \sum_{\sigma}\sigma|\Phi_{\mathbf{q}}\rangle = \sum_{\sigma}\sum_{\mathbf{j},\mathbf{i}}\Phi_{\mathbf{q}}(\mathbf{j},\mathbf{i})\sigma|\mathbf{j},\mathbf{i}\rangle. \tag{17}$$

This modification of the boundary state does not affect the conclusions of the paper [4]. Similarly, we pose

$$|W\rangle = \sum_{\sigma}\sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})\sigma|\mathbf{j},\mathbf{i}\rangle. \tag{18}$$

Before beginning the actual calculation of the propagator, consider what happens by contracting the vertex amplitude with the boundary state. We have the double sum over permutations

$$\langle W|\Psi\rangle = \sum_{\sigma\sigma'}(\sum_{\mathbf{j}\mathbf{i}\mathbf{j}'\mathbf{i}'}\overline{W(\mathbf{j},\mathbf{i})}\,\Phi(\mathbf{j}',\mathbf{i}')\langle\sigma(\mathbf{j},\mathbf{i})|\sigma'(\mathbf{j}',\mathbf{i}')\rangle). \tag{19}$$

The scalar product is

$$\langle\mathbf{j},\mathbf{i}|\mathbf{j}',\mathbf{i}'\rangle = \delta_{\mathbf{j},\mathbf{j}'}\prod_n\langle i_n|i'_n\rangle, \tag{20}$$

where $\langle i_n|i'_n\rangle$ is δ_{i_n,i'_n} if the two intertwiners are written in the same basis, and is the matrix of the change of basis, namely, a $6j$ symbol, otherwise. Now, it was observed in [4] that if one of these $6j$ symbols enters in a sum like (19) then the sum is suppressed in the large j_0 limit, because the $6j$ symbol contains a rapidly oscillating factor which is not compensated. Hence, in this limit we can effectively rewrite (19) in the form

$$\langle W|\Psi\rangle = \sum_{\sigma\sigma'}\left(\sum_{\mathbf{j}\mathbf{i}\mathbf{j}'\mathbf{i}'}\overline{W(\mathbf{j},\mathbf{i})}\,\Phi(\mathbf{j}',\mathbf{i}')\delta_{\sigma\mathbf{j},\sigma'\mathbf{j}'}\delta_{\sigma\mathbf{i},\sigma'\mathbf{i}'}\right), \tag{21}$$

where the second δ vanishes unless the two intertwiners have the same value and are written in the same basis. Up to accidental symmetry factors that we absorb in the state, we can then rewrite the scalar product in the form

$$\langle W|\Psi\rangle = \sum_{\sigma}\left(\sum_{\mathbf{j}\mathbf{i}}\overline{W(\mathbf{j},\mathbf{i})}\,\Phi(\mathbf{j},\mathbf{i})\right) = 5!\sum_{\mathbf{j}\mathbf{i}}\overline{W(\mathbf{j},\mathbf{i})}\,\Phi(\mathbf{j},\mathbf{i}). \tag{22}$$

We shall see that a similar simplification happens in the calculation of the matrix elements of the propagator.

IV. PROPAGATOR

Let us begin by recalling the action of the grasping operators. This was computed in [4], to which we refer for the notation. Consider the operators acting on a node n. The diagonal action is simply

$$E_n^{(i)}\cdot E_n^{(i)}|\Phi_{\mathbf{q}}\rangle = C^{ni}|\Phi_{\mathbf{q}}\rangle, \tag{23}$$

where C^{ni} is the Casimir of the representation associated to the link ni. The nondiagonal action depends on the pairing at the node n. We have three cases, depending on the three possible pairings. These are as follows. Say the node n is in the pairing (ij), (ef), with positive orientation at the two trivalent vertices (i_n, i, j) and (i_n, e, f). Then we have the diagonal double grasping

$$E_n^{(i)}\cdot E_n^{(j)}|\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}} D_n^{ij}\Phi(\mathbf{j},\mathbf{i})|\mathbf{j},\mathbf{i}\rangle, \tag{24}$$

while the two possible nondiagonal graspings give

$$E_n^{(i)}\cdot E_n^{(e)}|\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}}\Phi(\mathbf{j},\mathbf{i})(X_n^{ie}|\mathbf{j},\mathbf{i}\rangle - Y_n^{ie}|\mathbf{j},(i_n-1),\mathbf{i}'\rangle - Z_n^{ie}|\mathbf{j},(i_n+1),\mathbf{i}'\rangle) \tag{25}$$

and

$$E_n^{(i)}\cdot E_n^{(f)}|\Phi_{\mathbf{q}}\rangle = \sum_{\mathbf{j},\mathbf{i}}\Phi(\mathbf{j},\mathbf{i})(X_n^{if}|\mathbf{j},\mathbf{i}\rangle + Y_n^{if}|\mathbf{j},(i_n-1)\mathbf{i}'\rangle + Z_n^{if}|\mathbf{j},(i_n+1)\mathbf{i}'\rangle), \tag{26}$$

and so on cyclically. The quantities D_n^{ij}, X_n^{ij}, Y_n^{ij}, and Z_n^{ij} are defined in [4]. Here $\mathbf{i}'$ indicates the four intertwiners different from i_n.

