PHYSICAL REVIEW D **77,** 044023 (2008)

Dilaton cosmology, noncommutativity, and generalized uncertainty principle

Babak Vakili*

Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran

(Received 15 December 2007; published 14 February 2008)

The effects of noncommutativity and of the existence of a minimal length on the phase space of a dilatonic cosmological model are investigated. The existence of a minimum length results in the generalized uncertainty principle (GUP), which is a deformed Heisenberg algebra between the minisuperspace variables and their momenta operators. I extend these deformed commutating relations to the corresponding deformed Poisson algebra. For an exponential dilaton potential, the exact classical and quantum solutions in the commutative and noncommutative cases, and some approximate analytical solutions in the case of GUP, are presented and compared.

DOI: 10.1103/PhysRevD.77.044023 PACS numbers: 04.60.−m, 04.60.Ds, 04.60.Kz

I. INTRODUCTION

Since cosmology can test physics at energies that are much higher than those which the experiments on Earth can achieve, it seems natural that the effects of quantum gravity could be observed in this context. Therefore, until a completely satisfactory theory regarding cosmology can be afforded by string theory, the study of the general properties of quantum gravity through cosmological systems such as the Universe seems reasonably promising, and in recent years many efforts have been made in cosmology from the string theory point of view [1–4]. In the pre−big bang scenario, based on the string effective action [5], the birth of the Universe is described by a transition from the string perturbative vacuum with weak coupling, low curvature and cold state to the standard radiation dominated regime, passing through a high curvature and strong coupling phase. This transition is made by the kinetic energy term of the dilaton, a scalar field with which the Einstein-Hilbert action of general relativity is augmented; see [6] for a more modern review of string dilaton cosmology. One of the major features of the solutions of equations of motion in string dilaton cosmology (see for example [7] for some exact solutions in dilaton cosmology) is the duality, so that if $a(t)$, the scale factor, solves the equations of motion, $1/a(t)$ is also a solution. This means that the whole Universe behaves like a string, i.e. has a minimal size of order of string scale and also a maximal size of order of the inverse of string scale.

The existence of a minimal length is one of the most important predictions of the theories which deal with quantum gravity [8]. From the perturbative string theory point of view, such a minimal length is due to the fact that the strings cannot probe distances smaller than the string size. One of the interesting features of the existence of a minimal length described above is the modification it makes to the standard commutation relation between position and momentum in usual quantum mechanics [9,10], which is called the generalized uncertainty principle (GUP). In one dimension the simplest form of such relations can be written as

$$\triangle p \triangle x \geq \frac{\hbar}{2}(1 + \beta(\triangle p)^2 + \gamma), \tag{1}$$

where β and γ are positive and independent of $\triangle x$ and $\triangle p$, but may in general depend on the expectation values $<x>$ and $<p>$. The usual Heisenberg commutation relation can be recovered in the limit $\beta = \gamma = 0$. As is clear from Eq. (1), this equation implies a minimum position uncertainty of $(\triangle x)_{\rm min} = \hbar\sqrt{\beta}$, and hence β must be related to the Planck length. Now, it is possible to realize Eq. (1) from the following commutation relation between position and momentum operators:

$$[x, p] = i\hbar(1 + \beta p^2), \tag{2}$$

where I take $\gamma = \beta < p >^2$. More general cases of such commutation relations are studied in Ref. [11].

One of interesting features of GUP in more than one dimension is that it implies naturally a noncommutative geometric generalization of position space [9]. Noncommutativity between space-time coordinates was first introduced by Snyder [12], and in more recent times a great deal of interest has been generated in this area of research [13–15]. This interest has been gathering pace in recent years because of strong motivations in the development of string and M-theories, [16,17]. However, noncommutative theories may also be justified in their own right because of the interesting predictions they have made in particle physics, a few examples of which are the IR/UV mixing and nonlocality [18], Lorentz violation [19], and new physics at very short distance scales [19–21]. Noncommutative versions of ordinary quantum [22] and classical mechanics [23,24] have also been studied and shown to be equivalent to their commutative versions if an external magnetic field is added to the Hamiltonian.

In cosmological systems, since the scale factors, matter fields, and their conjugate momenta play the role of dynamical variables of the system, introduction of noncommutativity by adopting the approach discussed above is

*b-vakili@sbu.ac.ir

particularly relevant. The resulting noncommutative classical and quantum cosmology of such models have been studied in different works [25]. These and similar works have opened a new window through which some of problems related to cosmology can be looked at and, hopefully, resolved. For example, an investigation of the cosmological constant problem can be found in [26]. In [27] the same problem is carried over to the Kaluza-Klein cosmology. The problem of compactification and stabilization of the extra dimensions in multidimensional cosmology may also be addressed using noncommutative ideas in [28].

In this paper I deal with noncommutativity and GUP in a dilaton cosmological model with an exponential dilaton potential, and to facilitate solutions for the case under consideration, I choose a suitable metric. My approach to GUP is through its introduction in phase space constructed by minisuperspace fields and their conjugate momenta [29]. In general GUP in its original form (see [9,10]) implies a noncommutative underlying geometry for space-time. But formulation of gravity in a noncommutative space-time is highly nonlinear and setting up cosmological models is not an easy task. Here my aim is to study some aspects regarding the application of the GUP framework in quantum cosmology, i.e., in the context of a minisuperspace reduction of the dynamics. As is well-known in the minisuperspace approach of quantum cosmology, which is based on the canonical quantization procedure, one first freezes a large number of degrees of freedom by imposition of symmetries on the spatial part of the metric and then quantizes the remaining ones. Therefore, in the absence of a full theory of quantum gravity, quantum cosmology is a quantum mechanical toy model with finite degrees of freedom which is a simple arena to test ideas and constructions which can be introduced in quantum general relativity. In this respect, the GUP approach to quantum cosmology appears to have physical grounds. In fact, one notes that a deformation of the canonical Heisenberg algebra immediately leads to a generalized uncertainty principle. In other words, the GUP scheme relies on a modification of the canonical quantization prescriptions and, in this respect, it can be reliably applied to any dynamical system (see [30] for a more clear explanation on the GUP in the minisuperspace dynamics). Since my model has 2 degrees of freedom, the scale factor a and the dilaton ϕ, with a change of variables, I have a set of dynamical variables (x, y), which are suitable candidates for introducing noncommutativity and GUP in the phase space of the problem at hand. I present exact solutions of classical and quantum commutative and noncommutative cosmology. Also in the case when the minisuperspace variables obey the GUP commutating relations, I obtain approximate analytical solutions for the corresponding classical and quantum cosmology. Finally, I compare and contrast these solutions at both classical and quantum levels.

