Using full information when computing modes of post-Newtonian waveforms from inspiralling compact binaries in circular orbit

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The increasing sophistication and accuracy of numerical simulations of compact binaries (especially binary black holes) presents the opportunity to test the regime in which post-Newtonian (PN) predictions for the emitted gravitational waves are accurate. In order to confront numerical results with those of post-Newtonian theory, it is convenient to compare multipolar decompositions of the two waveforms. It is pointed out here that the individual modes can be computed to higher post-Newtonian order by examining the radiative multipole moments of the system, rather than by decomposing the 2.5PN polarization waveforms. In particular, the dominant ($l = 2$, $m = \pm 2$) mode *can be computed to 3PN order*. Individual modes are computed to as high a post-Newtonian order as possible given previous post-Newtonian results.

I. INTRODUCTION

The first generation of laser interferometric gravitational wave detectors is now operating at or near their design sensitivities [1–5]. One of the most promising sources that they may detect is the inspiral and merger of compact binary systems with black holes or neutron stars. One of the primary means of detecting and interpreting the signals from such systems will be the use of a matched filtering technique, which requires accurate templates with which to match a theoretical model to a noisy signal. Until the last several orbits prior to merger, the post-Newtonian (PN) approximation is expected to be sufficient to generate accurate templates, while numerical relativity will be needed to help construct accurate templates covering the late inspiral and merger phases.

Estimates of the accuracy of post-Newtonian templates and the effect of this accuracy on detecting gravitational waves have led the post-Newtonian expansion to be pushed to high order [6–9]. Currently the equations of motion and the gravitational wave energy flux are known through 3.5PN order [10,11], and the polarization waveform amplitudes are known through 2.5PN order [12,13]. As the post-Newtonian expansion is a slow-motion, weak-field approximation to general relativity, and the motion is not so slow and the fields not so weak during the last orbits prior to merger, it is legitimate to ask where the post-Newtonian expansion breaks down.

Recent breakthroughs in numerical relativity [14–17] have finally provided the possibility of testing the validity of the post-Newtonian expansion in the late inspiral. Initial studies focused on qualitative comparisons between post-Newtonian generated waveforms and numerical simulations of the final two to four orbits prior to merger [18–20]. More recently, attempts have been made to quantify the difference between post-Newtonian generated waveforms and numerical simulations of a nonspinning equal-mass binary inspiral lasting more than eight orbits [21–23]. As long inspiral simulations are computationally expensive to perform, it is important to quantify the errors in the post-Newtonian generated waveforms and determine how they influence the detection of gravitational waves. One would also like to know the accuracy with which the parameters such as the individual masses and spins of the binary can be determined from the observed waveform.

One of the principal ways of quantifying the accuracy of the post-Newtonian approximation is to compare the amplitude and phase of the gravitational waveform with that from numerical simulations. Unlike gravitational wave detection which is sensitive only to the waveform in one particular direction, comparisons between numerical simulations and post-Newtonian waveforms can be performed over the entire sphere. Therefore it is convenient to project the waveforms onto spin-weighted spherical harmonics and compare the individual components. Previous studies [18,19], however, have computed the spin-weighted spherical harmonic components of the post-Newtonian amplitude in such a way that has not made full use of current post-Newtonian results.

This situation has arisen primarily because of how the post-Newtonian gravitational wave amplitude is presented. Often the final result of the post-Newtonian computation of the waveform is given as the polarization waveforms as a function of the direction to the observer (as this is most useful when constructing templates). Currently the polarization waveforms are known to 2.5PN order in amplitude [12,13]. In previous comparisons between post-Newtonian and numerical waveforms, it was the polarization waveforms that were projected into spin-weighted spherical harmonics in order to determine the post-Newtonian components. The post-Newtonian polarization waveforms are computed by truncating an infinite sum of radiative multipole moments at a given post-Newtonian order [24]. As will be demonstrated in this paper, it makes more sense to project the waveform onto spin-weighted spherical harmonics *prior* to truncating the series as this retains the

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full information currently known. In particular, the dominant ($l = 2$, $m = 2$) component of the waveform *can be computed to 3PN order.*[1]

While knowledge of particular spin-weighted components of a gravitational waveform is of limited value for computing templates for detection and characterization of gravitational waves, it can be used to quantify the difference between post-Newtonian and numerical waveforms. In addition, if the 3PN correction to the amplitude of the ($l = 2$, $m = 2$) component significantly improves the agreement between the post-Newtonian and numerical waveforms, this would suggest it might be worth the effort needed to compute the polarization waveforms to 3PN order. It has been shown that higher post-Newtonian corrections in the amplitude can improve detection rates [27–29] as well as improve parameter extraction [30–36].

The relationship between the gravitational waveform and radiative multipole moments is reviewed in Sec. II. It is pointed out that the spin-weighted spherical harmonic components of the waveform can be computed directly from the radiative mulitpole moments of the system in Sec. II B. Current post-Newtonian results that affect the computation of the radiative multipole moments are summarized in Sec. III. The radiative multipole moments are computed to as high a post-Newtonian order as possible in Sec. IV. Conclusions are presented in Sec. V.

II. GRAVITATIONAL WAVEFORM AND SPIN-WEIGHTED SPHERICAL HARMONICS

A. Notation

Let $X^\mu = (cT, X, Y, Z)$ be coordinates in an asymptotically flat radiative coordinate system, with $\vec{e}_T$ the unit timelike normal and $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ the spatial orthonormal coordinate basis vectors. Let (cT, R, Θ, Φ) be the corresponding spherical coordinate system with corresponding basis vectors $(\vec{e}_R, \vec{e}_\Theta, \vec{e}_\Phi)$. Let $T_R = T - R/c$ denote retarded time in radiative coordinates.

Greek letters are used for spacetime indices, Latin letters for spatial indices. As indices can be raised or lowered using the Minkowski metric, components of tensors will typically have down indices. Let N_i be a component of the unit radial vector $\vec{e}_R$. A capital letter for an index denotes a multi-index (i.e. $T_L = T_{i_1 i_2 \cdots i_\ell}$). For a vector V, let V_L be a product of components of the vector (i.e. $V_L = V_{i_1} V_{i_2} \cdots V_{i_\ell}$). Repeated spatial indices (including multi-indices) are to be summed over. Tensors that are fully symmetric and trace-free (STF) will be denoted with capital script letters (e.g. $\mathcal{I}_{ij}$). Symmetrization, antisymmetrization, and STF projection of indices are denoted by (), [], and $\langle \rangle$, respectively, with underlined indices between the

delimiters to be excluded from the operation (e.g. $2T_{(a\underline{b}c)} = T_{abc} + T_{cba}$).

Let $(\vec{k}, \vec{l}, \vec{m})$ be a set of null vectors defined[2] by

$$\vec{k} = \frac{1}{\sqrt{2}}(\vec{e}_T + \vec{e}_R), \tag{1}$$

$$\vec{l} = \frac{1}{\sqrt{2}}(\vec{e}_T - \vec{e}_R), \tag{2}$$

$$\vec{m} = \frac{1}{\sqrt{2}}(\vec{e}_\Theta + i\vec{e}_\Phi). \tag{3}$$

The spin-weighted spherical harmonics are defined in terms of the Wigner d functions by

$$_{-s}Y^{\ell m}(\Theta, \Phi) = (-1)^s \sqrt{\frac{2\ell + 1}{4\pi}} d^\ell_{ms}(\Theta) e^{im\Phi}, \tag{4}$$

where

$$d^\ell_{ms}(\Theta) = \sqrt{(\ell + m)!(\ell - m)!(\ell + s)!(\ell - s)!}$$
$$\times \sum_{k=k_i}^{k_f} \frac{(-1)^k (\sin\frac{\Theta}{2})^{2k+s-m}(\cos\frac{\Theta}{2})^{2\ell+m-s-2k}}{k!(\ell + m - k)!(\ell - s - k)!(s - m + k)!}, \tag{5}$$

where $k_i = \max(0, m - s)$ and $k_f = \min(\ell + m, \ell - s)$.

B. Polarization waveforms and modes

The asymptotic waveform h^{TT}_{ij} can be decomposed into two sets of STF radiative multipole moments as [24]

$$h^{TT}_{ij} = \frac{4G}{c^2 R} \Pi_{ijmn} \sum_{\ell=2}^\infty \left\{ \frac{1}{c^\ell \ell!} \mathcal{U}_{mnL-2}(T_R) N_{L-2} \right.$$
$$\left. + \frac{2\ell}{c^{\ell+1}(\ell + 1)!} \epsilon_{pq(m} \mathcal{V}_{n)pL-2}(T_R) N_{qL-2} \right\}. \tag{6}$$

Here $\mathcal{U}_L(T_R)$ are the mass-type moments and $\mathcal{V}_L(T_R)$ are the current-type moments. In Sec. III A these radiative multipole moments will be related to multipole moments describing the source in the near zone. Note that higher multipole moments contribute to the waveform at higher post-Newtonian order, so that at any finite post-Newtonian order, only a finite number of multipoles contribute. The transverse-traceless (TT) projection operator Π_{ijmn} is given by

$$\Pi_{ijmn} = P_{im}P_{jn} - \tfrac{1}{2}P_{ij}P_{mn}, \tag{7}$$

where $P_{ij} = \delta_{ij} - N_i N_j$.