Inserting the expressions (17) and (18) in the expression (1) for the propagator, gives the double sum over permutations

$$\mathbf{G}_{\mathbf{q}n,m}^{ij,kl} = \sum_{\sigma'\sigma}\left[\sum_{\mathbf{j},\mathbf{i}}\overline{W(\sigma'(\mathbf{j}),\sigma'(\mathbf{i}))}(E_n^{(i)}\cdot E_n^{(j)} - n_n^{(i)}\cdot n_n^{(j)}) \times (E_m^{(k)}\cdot E_m^{(l)} - n_m^{(k)}\cdot n_m^{(l)})\Phi(\sigma(\mathbf{j}),\sigma(\mathbf{i}))\right]. \tag{27}$$

The E operators do not change the spin, and the argument at the end of the previous section can be repeated. This time, however, the residual sum over permutations remains, because the operators are not invariant under it:

$$\mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = \sum_{\sigma}\bigg(\sum_{\mathbf{j},\mathbf{i}} \overline{W(\sigma(\mathbf{j}),\sigma(\mathbf{i}))}(E^{(i)}_n\cdot E^{(j)}_n - n^{(i)}_n\cdot n^{(j)}_n) \times (E^{(k)}_m\cdot E^{(l)}_m - n^{(k)}_m\cdot n^{(l)}_m)\Phi(\sigma(\mathbf{j}),\sigma(\mathbf{i}))\bigg). \quad (28)$$

By changing variables, we can move the symmetrization to the operators, hence writing

$$\mathbf{G}\,^{ij,kl}_{\mathbf{q}n,m} = \sum_{\sigma}\tilde{\mathbf{G}}^{\sigma(i)\sigma(j),\sigma(k)\sigma(l)}_{\mathbf{q}\sigma(n),\sigma(m)}, \quad (29)$$

where

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}(E^{(i)}_n\cdot E^{(j)}_n - n^{(i)}_n\cdot n^{(j)}_n) \times (E^{(k)}_m\cdot E^{(l)}_m - n^{(k)}_m\cdot n^{(l)}_m)\Phi(\mathbf{j},\mathbf{i}). \quad (30)$$

In other words, we can first compute the propagator with unsymmetrized states and vertex, and then symmetrize the propagator.

We can now begin the actual calculation of the various terms of the propagator. It is useful to distinguish three cases: the diagonal-diagonal components $\tilde{\mathbf{G}}^{ii,kk}_{\mathbf{q}n,m}$, the diagonal-nondiagonal components $\tilde{\mathbf{G}}^{ii,kl}_{\mathbf{q}n,m}$, and the nondiagonal-nondiagonal components $\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m}$, where again different indices are distinct. Let us consider the three cases separately.

In the diagonal-diagonal case, from the expression of the previous section, we have

$$\tilde{\mathbf{G}}\,^{ii,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{ji}} \overline{W(\mathbf{j},\mathbf{i})}(C^{ni} - |n^{(i)}_n|^2)(C^{nk} - |n^{(k)}_m|^2)\Phi(\mathbf{j},\mathbf{i}). \quad (31)$$

As we have seen in [4] the background geometry determines the background link j^0

$$|n^{(i)}_n|^2 = C^2(j^0) = j^0(j^0+1) \quad (32)$$

and

$$C^{ni} = C^2(j^{ni}). \quad (33)$$

In the large j^0 limit we have at leading order

$$C^{ni} - |n^{(i)}_n|^2 \approx 2j^0\delta j^{ni}; \quad (34)$$

the propagator components are then

$$\tilde{\mathbf{G}}\,^{ii,kk}_{\mathbf{q}n,m} = 4j_0^2\sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}\,\delta j^{ni}\delta j^{mk}\Phi(\mathbf{j},\mathbf{i}). \quad (35)$$

The sum over permutations is now trivial. It only gives a 5! factor that cancels with the same factor in the normalization. We can therefore drop the tilde from (35).

In the diagonal-nondiagonal case, from (30) we have

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})(E^{(i)}_n\cdot E^{(j)}_n - n^{(i)}_n\cdot n^{(j)}_n) \times (E^{(k)}_m\cdot E^{(k)}_m - |n^{(k)}_m|)\Phi(\mathbf{j},\mathbf{i}). \quad (36)$$

Now the second operator is diagonal and gives (34) at leading order; the action of the first operator instead gives only one of the three terms (24)–(26) depending on how the two links ni and nj are paired at the node n. The possible results (at leading order) are

$$\tilde{\mathbf{G}}\,^{ij,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}} W(\mathbf{j},\mathbf{i})2j_0\delta j^{(mk)}\bigg(D^{(ij)}_n + \frac{j_0^2}{3}\bigg)\Phi(\mathbf{j},\mathbf{i}) \quad (37)$$

if the two links are paired. The second term in the parentheses comes from the fact that the background normals are fixed by the background geometry. In the large j^0 limit

$$n^{(i)}_n\cdot n^{(nj)}_n \approx -\tfrac{1}{3}(j_0)^2. \quad (38)$$

And

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}}\bigg(\overline{W(\mathbf{j},\mathbf{i})}\bigg(X^{ij}_n + \frac{j_0^2}{3}\bigg) - \overline{W(\mathbf{j},\mathbf{i}',i_n-1)}Y^{ij}_n - \overline{W(\mathbf{j},\mathbf{i}',i_n+1)}Z^{ij}_n\bigg)2j_0\delta j^{mk}\Phi(\mathbf{j},\mathbf{i}), \quad (39)$$