II. THE MODEL

In $D = 4$ dimensions the lowest order gravi-dilaton effective action, in the string frame, can be written as [31]

$$\mathcal{S} = -\frac{1}{2\lambda_s}\int d^4x\sqrt{-g}e^{-\phi}(\mathcal{R} + \partial_\mu\phi\partial^\mu\phi + V(\phi)), \tag{3}$$

where ϕ is the dilaton field, λ_s is the fundamental string length l_s parameter, and $V(\phi)$ is the dilaton potential. In the string frame the fundamental unit is the string length l_s, and thus the Planck mass, which is the effective coefficient of the Ricci scalar $\mathcal{R}$, varies with the dilaton. One can also write the action in the Einstein frame, for which the fundamental unit is the Planck length. Since the Planck length is more appropriate for my purpose, I prefer to work in the Einstein frame. In [4], it is shown in detail that the action (3) in the Einstein frame takes the form

$$\mathcal{S} = -\frac{M_4^2}{2}\int d^4x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right), \tag{4}$$

where now all quantities in the action are in the Einstein frame. I consider a spatially flat Friedmann-Robertson-Walker (FRW) space-time which, following [32], is specified by the metric

$$ds^2 = -\frac{N^2(t)}{a^2(t)}dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{5}$$

Here $N(t)$ is the lapse function and $a(t)$ represents the scale factor of the Universe. The square of the scale factor dividing the lapse function turns out to simplify the calculations and makes the Hamiltonian quadratic. Now, it is easy to show that the effective Lagrangian of the model can be written in the form

$$\mathcal{L} = \frac{1}{N}\left(-\frac{1}{2}a^2\dot{a}^2 + \frac{1}{2}a^4\dot{\phi}^2\right) - Na^2V(\phi). \tag{6}$$

To simplify the above Lagrangian, let me introduce a new set of variables [33]

$$x = \frac{a^2}{2}\cosh\alpha\phi, \qquad y = \frac{a^2}{2}\sinh\alpha\phi, \tag{7}$$

where α is a positive constant. In terms of these new variables the Lagrangian (6) takes the form

$$\mathcal{L} = \frac{1}{2N}(\dot{y}^2 - \dot{x}^2) - 2N(x - y)e^{\alpha\phi}V(\phi). \tag{8}$$

From now on, I choose an exponential potential

$$V(\phi) = \frac{V_0}{2}e^{-\alpha\phi}, \tag{9}$$

which simplifies the last term in the Lagrangian (8) leading to

$$\mathcal{L} = \frac{1}{2N}(\dot{y}^2 - \dot{x}^2) - NV_0(x - y), \tag{10}$$

with the corresponding Hamiltonian constraint written as

$$\mathcal{H} = -\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V_0(x - y). \tag{11}$$

Note that the minisuperspace of the above model is a two-dimensional manifold $0 < a < \infty$, $-\infty < \phi < +\infty$. According to [34], its nonsingular boundary is the line $a = 0$ with $|\phi| < +\infty$, while at the singular boundary, at least one of the two variables is infinite. In terms of the variables x and y, introduced in (7), the minisuperspace is recovered by $x > 0$, $x > |y|$, and the nonsingular boundary may be represented by $x = y = 0$.

III. CLASSICAL COSMOLOGY

The classical and quantum solutions of the model described by Hamiltonian (11) can be easily obtained. Since my aim here is to compare the commutative solutions with noncommutative and GUP solutions, in what follows I consider commutative, noncommutative, and GUP classical cosmologies, and compare the results with each other. In the next section I shall deal with the quantum cosmology of the model.

A. Commutative case

The Poisson brackets for the classical phase-space variables are

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}, \tag{12}$$

where $x_i(i = 1, 2) = x, y$ and $p_i(i = 1, 2) = p_x, p_y$. Therefore, the equation of motion becomes (in $N = 1$ gauge)

$$\dot{x} = \{x, \mathcal{H}\} = -p_x, \qquad \dot{p}_x = \{p_x, \mathcal{H}\} = -V_0, \tag{13}$$

$$\dot{y} = \{y, \mathcal{H}\} = p_y, \qquad \dot{p}_y = \{p_y, \mathcal{H}\} = V_0. \tag{14}$$

Equations (13) and (14) can be immediately integrated to yield

$$x(t) = \frac{1}{2}V_0t^2 - p_{0x}t + x_0, \qquad p_x(t) = -V_0t + p_{0x}, \tag{15}$$

$$y(t) = \frac{1}{2}V_0t^2 + p_{0y}t + y_0, \qquad p_y(t) = V_0t + p_{0y}. \tag{16}$$

Now, these solutions must satisfy the zero energy condition, $\mathcal{H} = 0$. Thus, substitution of Eqs. (15) and (16) into (11) gives a relation between integration constants as

$$p_{0y}^2 - p_{0x}^2 = 2V_0(y_0 - x_0). \tag{17}$$

Equations (15) and (16) are like the equation of motion for a particle moving in a plane with its acceleration components equal to V_0, while $-p_x(t)$ and $p_y(t)$ play the role of its velocity. Note that the condition $x > 0$ implies that $p_{0x}^2 - 2V_0x_0 < 0$, thus, Eq. (17) results in $p_{0y}^2 - 2V_0y_0 < 0$, which means that $y > 0$. Therefore, in classical cosmology only half of the minisuperspace $x > y > 0$ or ($a > 0$, $\phi > 0$) is recovered by the dynamical variables $x(t)$ and $y(t)$. Now, using relations (7) one can find the scale factor and dilaton field as [to get a more simple form I take $x_0 = y_0$ and $p_{0x} = p_{0y}$ which of course satisfy the condition (17)]

$$a(t) = [8|p_{0x}|V_0t^3 + 16x_0|p_{0x}|t]^{1/4}, \tag{18}$$

$$\phi(t) = \frac{1}{2\alpha}\ln\left(\frac{V_0t^2 + 2x_0}{2|p_{0x}|}\right). \tag{19}$$

The limiting behavior of $a(t)$ and $\phi(t)$ in the early and late times is then as follows:

$$a(t) \sim t^{1/4}, \qquad \phi(t) \sim \text{const.}, \qquad t \ll 1, \tag{20}$$

$$a(t) \sim t^{3/4}, \qquad \phi(t) \sim \ln t, \qquad t \gg 1. \tag{21}$$

A remark about the above analysis is that I use a nonstandard parametrization of the FRW metric; this is done in order to simplify the calculations and have a manageable Lagrangian for the noncommutative deformation. As is well-known usually the introduction of the lapse function gives a new parametrization of time, but if $N(t) = 1$ one returns to the usual cosmic time where in my parametrization this is not the case. Therefore, let me translate these results in terms of the cosmic time τ. Using its relationship with my time parameter t, that is

$$d\tau = \frac{1}{a(t)}dt, \tag{22}$$

I obtain

$$\tau \sim t^{3/4}, \qquad t \ll 1, \quad \text{and} \quad \tau \sim t^{1/4}, \qquad t \gg 1. \tag{23}$$