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[1]This was independently recognized by Damour and Nagar [25,26] who have derived a resummed 3PN quadrupolar waveform within the effective-one-body approach.

[2]The reader is hereby warned that different conventions exist in the literature for the definitions of null vectors, (spin-weighted) spherical harmonics, polarization triads, etc. I have attempted to be explicit in the definitions used here so that the results may be converted to the reader's favorite conventions.

Given an orthonormal triad $(\vec{N}, \vec{P}, \vec{Q})$, the polarization waveforms can be given by

$$h_+ = \tfrac{1}{2}(P_m P_n - Q_m Q_n)h_{mn}^{TT}, \tag{8}$$

$$h_\times = \tfrac{1}{2}(P_m Q_n + P_n Q_m)h_{mn}^{TT}. \tag{9}$$

A natural (but by no means unique[3]) choice for the triad is $\vec{P} = \vec{e}_\Theta$ and $\vec{Q} = \vec{e}_\Phi$. It is then straightforward to show that

$$h_+ - i h_\times = m_m^* m_n^* h_{mn}^{TT}, \tag{10}$$

where $*$ denotes complex conjugation. It will now be shown how $h_+ - i h_\times$ can be decomposed into modes using spin-weighted spherical harmonics of weight -2,

$$h_+ - i h_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} {}_{-2}Y^{\ell m}(\Theta, \Phi). \tag{11}$$

An alternative expression for the waveform is given by [24]

$$h_{ij}^{TT} = \frac{G}{c^2 R} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \frac{1}{c^\ell} U^{\ell m}(T_R) T_{ij}^{E2,\ell m} \right. $$
$$\left. + \frac{1}{c^{\ell+1}} V^{\ell m}(T_R) T_{ij}^{B2,\ell m} \right\}, \tag{12}$$

where $T_{ij}^{E2,\ell m}$ and $T_{ij}^{B2,\ell m}$ are pure-spin tensor harmonics, and where the mass multipole moments $U^{\ell m}(T_R)$ and current multipole moments $V^{\ell m}(T_R)$ are related to their STF counterparts by [24]

$$U^{\ell m} = \frac{16\pi}{(2\ell+1)!!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \mathcal{U}_L \mathcal{Y}_L^{\ell m*}, \tag{13}$$

$$V^{\ell m} = \frac{-32\pi\ell}{(2\ell+1)!!} \sqrt{\frac{(\ell+2)}{2\ell(\ell+1)(\ell-1)}} \mathcal{V}_L \mathcal{Y}_L^{\ell m*}, \tag{14}$$

where $\mathcal{Y}_L^{\ell m}$ are the STF spherical harmonics which are related to the scalar spherical harmonics by

$$Y^{\ell m}(\Theta, \Phi) = \mathcal{Y}_L^{\ell m} N_L. \tag{15}$$

The pure-spin tensor harmonics are related to the spin-weighted spherical harmonics by [24]

$$T_{ij}^{E2,\ell m} = \frac{1}{\sqrt{2}}({}_{-2}Y^{\ell m} m_i m_j + {}_2 Y^{\ell m} m_i^* m_j^*), \tag{16}$$

$$T_{ij}^{B2,\ell m} = \frac{-i}{\sqrt{2}}({}_{-2}Y^{\ell m} m_i m_j - {}_2 Y^{\ell m} m_i^* m_j^*). \tag{17}$$

Combining Eqs. (10), (12), and (16) yields

$$h_+ - i h_\times = \frac{G}{c^2 R} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \frac{1}{c^\ell \sqrt{2}} U^{\ell m}(T_R) \right. $$
$$\left. - \frac{i}{c^{\ell+1}\sqrt{2}} V^{\ell m}(T_R) \right\} {}_{-2}Y^{\ell m}(\Theta, \Phi). \tag{18}$$

Thus the spin-weighted spherical harmonic components of the waveform are given by

$$h^{\ell m} = \frac{G}{\sqrt{2}Rc^{\ell+2}} \left(U^{\ell m}(T_R) - \frac{i}{c} V^{\ell m}(T_R) \right). \tag{19}$$

Note that the spin-weighted spherical harmonic components can be computed *directly* from the radiative multipole moments. It is not necessary to compute the waveform as a function of position and then project out the components as is commonly done in the recent literature. In fact, not only are these extra steps unnecessary, they cause individual components to be truncated at a lower post-Newtonian order than they need to be. In particular, the dominant h^{22} component is truncated to 2.5PN order, when it can be computed (as will be shown below) to 3PN order directly from the radiative mass quadrupole moment.

This situation arises as the complete polarization waveforms h_+ and $h_\times$ are known only through 2.5PN order. But as discussed in [12], the computation of the 3PN waveform is not limited by the post-Newtonian order of the dominant mass quadrupole contribution, but by the order of higher

TABLE I. The post-Newtonian order to which each radiative multipole moment must be computed to obtain the full 2.5PN polarization waveforms h_+ and $h_\times$, and the order to which they are known.

Radiative multipole	PN order for $h_{+,\times}$	Known PN order
$\mathcal{I}_{ij}$	2.5	3
$\mathcal{J}_{ij}$	2	2
$\mathcal{I}_{ijk}$	2	2
$\mathcal{J}_{ijk}$	1.5	1.5[a]
$\mathcal{I}_{ijkl}$	1.5	2
$\mathcal{J}_{ijkl}$	1	1
$\mathcal{I}_{ijklm}$	1	1
$\mathcal{J}_{ijklm}$	0.5	1
$\mathcal{I}_{ijklmn}$	0.5	1
$\mathcal{J}_{ijklmn}$	0	1
$\mathcal{I}_{ijklmno}$	0	1
$\mathcal{I}_L(\ell > 7)$	$\cdots$	1
$\mathcal{J}_L(\ell > 6)$	$\cdots$	1

[a]The radiative current octupole could be computed to 2PN order by a computation of the 2PN correction to the source current octupole.

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[3]The 2.5PN polarization waveforms in [12] are defined with $\vec{P} = -\vec{e}_\Phi$ and $\vec{Q} = \vec{e}_\Theta$ evaluated at $\Theta = i$, $\Phi = \pi/2$.

multipoles. Therefore to obtain the spin-weighted spherical harmonic components of the waveform to as high a post-Newtonian order as possible, they should be computed directly from the radiative multipole moments. As discussed in Sec. V below, this will allow post-Newtonian results to be probed to a higher order when comparing with numerical simulations. Table I summarizes the post-Newtonian order to which each radiative multipole moment needs to computed to obtain the full 2.5PN order polarization waveforms, and the post-Newtonian order to which they are currently known.

III. SUMMARY OF CURRENT POST-NEWTONIAN RESULTS

The post-Newtonian approximation is a slow-motion, weak-field approximation to general relativity with an expansion parameter $\epsilon \sim (v/c)^2 \sim (Gm/rc^2)$. For a binary system of two point masses m_1 and m_2, v is the magnitude of the relative velocity, $m = m_1 + m_2$, and r is the separation. In order to produce a post-Newtonian waveform, it is necessary to solve both the post-Newtonian equations of motion describing the binary, and the post-Newtonian equations describing the generation of gravitational waves. For a complete review of post-Newtonian methods applied to inspiralling compact binaries, see [37].

A. Post-Newtonian computation of radiative multipoles

In Sec. II B the spin-weighted spherical harmonic components have been related to the radiative multipole moments. In order to use Eq. (19) it is necessary to relate the radiative moments to a description of the compact binary system. The post-Newtonian wave generation formalism

has been developed [38–47] in a systematic manner to relate the radiative multipole moments $\{\mathcal{U}_L, \mathcal{V}_L\}$ to a set of six STF source moments $\{\mathcal{I}_L, \mathcal{J}_L, \mathcal{W}_L, \mathcal{X}_L, \mathcal{Y}_L, \mathcal{Z}_L\}$, which can be computed from the stress-energy pseudotensor of the material and gravitational fields of the source. A set of two canonical source moments $\{\mathcal{M}_L, \mathcal{S}_L\}$ can be computed as a nonlinear functional of the source moments as an intermediate step between the source moments and the radiative moments. The radiative moments are then given as nonlinear (and even nonlocal) functionals of the canonical moments. It turns out that two of the source moments, the source mass moments $\mathcal{I}_L$ and the source current moments $\mathcal{J}_L$, are dominant, while the other four parametrize a gauge transformation and only make a contribution to the canonical source moments starting at 2.5PN order. To the post-Newtonian order needed in this paper, only the canonical mass quadrupole moment contains a correction from its corresponding source moment [48,49]

$$\mathcal{M}_{ij} = \mathcal{I}_{ij} + \frac{4G}{c^5}(\mathcal{W}^{(2)}\mathcal{I}_{ij} - \mathcal{W}^{(1)}\mathcal{I}_{ij}^{(1)}) + O(\epsilon^{7/2}),$$
(20)

where $\mathcal{T}_L^{(p)}$ denotes the pth time derivative of $\mathcal{T}_L$ and $O(\epsilon^k)$ denotes the kth order and higher post-Newtonian corrections that are unknown. The other canonical source moments are simply related to the source moments

$$\mathcal{M}_L = \mathcal{I}_L + O(\epsilon^{5/2}),$$
(21)

$$\mathcal{S}_L = \mathcal{J}_L + O(\epsilon^{5/2}).$$
(22)