or

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}}\bigg(\overline{W(\mathbf{j},\mathbf{i})}\bigg(X^{ij}_n + \frac{j_0^2}{3}\bigg) + \overline{W(\mathbf{j},\mathbf{i}',i_n-1)}Y^{ij}_n + \overline{W(\mathbf{j},\mathbf{i}',i_n+1)}Z^{ij}_n\bigg)2j_0\delta j^{mk}\Phi(\mathbf{j},\mathbf{i}), \quad (40)$$

according to orientation, if they are not paired.

In (39) and (40) the terms in Y and Z cancel at the leading order for the following reason. First, recall from [4] that Y and Z are equal at leading order. The difference between the Y term and the Z term is then only given by the ±1 in the argument of W. Now, as in [4], and as we shall see below, only the first of the two terms in the right-hand side of (4) contribute to the propagator. From Eqs. (4) and (5), notice that

$$w(\mathbf{j},\mathbf{i}',i_n+1) = -w(\mathbf{j},\mathbf{i}',i_n-1)e^{2i(2G_{nn}\delta i_n + G_{n(nb)}\delta j_{nb} + G_{nm}\delta i_m)}, \quad (41)$$

but the exponential contributes to the propagator only at subleading order in j_0, because G is proportional to $1/j_0$. Hence at leading order the terms in $W(\mathbf{j},\mathbf{i}',i+1)$ and $W(\mathbf{j},\mathbf{i}',i-1)$ give contributions to the propagator that cancel. Thus we have

$$\tilde{\mathbf{G}}\,^{ij,kk}_{\mathbf{q}n,m} = \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}\bigg(X^{ij}_n + \frac{j_0^2}{3}\bigg)2j_0\delta j^{mk}\Phi(\mathbf{j},\mathbf{i}), \quad (42)$$

anytime ni and nj are not paired.

In the large-distance limit we have (38) and

$$D_n^{ij} - n_n^{(i)} \cdot n_n^{(j)} = \frac{C^2(i_n) - C^2(j^{(ni)}) - C^2(j^{(nj)})}{2} + \frac{1}{3}(j_0)^2. \tag{43}$$

Introduce the fluctuation variables $\delta j_{nj} = j_{nj} - j_0$ and $\delta i_n = i_n - i_0$ and expand around the background values j^0 and i^0. In the large j_0 limit (which is also large i_0), the dominant term of the (43) is

$$D_n^{ij} - n^{(ni)} \cdot n^{(nj)} = \delta i_n i_0 - \delta j^{ni} j_0 - \delta j^{nj} j_0. \tag{44}$$

Similarly, using the results of [4], the X terms are approximated substituting $C^2(j) \approx j^2$ and keeping the dominant terms

$$X_n^{ij} = -\tfrac{1}{4}((i_0)^2 + 2j_0\delta j^{ni} + 2j_0\delta j^{nj} - 2j_0\delta j^{nf} - 2j_0\delta j^{ne} + 2i_0\delta i_n), \tag{45}$$

where nf and ne indicate the other two links of the node n. The relation between i_0 and j_0 is easy to compute [see Eqs. (27) and (28) of [4]]: in the large j_0 limit, j_0^2 is the eigenvalue of $|\vec{J}|^2$ and i_0^2 is the eigenvalue of $|\vec{J} + \vec{J}'|^2 = |\vec{J}|^2 + |\vec{J}'|^2 + 2\vec{J} \cdot \vec{J}' = |\vec{J}|^2(1 + 1 + 2(-1/3))$; hence, $i_0 = \frac{2}{\sqrt{3}} j_0$. Therefore, the first term of the sum cancels the norm of the n, leaving

$$X_n^{ij} + \frac{j_0^2}{3} = -\frac{1}{4}(2j_0\delta j^{ni} + 2j_0\delta j^{nj} - 2j_0\delta j^{nf} - 2j_0\delta j^{ne} + 2i_0\delta i_n). \tag{46}$$

In conclusion, we have for the paired case

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = 2j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}\left(\frac{2}{\sqrt{3}}\delta i_n - \delta j^{ni} - \delta j^{nj}\right)\delta j^{mk}\Phi(\mathbf{j},\mathbf{i}), \tag{47}$$

and for the unpaired one

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})} \times \left(-\delta j^{ni} - \delta j^{nj} + \delta j^{nf} + \delta j^{ne} - \frac{2}{\sqrt{3}}\delta i_n\right) \times \delta j^{mk}\Phi(\mathbf{j},\mathbf{i}). \tag{48}$$

Finally, the nondiagonal-nondiagonal case is

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = \langle W|(E_n^{(i)} \cdot E_n^{(j)} - n_n^{(i)} \cdot n_n^{(j)}) \times (E_m^{(k)} \cdot E_m^{(l)} - n_m^{(k)} \cdot n_m^{(l)})|\Phi\rangle. \tag{49}$$

The calculations are clearly the same as above.