Therefore, the behavior of the scale factor and the dilatonic field in the early and late (cosmic) times is as

$$a(\tau) \sim \tau^{1/3}, \qquad \phi(\tau) \sim \text{const.}, \qquad \tau \ll 1, \tag{24}$$

$$a(\tau) \sim \tau^3, \qquad \phi(\tau) \sim \ln\tau, \qquad \tau \gg 1. \tag{25}$$

I see that in the usual commutative phase space of my model the scale factor has a decelerated expansion in early times while it undergoes an accelerated phase in its late time evolution due to a constant and growing with time dilatonic field, respectively. These results are comparable with those that are presented in the last paper of [25] in which the authors used the gauge $d\tau = a^3dt$.

B. Noncommutative case

Let me now concentrate on the noncommutativity concepts in classical cosmology. Noncommutativity in classical physics [23] is described by a deformed product, also known as the Moyal product law between two arbitrary functions of position and momentum as

$$(f *_\alpha g)(x) = \exp\left[\frac{1}{2}\alpha^{ab}\partial_a^{(1)}\partial_b^{(2)}\right]f(x_1)g(x_2)|_{x_1=x_2=x}, \tag{26}$$

such that

$$\alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij}+\sigma_{ij} \\ -\delta_{ij}-\sigma_{ij} & \beta_{ij} \end{pmatrix}, \tag{27}$$

where the $N \times N$ matrices θ and β are assumed to be antisymmetric with $2N$ being the dimension of the classical phase space, represents the noncommutativity in coordinates and momenta, respectively. With this product law, the deformed Poisson brackets can be written as

$$\{f, g\}_\alpha = f *_\alpha g - g *_\alpha f. \tag{28}$$

A simple calculation shows that

$$\{x_i, x_j\}_\alpha = \theta_{ij}, \qquad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij}, \qquad \{p_i, p_j\}_\alpha = \beta_{ij}. \tag{29}$$

Now, consider the following transformations on the classical phase space:

$$x_i' = x_i - \frac{1}{2}\theta_{ij}p^j, \qquad p_i' = p_i + \frac{1}{2}\beta_{ij}x^j. \tag{30}$$

It can easily be checked that if (x_i, p_j) obey the usual Poisson algebra (12), then

$$\{x_i', x_j'\} = \theta_{ij}, \qquad \{x_i', p_j'\} = \delta_{ij} + \sigma_{ij}, \qquad \{p_i', p_j'\} = \beta_{ij}, \tag{31}$$

where $\sigma_{ij} = -\frac{1}{8}(\theta_i^k\beta_{kj} + \beta_i^k\theta_{kj})$. These commutative relations are the same as (29). Consequently, for introducing noncommutativity, it is more convenient to work with Poisson brackets (31) than α-star deformed Poisson brackets (29). It is important to note that the relations represented by Eq. (29) are defined in the spirit of the Moyal product given above. However, in the relations defined by (31), the variables (x_i, p_j) obey the usual Poisson bracket relations so that the two sets of deformed and ordinary Poisson brackets represented by relations (29) and (31) should be considered as distinct.

In this work I consider a noncommutative phase space in which $\beta_{ij} = 0$ and so that $\sigma_{ij} = 0$, i.e. the Poisson brackets of the phase-space variables are as follows:

$$\{x_{nc}, y_{nc}\} = \theta, \qquad \{x_{inc}, p_{jnc}\} = \delta_{ij}, \qquad \{p_{inc}, p_{jnc}\} = 0. \tag{32}$$

With the noncommutative phase space defined above, I consider the Hamiltonian of the noncommutative model as having the same functional form as Eq. (11), but in which the dynamical variables satisfy the above-deformed Poisson brackets, that is

$$\mathcal{H}_{nc} = -\frac{1}{2}p_{xnc}^2 + \frac{1}{2}p_{ync}^2 + V_0(x_{nc} - y_{nc}). \tag{33}$$

Therefore, the equations of motion read

$$\dot{x_{nc}} = \{x_{nc}, \mathcal{H}_{nc}\} = -p_{xnc} - \theta V_0, \qquad \dot{p_{xnc}} = \{p_{xnc}, \mathcal{H}_{nc}\} = -V_0, \tag{34}$$

$$\dot{y_{nc}} = \{y_{nc}, \mathcal{H}_{nc}\} = p_{ync} - \theta V_0, \qquad \dot{p_{ync}} = \{p_{ync}, \mathcal{H}_{nc}\} = V_0. \tag{35}$$

The above equations are similar to Eqs. (13) and (14) in the commutative case. Their solutions are therefore as follows:

$$x_{nc} = \frac{1}{2}V_0t^2 - (p_{0x} + \theta V_0)t + x_0, \qquad p_{xnc} = -V_0t + p_{0x}, \tag{36}$$

$$y_{nc} = \frac{1}{2}V_0t^2 + (p_{0y} - \theta V_0)t + y_0, \qquad p_{ync} = V_0t + p_{0y}. \tag{37}$$

The requirement that these solutions must satisfy the noncommutative Hamiltonian constraint $\mathcal{H}_{nc} = 0$, gives again the relation (17) between integration constants. As mentioned before, instead of dealing with the noncommutative variables I can construct, with the help of transformations (30), a set of commutative dynamical variables x, y obeying the usual Poisson brackets (12) which, for the problem at hand read

$$p_{x_{nc}} = p_x, \qquad p_{y_{nc}} = p_y, \qquad x_{nc} = x - \frac{1}{2}\theta p_y, \qquad y_{nc} = y + \frac{1}{2}\theta p_x. \tag{38}$$

In terms of these commutative variables the Hamiltonian takes the form

$$\mathcal{H} = -\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V_0(x - y) - \frac{1}{2}\theta V_0(p_x + p_y). \tag{39}$$

Therefore, I have the following equations of motion

$$\dot{x} = \{x, \mathcal{H}\} = -p_x - \frac{1}{2}\theta V_0, \qquad \dot{p}_x = \{p_x, \mathcal{H}\} = -V_0, \tag{40}$$

$$\dot{y} = \{y, \mathcal{H}\} = p_y - \frac{1}{2}\theta V_0, \qquad \dot{p}_y = \{p_y, \mathcal{H}\} = V_0. \tag{41}$$