The radiative mass quadrupole is given by

$$\mathcal{U}_{ij}(T_R) = \mathcal{M}_{ij}^{(2)}(T_R) + \frac{2GM}{c^3}\int_0^\infty d\tau \mathcal{M}_{ij}^{(4)}(T_R - \tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] - \frac{2}{7}\frac{G}{c^5}\int_0^\infty d\tau \mathcal{M}_{a\langle i}^{(3)}(T_R - \tau)\mathcal{M}_{j\rangle a}^{(3)}(T_R - \tau)$$
$$+ \frac{G}{c^5}\left\{\frac{1}{7}\mathcal{M}_{a\langle i}^{(5)}\mathcal{M}_{j\rangle a} - \frac{5}{7}\mathcal{M}_{a\langle i}^{(4)}\mathcal{M}_{j\rangle a}^{(1)} - \frac{2}{7}\mathcal{M}_{a\langle i}^{(3)}\mathcal{M}_{j\rangle a}^{(2)} + \frac{1}{3}\epsilon_{ab\langle i}\mathcal{M}_{j\rangle a}^{(4)}\mathcal{S}_b\right\}$$
$$+ \frac{2G^2M^2}{c^6}\int_0^\infty d\tau \mathcal{M}_{ij}^{(5)}(T_R - \tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124\,627}{44\,100}\right] + O(\epsilon^{7/2}).$$
(23)

The first integral in the above expression is the dominant radiation tail at 1.5PN order obtained in [43]. The 2.5PN nonlinear memory integral has been obtained in [43,45,50–52]. The other nonlinear interactions at 2.5PN order were obtained in [45]. Finally the 3PN "tail of tail" integral was derived in [46]. The tail integrals involve nonlinear interactions with the mass monopole M (equivalent to the Arnowitt-Deser-Misner mass) of the system. The tail integrals also contain a freely specifiable constant r_0 that corresponds to the choice of the origin of radiative time T with respect to harmonic time t, and enters the relation between the retarded time in radiative coordinates and the retarded time $t - r/c$ in harmonic coordinates (the

coordinates in which the equations of motion are given) [53,54]:

$$T_R = t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right).$$
(24)

The remaining radiative multipole moments are given by [43,45,46]

$$\mathcal{U}_{ijk}(T_R) = \mathcal{M}_{ijk}^{(3)}(T_R) + \frac{2GM}{c^3}\int_0^\infty d\tau \mathcal{M}_{ijk}^{(5)}(T_R - \tau)$$
$$\times \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] + O(\epsilon^{5/2}),$$
(25)

$$\mathcal{U}_{ijkl}(T_R) = \mathcal{M}_{ijkl}^{(4)}(T_R) + \frac{2GM}{c^3}\int_0^\infty d\tau \mathcal{M}_{ijkl}^{(6)}(T_R - \tau)$$
$$\times \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right]$$
$$+ \frac{2}{5}\frac{G}{c^3}\int_0^\infty d\tau \mathcal{M}_{\langle ij}^{(3)}(T_R - \tau)\mathcal{M}_{kl\rangle}^{(3)}(T_R - \tau)$$
$$- \frac{G}{c^3}\left\{\frac{21}{5}\mathcal{M}_{\langle ij}^{(5)}\mathcal{M}_{kl\rangle} + \frac{63}{5}\mathcal{M}_{\langle ij}^{(4)}\mathcal{M}_{kl\rangle}^{(1)}\right.$$
$$\left. + \frac{102}{5}\mathcal{M}_{\langle ij}^{(3)}\mathcal{M}_{kl\rangle}^{(2)}\right\} + O(\epsilon^{5/2}), \quad (26)$$

$$\mathcal{U}_L = \mathcal{M}_L^{(\ell)}(T_R) + O(\epsilon^{3/2}), \quad (27)$$

$$\mathcal{V}_{ij}(T_R) = \mathcal{S}_{ij}^{(2)}(T_R) + \frac{2GM}{c^3}\int_0^\infty d\tau \mathcal{S}_{ij}^{(4)}(T_R - \tau)$$
$$\times \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] + O(\epsilon^{5/2}), \quad (28)$$

$$\mathcal{V}_{ijk}(T_R) = \mathcal{S}_{ijk}^{(3)}(T_R) + \frac{2GM}{c^3}\int_0^\infty d\tau \mathcal{S}_{ijk}^{(5)}(T_R - \tau)$$
$$\times \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3}\right] + \frac{G}{c^3}\left\{\frac{1}{10}\epsilon_{ab\langle i}\mathcal{M}_{j\underline{a}}^{(5)}\mathcal{M}_{k\rangle b}\right.$$
$$\left. - \frac{1}{2}\epsilon_{ab\langle i}\mathcal{M}_{j\underline{a}}^{(4)}\mathcal{M}_{k\rangle b}^{(1)} - 2\mathcal{S}_{\langle i}\mathcal{M}_{jk\rangle}^{(4)}\right\} + O(\epsilon^{5/2}), \quad (29)$$

$$\mathcal{V}_L = \mathcal{S}_L^{(\ell)}(T_R) + O(\epsilon^{3/2}). \quad (30)$$

B. Adiabatic inspiral of quasicircular orbits

Solving the equations of motion yields explicit expressions for the accelerations of each body in terms of the positions and velocities of the two bodies [55–65]. The two-body equations of motion can then be reduced to relative equations of motion in the center-of-mass frame in terms of the relative position $\vec{x}$ and velocity $\vec{v}$ [66]. The relative acceleration $\vec{a}$ is currently known through 3.5PN order [10,11]. The effects of radiation reaction (due to the emission of gravitational waves) enter the relative acceleration starting at 2.5PN order.

The emission of gravitational radiation causes the orbits of an isolated binary system to circularize [67]. Thus, for the remainder of this paper the orbital evolution of the binary will be modeled as a slow adiabatic inspiral of a quasicircular orbit. In addition, it will be assumed that the individual compact objects are nonspinning.

The orbital plane is chosen to be the X-Y plane with the orbital phase ϕ defining the direction of the unit vector $\vec{n} = \vec{x}/r$ along the relative separation vector by

$$\mathbf{n} = \cos\phi\,\vec{e}_X + \sin\phi\,\vec{e}_Y. \quad (31)$$

Then the motion of the binary can be described using the rotating orthonormal triad $(\vec{n}, \vec{\lambda}, \vec{e}_Z)$ with $\vec{\lambda} = \vec{e}_Z \times \vec{n}$.

The relative position, velocity, and acceleration are given by

$$\vec{x} = r\vec{n}, \quad (32)$$

$$\vec{v} = \dot{r}\vec{n} + r\omega\vec{\lambda}, \quad (33)$$

$$\vec{a} = (\ddot{r} - r\omega^2)\vec{n} + (r\dot{\omega} + 2\dot{r}\omega)\vec{\lambda}, \quad (34)$$

where the orbital frequency $\omega = \dot{\phi}$. Through 2PN order, it is possible to model the motion of the binary as a circular orbit with the solution $\ddot{r} = \dot{r} = \dot{\omega} = 0$ and $r\omega^2 = -\vec{n}\cdot\vec{a}$.

At 2.5PN order, however, the inspiral motion must be taken into account. The leading order contribution to the inspiral of a quasicircular orbit can be obtained by examining the Newtonian orbital energy of a circular orbit,

$$E = -\frac{1}{2}\nu\frac{Gm^2}{r} + O(\epsilon), \quad (35)$$

where $\nu = m_1 m_2/m^2$, and the leading order gravitational luminosity from a circular orbit,

$$\mathcal{L} = \frac{32}{5}\nu^2\frac{G^4 m^5}{r^5 c^5} + O(\epsilon), \quad (36)$$

and assuming that the energy radiated by the gravitational waves is balanced by the change in the orbital energy (i.e. $dE/dt = -\mathcal{L}$). Then

$$\dot{r} = \left(\frac{dE}{dt}\Big/\frac{dE}{dr}\right) = -\frac{64}{5}\nu\frac{G^3 m^3}{r^3 c^5} + O(\epsilon^{7/2}), \quad (37)$$

and similarly the orbital frequency changes by

$$\dot{\omega} = \left(\frac{d\omega}{dr}\Big/\frac{dr}{dt}\right) = \frac{96}{5}\nu\left(\frac{G^7 m^7}{c^{10} r^{11}}\right)^{1/2} + O(\epsilon^{7/2}). \quad (38)$$

Substituting (37) and (38) into Eqs. (33) and (34) and noting that $\ddot{r} = O(\epsilon^5)$, the following expressions for the 3PN inspiral relative velocity and relative acceleration are obtained:

$$\vec{v} = r\omega\vec{\lambda} - \frac{64}{5}\nu\frac{G^3 m^3}{r^3 c^5}\vec{n} + O(\epsilon^{7/2}), \quad (39)$$

$$\vec{a} = -\omega^2\vec{x} - \frac{32}{5}\nu\frac{G^3 m^3}{c^5 r^4}\vec{v} + O(\epsilon^{7/2}), \quad (40)$$

where the 3PN orbital angular frequency is [59,60]

$$\omega^2 = \frac{Gm}{r^3}\left\{1 + \gamma(-3 + \nu) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right)\right.$$
$$+ \gamma^3\left(-10 + \left[-\frac{75\,707}{840} + \frac{41}{64}\pi^2 + 22\ln\left(\frac{r}{r_0'}\right)\right]\nu\right.$$
$$\left.\left. + \frac{19}{2}\nu^2 + \nu^3\right)\right\} + O(\epsilon^4), \quad (41)$$

where the post-Newtonian parameter

$$\gamma \equiv \frac{Gm}{rc^2}, \qquad (42)$$

and r_0' is a gauge constant.