The final result is

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} \overline{W(\mathbf{j},\mathbf{i})}\, K_n^{ij} K_m^{kl} \Phi(\mathbf{j},\mathbf{i}), \tag{50}$$

where

$$K_n^{ij} = \frac{2}{\sqrt{3}}\delta i_n - \delta j^{ni} - \delta j^{nj} \tag{51}$$

if ni and nj are paired at n and

$$K_n^{ij} = \frac{1}{2}\left(-\delta j^{ni} - \delta j^{nj} + \delta j^{nf} + \delta j^{ne} - \frac{2}{\sqrt{3}}\delta i_n\right) \tag{52}$$

if they are not; while

$$K_n^{ii} = 2\delta j^{ni}. \tag{53}$$

Both the state coefficients $\Phi(\mathbf{j},\mathbf{i})$ and the vertex coefficients $W(\mathbf{j},\mathbf{i})$ are given by a Gaussian in δI_α. The phases in the boundary state cancel with the phase of one of the two terms of W, while the other term is suppressed for large j_0. This is where the phase factor (5) plays an essential role. Thus, (50) reduces to

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = j_0^2 \sum_{\mathbf{j},\mathbf{i}} e^{-(1/2j_0)M_{\alpha\beta}\delta I_\alpha \delta I_\beta} K_n^{ij} K_m^{kl}, \tag{54}$$

where $M = A + ij_0 G$. As in [4], we approximate the sum by a Gaussian integral with quadratic insertions. The result of the integral is easily expressed in terms of the matrix M^{-1} obtained inverting the 15×15 covariance matrix M, in the 10 spin variables δj_{nm} and the five intertwiner variables δi_n.

The symmetries of the matrix M^{-1} are the same as the symmetries of M, and are dictated by the symmetries of the problem. Which ones are these symmetries? At first sight, one is tempted to say that M^{-1} must respect the symmetries of the 4-simplex, and therefore it must be invariant under any permutation of the five vertices n. Therefore, it can have only seven independent components:

$$M^{-1}_{(ij)(ij)} = c_2, \quad M^{-1}_{(ij)(ik)} = c_1, \quad M^{-1}_{(ij)(kl)} = c_3, \quad M^{-1}_{ii} = c_4, \quad M^{-1}_{ij} = c_5, \quad M^{-1}_{(ij)i} = c_6, \quad M^{-1}_{(ij)k} = c_7, \tag{55}$$

where different indices are distinct. The ratio for this being for instance that M_{11} must be equal to M_{22} because of the symmetry under the exchange of the vertex 1 and the vertex 2. However, this argument is incorrect.

The reason is that the vertex function and the state function are written as a function of intertwiner variables i_n which are tied to a given choice of pairing at each node. Specifically, we have chosen the pairing $i_1^{(23)(45)}$, $i_2^{(34)(51)}$, $i_3^{(45)(12)}$, $i_4^{(51)(23)}$, $i_5^{(12)(34)}$. This choice breaks the symmetry under the permutations of the vertices, although this is not immediately evident. To see this, consider for instance the two matrix elements $M^{-1}_{(12)3}$ and $M^{-1}_{(12)4}$. According to (55), they should be equal (both be equal to c_7 by symmetry). But notice that 1 and 2 are paired at the node 3, while they are not paired at the node 4. Therefore the two are not equal under the symmetries of the paired 4-simplex. To see this more formally, let us indicate explicitly the pairing in which the intertwiner is written by writing $i_n^{(ij)(ef)}$ instead of i_n. Then we see that $M^{-1}_{(12)3}$ is of the form $M^{-1}_{(ij)i_n^{(ij)(kl)}}$ while $M^{-1}_{(12)4}$ is of the form $M^{-1}_{(ij)i_n^{(ik)(jl)}}$, which makes it obvious that a permutation $ijklm \to i'j'k'l'm'$ cannot transform one into the other, since it cannot undo the fact that the ij indices of the link are paired at the node. As a consequence, we must for instance replace the last entry of (55) by

$$M^{-1}_{(ij)i_n^{(ij)(kl)}} = c_7, \qquad M^{-1}_{(ij)i_n^{(ik)(jl)}} = c_8, \tag{56}$$

and so on. Thus, the matrix M^{-1} may in general have a more complicated structure than (55).

Now, the details of this structure depend on the pairing chosen. In fact, there are five possible inequivalent ways of choosing the pairings at the nodes, which do not transform into one another under permutations. These are illustrated in Fig. 1. The fact that they cannot be transformed into one another by a permutation can be deduced from the following consideration. In each diagram (Fig. 1), consider the sequences of links that can be followed without ever crossing an intertwiner. Observe that in the first case all links are clustered in a single cluster of length 10. In the second, they are clustered in two diagrams of length (5, 5), and so on as indicated. Clearly a permutation cannot change the structure of these clusterings, and therefore these pairing choices cannot be transformed into one another under permutations. The five cases illustrated correspond to the five different $15j$ Wigner symbols illustrated in [13]. These five classes define, therefore, distinct possibilities for the definitions of vertex and the state. As here we are not interested in generality, we have just picked one of these: the first case. Also, since we are not interested in the full generality of an arbitrary Gaussian vertex and state, we just assume a particular form, compatible with the symmetries, for the matrix M^{-1}. Specifically, we assume that M^{-1} has the form (55) with the last entry replaced by (56). That is, we assume the state depends on (at least) eight independent parameters that determine $\mathbf{c} = (c_1, \ldots, c_8)$. The symmetries of the 4-simplex equivalence class admit a greater number of free parameters, but we do not need the most general possible Gaussian state for what follows. Assuming thus this form for M^{-1}, we can then proceed with the calculation of (54).