The solutions of the above equations can be straightforwardly obtained in the same manner as that of system (13) and (14). It is easy to check that the action of transformations (38) on the solutions of system (40) and (41) is to recover solutions (36) and (37). One sees that the effects of the noncommutative parameter θ appear only in the initial velocity of the evolution. This means that noncommutativity in phase space shows itself in the early epoch of the cosmic evolution and when time grows the differences between commutative solutions (15) and (16) and noncommutative solutions (36) and (37) disappear. To make this issue more clear, let me return to the variables $a(t)$ and $\phi(t)$ using the transformation (7). Choosing again $x_0 = y_0$ and $p_{0x} = p_{0y}$ I obtain

$$a_{nc}(t) = [8|p_{0x}|V_0 t^3 + 16\theta|p_{0x}|t^2 + 16x_0|p_{0x}|t]^{1/4}, \tag{42}$$

$$\phi_{nc}(t) = \frac{1}{2\alpha}\ln\left|\frac{V_0 t^2 - 2\theta V_0 t + 2x_0}{2p_{0x}}\right|. \tag{43}$$

The late time ($t \gg 1$) behavior of $a_{nc}(t)$ and $\phi_{nc}(t)$ is the same as (21). On the other hand in the regime $t \ll 1$, considering the θ-term in (42) and (43) I obtain

$$a_{nc}(t) \sim \theta^{1/4}t^{1/2}, \qquad \phi_{nc}(t) \sim \ln(\theta t), \qquad t \ll 1. \tag{44}$$

In this limit the cosmic time $d\tau = \frac{1}{a}dt$ takes the form

$$\tau \sim \theta^{-1/4}t^{1/2}, \tag{45}$$

and then the early (cosmic) time behavior of the scale factor and the dilatonic field is as follows:

$$a_{nc}(\tau) \sim \theta^{1/2}\tau, \qquad \phi_{nc}(\tau) \sim \ln(\theta^{3/2}\tau^2), \qquad \tau \ll 1. \tag{46}$$

I see that noncommutativity causes a uniform expansion (not decelerated expansion) in the early times of cosmic evolution.

C. Classical cosmology with GUP

In more than one dimension a natural generalization of Eq. (2) is defined by the following commutation relations [9]:

$$[x_i, p_j] = i(\delta_{ij} + \beta\delta_{ij}p^2 + \beta' p_i p_j), \tag{47}$$

where $p^2 = \sum p_i p_i$ and $\beta, \beta' > 0$ are considered as small quantities of first order. Also, assuming that

$$[p_i, p_j] = 0, \tag{48}$$

the commutation relations for the coordinates are obtained as

$$[x_i, x_j] = i\frac{(2\beta - \beta') + (2\beta + \beta')\beta p^2}{1 + \beta p^2}(p_i x_j - p_j x_i). \tag{49}$$

As it is clear from the above expression, the coordinates do not commute. This means that to construct the Hilbert space representations, one cannot work in position space. It is therefore more convenient to work in momentum space. However, since in quantum cosmology the wave function of the Universe in momentum space has no suitable interpretation, I restrict myself to the special case $\beta' = 2\beta$. As one can see immediately from Eq. (49), the coordinates commute to first order in β and thus a coordinate representation can be defined. Now, it is easy to show that the following representation of the momentum operator in position space satisfies relations (47) and (48) (with $\beta' = 2\beta$) to first order in β

$$p_i = -i\left(1 - \frac{\beta}{3}\frac{\partial^2}{\partial x_i^2}\right)\frac{\partial}{\partial x_i}. \tag{50}$$

A comment on the above issue is that applying the GUP to a curved background such as a cosmological model needs some modifications [35]. Here, since I apply the GUP to the minisuperspace variables x, y which correspond to a Minkowskian metric, I can safely use the above expressions without any modifications. Now, it is possible to realize Eqs. (47)–(50) from the following commutation relations between position and momentum operators:

$$[x, p_x] = i(1 + \beta p^2 + 2\beta p_x^2), \qquad [y, p_y] = i(1 + \beta p^2 + 2\beta p_y^2), \tag{51}$$

$$[x, p_y] = [y, p_y] = 2i\beta p_x p_y, \tag{52}$$

$$[x_i, x_j] = [p_i, p_j] = 0, \qquad x_i(i = 1, 2) = x, y, \qquad p_i(i = 1, 2) = p_x, p_y. \tag{53}$$

Now, before quantizing the model in the GUP framework in the next section, I would like to investigate the effects of the classical version of GUP, i.e., the classical version of commutation relations (51)–(53) on the above cosmology. As is well-known, in the classical limit the quantum mechanical commutators should be replaced by the classical Poisson brackets as $[P, Q] \to i\hbar\{P, Q\}$. Thus, the GUP in classical phase space changes the Poisson algebra (12) into their deformed forms as[1]

$$\{x, p_x\} = 1 + \beta p^2 + 2\beta p_x^2, \qquad \{y, p_y\} = 1 + \beta p^2 + 2\beta p_y^2, \tag{54}$$

$$\{x, p_y\} = \{y, p_x\} = 2\beta p_x p_y, \tag{55}$$

[1]Such deformed Poisson algebra is used in [36] to investigate effects of the deformation on the classical orbits of particles in a central force field and on the Kepler third law. Also, the stability of planetary circular orbits in the framework of such deformed Poisson brackets is considered in [37]. Note that here I deal with modifications of a classical cosmology that become important only at the Planck scale, where the classical description is no longer appropriate and a quantum model is required. However, before quantizing the model I shall provide a deformed classical cosmology. In this classical description of the Universe in transition from commutation relation (2) to its Poisson bracket counterpart I keep the parameter β fixed as $\hbar \to 0$. In string theory this means that the string momentum scale is fixed when its length scale approaches the zero.

