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Multiscale analysis of the electromagnetic self-force in a weak gravitational field

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We examine the motion of a charged particle in a weak gravitational field. In addition to the Newtonian gravity exerted by a large central body, the particle is subjected to an electromagnetic self-force that contains both a conservative piece and a radiation-reaction piece. This toy problem shares many of the features of the strong-field gravitational self-force problem, and it is sufficiently simple that it can be solved exactly with numerical methods, and approximately with analytical methods. We submit the equations of motion to a multiscale analysis, and we examine the roles of the conservative and radiation-reaction pieces of the self-force. We show that the radiation-reaction force drives secular changes in the orbit's semilatus rectum and eccentricity, while the conservative force drives a secular regression of the periapsis and affects the orbital time function; neglect of the conservative term can hence give rise to an important phasing error. We next examine what might be required in the formulation of a reliable secular approximation for the orbital evolution; this would capture all secular changes in the orbit and discard all irrelevant oscillations. We conclude that such an approximation would be very difficult to formulate without prior knowledge of the exact solution.

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I. INTRODUCTION

The gravitational inspiral of a solar-mass compact object into a massive black hole residing in a galactic center has been identified as one of the most promising sources of gravitational waves for the Laser Interferometer Space Antenna [1]. The need for accurate theoretical models of the expected signal, for the purposes of signal detection and source identification, has motivated an intense effort from many workers to determine the motion of the small body in the field of the large black hole. This is done in a treatment that goes beyond the geodesic approximation and takes into account the body's own gravitational field, which is a small perturbation over the field of the black hole. In this treatment the small body can be described as moving on an accelerated world line in the background spacetime of the large black hole; the body is said to move under the influence of its own gravitational self-force [2,3], and this force is derived from the retarded gravitational perturbation produced by the moving body. For a review of the self-force formalism, see Ref. [4] and the special issue of *Classical and Quantum Gravity* devoted to this topic [5].

The concrete evaluation of the gravitational self-force acting on a small body moving in the Kerr spacetime is a challenging project that has not yet been completed (although progress has been steady). Given the severity of the challenge, a number of authors [6–16] have attempted to formulate various simple schemes that would allow them to reproduce the effects of the self-force; their hope is that these schemes will be simple enough for rapid implementation in numerical codes, and accurate enough to describe faithfully the orbital evolution of a body subjected to its gravitational self-force. One such scheme is *Mino's radiative approximation* [6–8], which is based on an approximate self-force constructed from the half-retarded minus half-advanced gravitational perturbation associated with the moving body. Mino proved that while his version of the self-force neglects all conservative corrections to the motion, it correctly accounts for the long-term dissipative effects associated with the true self-force. This led to the widespread belief that all long-term secular changes in the orbital motion would be captured by the radiative approximation, and that conservative effects would produce only short-term changes that would not accumulate in the long run. The simplicity of Mino's scheme made it attractive, and it was adopted in a number of works that aimed to model the inspiral of a small body into a rapidly rotating black hole [9–16].

Mino's radiative approximation was criticized, however, in an earlier work by the present authors [17], hereafter referred to as "paper I." Building on an analogy between the gravitational self-force and its electromagnetic counterpart, we showed that conservative terms in the true self-force do lead to long-term secular changes in the orbital motion. These changes are not captured by the radiative approximation, and we concluded that Mino's scheme has severe limitations. This conclusion was supported by a recent analysis by Drasco and Hughes [12], and the general attitude currently is that while the radiative approximation may be useful to generate template waveforms for signal detection, it is probably insufficient for reliable parameter estimation.

Our purpose in this paper is to revisit the analysis presented in paper I. There are three reasons for reopening the case. The first is that our original analysis of the electromagnetic self-force employed rather crude mathematical tools, and we wish to present here a more thorough and rigorous treatment. The second is that the main source of discrepancy between the effects of the radiative self-force and those of the true self-force was not correctly identified in paper I. In the original paper we claim that the discrepancy is mainly due to the secular regression in the orbit's periapsis, an effect that is produced by the true self-force but not accounted for by the radiative approximation. In this paper we show that while this is indeed a source of discrepancy, it is not the most important one. As we shall explain in Sec. IV C, the most important conservative effect is actually associated with the time function on the orbit. The third reason is that we wish to introduce here a clear distinction between the *radiative approximation* to the self-force and the notion of a *secular approximation* to an orbital evolution; our secular approximation is a specific implementation of the general idea of capturing the long-term orbital evolution through an *adiabatic approximation* that allows the orbit to evolve slowly. The phrases ''radiative approximation'' and ''adiabatic approximation'' are used synonymously in paper I (and indeed, in most of the literature on this topic), but we feel that this is a highly misleading practice. A large portion of this paper is devoted to the task of identifying what should be required of a good secular approximation, and we shall see that the radiative approximation does not meet those requirements.