Each term of the normalized propagator is a sum of individual elements of the matrix M^{-1}. The overall dependence on j_0 is as in the diagonal case, and gives the expected inverse-square dependence. The normalization factor is

$$\mathcal{N}^{-1} = j_0^2 \int d(\delta I_\alpha) e^{-(1/2j_0)M_{\alpha\beta}\delta I_\alpha \delta I_\beta}. \tag{57}$$

The diagonal-diagonal term gives

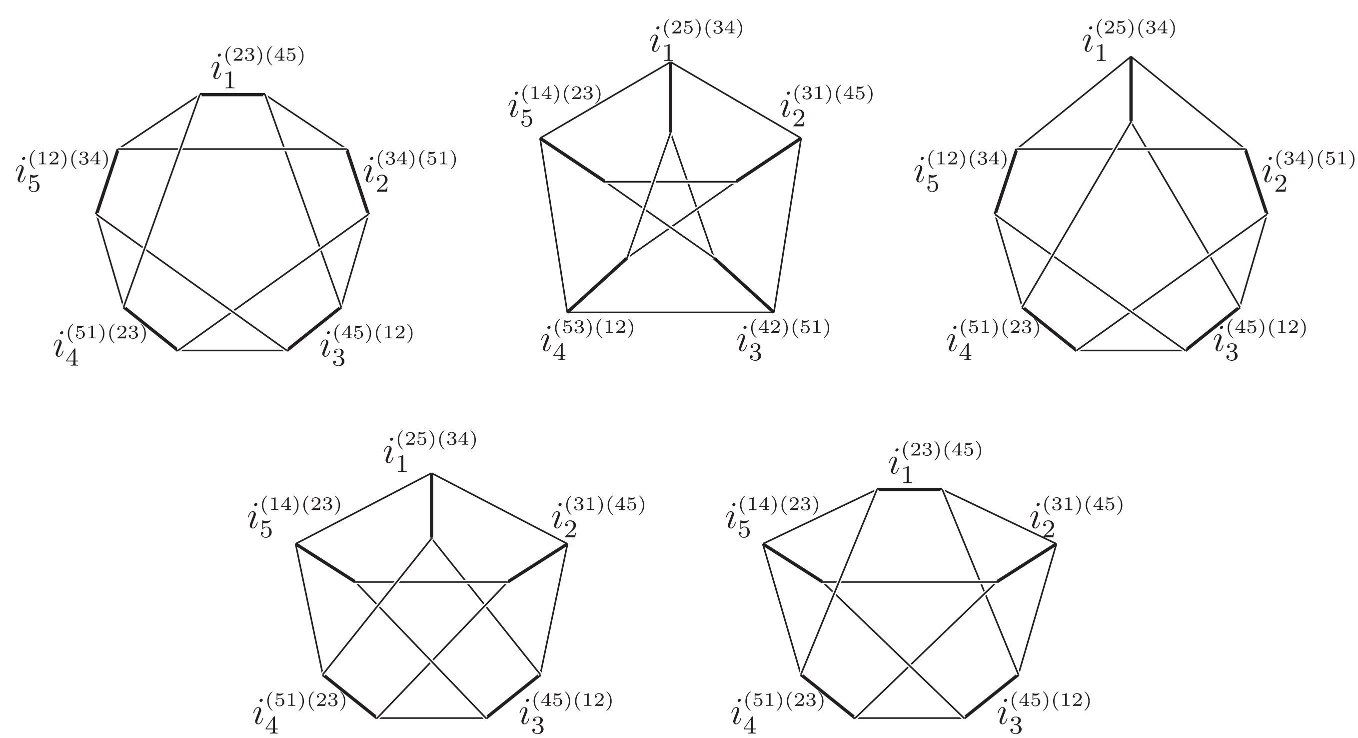

FIG. 1. The five classes of pairings: from the upper left: (10), (5, 5), (7, 3), (6, 4), and (4, 3, 3).

$$\tilde{\mathbf{G}}\,^{ii,kk}_{\mathbf{q}n,m} = \mathcal{N} j_0^2 \int d(\delta I_\alpha) e^{-(1/2j_0)M_{\alpha\beta}\delta I_\alpha \delta I_\beta} 2\delta j_{ni} 2\delta j_{mk} = \frac{4}{j^0} M^{-1}_{(ni)(mk)} = \begin{cases} \frac{4}{j^0} c_1 & \text{if } i=k \text{ or } i=m, \\ \frac{4}{j^0} c_3 & \text{otherwise.} \end{cases} \tag{58}$$

In this case $\tilde{G}$ gives immediately G since the permutation does not mix c_1 and c_3 terms.