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \qquad x_i(i = 1, 2) = x, y, \qquad p_i(i = 1, 2) = p_x, p_y. \tag{56}$$

Therefore, the equations of motion read

$$\dot{x} = \{x, \mathcal{H}\} = -p_x(1 - \beta p^2), \qquad \dot{p_x} = \{p_x, \mathcal{H}\} = -V_0[1 + \beta(p_y - p_x)^2], \tag{57}$$

$$\dot{y} = \{y, \mathcal{H}\} = p_y(1 + 3\beta p^2), \qquad \dot{p_y} = \{p_y, \mathcal{H}\} = V_0[1 + \beta(p_y - p_x)(3p_y + p_x)]. \tag{58}$$

I see that the deformed classical cosmology forms a system of nonlinear coupled differential equations, which are not easy to solve. Thus, to simplify it, I may make some approximations. From Eqs. (57) and (58) I get

$$\dot{p_x} + \dot{p_y} = 2\beta V_0(p_y^2 - p_x^2), \tag{59}$$

if in the first approximation I neglect the right-hand side of the above equation, I obtain

$$\dot{p_x} + \dot{p_y} = 0 \Rightarrow p_x + p_y = p_0 = \text{Const.} \tag{60}$$

Substituting this result in Eqs. (57) and (58), I am led to the following decoupled equations for p_x and p_y:

$$\dot{p_x} = -V_0[1 + \beta(p_0 - 2p_x)^2], \tag{61}$$

$$\dot{p_y} = V_0[1 + \beta(4p_y^2 - p_0^2)], \tag{62}$$

which are immediately integrable with the results

$$p_x(t) = \frac{1}{2}p_0 - \frac{1}{2\sqrt{\beta}}\tan 2\sqrt{\beta}V_0(t + t_0), \tag{63}$$

$$p_y(t) = \frac{1}{2}p_0 + \frac{1}{2\sqrt{\beta}}\tan 2\sqrt{\beta}V_0(t + t_0). \tag{64}$$

Substituting these results into the first equations of the system (57) and (58), I can obtain $x(t)$ and $y(t)$ as

$$x(t) = \frac{p_0}{4}(p_0^2\beta - 3)t + \frac{p_0}{8V_0\sqrt{\beta}}\tan 2V_0\sqrt{\beta}(t + t_0) + \left(\frac{p_0^2}{8V_0} - \frac{3}{8V_0\beta}\right)\ln[\cos 2V_0\sqrt{\beta}(t + t_0)] - \frac{1}{16\beta V_0}\tan^2 2V_0\sqrt{\beta}(t + t_0), \tag{65}$$

$$y(t) = \frac{p_0}{4}(3p_0^2\beta - 1)t + \frac{3p_0}{8V_0\sqrt{\beta}}\tan 2V_0\sqrt{\beta}(t + t_0) - \left(\frac{3p_0^2}{8V_0} - \frac{1}{8V_0\beta}\right)\ln[\cos 2V_0\sqrt{\beta}(t + t_0)] + \frac{3}{16\beta V_0}\tan^2 2V_0\sqrt{\beta}(t + t_0). \tag{66}$$

It is easy to see that in the limit $\beta \rightarrow 0$, with a suitable choice of t_0 in terms of p_{0x}, p_{0y}, and V_0, one can recover the ordinary classical cosmology (15) and (16). A comment on the above solutions is that the effects of GUP are important not only in the early but also at late times of the cosmic evolution. In fact, these solutions show that in the GUP framework the quantum gravitational effects may be detected also in large scales.

IV. QUANTIZATION OF THE MODEL

Now, let me quantize the model described above. As in the classical cosmology, here for comparison purposes between ordinary commutative, noncommutative, and GUP, I study the quantum cosmology of the model in these frameworks separately and compare the results.

A. Commutative quantum cosmology

I first discuss the commutative quantum cosmology of my model. For this purpose I quantize the dynamical variables of the model with the use of a canonical quantization procedure that leads to the Wheeler-DeWitt (WD) equation, $\mathcal{H}\Psi = 0$. Here, $\mathcal{H}$ is the operator form of the Hamiltonian given by (11), and Ψ is the wave function of the Universe, a function of spatial geometry and matter fields, if they exist. With replacement $p_x \rightarrow -i\partial/\partial x$ and similarly for p_y in (11), the WD equation reads

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2V_0(x - y)\right]\Psi(x, y) = 0. \tag{67}$$

The solutions of the above differential equation are separable and may be written in the form $\Psi(x, y) = X(x)Y(y)$, leading to

$$\frac{d^2X}{dx^2} + (2V_0x - \nu)X = 0, \qquad \frac{d^2Y}{dy^2} + (2V_0y - \nu)Y = 0, \tag{68}$$

where ν is a separation constant. Eqs. (68) have well-known solutions in terms of Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. The functions $\mathrm{Bi}(z)$ are usually omitted because of their divergent behavior in the limit $z \rightarrow \infty$. Therefore, the eigenfunctions of the WD equation can be written as

$$\Psi_\nu(x, y) = \mathrm{Ai}\left(\frac{\nu - 2V_0x}{(2V_0)^{2/3}}\right)\mathrm{Ai}\left(\frac{\nu - 2V_0y}{(2V_0)^{2/3}}\right). \tag{69}$$

Now, I impose the boundary condition on these solutions such that at the nonsingular boundary (at $a = 0$ and $|\phi| < \infty$) the wave function vanishes [34],

$$\Psi(a = 0, \phi) = 0 \Rightarrow \Psi(x = 0, y = 0) = 0, \tag{70}$$

which yields

$$\mathrm{Ai}\left(\frac{\nu}{(2V_0)^{2/3}}\right) = 0 \Rightarrow \nu_n = (2V_0)^{2/3}\alpha_n, \tag{71}$$

where α_n is the nth zero of the Airy function $\mathrm{Ai}(z)$. I may now write the general solution of the WD equation as a

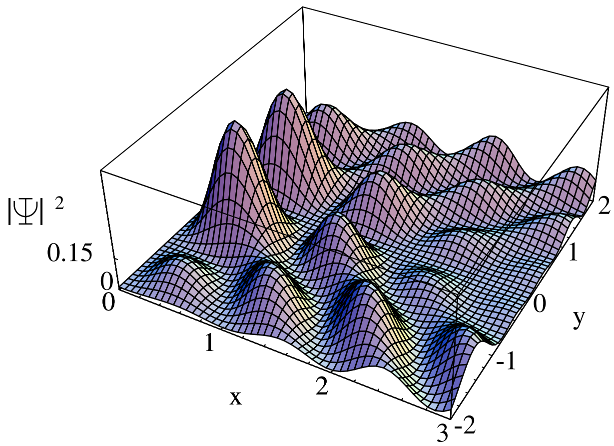

FIG. 1 (color online). The square of the wave function in the commutative case. I take the numerical value $V_0 = 1$.

superposition of its eigenfunctions

$$\Psi(x, y) = \sum_{n=1}^{\infty} c_n \mathrm{Ai}\left(\frac{\nu_n - 2V_0 x}{(2V_0)^{2/3}}\right)\mathrm{Ai}\left(\frac{\nu_n - 2V_0 y}{(2V_0)^{2/3}}\right). \quad (72)$$

Figure 1 shows the square of the wave function of the commutative quantum Universe. As is clear from this figure the wave function peaks symmetrically around $y = 0$. The largest peaks correspond to some nonzero values x_0 for x and $\pm y_0$ for y. This means that there are different possible states (corresponding to positive and negative dilaton) from which our present Universe could have evolved and tunneled in the past, from one state to another.