The precise meaning of an adiabatic approximation is somewhat ambiguous in the literature. In all cases, the basic assumption is that the secular effects of the self-force occur on a time scale that is long compared to the orbital period. From this assumption, numerous approximations have been formulated: (1) Since the particle's orbit deviates only slowly from geodesic motion, the self-force can be calculated as if the particle travels on a geodesic (or, in the post-Newtonian case, the radiation reaction can be calculated as if the particle's dynamics were conservative); (2) Since the radiation-reaction time scale is much longer than the orbital period, periodic effects can be neglected; and (3) Based on various arguments, conservative effects can be neglected. Although each one of these three approximations has been called an adiabatic approximation, we believe that they should be distinguished from one another. To discuss the first approximation is beyond the scope of this paper. We focus instead on the latter two approximations: number (2) above, which we call the *secular approximation*, and which neglects periodic effects; and number (3), which we shall call the *radiative approximation*, and which neglects conservative effects.

The main idea behind the construction of a secular approximation is the following. We consider an orbital evolution under the action of a self-force, and we wish to simplify the equations of motion in such a way that the long-term, secular changes will be captured, at the cost of discarding irrelevant, short-term effects. Suppose that we describe the orbital evolution in terms of a set of orbital elements $I^A(t)$, where A is an index that labels each element. (This description is introduced in Sec. III, and explained fully in Appendices A and B.) The orbital elements would be constant in the absence of a perturbing force, but they evolve in time as a result of the force's action. It is expected that each orbital element will display a behavior that can be decomposed into a secular change that accumulates monotonically over time, and an oscillation that averages to zero in the long run. We thus write $I^A(t) = I^A_{\rm sec}(t) + I^A_{\rm osc}(t)$, and a secular approximation for the orbital elements would keep the secular terms and discard the oscillations. We would write $I^A(t) \simeq I^A_{\rm sec}(t)$, and seek a method to obtain $I^A_{\rm sec}(t)$ in the most direct and convenient way possible. Presuming that this must be done in a context in which the exact solution $I^A(t)$ would be too difficult to obtain, we would seek to formulate equations of motion directly for $I^A_{\rm sec}(t)$, and we would hope that those equations are sufficiently simple that a solution could easily be found (analytically or numerically). This is the main idea, and the task ahead appears to be clearly identified. But to turn the idea into a precise algorithm may not be easy. To illustrate the difficulty we shall examine, in a specific context in which we can make progress (the electromagnetic self-force of paper I), what would be required in the construction of a secular approximation for the orbital evolution.

The secular approximation is logically distinct from the radiative approximation, in which the true self-force is truncated so as to discard all conservative terms. In the radiative approximation, one writes $I^A(t) \simeq I^A_{\rm rad}(t)$, and one calculates $I^A_{\rm rad}(t)$ on the basis of the truncated self-force. It is known, as Mino has shown [6–8], that the radiative self-force correctly accounts for the long-term, *dissipative changes* in the orbital elements. If it correctly produced the long-term, *conservative changes* as well, we would conclude that the radiative approximation captures the idea of a secular approximation. But, as we have shown in paper I [17], and as we intend to show even more convincingly here, the radiative approximation fails to account for secular changes in $I^A(t)$ that are produced by the conservative piece of the self-force. The radiative and secular approximations are therefore distinct, and we consider it important to distinguish these terms carefully.