Proceeding in the same way for the other cases, we get for the diagonal-nondiagonal term the two cases

$$\tilde{\mathbf{G}}^{ij,kk}_{\mathbf{q}n,m} = \frac{1}{j^0}\left(-2M^{-1}_{(mk)(ni)} - 2M^{-1}_{(mk)(nj)} + \frac{4}{\sqrt{3}} M^{-1}_{(mk)n}\right) = \begin{cases} -\frac{4}{j^0}(c_1 - \frac{1}{\sqrt{3}} c_7) & \text{if } i=k \text{ and } j=m, \\ -\frac{4}{j^0}(c_3 - \frac{1}{\sqrt{3}} c_7) & \text{if } i\neq k \text{ and } j\neq k, m, \\ -\frac{2}{j^0}(c_1 + c_3 - \frac{2}{\sqrt{3}} c_8) & \text{otherwise,} \end{cases} \tag{59}$$

and

$$\tilde{\mathbf{G}}\,^{ij,kk}_{\mathbf{q}n,m} = \frac{1}{j^0}\left(-M^{-1}_{(mk)(ni)} - M^{-1}_{(mk)(nj)} + M^{-1}_{(mk)(np)} + M^{-1}_{(mk)(nq)} - \frac{2}{\sqrt{3}} M^{-1}_{(mk)(n)}\right) \tag{60}$$

$$= \begin{cases} \frac{2}{j^0}(-c_1 + c_3 - \frac{1}{\sqrt{3}} c_8) & \text{if } i=k \text{ and } j=m \\ \frac{2}{j^0}(-c_3 + c_1 - \frac{1}{\sqrt{3}} c_8) & \text{if } i\neq k \text{ and } j\neq k, m \\ -\frac{2}{\sqrt{3}j^0} c_8 \quad \text{or} \quad -\frac{2}{\sqrt{3}j^0} c_7 & \text{otherwise,} \end{cases} \tag{61}$$

depending on the pairing of the node n. For the nondiagonal-nondiagonal terms, we have the three possibilities: diagonal double grasping on the two nodes

$$\tilde{\mathbf{G}}\,^{ij,kl}_{\mathbf{q}n,m} = \frac{1}{j_0}\left(\frac{4}{3} M^{-1}_{mn} - \frac{2}{\sqrt{3}}(M^{-1}_{nmk} + M^{-1}_{nml} + M^{-1}_{mni} + M^{-1}_{mnj}\right) + M^{-1}_{nimk} + M^{-1}_{niml} + M^{-1}_{njmk} + M^{-1}_{njml}); \tag{62}$$

diagonal double grasping on one node and nondiagonal on the other one

$$\tilde{\mathbf{G}}\,^{ij,kl}_{\mathbf{q}n,m} = \frac{1}{2j_0}\left(-\frac{4}{3} M^{-1}_{nm} + -\frac{2}{\sqrt{3}} M^{-1}_{nmk} - \frac{2}{\sqrt{3}} M^{-1}_{nml} + \frac{2}{\sqrt{3}} M^{-1}_{nmp} + \frac{2}{\sqrt{3}} M^{-1}_{nmq} + \frac{2}{\sqrt{3}} M^{-1}_{nim} + M^{-1}_{nimk} + M^{-1}_{niml} - M^{-1}_{nimp} \right.$$
$$\left. - M^{-1}_{nimq} + \frac{2}{\sqrt{3}} M^{-1}_{njm} + M^{-1}_{njmk} + M^{-1}_{njml} - M^{-1}_{njmp} - M^{-1}_{njmq}\right), \tag{63}$$

and nondiagonal on both nodes

$$\tilde{\mathbf{G}}^{ij,kl}_{\mathbf{q}n,m} = \frac{1}{4j_0}\left(\frac{4}{3} M^{-1}_{nm} + \frac{2}{\sqrt{3}} M^{-1}_{nmk} + \frac{2}{\sqrt{3}} M^{-1}_{nml} - \frac{2}{\sqrt{3}} M^{-1}_{nmp} - \frac{2}{\sqrt{3}} M^{-1}_{nmq} + \frac{2}{\sqrt{3}} M^{-1}_{nim} + M^{-1}_{nimk} + M^{-1}_{niml} - M^{-1}_{nimp} - M^{-1}_{nimq} \right.$$
$$+ \frac{2}{\sqrt{3}} M^{-1}_{njm} + M^{-1}_{njmk} + M^{-1}_{njml} - M^{-1}_{njmp} - M^{-1}_{njmq} - \frac{2}{\sqrt{3}} M^{-1}_{ne,m} - M^{-1}_{nemk} - M^{-1}_{neml} + M^{-1}_{nemp} + M^{-1}_{nemq}$$
$$\left. - \frac{2}{\sqrt{3}} M^{-1}_{nf,m} - M^{-1}_{nfmk} - M^{-1}_{nfml} + M^{-1}_{nfmp} + M^{-1}_{nfmq}\right), \tag{64}$$

whose expression in terms of the c coefficients in turn depends on pairings, and so on. Notice that only the seven parameters c_1, c_2, c_3 and c_5 c_6, c_7, c_8 enter the components of the propagator. The other one, namely c_4, does not, because we are only looking at the propagator between points on different tetrahedra.