B. Noncommutative quantum cosmology

To study noncommutativity at the quantum level, I follow the same procedure as before, namely, the canonical transition from classical to quantum mechanics by replacing the Poisson brackets with the corresponding Dirac commutators $\{,\} \to -i[,]$. Thus, the commutation relations between my dynamical variables should be modified as follows:

$$[x_{nc}, y_{nc}] = i\theta, [x_{nc}, p_x] = [y_{nc}, p_y] = i. \quad (73)$$

The corresponding WD equation can be obtained by modification of the operator product in (67) with the Moyal deformed product [25]

$$\left[-\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V_0(x - y)\right] * \Psi(x, y) = 0, \quad (74)$$

Using the definition of the Moyal product (26), it may be shown that

$$f(x, y) * \Psi(x, y) = f(x_{nc}, y_{nc})\Psi(x, y), \quad (75)$$

where the relations between the noncommutative variables x_{nc}, y_{nc} and commutative variables x, y are given by (38). Therefore, the noncommutative version of the WD equation can be written as

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + i\theta V_0\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + 2V_0(x - y)\right]\Psi(x, y) = 0. \quad (76)$$

I again separate the solutions into the form $\Psi(x, y) = X(x)Y(y)$, which leads to the following equations for the functions $X(x)$ and $Y(y)$ with a separation constant ν:

$$\frac{d^2X}{dx^2} + i\theta V_0\frac{dX}{dx} + (2V_0 x - \nu)X = 0,$$
$$\frac{d^2Y}{dy^2} - i\theta V_0\frac{dY}{dy} + (2V_0 y - \nu)Y = 0. \quad (77)$$

The solutions of Eq. (77) can be written in terms of Airy functions as

$$X(x) = e^{-(i/2)V_0\theta x}\mathrm{Ai}\left(\frac{\nu - \frac{1}{4}V_0^2\theta^2 - 2V_0 x}{(2V_0)^{2/3}}\right),$$
$$Y(y) = e^{(i/2)V_0\theta y}\mathrm{Ai}\left(\frac{\nu - \frac{1}{4}V_0^2\theta^2 - 2V_0 y}{(2V_0)^{2/3}}\right), \quad (78)$$

where to recover the commutative solutions in the case of $\theta = 0$, I have omitted the functions $\mathrm{Bi}(z)$. Thus the eigenfunctions of the noncommutative WD equation are as follows:

$$\Psi_\nu(x, y) = e^{(i/2)V_0\theta(y-x)}\mathrm{Ai}\left(\frac{\nu - \frac{1}{4}V_0^2\theta^2 - 2V_0 x}{(2V_0)^{2/3}}\right) \times \mathrm{Ai}\left(\frac{\nu - \frac{1}{4}V_0^2\theta^2 - 2V_0 y}{(2V_0)^{2/3}}\right). \quad (79)$$

Note that in the context of my noncommutative model choosing the boundary condition (70) is not trivial and instead I construct the general solution of the WD equation

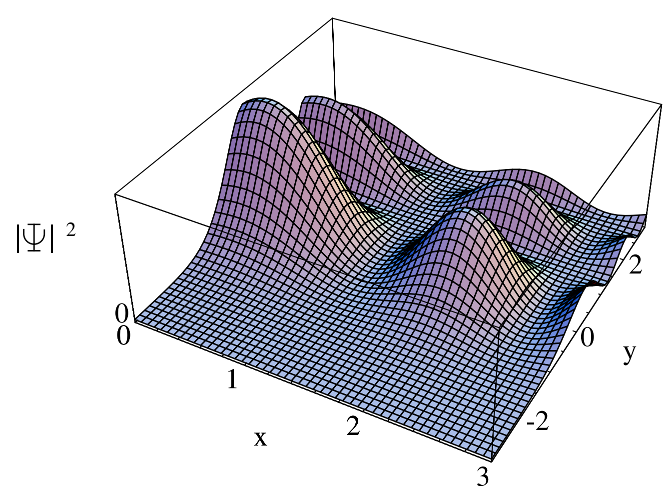

FIG. 2 (color online). The square of the wave function in the noncommutative case. I take the numerical values $V_0 = 1$ and $\theta = 2$.

as a superposition of eigenfunctions in the form

$$\Psi(x, y) = \int_{-\infty}^{+\infty} C(\nu)\Psi_\nu(x, y)d\nu, \tag{80}$$

where $C(\nu)$ can be chosen as a shifted Gaussian weight function $e^{-a(\nu-b)^2}$; see Ref. [25]. Figure 2 shows the square of the wave function in the noncommutative case. I see that in this case the peaks occur only for positive values of y (or positive values of dilaton). Also, noncommutivity causes a shift in the minimum of the values of x corresponding to the spacial volume.

C. Quantum cosmology in GUP framework

In this subsection I focus attention on the study of the quantum cosmology of my model based on the GUP formalism reviewed in the previous section. The corresponding commutation relations are given by (51)–(53). As I have mentioned in the previous section, in the special case when $\beta' = 2\beta$, I have the following representations for p_x and p_y in the $x - y$ space which fulfill the commutation relations (51)–(53):

$$p_x = -i\left(1 - \frac{\beta}{3}\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}, \qquad p_y = -i\left(1 - \frac{\beta}{3}\frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial y}. \tag{81}$$

Now, using these representations for the momenta in Hamiltonian (11), the WD equation can be written, up to the first order in β as

$$\left[-\frac{2}{3}\beta\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} + \frac{2}{3}\beta\frac{\partial^4}{\partial y^4} - \frac{\partial^2}{\partial y^2} + 2V_0(x - y)\right]\Psi(x, y) = 0. \tag{82}$$

I again separate the solutions into the form $\Psi(x, y) = X(x)Y(y)$, leading to

$$-\frac{2}{3}\beta\frac{d^4Z_i}{dz_i^4} + \frac{d^2Z_i}{dz_i^2} + (2V_0z_i - \nu)Z_i = 0,$$
$$Z_i(i = 1, 2) = X, Y, \qquad z_i(i = 1, 2) = x, y, \tag{83}$$

where ν is the separation constant as before. I cannot solve the above fourth order equations analytically, but I can provide an approximation method which in its domain of validity, I need to solve a second order differential equation. Taking $\beta = 0$ in Eq. (83) yields the ordinary WD equation where the solutions are given by (69). In the case when $\beta \neq 0$, note that the effects of β are important at the Planck scales, i.e., in cosmology language in the very early Universe, that is, when the scale factor is small, which in my model means $x, y \sim 0$. Thus, if I use the solutions (69) in the β-term of (83), I may obtain some approximate analytical solutions in the region $x, y \to 0$. To this end, I write the limiting behavior of the solutions (69) in the region $x, y \sim 0$ as

$$\mathrm{Ai}\left(\frac{\nu - 2V_0z}{(2V_0)^{2/3}}\right) \to c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \mathcal{O}(z^5). \tag{84}$$