We will examine the limitations of the radiative approximation, and attempt to construct a faithful secular approximation, in the specific context of an electromagnetic self-force acting on a charged particle moving in a weak gravitational field. The motion of the particle is governed by the equations

$$\boldsymbol{a} = \boldsymbol{g} + \boldsymbol{f}_{\rm self}, \tag{1.1}$$

where $\boldsymbol{a} = d^2\boldsymbol{r}/dt^2$ is the particle's acceleration vector, $\boldsymbol{g} = -M\hat{\boldsymbol{r}}/r^2$ is the Newtonian gravitational field of a body of mass M, and

$$\boldsymbol{f}_{\rm self} = \lambda_{\rm c}\frac{q^2}{\mu}\frac{M}{r^3}\hat{\boldsymbol{r}} + \lambda_{\rm rr}\frac{2}{3}\frac{q^2}{\mu}\frac{d\boldsymbol{g}}{dt} \tag{1.2}$$

is the electromagnetic self-force divided by the particle's mass μ. (The particle is treated as a test mass. In this treatment, the central body does not move, its mass M is the system's total mass, and μ is the system's reduced mass.) This self-force was calculated [18,19] for the weakly curved spacetime produced by the central body, assuming that the charged particle moves slowly. Here q is the particle's charge, and $\boldsymbol{r}(t)$ is its position vector relative to the central body; we have also introduced the distance $r = |\boldsymbol{r}|$ and the unit vector $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The constants $\lambda_{\rm c}$ and $\lambda_{\rm rr}$ in Eq. (1.2) are both equal to unity; they serve to remind us that the first term in Eq. (1.2) is the conservative piece of the self-force, while the second term is the dissipative (or radiation-reaction) piece. By keeping these constants in our calculations we will be able to distinguish conservative effects from dissipative effects; the radiative approximation is obtained by setting $\lambda_{\rm c} = 0$ and keeping $\lambda_{\rm rr} = 1$. Throughout the paper we use the usual vectorial notation of three-dimensional flat space, and we work in units such that $G = c = 1$.

We work in the specific context of Eqs. (1.1) and (1.2) for two reasons. First, our toy problem is an actual example of a self-force acting on an orbiting body. While the force has an electromagnetic origin instead of a gravitational origin, and while the motion takes place in a weak (Newtonian) gravitational field instead of a strong field, the self-force of Eq. (1.2) nevertheless contains conservative and dissipative terms that will have different effects on the orbital motion. In the usual Lorenz gauge, the gravitational self-force in strong fields will also contain conservative and dissipative pieces. In addition, each self-force comes with a similar post-Newtonian counting. From Eq. (1.2) we see that the conservative term in the electromagnetic self-force is a correction of order $q^2/(\mu r)$ relative to $\boldsymbol{g}$, and taking q to be of order μ, we recognize this as a correction of first post-Newtonian (1PN) order; the dissipative term is a correction of order $q^2 v/(\mu r)$, where v is the orbital velocity, and we recognize this as a correction of 1.5PN order. On the other hand, in a post-Newtonian context the gravitational self-force presents conservative pieces at orders 0PN, 1PN, 2PN, 3PN, and so on, and dissipative pieces at orders 2.5PN, 3.5PN, and so on. While the post-Newtonian counting is not identical, in each case we have dominance of the conservative effects over the dissipative effects, and all in all, this gives us good reasons to believe that the electromagnetic problem captures the essential physics of the more complicated, gravitational problem when it is formulated in the usual Lorenz gauge.

Second, Eq. (1.1) is far simpler than the realistic self-force equation (for which the force must be obtained numerically), and this permits a very thorough and rigorous mathematical analysis. We shall therefore be able to extract very precise consequences of Eqs. (1.1) and (1.2), and examine closely the issues that concern us regarding the radiative and secular approximations. The simple mathematics of the toy problem will allow us to draw firm and clear conclusions, and the proximity of its physics to that of the realistic problem will give us confidence that these conclusions extend from the toy problem to the realistic case.

We begin in Sec. II with a simple illustration of the themes to be explored in this paper. The mathematical analysis of Eqs. (1.1) and (1.2) is carried out in Sec. III, in the framework of osculating orbital elements summarized in Appendices A and B. The mathematical details are presented in Sec. III with minimum commentary, but we present a full discussion of our results in Sec. IV. We summarize our conclusions in Sec. V.

II. ILLUSTRATION

Before we proceed with our mathematical analysis of Eqs. (1.1) and (1.2), it is helpful to describe a very simple problem that illustrates rather well the issues we shall encounter.