The last step is to symmetrize the propagator under permutations. The only terms that change under permutations, at this point, are those due to the pairing. Hence, the only result of a sum over permutation is to combine the two coefficients c_7 and c_8, which are the only pairing dependent ones. For instance, a straightforward calculation gives the diagonal-nondiagonal term (which has the peculiarity of not depending on the pairing class)

$$\mathbf{G}^{ij,kk}_{\mathbf{q}n,m} = \sum_\sigma \tilde{\mathbf{G}}^{\sigma i\sigma j,\sigma k\sigma k}_{\mathbf{q}\sigma n,\sigma m} = \begin{cases} \frac{1}{3j_0}[4(-c_1+c_3)-4c_1+\frac{4}{\sqrt{3}}(c_7-c_8)] & \text{if } i=k \text{ and } j=m, \\ \frac{1}{3j_0}[4(-c_3+c_1)-4c_3+\frac{4}{\sqrt{3}}(c_7-c_8)] & \text{if } i\neq k \text{ and } j\neq k, m, \\ \frac{1}{3j_0}[-2(c_1+c_3)-\frac{2}{\sqrt{3}}(c_7-c_8)] & \text{otherwise.} \end{cases} \tag{65}$$

Notice that c_7 and c_8 enter this expression only through the linear combination $(c_7 - c_8)$.

The expression of the propagator components nondiagonal on both nodes depends on the class of pairing chosen for the state and the vertex. As an example we give the expressions of the two relevant cases for the first class of Fig. 1:

$$\mathbf{G}^{ij,ij}_{\mathbf{q}n,m} = \frac{1}{6}\left[c_5 - \frac{1}{\sqrt{3}}c_6 + \frac{7}{\sqrt{3}}c_7 - \frac{10}{\sqrt{3}}c_8 + \frac{11}{4}c_1 + \frac{3}{4}c_2 + \frac{1}{2}c_3\right] \tag{66}$$

and

$$\mathbf{G}^{ij,ik}_{\mathbf{q}n,m} = \frac{1}{120}\left[-10c_5 + \frac{10}{\sqrt{3}}c_6 - \frac{74}{\sqrt{3}}c_7 + \frac{104}{\sqrt{3}}c_8 - \frac{175}{2}c_1 + \frac{25}{2}c_2 + 115c_3\right]. \tag{67}$$

After the sum over permutations, the final expression of all components of the propagator depends on the parameters c_1, c_2, c_3, c_5, c_6, c_7, and c_8.

In conclusion, varying the parameters in the state $\mathbf{G}^{ij,kl}_{\mathbf{q}n,m}$ turns out to be a matrix with the symmetries of the 4-simplex, freely depending on seven arbitrary parameters. Can this give the same propagator as the linearized theory?

V. COMPARISON WITH THE LINEARIZED THEORY

The number of components of $\mathbf{G}^{ij,kl}_{\mathbf{q}n,m}$ is large, and it may seem hard to believe that the five-parameters freedom in the state could be sufficient to recover the tensorial structure of the linearized propagator. However, there are two properties of the propagator that strongly constrain it. First, the symmetrization of the 4-simplex symmetries largely reduces the number of independent components. Second, as proven in [4], the propagator satisfies the closure relation

$$\sum_i \mathbf{G}^{ij,kl}_{\mathbf{q}n,m} = 0. \tag{68}$$

[For instance this relation can be explicitly verified in a particular case using Eqs. (65) and (58).] Let us count the number of free parameters of an arbitrary tensor $\mathbf{G}^{ij,kl}_{\mathbf{q}n,m}$ satisfying these requirements. Using (68), we can always express a term in which any of the four indices i, j, k, l is equal to either n or m as sum of terms not of this kind. This reduces the independent terms to, say, $\mathbf{G}^{ij,kl}_{\mathbf{q}1,2}$ where $i, j, k, l = 3, 4, 5$. A few pictures and a moment of reflection will convince the reader that the only independent ones of these are

$$\mathbf{G}^{ii,ii}_{\mathbf{q}n,m}, \mathbf{G}^{ii,kk}_{\mathbf{q}n,m}, \mathbf{G}^{ij,kk}_{\mathbf{q}n,m}, \mathbf{G}^{ij,ij}_{\mathbf{q}n,m}, \mathbf{G}^{ij,ik}_{\mathbf{q}n,m}. \tag{69}$$

All the other terms can be obtained from these by a permutation of the indices. Therefore a tensor with these symmetries depends only on five parameters. This implies that adjusting five parameters in the state, we can match any such tensor, and in particular the propagator.

This can be checked by an explicit calculation of the propagator of the linearized theory in the harmonic gauge (on the compatibility of the radial and harmonic gauge, see [14]). The quantity $\mathbf{G}^{ij,kl}_{\mathbf{q}n,m}$ is the propagator projected in the directions normal to the faces of the tetrahedra. The $4d$ linearized graviton propagator is

$$G_{\mu\nu\rho\sigma} = \frac{1}{2L^2}(\delta_{\mu\rho}\delta_{\beta\gamma} + \delta_{\mu\sigma}\delta_{\beta\gamma} - \delta_{\mu\nu}\delta_{\rho\sigma}) \tag{70}$$

and its projection on the four linear dependent normals to the faces of each tetrahedron reads

$$G^{ij,kl}_{nm} \equiv G^{\mu\nu\rho\sigma}(n^{(i)}_n)_\mu(n^{(j)}_n)_\nu(n^{(k)}_m)_\rho(n^{(l)}_m)_\sigma. \tag{71}$$