Therefore, I can replace the fourth derivative of $X(x)$ and $Y(y)$ in Eq. (83) with a constant and thus are led to the following equations:

$$\frac{d^2Z_i}{dz_i^2} + (2V_0z_i - \nu)Z_i = \beta_0, \qquad Z_i(i = 1, 2) = X, Y, \qquad z_i(i = 1, 2) = x, y, \tag{85}$$

in which $\beta_0 = 16c_4\beta$. The solutions of the above equation can be written in terms of Airy functions and hypergeometric functions ${}_pF_q(\{a_1, \ldots, a_p\}; \{b_1, \ldots, b_q\}; z)$ as

$$Z(z) = \mathrm{Ai}\left(\frac{\nu - 2V_0z}{(2V_0)^{2/3}}\right) + \mathcal{A}V_0\nu\beta_0\mathrm{Ai}\left(\frac{\nu - 2V_0z}{(2V_0)^{2/3}}\right){}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{(\nu - 2V_0z)^3}{36V_0^2}\right) + \ldots, \tag{86}$$

where $\mathcal{A}$ is

$$\mathcal{A} = \frac{32^{2/3}3^{5/6}\pi}{36\Gamma(2/3)},$$

and ..., denotes the terms that I have neglected in my approximation proposal. I have also removed the Airy functions $\mathrm{Bi}(z)$ from the solutions to recover the solutions (69) in the limit $\beta \to 0$. Thus, the eigenfunctions of WD equation (82) read

$$\Psi_\nu(x, y) = \left[\mathrm{Ai}\left(\frac{\nu - 2V_0x}{(2V_0)^{2/3}}\right) + \mathcal{A}V_0\nu\beta_0\mathrm{Ai}\left(\frac{\nu - 2V_0x}{(2V_0)^{2/3}}\right){}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{(\nu - 2V_0x)^3}{36V_0^2}\right)\right] \times \left[\mathrm{Ai}\left(\frac{\nu - 2V_0y}{(2V_0)^{2/3}}\right) + \mathcal{A}V_0\nu\beta_0\mathrm{Ai}\left(\frac{\nu - 2V_0y}{(2V_0)^{2/3}}\right){}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{(\nu - 2V_0y)^3}{36V_0^2}\right)\right]. \tag{87}$$

Now, bearing in the mind that in my GUP framework, I have chosen the GUP parameters $\beta' = 2\beta$ such that the coordinates commute, I can apply the boundary condition (70), also on the GUP wave function, which yields

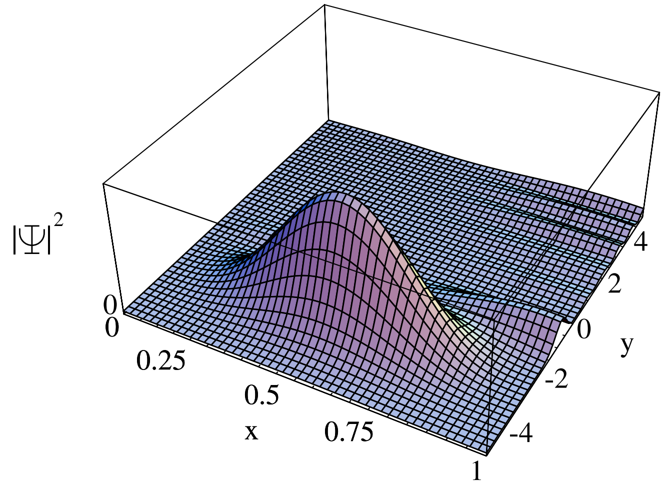

FIG. 3 (color online). The square of the wave function in the GUP case. I take the numerical values $V_0 = 1$ and $\beta_0 = 1$.

$$\mathrm{Ai}\left(\frac{\nu}{(2V_0)^{2/3}}\right) + \mathcal{A} V_0 \nu \beta_0 \mathrm{Ai}\left(\frac{\nu}{(2V_0)^{2/3}}\right) {}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{\nu^3}{36V_0^2}\right) = 0. \tag{88}$$

Therefore, the general solution of the WD equation can be written as

$$\Psi(x,y) = \sum_n c_n \left[\mathrm{Ai}\left(\frac{\nu_n - 2V_0 x}{(2V_0)^{2/3}}\right) + \mathcal{A} V_0 \nu_n \beta_0 \mathrm{Ai}\left(\frac{\nu_n - 2V_0 x}{(2V_0)^{2/3}}\right) {}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{(\nu_n - 2V_0 x)^3}{36V_0^2}\right)\right] \times \left[\mathrm{Ai}\left(\frac{\nu_n - 2V_0 y}{(2V_0)^{2/3}}\right) + \mathcal{A} V_0 \nu_n \beta_0 \mathrm{Ai}\left(\frac{\nu_n - 2V_0 y}{(2V_0)^{2/3}}\right) {}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{(\nu_n - 2V_0 y)^3}{36V_0^2}\right)\right], \tag{89}$$

where ν_n are the zeros of Eq. (88). In Fig. 3 I have plotted the square of wave function when the phase-space variables obey GUP relations, for small values of x and y. This figure shows only a possible state in the early Universe with a negative value for y and a nonzero positive value of x. Thus, in the context of GUP quantum cosmology our Universe emerges from a nonsingular state where the dilaton field has a negative value.

V. CONCLUSIONS AND COMPARISON OF THE RESULTS

In this paper I have studied the effects of noncommutativity and generalized uncertainty relations in phase space, on classical and quantum cosmology of a dilaton model with an exponential dilaton potential. In the case of commutative phase space, the evolution of the classical Universe is like the motion of a particle (Universe) moving on a plane with a constant acceleration. I have shown that in this case both dynamical variables x and y should be positive which means that only half of the minisuperspace is recovered through the evolution of the Universe. In the case when quantum cosmology is considered in the commutative phase space, I have seen that the wave function of the Universe peaks symmetrically around $y = 0$, which means that the present Universe could have evolved from different states with the same values for x but different symmetric values for y. In the case of noncommutative classical cosmology, the solutions are like the commutative case with a little difference, that is, the noncommutative parameter shows its effect on the initial velocity of the evolution. On the other hand noncommutative quantum cosmology predicts the emergence of the Universe from a positive value of y, that is, from a positive value of the dilaton field. Finally, when the phase-space variables obey the GUP relations, the classical cosmology is described by Eqs. (65) and (66), which are more complicated than the commutative case. Also, I have presented approximate analytical solutions of quantum cosmology in the GUP framework. These solutions show only one possible state in the early Universe with a negative value for y and a nonzero positive value of x. Thus, in the context of GUP quantum cosmology the early Universe emerges from a nonsingular state where the dilaton field has a negative value.