As γ is defined with respect to the harmonic-coordinate separation r, it is convenient to introduce the frequency-related post-Newtonian parameter

$$x \equiv \left(\frac{Gm\omega}{c^3}\right)^{2/3}. \qquad (43)$$

Inverting (41) order by order yields

$$\gamma = x\left\{1 + x\left(1 - \frac{1}{3}\nu\right) + x^2\left(1 - \frac{65}{12}\nu\right)\right.$$
$$+ x^3\left(1 + \left[-\frac{2203}{2520} - \frac{41}{192}\pi^2 - \frac{22}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu\right.$$
$$\left.\left. + \frac{229}{36}\nu^2 + \frac{1}{81}\nu^3\right)\right\} + O(\epsilon^4). \qquad (44)$$

C. Source multipole moments

Defining $\delta m = m_1 - m_2$, the source mass multipole moments for circular orbits are given by [40,49,68]

$$I = m\{1 - \tfrac{1}{2}\nu\gamma + \tfrac{1}{8}\nu(7 - \nu)\gamma^2\} + O(\epsilon^3), \qquad (45)$$

$$I_i = 0, \qquad (46)$$

$$I_{ij} = \nu m\left\{\left[1 - \gamma\left(\frac{1}{42} + \frac{13}{14}\nu\right) - \gamma^2\left(\frac{461}{1512} + \frac{18\,395}{1512}\nu + \frac{241}{1512}\nu^2\right)\right.\right.$$
$$+ \gamma^3\left(\frac{395\,899}{13\,200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3\,304\,319}{166\,320} - \frac{44}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \frac{162\,539}{16\,632}\nu^2 + \frac{2351}{33\,264}\nu^3\right)\right]x_{\langle ij\rangle}$$
$$+ \left[\frac{11}{21}(1 - 3\nu) + \gamma\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) + \gamma^2\left(\frac{428}{105}\ln\left(\frac{r}{r_0}\right) - \frac{357\,761}{19\,800} - \frac{92\,339}{5544}\nu + \frac{35\,759}{924}\nu^2 + \frac{457}{5544}\nu^3\right)\right]$$
$$\left.\times \frac{r^2}{c^2}v_{\langle ij\rangle} + \frac{48}{7}\frac{G^2m^2\nu}{rc^5}x_{\langle i}v_{j\rangle}\right\} + O(\epsilon^{7/2}), \qquad (47)$$

$$I_{ijk} = -\nu\delta m\left\{\left[1 - \gamma\nu - \gamma^2\left(\frac{139}{330} + \frac{11\,923}{660}\nu + \frac{29}{110}\nu^2\right)\right]x_{\langle ijk\rangle} + \left[1 - 2\nu + \gamma\left(\frac{1066}{165} - \frac{1433}{330}\nu + \frac{21}{55}\nu^2\right)\right]\frac{r^2}{c^2}x_{\langle i}v_{jk\rangle}\right\}$$
$$+ O(\epsilon^{5/2}), \qquad (48)$$

$$I_{ijkl} = \nu m\left\{\left[1 - 3\nu + \gamma\left(\frac{3}{110} - \frac{25}{22}\nu + \frac{69}{22}\nu^2\right) - \gamma^2\left(\frac{126\,901}{200\,200} + \frac{58\,101}{2600}\nu - \frac{204\,153}{2860}\nu^2 - \frac{1149}{1144}\nu^3\right)\right]x_{\langle ijkl\rangle}\right.$$
$$+ \left[\frac{78}{55}(1 - 5\nu + 5\nu^2) + \gamma\left(\frac{30\,583}{3575} - \frac{107\,039}{3575}\nu + \frac{8792}{715}\nu^2 - \frac{639}{715}\nu^3\right)\right]\frac{r^2}{c^2}x_{\langle ij}v_{kl\rangle}$$
$$\left.+ \left[\frac{71}{715}(1 - 7\nu + 14\nu^2 - 7\nu^3)\right]\frac{r^4}{c^4}v_{\langle ijkl\rangle}\right\} + O(\epsilon^{5/2}), \qquad (49)$$

$$I_{ijklm} = -\nu\delta m\left\{\left[1 - 2\nu + \gamma\left(\frac{2}{39} - \frac{47}{39}\nu + \frac{28}{13}\nu^2\right)\right]x_{\langle ijklm\rangle} + \frac{70}{39}(1 - 4\nu + 3\nu^2)\frac{r^2}{c^2}x_{\langle ijk}v_{lm\rangle}\right\} + O(\epsilon^2), \qquad (50)$$

$$I_{ijklmn} = \nu m\left\{\left[1 - 5\nu + 5\nu^2 + \gamma\left(\frac{1}{14} - \frac{3}{2}\nu + 6\nu^2 - \frac{11}{2}\nu^3\right)\right]x_{\langle ijklmn\rangle} + \frac{15}{7}(1 - 7\nu + 14\nu^2 - 7\nu^3)\frac{r^2}{c^2}x_{\langle ijkl}v_{mn\rangle}\right\}$$
$$+ O(\epsilon^2), \qquad (51)$$

$$I_{ijklmno} = -\nu\delta m\left\{\left[1 - 4\nu + 3\nu^2 + \gamma\left(\frac{3}{34} - \frac{26}{17}\nu + \frac{83}{17}\nu^2 - \frac{57}{17}\nu^3\right)\right]x_{\langle ijklmno\rangle}\right.$$
$$\left.+ \frac{42}{17}(1 - 6\nu + 10\nu^2 - 4\nu^3)\frac{r^2}{c^2}x_{\langle ijklm}v_{no\rangle}\right\} + O(\epsilon^2), \qquad (52)$$

$$I_{ijklmnop} = \nu m \left\{ \left[1 - 7\nu + 14\nu^2 - 7\nu^3 + \gamma \left(\frac{35}{342} - \frac{73}{38}\nu + \frac{317}{38}\nu^2 - \frac{973}{57}\nu^3 + \frac{301}{38}\nu^4 \right) \right] x_{\langle ijklmnop \rangle} \right.$$
$$\left. + \frac{476}{171}(1 - 9\nu + 27\nu^2 - 30\nu^3 + 9\nu^4)\frac{r^2}{c^2} x_{\langle ijklmn}v_{op\rangle} \right\} + O(\epsilon^2), \tag{53}$$

through $\ell = 8$. In Appendix A an expression is given for the source mass moments to 1PN order for arbitrary ℓ.

The source current multipole moments for circular orbits are [42,44,49]

$$\mathcal{J}_i = L_i + O(\epsilon), \tag{54}$$

$$\mathcal{J}_{ij} = -\frac{\delta m}{m} \left[1 + \gamma \left(\frac{67}{28} - \frac{2}{7}\nu \right) \right.$$
$$\left. + \gamma^2 \left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2 \right) \right] L_{\langle i}x_{j\rangle} + O(\epsilon^{5/2}), \tag{55}$$

$$\mathcal{J}_{ijk} = \left[1 - 3\nu + \gamma \left(\frac{181}{90} - \frac{109}{18}\nu + \frac{13}{18}\nu^2 \right) \right] L_{\langle i}x_{jk\rangle}$$
$$+ \frac{7}{45}(1 - 5\nu + 5\nu^2)\frac{r^2}{c^2} L_{\langle i}v_{jk\rangle} + O(\epsilon^2), \tag{56}$$

$$\mathcal{J}_{ijkl} = -\frac{\delta m}{m} \left\{ \left[1 - 2\nu + \gamma \left(\frac{20}{11} - \frac{155}{44}\nu + \frac{5}{11}\nu^2 \right) \right] L_{\langle i}x_{jkl\rangle} \right.$$
$$\left. + \frac{4}{11}(1 - 4\nu + 3\nu^2)\frac{r^2}{c^2} L_{\langle i}x_j v_{kl\rangle} \right\} + O(\epsilon^2), \tag{57}$$

$$\mathcal{J}_{ijklm} = \left[1 - 5\nu + 5\nu^2 \right.$$
$$+ \gamma \left(\frac{1549}{910} - \frac{1081}{130}\nu + \frac{107}{13}\nu^2 - \frac{29}{26}\nu^3 \right) \right] L_{\langle i}x_{jklm\rangle}$$
$$+ \frac{54}{91}(1 - 7\nu + 14\nu^2 - 7\nu^3)\frac{r^2}{c^2} L_{\langle i}x_{jk}v_{lm\rangle}$$
$$+ O(\epsilon^2), \tag{58}$$

$$\mathcal{J}_{ijklmn} = -\frac{\delta m}{m} \{ 1 - 4\nu + 3\nu^2 \} L_{\langle i}x_{jklmn\rangle} + O(\epsilon), \tag{59}$$

$$\mathcal{J}_{ijklmno} = \{ 1 - 7\nu + 14\nu^2 - 7\nu^3 \} L_{\langle i}x_{jklmno\rangle} + O(\epsilon), \tag{60}$$

$$\mathcal{J}_{ijklmnop} = -\frac{\delta m}{m} \{ 1 - 6\nu + 10\nu^2 - 4\nu^3 \} L_{\langle i}x_{jklmnop\rangle} + O(\epsilon), \tag{61}$$

where $L_i = \nu m \epsilon_{iab}x_a v_b$. In Appendix B an expression is given for the source current moments to leading order for arbitrary ℓ.