Suppose that we are interested in a quantity $q(t)$ that is governed by the system of dynamical equations

$$\frac{dq}{d\lambda} = \epsilon_1 - \epsilon_2 \sin\lambda, \qquad \frac{dt}{d\lambda} = 1 + \cos\lambda + \epsilon_3, \tag{2.1}$$

where λ is a running parameter, and ϵ_1, ϵ_2, and ϵ_3 are small constants. The differential equations come with the initial conditions

$$q(0) = 1, \qquad t(0) = 0. \tag{2.2}$$

In this example, $q(t)$ is analogous to the set of orbital elements $I^A(t)$ that were introduced previously, and the equation for $dq/d\lambda$ is analogous to Eq. (3.10) below, with ϵ_1 and ϵ_2 playing the roles of $\epsilon_{\rm rr}$ and $\epsilon_{\rm c}$, respectively. The quantity q is constant when $\epsilon_1 = \epsilon_2 = 0$ (the unperturbed situation), and it acquires a time dependence when the perturbation is turned on. The equation for $dt/d\lambda$ is analogous to the first line of Eq. (3.34) below, with ϵ_3 playing the role of $\epsilon_{\rm c}$. As we explain in Sec. III C, the first term proportional to $\epsilon_{\rm c}$ in Eq. (3.34) is generated by oscillatory terms in the orbital elements and oscillatory terms in the unperturbed equation for t; these combine to give rise to a secular term in the perturbed equation. To ignore the oscillations would produce the significant mistake of dropping the ϵ_3 term in Eq. (2.1). The variable λ gives a convenient parametrization of the motion; its use is motivated by the fact that the "force" can be expressed as a simple function of λ, while it would be very difficult to express it in terms of the time variable t. The parameter λ is analogous to the orbital parameter ϕ that will be introduced in Sec. III. (It is also analogous to "Mino time" [6,20], a convenient parametrization of geodesics in Kerr spacetime.)

The exact solution to the system of equations is

$$q(\lambda) = 1 + \epsilon_1 \lambda + \epsilon_2(\cos\lambda - 1), \tag{2.3}$$

$$t(\lambda) = \lambda + \sin\lambda + \epsilon_3 \lambda. \tag{2.4}$$

We see that ϵ_1 produces a secular growth in $q(\lambda)$, ϵ_2 is associated with oscillations, and ϵ_3 produces a secular drift in the time function $t(\lambda)$.

Suppose that we are interested only in the long-term, secular changes in q, and that we wish to construct a secular approximation for it. Because we have access to the exact solution, it is a simple matter to remove the oscillations by subjecting it to an averaging procedure. Our first option is to introduce

$$\langle q\rangle_\lambda := \frac{1}{2\pi}\int_{\lambda-\pi}^{\lambda+\pi} q(\lambda')d\lambda' \tag{2.5}$$

and to define the secular approximation as $q_{\rm sec} = \langle q\rangle_\lambda$. Our choice here is therefore to remove the oscillations with respect to λ, and a calculation based on Eqs. (2.3) and (2.5) gives

$$\langle q\rangle_\lambda = 1 + \epsilon_1\lambda - \epsilon_2 + O(\epsilon^2). \tag{2.6}$$

This version of the secular approximation is a solution to the modified differential equation

$$\frac{d}{d\lambda}\langle q\rangle_\lambda = \epsilon_1 + O(\epsilon^2) \tag{2.7}$$

with the modified initial condition

$$\langle q\rangle_\lambda(0) = 1 - \epsilon_2 + O(\epsilon^2). \tag{2.8}$$

If we did not have access to the exact solution, we might still have guessed that the correct differential equation for $q_{\rm sec}$ is Eq. (2.7), because it can be obtained directly from Eq. (2.1) by averaging over the oscillatory term. But we would be hard pressed to guess that the correct initial condition is given by Eq. (2.8). Using the approximate differential equation with the exact initial condition $q_{\rm sec}(0) = 1$ would produce a function that is offset by ϵ_2 relative to $\langle q\rangle_\lambda$.

Our first message is that a faithful secular approximation can be based on an averaged version of the differential equation, but that it must come also with a corresponding change of initial condition. To obtain the approximate differential equation might be easy, but to identify the correct initial condition is impossible when the exact solution is unknown. The formulation of a secular approximation therefore suffers from an ambiguity regarding the correct choice of initial condition. In this example the consequence of missing the ϵ_2 term in the initial condition is not severe: The difference between the solutions $1 + \epsilon_1\lambda$ and $1 + \epsilon_1\lambda - \epsilon_2$ becomes relatively small as λ increases and each solution grows secularly. In other situations, however, the difference in initial conditions could lead to more serious discrepancies.