ACKNOWLEDGMENTS

The author would like to thank H. R. Sepangi for a careful reading of the manuscript and useful comments.

[1] M. Gasperini and G. Veneziano, Mod. Phys. Lett. A **8**, 3701 (1993); M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).
[2] M. Gasperini, arXiv:gr-qc/9706037.
[3] J. Lidsey, D. Wands, and E. Copeland, Phys. Rep. **337**, 343 (2000); G. Veneziano, arXiv:hep-th/0002094; F. Quevedo, Classical Quantum Gravity **19**, 5721 (2002).
[4] U.H. Danielsson, Classical Quantum Gravity **22**, S1 (2005).
[5] G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Astropart. Phys. **1**, 317 (1993).
[6] M. Gasperini, arXiv:hep-th/0702166.
[7] A. A. Tseytlin and C. Vafa, Nucl. Phys. **B372**, 443 (1992); S. Capozziello and R. de Ritis, Int. J. Mod. Phys. D **2**, 373 (1993); E.J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **50**, 4868 (1994); S. Capozziello, G. Lambiase, and R. Capaldo, Int. J. Mod. Phys. D **8**, 213 (1999).
[8] D. J. Gross and P. F. Mende, Nucl. Phys. **B303**, 407 (1988); D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B **216**, 41 (1989); M. Kato, Phys. Lett. B **245**, 43 (1990); S. Haro, J. High Energy Phys. 10 (1998) 023; L. G. Garay, Int. J. Mod. Phys. A **10**, 145 (1995); K. Konishi, G. Paffuti, and P. Provero, Phys. Lett. B **234**, 276 (1990).
[9] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995).
[10] A. Kempf and G. Mangano, Phys. Rev. D **55**, 7909 (1997).
[11] M. Maggiore, Phys. Lett. B **319**, 83 (1993); Phys. Rev. D **49**, 5182 (1994); A. Kempf, J. Math. Phys. (N.Y.) **35**, 4483 (1994); R.J. Adler and D. I. Santiago, Mod. Phys. Lett. A **14**, 1371 (1999); A. Kempf, Phys. Rev. D **63**, 083514 (2001); A. Kempf and J.C. Niemeyer, Phys. Rev. D **64**, 103501 (2001); L.N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125027 (2002); H. Hinvichsen and A. Kempf, J. Math. Phys. (N.Y.) **37**, 2121 (1996).
[12] H. Snyder, Phys. Rev. **71**, 38 (1947).
[13] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994); J. Math. Phys. (N.Y.) **41**, 3832 (2000).
[14] J.C. Varilly, arXiv:physics/9709045.
[15] M.R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[16] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[17] N. Seiberg and E. Witten, J. High Energy Phys. 02 (1999) 032.
[18] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, J. High Energy Phys. 02 (2000) 020.
[19] S. M. Carroll, J. A. Harvey, V. A. Kostelesky, C. D. Lane, and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001); C. E. Carlson, C. D. Carone, and R. F. Lebed, Phys. Lett. B **518**, 201 (2001).**549**, 337 (2002).
[20] R. J. Szabo, Phys. Rep. **378**, 207 (2003); M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[21] I. Bars, arXiv:hep-th/0109132.
[22] J. Gamboa, M. Loewe, and J. S. Rojas, Phys. Rev. D **64**, 067901 (2001); A. E. F. Djemai and H. Smail, Commun. Theor. Phys. **41**, 837 (2004); L. Mezincescu, arXiv:hep-th/0007046.
[23] J. M. Romero, J. A. Satiago, and D. Vergara, Phys. Lett. A **310**, 9 (2003); M. Przanowski and J. Tosiek, Acta Phys. Pol. B **30**, 179 (1999).
[24] A. E. F. Djemai, arXiv:hep-th/0309034.
[25] H. Garcia-Compean, O. Obregon, and C. Ramirez, Phys. Rev. Lett. **88**, 161301 (2002); G. D. Barbosa and N. Pinto-Neto, Phys. Rev. D **70**, 103512 (2004); G. D. Barbosa, Phys. Rev. D **71**, 063511 (2005); L. O. Pimentel and C. Mora, Gen. Relativ. Gravit. **37**, 817 (2005); L. O. Pimentel and O. Obregon, Gen. Relativ. Gravit. **38**, 553 (2006); B. Vakili, N. Khosravi, and H.R. Sepangi, Classical Quantum Gravity **24**, 931 (2007); E. Mena, O. Obregon, and M. Sabido, arXiv:gr-qc/0701097.
[26] M. I. Beciu, arXiv:gr-qc/0305077.
[27] F. Darabi, A. Rezaei-Aghdam, and A. R. Rastkar, Phys. Lett. B **615**, 141 (2005).
[28] N. Khosravi, S. Jalalzadeh, and H. R. Sepangi, J. High Energy Phys. 01 (2006) 134.
[29] M. V. Battisti and G. Montani, Phys. Lett. B **656**, 96 (2007); arXiv:0707.2726; B. Vakili and H. R. Sepangi, Phys. Lett. B **651**, 79 (2007).
[30] M. V. Battisti and G. Montani, AIP Conf. Proc. **966**, 219 (2008).
[31] M. Gasperini, J. Maharana, and G. Veneziano, Nucl. Phys. **B472**, 349 (1996).
[32] J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206 (1989); L. G. Garay, J. J. Halliwell, and G. A. Mena Marugan, Phys. Rev. D **43**, 2572 (1991).
[33] J. Maharana, Phys. Rev. D **65**, 103504 (2002).
[34] A. Vilenkin, Phys. Rev. D **37**, 888 (1988).
[35] S. F. Hassan and M. S. Sloth, Nucl. Phys. B **674**, 434 (2003).
[36] S. Benczik *et al.*, Phys. Rev. D **66**, 026003 (2002); S. Benczik *et al.*, arXiv:hep-th/0209119.
[37] K. Nozari and S. Akhshabi, arXiv:gr-qc/0608076.