Finally, the gauge monopolar moment $\mathcal{W}$ for a circular orbit is

$$\mathcal{W} = \tfrac{1}{3}\mu x_a v_a, \tag{62}$$

which is proportional to $\dot{r} \sim O(\epsilon^{5/2})$. As $\mathcal{W}$ is already a 2.5PN correction to the source mass quadrupole, it gives no contribution in the present case, and the canonical moments will simply be given by the source moments.

IV. RESULTS

The computation of the spin-weighted spherical harmonic components is now an exercise in algebra. The evaluation of the tail, memory, and tail of tail integrals requires special treatment that is described in Sec. IV A. The spin-weighted spherical harmonic components are listed in Sec. IV B. They will contain terms that depend upon the freely specifiable constant r_0 [see the discussion between Eqs. (23) and (24)]. These terms can be absorbed into a redefinition of the phase as will be discussed in Sec. IV C.

A. Evaluation of hereditary terms

The hereditary terms in the radiative multipole moments involve integrals over the entire past history of the binary. These integrals fall into two types, the tail integrals (including the tail of tail term) that have logarithmic terms and physically correspond to the backscattering of the gravitational waves off the background curvature [43]; and the memory integrals that can physically be thought of as the reradiation of the stress energy of the propagating waves [50–52].

Inserting the canonical multipole moments into the tail integrals of the radiative multipole moments yields expressions of the form

$$I_1(T_R) = \int_0^\infty d\tau F(T_R - \tau)e^{-ik\omega(T_R-\tau)} \left[\ln \left(\frac{c\tau}{2r_0} \right) + b \right], \tag{63}$$

where k is an integer (the index m of the $h^{\ell m}$ being computed), b is a rational number, and $F(T_R - \tau)$ represents a function whose time dependence enters only through its dependence on the orbital frequency. In [12,54], it has been shown that the oscillatory term in the integrand combined with the slow adiabatic evolution of the orbital frequency allow the integral to be approximated by using the orbital frequency of a fixed circular orbit at the current value of T_R, so that

$$I_1(T_R) = F(T_R)e^{-ik\phi}\left\{\int_0^\infty d\tau e^{ik\omega\tau}\left[\ln\left(\frac{c\tau}{2r_0}\right) + b\right]\right.$$
$$\left. + O(\xi\ln\xi)\right\}, \tag{64}$$

where $\xi \sim \dot{\omega}/\omega^2$ is the adiabatic parameter describing the decay of the orbit which is $O(\epsilon^{5/2})$. Equation (64) can be evaluated using the identity

$$\int_0^\infty d\tau \ln\tau e^{-\sigma\tau} = -\frac{1}{\sigma}(\gamma_E + \ln\sigma), \tag{65}$$

where γ_E is Euler's constant. Using (65) to evaluate (64) yields

$$I_1(T_R) = -\frac{1}{k\omega}F(T_R)e^{-ik\phi}\left\{\frac{\pi}{2} + i\left[\gamma_E + \ln\left(\frac{2k\omega r_0}{c}\right) - b\right]\right\}. \tag{66}$$

For reasons that will be explained in Sec. IV C, it is convenient to express the above result as

$$I_1(T_R) = -\frac{1}{k\omega}F(T_R)e^{-ik\phi}\left\{\frac{\pi}{2} + i\left[\frac{3}{2}\ln\left(\frac{x}{x_0}\right) + \ln\frac{k}{2}\right.\right.$$
$$\left.\left. + \frac{11}{12} - b\right]\right\}, \tag{67}$$

where

$$\ln x_0 \equiv \frac{11}{18} - \frac{2}{3}\gamma_E - \frac{4}{3}\ln 2 + \frac{2}{3}\ln\left(\frac{Gm}{c^2 r_0}\right). \tag{68}$$

A similar argument holds for the tail of tail integral in Eq. (23) where a term of the form

$$I_2(T_R) = \int_0^\infty d\tau F(T_R - \tau)e^{-ik\omega(T_R-\tau)}\left[\ln\left(\frac{c\tau}{2r_0}\right)\right]^2 \tag{69}$$

is found. In this case the integral can be evaluated with the aid of the identity

$$\int_0^\infty d\tau(\ln\tau)^2 e^{-\sigma\tau} = \frac{1}{\sigma}\left[\frac{\pi^2}{6} + (\gamma_E + \ln\sigma)^2\right], \tag{70}$$

to yield

$$I_2(T_R) = \frac{1}{k\omega}F(T_R)e^{-ik\phi}\left\{\pi\left[\frac{3}{2}\ln\left(\frac{x}{x_0}\right) + \ln\frac{k}{2} + \frac{11}{12}\right]\right.$$
$$\left. + i\left[\frac{3}{2}\ln\left(\frac{x}{x_0}\right) + \ln\frac{k}{2} + \frac{11}{12}\right]^2\right\}. \tag{71}$$

The memory integral in Eq. (26) leads to an integral of the form

$$I_3(T_R) = \int_0^\infty F(T_R - \tau)e^{-ik\omega(T_R-\tau)}d\tau. \tag{72}$$

Using similar arguments as for the tail integrals, Ref. [12] has shown that this integral can be approximated by

$$I_3(T_R) = \frac{i}{k\omega}F(T_R)e^{-ik\phi}[1 + O(\xi)]. \tag{73}$$

The memory integrals also lead to an integral of the form

$$I_4(T_R) = \int_0^\infty x^5(T_R - \tau)d\tau. \tag{74}$$

Unlike the other integrals, it does not depend upon the orbital phase, and thus is much more sensitive to the past history of the binary. This is the nonlinear memory effect described in [50–52]. This integral has been evaluated by [12,51] using the evolution of the frequency parameter x found by integrating Eq. (38),

$$x(t) = \frac{1}{4}\left[\left(\frac{\nu c^3}{5Gm}\right)(t_c - t)\right]^{-1/4}[1 + O(\epsilon)] \tag{75}$$

where t_c denotes the time of coalescence. With this model,

$$I_4(T_R) = \frac{5}{64}\frac{Gm}{\nu c^3}x[1 + O(\epsilon)]. \tag{76}$$

As discussed in [12,43,51], this contribution to the waveform will be very difficult to detect as it is essentially DC in character, corresponding to a steadily growing part of the waveform with no dependence on the orbital phase. It does, however, build up during the entire inspiral which leads to its magnitude being comparable with the 0PN quadrupolar term despite arising formally as a 2.5PN contribution. See [52] for a discussion on strategies for detecting the memory piece of the waveform.

B. Spin-weighted spherical harmonic components

Examining Eq. (19) seems to show that the individual spin-weighted spherical harmonic components could obtain contributions from both mass and current radiative multipoles. It turns out for the nonspinning case, however, that the mass (current) multipoles only contribute to components with $\ell + m$ even (odd). Because of this separation, and since [24] $U^{\ell m*} = (-1)^m U^{\ell, -m}$ and $V^{\ell m*} = (-1)^m V^{\ell, -m}$, it follows from Eq. (19) that

$$h^{\ell, -m} = \begin{cases} (-1)^m h^{\ell m*} & (\ell + m \text{ even}), \\ (-1)^{m+1} h^{\ell m*} & (\ell + m \text{ odd}), \end{cases} \tag{77}$$

which simply reduces to

$$h^{\ell, -m} = (-1)^\ell h^{\ell m*} \tag{78}$$

Substituting the source mass quadrupole (47) into the expression for the radiative mass quadrupole (23), taking the appropriate time derivatives, substituting the equations of motion (40), and evaluating the hereditary integrals using the techniques described in Sec. IV A, the dominant component of the waveform is given to 3PN order as

$$h^{22} = -8\sqrt{\frac{\pi}{5}}\frac{G\nu m}{c^2 R} e^{-2i\phi} x\Bigg\{1 - x\Big(\frac{107}{42} - \frac{55}{42}\nu\Big) + x^{3/2}\Big[2\pi + 6i\ln\Big(\frac{x}{x_0}\Big)\Big] - x^2\Big(\frac{2173}{1512} + \frac{1069}{216}\nu - \frac{2047}{1512}\nu^2\Big)$$

$$- x^{5/2}\Big[\Big(\frac{107}{21} - \frac{34}{21}\nu\Big)\pi + 24i\nu + \Big(\frac{107i}{7} - \frac{34i}{7}\nu\Big)\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$+ x^3\Big[\frac{27\,027\,409}{646\,800} - \frac{856}{105}\gamma_E + \frac{2}{3}\pi^2 - \frac{1712}{105}\ln 2 - \frac{428}{105}\ln x$$

$$- 18\Big[\ln\Big(\frac{x}{x_0}\Big)\Big]^2 - \Big(\frac{278\,185}{33\,264} - \frac{41}{96}\pi^2\Big)\nu - \frac{20\,261}{2772}\nu^2 + \frac{114\,635}{99\,792}\nu^3 + \frac{428i}{105}\pi + 12i\pi\ln\Big(\frac{x}{x_0}\Big)\Big] + O(\epsilon^{7/2})\Bigg\}, \quad (79)$$

where the constant r_0 has been eliminating from the expression by using Eq. (68). The remaining components are given by