In Eq. (2.5) we removed the oscillations of the exact solution by averaging over the parameter λ. Because the observer might be more interested in the time behavior of the function q, an alternative choice is to perform the averaging over t instead of λ. And since $t(\lambda)$ contains oscillations, it should be expected that this alternative method of averaging will lead to a distinct formulation of the secular approximation. Our second option is therefore to introduce

$$\langle q\rangle_t := \frac{\int_{\lambda-\pi}^{\lambda+\pi} q(\lambda')(dt/d\lambda')d\lambda'}{\int_{\lambda-\pi}^{\lambda+\pi}(dt/d\lambda')d\lambda'} \tag{2.9}$$

and to define version 2 of the secular approximation as $q_{\rm sec} = \langle q\rangle_t$. A calculation based on Eqs. (2.1), (2.3), and (2.9) gives

$$\langle q\rangle_t = 1 + \epsilon_1(\lambda - \sin\lambda) - \frac{1}{2}\epsilon_2 + O(\epsilon^2). \tag{2.10}$$

This is a solution to the modified differential equation

$$\frac{d}{d\lambda}\langle q\rangle_t = \epsilon_1(1 + \cos\lambda) + O(\epsilon^2) \tag{2.11}$$

and the modified initial condition

$$\langle q\rangle_t(0) = 1 - \frac{1}{2}\epsilon_2 + O(\epsilon^2). \tag{2.12}$$

Here the situation is more interesting. If we did not have access to the exact solution, we would never have guessed that the correct differential equation for the secular approximation is Eq. (2.11), and we would also never have arrived at Eq. (2.12).

Our second message is that this new secular approximation (version 2, which removes the oscillations in t instead of the oscillations in λ) must be based on an approximate differential equation and an approximate initial condition that are impossible to identify without knowing the solution to the exact problem. The ambiguity of the first method extends from the choice of initial condition to the specification of the differential equation.

Our third message is that while the idea of formulating a secular approximation is clear enough, it is difficult to turn it into a precise algorithm. To remove the oscillations of an exact solution is easy enough. But to reformulate the system of differential equations and initial conditions into a set of approximate equations that would achieve the same result is difficult; it might well be impossible in most cases.

Our fourth message is concerned with the analogue here of formulating a radiative approximation to Eqs. (2.1). This is obtained by setting $\epsilon_2 = \epsilon_3 = 0$ while leaving ϵ_1 unchanged. This produces the functions

$$q_{\rm rad}(\lambda) = 1 + \epsilon_1\lambda, \qquad t_{\rm rad}(\lambda) = \lambda + \sin\lambda. \tag{2.13}$$

After a long time, when $\lambda \gg 1$, $q_{\rm rad}(\lambda)$ becomes very nearly equal to $q(\lambda)$, and the radiative approximation is accurate when q is expressed in terms of the orbital parameter. At late times, however, we have that $t_{\rm rad}(\lambda) \simeq \lambda$ while $t(\lambda) \simeq (1 + \epsilon_3)\lambda$, and we see that the radiative approximation produces a shift in the time function that becomes important when λ increases beyond $1/\epsilon_3$; we shall see that this feature is present also in the context of the electromagnetic self-force in Sec. III. When q is expressed as a function of time, we get that $q_{\rm rad}(t) \simeq 1 + \epsilon_1 t$, while the exact solution behaves as $q(t) \simeq 1 + \epsilon_1 t/(1 + \epsilon_3)$. The difference is equal to $\epsilon_1\epsilon_3 t/(1+\epsilon_3)$. While this appears to be small because of the first factor of order ϵ^2, it is steadily growing because of the additional factor of t. The radiative approximation, therefore, produces a secular drift in the time function, and a corresponding drift in q.

III. MULTISCALE ANALYSIS OF THE ELECTROMAGNETIC SELF-FORCE

We now proceed with our mathematical analysis of Eqs. (1.1) and (1.2). This section, unlike all others in this paper, is highly technical, and we intend to deal with the technical issues while keeping the commentary to a minimum. The implications of our results, in the light of the themes introduced in Sec. I, will be fully detailed in Sec. IV. The reader who may not wish to delve into the technical details, and who would prefer to pick up the story where we left off at the end of Sec. II, can omit reading this section and proceed directly to Sec. IV.

A. System of equations

We wish to integrate the equations of motion (1.1) for the electromagnetic self-force of Eq. (1.2). We shall do so by employing the method of osculating orbital elements developed in Appendix B.