We need the explicit expressions of the normals. To this aim, fix the coordinate of a 4-simplex giving the 5 vertices of a 4-simplex fixing the $4d$ vectors e^μ_I where μ is the $4d$ space index and I ($I = 1, \ldots, 5$) is the label of the vertex. The easiest way to deal with this $4d$ geometry is to introduce the bivectors $B^{\mu\nu}_{IJ}$

$$B^{\mu\nu}_{IJ} = e^\mu_K \wedge e^\nu_L + e^\mu_L \wedge e^\nu_M + e^\mu_M \wedge e^\nu_K, \tag{72}$$

where the indices $IJKLM$ form an even permutation of 1, 2, 3, 4, 5. If t_1 is the tetrahedron with vertices e_2, e_3, e_4, e_5, and so on cyclically, the bivector $B^{\mu\nu}_{nm}$ will be the bivector normal to the triangle t_{nm} shared by the tetrahedra t_n and t_m. The normal n^m_n to this triangle, in the 3 surface determined by the tetrahedron t_n is $(n^m_n)^\nu = B^{\mu\nu}_{nm}(t_n)_\mu$, where $(t_n)_\mu$ is the normal to the tetrahedron. Using this, it is a tedious but straightforward exercise to compute the components of the projected linearized propagator. Writing the bimatrix $G^{ij,kl}_{\text{linearized }1,2} = (G^{kl})^{ij}$, where $ijkl = 3, 4, 5$ we have

$$(G^{kl})^{ij} \sim \frac{1}{512}\begin{pmatrix} \begin{pmatrix} -16 & 6 & 6 \\ 6 & -28 & 16 \\ 6 & 16 & -28 \end{pmatrix} & \begin{pmatrix} 6 & 4 & -7 \\ 4 & 6 & -7 \\ -7 & -7 & 16 \end{pmatrix} & \begin{pmatrix} 6 & -7 & 4 \\ -7 & 16 & -7 \\ 4 & -7 & 6 \end{pmatrix} \\ \begin{pmatrix} 6 & 4 & -7 \\ 4 & 6 & -7 \\ -7 & -7 & 16 \end{pmatrix} & \begin{pmatrix} -28 & 6 & 16 \\ 6 & -16 & 6 \\ 16 & 6 & -28 \end{pmatrix} & \begin{pmatrix} 16 & -7 & -7 \\ -7 & 6 & 4 \\ -7 & 4 & 6 \end{pmatrix} \\ \begin{pmatrix} 6 & -7 & 4 \\ -7 & 16 & -7 \\ 4 & -7 & 6 \end{pmatrix} & \begin{pmatrix} 16 & -7 & -7 \\ -7 & 6 & 4 \\ -7 & 4 & 6 \end{pmatrix} & \begin{pmatrix} -28 & 16 & 6 \\ 16 & -28 & 6 \\ 6 & 6 & -16 \end{pmatrix} \end{pmatrix} \tag{73}$$

which displays the equality of the various terms. The five different components have values $(-16, 6, -28, -7, 4)/512$. A judicious choice of the parameters $c_2, c_3, c_5, (c_7 - c_8)$ can match these values. In particular, from (58) and (65) straightforward algebra gives $c_1 = -\frac{4}{512}$, $c_3 = -\frac{7}{512}$, $(c7 - c8) = \frac{2\sqrt{3}}{512}$, while the solution of (66) and (67) gives $c_2 = \frac{1}{512}\frac{2}{5}$, $c_5 = (\frac{1}{512}\frac{91}{5} + \frac{c_6}{\sqrt{3}} + \sqrt{3}c_7)$.

VI. CONCLUSION AND PERSPECTIVES

We have shown that a vertex with an appropriate asymptotic expansion, combined with a suitable boundary state, can yield the full tensorial structure of the propagator.

In doing so, we have also learned several lessons. The main lesson is that the noncommutativity of the angles requires a semiclassical state to have an oscillatory behavior in the intertwiners. In order to match this behavior, and approximate the semiclassical dynamics, the vertex must have a similar *oscillatory dependence on the intertwiners*. Such a phase factor allows the phase cancellation leading to Eq. (54), and hence prevents the suppression of the integral. The addition of this phase should not interfere with possible finiteness properties of the model [15].

The second lesson is that the symmetries of the boundary state must be considered with care, if we do not want to lose relevant dynamical information. Symmetrizing over the permutation of the vertices is a simple way of achieving a symmetric state without inserting additional unwanted symmetries. In doing so, however, one must take into account that a choice of pairing breaks the 4-simplex symmetry.

The most interesting open question, in our opinion, is whether other vertex amplitudes considered (such as [16]) and, in particular, the vertex amplitude recently studied in [6,7], satisfy the requirements for yielding the correct full tensorial structure of the graviton propagator. In particular, whether there is an oscillation in the intertwiners. This issue can be addressed analytically, via a saddle point analysis of the asymptotic of the new vertex, or numerically, using the technology developed in [17]. Some preliminary numerical indications appear to be optimistic [18]. Also, we think that the role of the five inequivalent structures illustrated in Fig. 1 deserve to be better understood.

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