$$h^{21} = -\frac{8i}{3}\sqrt{\frac{\pi}{5}}\frac{G\nu\delta m}{c^2 R} e^{-i\phi} x^{3/2}\Bigg\{1 - x\Big(\frac{17}{28} - \frac{5}{7}\nu\Big) + x^{3/2}\Big[\pi - \frac{i}{2} - 2i\ln 2 + 3i\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$- x^2\Big(\frac{43}{126} + \frac{509}{126}\nu - \frac{79}{168}\nu^2\Big) + O(\epsilon^{5/2})\Bigg\}, \quad (80)$$

$$h^{20} = \frac{2}{7}\sqrt{\frac{10\pi}{3}}\frac{G\nu m}{c^2 R} x[1 + O(\epsilon)], \quad (81)$$

$$h^{33} = 3i\sqrt{\frac{6\pi}{7}}\frac{G\nu\delta m}{c^2 R} e^{-3i\phi} x^{3/2}\Bigg\{1 - x(4 - 2\nu) + x^{3/2}\Big[3\pi - \frac{21i}{5} + 6i\ln\frac{3}{2} + 9i\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$+ x^2\Big(\frac{123}{110} - \frac{1838}{165}\nu + \frac{887}{330}\nu^2\Big) + O(\epsilon^{5/2})\Bigg\}, \quad (82)$$

$$h^{32} = -\frac{8}{3}\sqrt{\frac{\pi}{7}}\frac{G\nu m}{c^2 R} e^{-2i\phi} x^2\Bigg\{1 - 3\nu - x\Big(\frac{193}{90} - \frac{145}{18}\nu + \frac{73}{18}\nu^2\Big) + x^{3/2}\Big[2\pi(1 - 3\nu) - 3i + \frac{66i}{5}\nu + 6i(1 - 3\nu)\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$+ O(\epsilon^2)\Bigg\}, \quad (83)$$

$$h^{31} = -\frac{i}{3}\sqrt{\frac{2\pi}{35}}\frac{G\nu\delta m}{c^2 R} e^{-i\phi} x^{3/2}\Bigg\{1 - x\Big(\frac{8}{3} + \frac{2}{3}\nu\Big) + x^{3/2}\Big[\pi - \frac{7i}{5} - 2i\ln 2 + 3i\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$+ x^2\Big(\frac{607}{198} - \frac{136}{99}\nu - \frac{247}{198}\nu^2\Big) + O(\epsilon^{5/2})\Bigg\}, \quad (84)$$

$$h^{30} = \frac{16i}{5}\sqrt{\frac{6\pi}{35}}\frac{G\nu m}{c^2 R}\nu x^{7/2}[1 + O(\epsilon)], \quad (85)$$

$$h^{44} = \frac{64}{9}\sqrt{\frac{\pi}{7}}\frac{G\nu m}{c^2 R} e^{-4i\phi} x^2\Bigg\{1 - 3\nu - x\Big(\frac{593}{110} - \frac{1273}{66}\nu + \frac{175}{22}\nu^2\Big)$$

$$+ x^{3/2}\Big[4\pi(1 - 3\nu) - \frac{42i}{5} + \frac{1193i}{40}\nu + 8i(1 - 3\nu)\ln 2 + 12i(1 - 3\nu)\ln\Big(\frac{x}{x_0}\Big)\Big]$$

$$+ x^2\Big(\frac{1\,068\,671}{200\,200} - \frac{1\,088\,119}{28\,600}\nu + \frac{146\,879}{2340}\nu^2 - \frac{226\,097}{17\,160}\nu^3\Big) + O(\epsilon^{5/2})\Bigg\}, \quad (86)$$

$$h^{43} = \frac{9i}{5}\sqrt{\frac{2\pi}{7}}\frac{G\nu\delta m}{c^2 R} e^{-3i\phi} x^{5/2}\Bigg\{1 - 2\nu - x\Big(\frac{39}{11} - \frac{1267}{132}\nu + \frac{131}{33}\nu^2\Big) + O(\epsilon^{3/2})\Bigg\}, \quad (87)$$

$$h^{42} = -\frac{8}{63}\sqrt{\pi}\frac{G\nu m}{c^2 R}e^{-2i\phi}x^2\left\{1 - 3\nu - x\left(\frac{437}{110} - \frac{805}{66}\nu + \frac{19}{22}\nu^2\right)\right.$$
$$+ x^{3/2}\left[2\pi(1 - 3\nu) - \frac{21i}{5} + \frac{84i}{5}\nu + 6i(1 - 3\nu)\ln\left(\frac{x}{x_0}\right)\right] + x^2\left(\frac{1\,038\,039}{200\,200} - \frac{606\,751}{28\,600}\nu + \frac{400\,453}{25\,740}\nu^2 + \frac{25\,783}{17\,160}\nu^3\right)$$
$$\left. + O(\epsilon^{5/2})\right\}, \tag{88}$$

$$h^{41} = -\frac{i}{105}\sqrt{2\pi}\frac{G\nu\delta m}{c^2 R}e^{-i\phi}x^{5/2}\left\{1 - 2\nu - x\left(\frac{101}{33} - \frac{337}{44}\nu + \frac{83}{33}\nu^2\right) + O(\epsilon^{3/2})\right\}, \tag{89}$$

$$h^{40} = \frac{1}{63}\sqrt{\frac{\pi}{10}}\frac{G\nu m}{c^2 R}x[1 + O(\epsilon)], \tag{90}$$

$$h^{55} = -\frac{125i}{12}\sqrt{\frac{5\pi}{66}}\frac{G\nu\delta m}{c^2 R}e^{-5i\phi}x^{5/2}\left\{1 - 2\nu - x\left(\frac{263}{39} - \frac{688}{39}\nu + \frac{256}{39}\nu^2\right) + O(\epsilon^{3/2})\right\}, \tag{91}$$

$$h^{54} = \frac{256}{45}\sqrt{\frac{\pi}{33}}\frac{G\nu m}{c^2 R}e^{-4i\phi}x^3\left\{1 - 5\nu + 5\nu^2 - x\left(\frac{4451}{910} - \frac{3619}{130}\nu + \frac{521}{13}\nu^2 - \frac{339}{26}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{92}$$

$$h^{53} = \frac{9i}{20}\sqrt{\frac{3\pi}{22}}\frac{G\nu\delta m}{c^2 R}e^{-3i\phi}x^{5/2}\left\{1 - 2\nu - x\left(\frac{69}{13} - \frac{464}{39}\nu + \frac{88}{39}\nu^2\right) + O(\epsilon^{3/2})\right\}, \tag{93}$$

$$h^{52} = -\frac{16}{135}\sqrt{\frac{\pi}{11}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x^3\left\{1 - 5\nu + 5\nu^2 - x\left(\frac{3911}{910} - \frac{3079}{130}\nu + \frac{413}{13}\nu^2 - \frac{231}{26}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{94}$$

$$h^{51} = -\frac{i}{180}\sqrt{\frac{\pi}{77}}\frac{G\nu\delta m}{c^2 R}e^{-i\phi}x^{5/2}\left\{1 - 2\nu - x\left(\frac{179}{39} - \frac{352}{39}\nu + \frac{4}{39}\nu^2\right) + O(\epsilon^{3/2})\right\}, \tag{95}$$

$$h^{66} = -\frac{432}{5}\sqrt{\frac{\pi}{715}}\frac{G\nu m}{c^2 R}e^{-6i\phi}x^3\left\{1 - 5\nu + 5\nu^2 - x\left(\frac{113}{14} - \frac{91}{2}\nu + 64\nu^2 - \frac{39}{2}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{96}$$

$$h^{65} = -\frac{625i}{63}\sqrt{\frac{5\pi}{429}}\frac{G\nu\delta m}{c^2 R}e^{-5i\phi}x^{7/2}\{1 - 4\nu + 3\nu^2 + O(\epsilon)\}, \tag{97}$$

$$h^{64} = \frac{1024}{495}\sqrt{\frac{2\pi}{195}}\frac{G\nu m}{c^2 R}e^{-4i\phi}x^3\left\{1 - 5\nu + 5\nu^2 - x\left(\frac{93}{14} - \frac{71}{2}\nu + 44\nu^2 - \frac{19}{2}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{98}$$

$$h^{63} = \frac{81i}{385}\sqrt{\frac{\pi}{13}}\frac{G\nu\delta m}{c^2 R}e^{-3i\phi}x^{7/2}\{1 - 4\nu + 3\nu^2 + O(\epsilon)\}, \tag{99}$$

$$h^{62} = -\frac{16}{1485}\sqrt{\frac{\pi}{13}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x^3\left\{1 - 5\nu + 5\nu^2 - x\left(\frac{81}{14} - \frac{59}{2}\nu + 32\nu^2 - \frac{7}{2}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{100}$$

$$h^{61} = -\frac{i}{2079}\sqrt{\frac{2\pi}{65}}\frac{G\nu\delta m}{c^2 R}e^{-i\phi}x^{7/2}\{1 - 4\nu + 3\nu^2 + O(\epsilon)\}, \tag{101}$$

$$h^{77} = \frac{16\,807i}{180}\sqrt{\frac{7\pi}{4290}}\frac{G\nu\delta m}{c^2 R}e^{-7i\phi}x^{7/2}\left\{1 - 4\nu + 3\nu^2 - x\left(\frac{319}{34} - \frac{2225}{51}\nu + \frac{2558}{51}\nu^2 - \frac{230}{17}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{102}$$

$$h^{76} = -\frac{648}{35}\sqrt{\frac{3\pi}{715}}\frac{G\nu m}{c^2 R}e^{-6i\phi}x^4\{1 - 7\nu + 14\nu^2 - 7\nu^3 + O(\epsilon)\}, \tag{103}$$

$$h^{75} = -\frac{3125i}{3276}\sqrt{\frac{5\pi}{66}}\frac{G\nu\delta m}{c^2 R}e^{-5i\phi}x^{7/2}\left\{1 - 4\nu + 3\nu^2 - x\left(\frac{271}{34} - \frac{1793}{51}\nu + \frac{1838}{51}\nu^2 - \frac{134}{17}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{104}$$

$$h^{74} = \frac{1024}{1365}\sqrt{\frac{2\pi}{165}}\frac{G\nu m}{c^2 R}e^{-4i\phi}x^4\{1 - 7\nu + 14\nu^2 - 7\nu^3 + O(\epsilon)\}, \tag{105}$$

$$h^{73} = \frac{243i}{20\,020}\sqrt{\frac{3\pi}{10}}\frac{G\nu\delta m}{c^2 R}e^{-3i\phi}x^{7/2}\left\{1 - 4\nu + 3\nu^2 - x\left(\frac{239}{34} - \frac{1505}{51}\nu + \frac{1358}{51}\nu^2 - \frac{70}{17}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{106}$$

$$h^{72} = -\frac{8}{3003}\sqrt{\frac{\pi}{15}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x^4\{1 - 7\nu + 14\nu^2 - 7\nu^3 + O(\epsilon)\}, \tag{107}$$

$$h^{71} = -\frac{i}{108\,108}\sqrt{\frac{\pi}{10}}\frac{G\nu\delta m}{c^2 R}e^{-i\phi}x^{7/2}\left\{1 - 4\nu + 3\nu^2 - x\left(\frac{223}{34} - \frac{1361}{51}\nu + \frac{1118}{51}\nu^2 - \frac{38}{17}\nu^3\right) + O(\epsilon^{3/2})\right\}, \tag{108}$$

$$h^{88} = \frac{131\,072}{315}\sqrt{\frac{2\pi}{17\,017}}\frac{G\nu m}{c^2 R}e^{-8i\phi}x^4\left\{1 - 7\nu + 14\nu^2 - 7\nu^3 \right.$$
$$\left. - x\left(\frac{3653}{342} - \frac{9325}{114}\nu + \frac{22\,351}{114}\nu^2 - \frac{9107}{57}\nu^3 + \frac{4081}{114}\nu^4\right) + O(\epsilon^{3/2})\right\}, \tag{109}$$

$$h^{87} = \frac{117\,649i}{3240}\sqrt{\frac{7\pi}{4862}}\frac{G\nu\delta m}{c^2 R}e^{-7i\phi}x^{9/2}\{1 - 6\nu + 10\nu^2 - 4\nu^3 + O(\epsilon)\}, \tag{110}$$

$$h^{86} = -\frac{1944}{35}\sqrt{\frac{3\pi}{85\,085}}\frac{G\nu m}{c^2 R}e^{-6i\phi}x^4\left\{1 - 7\nu + 14\nu^2 - 7\nu^3 \right.$$
$$\left. - x\left(\frac{353}{38} - \frac{7897}{114}\nu + \frac{18\,067}{114}\nu^2 - \frac{6727}{57}\nu^3 + \frac{2653}{114}\nu^4\right) + O(\epsilon^{3/2})\right\}, \tag{111}$$

$$h^{85} = -\frac{15\,625i}{4536}\sqrt{\frac{5\pi}{4862}}\frac{G\nu\delta m}{c^2 R}e^{-5i\phi}x^{9/2}\{1 - 6\nu + 10\nu^2 - 4\nu^3 + O(\epsilon)\}, \tag{112}$$

$$h^{84} = \frac{1024}{4095}\sqrt{\frac{2\pi}{935}}\frac{G\nu m}{c^2 R}e^{-4i\phi}x^4\left\{1 - 7\nu + 14\nu^2 - 7\nu^3 - x\left(\frac{2837}{342} - \frac{6877}{114}\nu + \frac{15\,007}{114}\nu^2 - \frac{5027}{57}\nu^3 + \frac{1633}{114}\nu^4\right) + O(\epsilon^{3/2})\right\}, \tag{113}$$

$$h^{83} = \frac{81i}{3640}\sqrt{\frac{3\pi}{374}}\frac{G\nu\delta m}{c^2 R}e^{-3i\phi}x^{9/2}\{1 - 6\nu + 10\nu^2 - 4\nu^3 + O(\epsilon)\}, \tag{114}$$

$$h^{82} = -\frac{8}{45\,045}\sqrt{\frac{\pi}{17}}\frac{G\nu m}{c^2 R}e^{-2i\phi}x^4\left\{1 - 7\nu + 14\nu^2 - 7\nu^3 \right.$$
$$\left. - x\left(\frac{2633}{342} - \frac{6265}{114}\nu + \frac{13\,171}{114}\nu^2 - \frac{4007}{57}\nu^3 + \frac{1021}{114}\nu^4\right) + O(\epsilon^{3/2})\right\}, \tag{115}$$

$$h^{81} = -\frac{i}{92\,664}\sqrt{\frac{\pi}{1190}}\frac{G\nu\delta m}{c^2 R}e^{-i\phi}x^{9/2}\{1 - 6\nu + 10\nu^2 - 4\nu^3 + O(\epsilon)\}. \tag{116}$$

In Appendix C, spin-weighted harmonic components for even (odd) $\ell + m$ are given to 1PN (0PN) order for arbitrary ℓ. Again, the $m < 0$ components are given by (80).

C. Absorbing amplitude terms into the phase

The $\ln(x/x_0)$ terms that appear in the spin-weighted spherical harmonic components can be absorbed into a redefinition of the phase by introducing an auxiliary phase variable $\psi = \phi + \delta$. Since the $\ln(x/x_0)$ terms first enter at 1.5PN order, it is straightforward to show that choosing [12,48]

$$\delta = -3\frac{M}{m}x^{3/2}\ln\left(\frac{x}{x_0}\right), \tag{117}$$

where $M = I$ (the mass monopole of the source), will eliminate the $\ln(x/x_0)$ terms from the components. This follows from

$$h^{\ell m} = \tilde{h}^{\ell m}e^{-im\psi} = \tilde{h}^{\ell m}e^{-im\phi}e^{-im\delta}$$
$$= \tilde{h}^{\ell m}e^{-im\phi}[1 - im\delta - \tfrac{1}{2}m^2\delta^2 + O(x^{9/2})],$$

where $\tilde{h}^{\ell m}$ is $h^{\ell m}$ omitting the $\ln(x/x_0)$ terms.[4] Furthermore, since the orbital phase as a function of frequency goes as

$$\phi = -\frac{1}{32\nu}x^{-5/2} + O(\epsilon) \tag{118}$$

at leading order, the $\ln(x/x_0)$ terms, which were 1.5PN, 2.5PN, and 3PN order in the amplitude terms, now appear as phase corrections at relative order 4PN, 5PN, and 5.5PN. As these terms are beyond the order to which the orbital phase evolution is known (3.5PN order), it can be argued that these terms can be ignored. Note that the choices of x_0 in Eq. (68) and δ in Eq. (117) are not unique, as other amplitude terms can be absorbed into the phase (e.g. see [13]); these choices were made to gather all logarithmic terms (dependent upon x_0) into one term, as well as to simplify the waveform [48].

In order to recover the 2.5PN polarization waveforms in [12,13] from the components listed above, it is necessary to substitute $\ln(x/x_0) \to 0$. After substituting the coefficients into Eq. (11) and truncating the sum at 2.5PN order, the result must be evaluated at $\Theta = i$, $\Phi = \pi/2$. Furthermore, there is an overall sign difference due to a different choice of the polarization triad $(\vec{N}, \vec{P}, \vec{Q})$.

The polarization waveforms have been computed in the limit $\nu \to 0$ in [7] using black hole perturbation theory. In order to compare with their results, it is necessary to substitute

$$\ln\left(\frac{x}{x_0}\right) \to -\frac{17}{18} + \frac{2}{3}\ln 2$$

into the $h^{\ell m}$ listed above, as [7] makes a different choice in redefining the phase variable (and works in Schwarzschild coordinates as opposed to harmonic coordinates). After this substitution, and setting $\nu = 0$ and $\delta m/m = -1$, it is found that the $h^{\ell m}$ above agree with the results of [7,69].[5]

V. CONCLUSIONS

It has been shown that the spin-weighted spherical harmonic components of the waveform can be computed to higher post-Newtonian order by computing them directly from the radiative multipole moments rather than by projecting them from the full polarization waveforms. In particular, this allows the dominant h^{22} component to be computed to 3PN order. Since numerical simulations can compute the waveform over the entire sphere, it is possible to compare the spin-weighted spherical harmonic components of the waveform from the simulation with those predicted by a quasiadiabatic post-Newtonian inspiral. Thus, by examining the h^{22} component, it can be determined whether or not the 3PN contribution improves the agreement between the post-Newtonian waveform and the numerical waveform. If significant improvement is found, this would suggest that it would be worth the effort of computing the full 3PN waveform in order to improve detection of marginal signals [27–29] as well as improve parameter extraction [30–36].

In [23] a high-accuracy comparison is made between post-Newtonian generated waveforms and waveforms from a numerical simulation of 15 orbits of an inspiral of an equal-mass nonspinning binary black hole system. For this case, [23] finds that the 3PN contributions to the amplitude of the h^{22} mode improve the accuracy with respect to the numerical waveforms. This suggests that, for accurate parameter estimation, it may be desirable to compute the full 3PN amplitude for the polarization waveforms.

For an equal-mass, nonspinning binary, only the source current octupole needs to be computed to have the full polarization waveform. For a nonspinning binary with an arbitrary mass ratio, much more effort is required as explained at the end of [12]. But for comparison with numerical simulations, post-Newtonian theorists should keep in mind that extending the PN order of a given radiative multipole moment will result in corresponding improve-

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[4]Note that there remains one logarithmic term, $-\frac{428}{105}x^3\ln x$, in Eq. (81) that is not absorbed into the phase.

[5]There is a sign difference in $\zeta_+^{\ell,m}$ between [7,69]. The results presented here agree with the sign of [69].

ments in the spin-weighted spherical harmonic components of the waveform, and these corrections will be of interest even if all the corrections needed to improve the polarization waveforms to the next order have yet to be computed.

APPENDIX A: 1PN SOURCE MASS MOMENTS

The source mass multipole moments for a system of N (nonrotating) compact point masses to 1PN order are given by [40,49,68]

$$I_L = \sum_{A=1}^{N}\left\{\tilde{\mu}_A y_A^{\langle L\rangle} + \frac{1}{2(2\ell+3)c^2}\frac{d^2}{dt^2}[m_A y_A^2 y_A^{\langle L\rangle}]\right.$$
$$\left. - \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2}\frac{d}{dt}[m_A v_A^a y_A^{\langle aL\rangle}] + O(\epsilon^2)\right\},$$
(A1)

where m_A, $\vec{y}_A$, and $\vec{v}_A$ are the mass, position, and velocity, respectively, of the Ath point mass, and

$$\tilde{\mu}_A = m_A\left[1 + \frac{3}{2}\frac{v_A^2}{c^2} - \frac{G}{c^2}\sum_{B\neq A}\frac{m_B}{|\vec{y}_A - \vec{y}_B|} + O(\epsilon^2)\right].$$
(A2)

Restricting to the case of two bodies in a quasicircular orbit, transforming to the center-of-mass frame using

$$\vec{y}_1 = \frac{m_2}{m}\vec{x}[1 + O(\epsilon^2)],$$
(A3)

$$\vec{y}_2 = -\frac{m_1}{m}\vec{x}[1 + O(\epsilon^2)],$$
(A4)

and using the relative equation of motion (34) to eliminate the relative acceleration from the derivatives yields

$$I_L = \nu\tilde{m}\left\{\left[f_{\ell-1}(\nu) - \gamma f_\ell(\nu)\right.\right.$$
$$\left. + \gamma\frac{5\ell^2 + 6\ell + 9}{2(\ell+1)(2\ell+3)}f_{\ell+1}(\nu)\right]x_{\langle L\rangle}$$
$$+ \frac{\ell(\ell-1)(\ell+9)}{2(\ell+1)(2\ell+3)}f_{\ell+1}(\nu)\frac{r^2}{c^2}x_{\langle L-2}v_{i_{\ell-1}i_\ell\rangle}$$
$$\left. + O(\epsilon^2)\right\}$$
(A5)

where

$$\{\tilde{m}, f_k(\nu)\} = \begin{cases}\{m, s_k(\nu)\} & \text{for } \ell \text{ even,}\\ \{-\delta m, d_k(\nu)\} & \text{for } \ell \text{ odd,}\end{cases}$$
(A6)

where $s_\ell = (m_1^\ell + m_2^\ell)/m^\ell$ and $d_\ell = (m_1^\ell - m_2^\ell)/m^\ell$, which themselves can be rewritten as polynomials in ν as

$$s_\ell(\nu) = 1 + \sum_{k=1}^{\ell/2}\left[\binom{\ell-k-1}{k-1} + \binom{\ell-k}{k}\right](-\nu)^k, \quad (A7)$$

$$d_\ell(\nu) = \sum_{k=0}^{\ell/2}\binom{\ell-k-1}{k}(-\nu)^k.$$
(A8)

APPENDIX B: 0PN SOURCE CURRENT MOMENTS

The source current multipole moments for a system of N (nonspinning) compact point masses is given to 0PN order by [42,49]

$$\mathcal{J}_L = \sum_{A=1}^{N} m_A \epsilon^{ab\langle i_\ell} y_A^{L-1\rangle a} v_A^b + O(\epsilon).$$
(B1)

Restricting to the case of two bodies in a quasicircular orbit, and transforming to the center-of-mass frame as for the mass moments in Appendix A, it is straightforward to show that the source current multipole moments to leading order are given by

$$\mathcal{J}_L = \frac{\hat{m}}{m}g_\ell(\nu)L_{\langle i_\ell}x_{L-1\rangle} + O(\epsilon),$$
(B2)

where

$$\{\hat{m}, g_k(\nu)\} = \begin{cases}\{-\delta m, d_k(\nu)\} & \text{for } \ell \text{ even,}\\ \{m, s_k(\nu)\} & \text{for } \ell \text{ odd.}\end{cases}$$
(B3)

APPENDIX C: 1PN SPIN-WEIGHTED SPHERICAL HARMONIC COMPONENTS

1. $\ell + m$ even

For even $\ell + m$ (for nonspinning binaries), the spin-weighted spherical harmonic components are due to contributions from the radiative mass multipoles. To 1PN order, the radiative moments are given simply by ℓ time derivatives of the source moments. For arbitrary ℓ, the mass moment is given by Eq. (A5). In evaluating the spin-weighted spherical harmonic components from the source mass moments, the following identity is useful:

$$x_L \mathcal{Y}_L^{\ell m} = r^\ell n_L \mathcal{Y}_L^{\ell m} = r^\ell Y^{\ell m}\left(\frac{\pi}{2}, \phi\right),$$
(C1)

where ϕ is the orbital phase (as opposed to an angle on the sphere). Since $\phi = \omega t$,

$$\frac{d^k}{dt^k}Y^{\ell m}\left(\frac{\pi}{2}, \phi\right) = (-im\omega)^k Y^{\ell m}\left(\frac{\pi}{2}, \phi\right).$$
(C2)

Therefore, it can be shown that

$$x_{L-2}v_{i_{\ell-1}i_\ell}\mathcal{Y}_L^{\ell m} = \frac{\ell - m^2}{\ell(\ell - 1)}\omega^2 r^\ell Y^{\ell m}\left(\frac{\pi}{2}, \phi\right),$$ (C3)

using that $\dot{r}$ and $\dot{\omega}$ are of 2.5 order. Finally, using that

$$\left(\frac{r\omega}{c}\right)^\ell = x^{\ell/2}\left[1 - x\ell\left(1 - \frac{1}{3}\nu\right)\right],$$ (C4)

it can be shown that

$$h^{\ell m} = (im)^\ell \frac{8\pi}{(2\ell + 1)!!}\sqrt{\frac{(\ell + 1)(\ell + 2)}{\ell(\ell - 1)}}\frac{G\nu\tilde{m}}{c^2 R}x^{\ell/2}Y^{\ell, -m}\left(\frac{\pi}{2}, \phi\right)\left\{f_{\ell-1}(\nu)\left[1 - x\ell\left(1 - \frac{1}{3}\nu\right)\right] + \frac{3}{2}xf_{\ell+1}(\nu) - xf_\ell(\nu)\right.$$
$$\left. - x\frac{m^2(\ell + 9)}{2(\ell + 1)(2\ell + 3)}f_{\ell+1}(\nu) + O(\epsilon^{3/2})\right\} \quad \text{(for } \ell + m \text{ even)},$$ (C5)

where $\tilde{m}$ and $f_k(\nu)$ are given by (A6)
.

2. $\ell + m$ odd

For odd $\ell + m$ (for nonspinning binaries), the spin-weighted spherical harmonic components are due to contributions from the radiative current multipoles. To 0PN order, the radiative moments are given simply by ℓ time derivatives of the source moments. For arbitrary ℓ, the current moment is given by Eq. (B2). In evaluating the spin-weighted spherical harmonic components from the source current moments, the following identity is useful:

$$\delta_{zi_\ell}n_{L-1}\mathcal{Y}_L^{\ell m} = \frac{1}{\ell}\sqrt{\frac{(2\ell + 1)(\ell^2 - m^2)}{2\ell - 1}}Y^{\ell-1, m}\left(\frac{\pi}{2}, \phi\right).$$ (C6)

Following the same steps as above, it can be shown that

$$h^{\ell m} = -(im)^\ell \frac{16\pi i}{(2\ell + 1)!!}\sqrt{\frac{(2\ell + 1)(\ell + 2)(\ell^2 - m^2)}{\ell(2\ell - 1)(\ell + 1)(\ell - 1)}}\frac{G\nu\hat{m}}{c^2 R}x^{(\ell+1)/2}Y^{\ell-1, -m}\left(\frac{\pi}{2}, \phi\right)[g_\ell(\nu) + O(\epsilon)] \quad \text{(for } \ell + m \text{ odd)},$$ (C7)

where $\hat{m}$ and $g_k(\nu)$ are given by (B3).

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