The starting point of the method is the unperturbed situation described by the equations $\boldsymbol{a} = \boldsymbol{g}$, in which the particle follows a Keplerian orbit characterized by a number of orbital elements. (Kepler's problem is reviewed in Appendix A.) The orbital elements are constants of the Keplerian motion; they are related to the initial conditions placed on the particle's position and velocity vectors, but they are defined so as to provide the most useful information regarding the geometric properties of the orbit. The elliptical shape of the Keplerian orbit is described by

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \omega)}, \tag{3.1}$$

where r is the distance between the particle and the central mass M, and ϕ is the longitude. The orbital elements are the semilatus rectum p, the eccentricity e, and the longitude at periapsis ω. The elements p and e determine on which ellipse the particle is moving, and we shall call them the *principal orbital elements*. The element ω determines the particle's initial position on the selected ellipse, and we shall refer to it as a *positional orbital element*. The position of the particle as a function of time is determined by integrating

$$t' = \sqrt{\frac{p^3}{M}} \frac{1}{[1 + e\cos(\phi - \omega)]^2} \tag{3.2}$$

for the time function $t(\phi)$; the prime indicates differentiation with respect to ϕ. The motion in the orbital plane is then fully described (in parametric form) by the functions $r(\phi)$ and $t(\phi)$. It is an important fact that Eq. (3.2) does not admit a closed-form solution; a convenient way to handle it is by straightforward numerical integration.

We next move on to the equations $\boldsymbol{a} = \boldsymbol{g} + \boldsymbol{f}_{\rm self}$ and a description of the perturbed motion. In the method of osculating orbital elements, the motion continues to be described by Eqs. (3.1) and (3.2), but the orbital elements (p, e, ω) acquire a ϕ dependence that accounts for the perturbation. Their evolution equations are given by Eqs. (B16)–(B18) in Appendix B. They rely on a decomposition of the self-force according to $\boldsymbol{f}_{\rm self} = R\hat{\boldsymbol{r}} + S\hat{\boldsymbol{\phi}}$, with R denoting its radial component and S its tangential component; the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ point in the directions of increasing r and ϕ, respectively. The self-force does not contain a component normal to the orbital plane, and indeed, our version of the method of osculating elements is restricted to perturbing forces that are tangent to the plane.

From the expression given in Eq. (1.2), we find that the radial and tangential components of the self-force are

$$R = \frac{q^2 M}{\mu}\left(\lambda_{\rm c} + \frac{4}{3}\lambda_{\rm rr}\dot{r}\right)\frac{1}{r^3} \tag{3.3}$$

and

$$S = \frac{q^2 M}{\mu}\left(-\frac{2}{3}\lambda_{\rm rr}\dot{\phi}\right)\frac{1}{r^2}, \tag{3.4}$$

respectively. Here an overdot indicates differentiation with respect to t, and time derivatives can be converted into ϕ derivatives by involving Eq. (3.2). After differentiating Eq. (3.1) to obtain r', we find that Eqs. (3.3) and (3.4) become

$$R = \frac{q^2 M}{\mu}\frac{(1 + ec)^3}{p^3}\left(\lambda_{\rm c} + \frac{4}{3}\lambda_{\rm rr}\sqrt{\frac{M}{p}}es\right) \tag{3.5}$$

and

$$S = \frac{q^2 M}{\mu}\frac{(1 + ec)^4}{p^3}\left(-\frac{2}{3}\lambda_{\rm rr}\sqrt{\frac{M}{p}}\right), \tag{3.6}$$

where $c := \cos(\phi - \omega)$ and $s := \sin(\phi - \omega)$. These expressions are ready to be inserted within Eqs. (B16)–(B18).

The evolution equations come with the initial conditions

$$p(\phi = 0) =: p^*, \qquad e(\phi = 0) =: e^*, \\ \omega(\phi = 0) =: \omega^* \equiv 0, \qquad t(\phi = 0) =: t^* \equiv 0; \tag{3.7}$$

the values selected for ω^* and t^* produce no loss of generality. To facilitate the integrations we introduce the dimensionless semilatus rectum p and dimensionless time t, as well as the dimensionless parameters $\epsilon_{\rm c}$ and $\epsilon_{\rm rr}$ that characterize the strength of the perturbing force. These are defined by

$$\mathsf{p} := \frac{p}{p^*}, \qquad \mathsf{t} := \sqrt{\frac{M}{p^{*3}}}t, \qquad \epsilon_c := \lambda_c \frac{q^2}{\mu p^*}, \qquad \epsilon_{rr} := \frac{2}{3}\lambda_{rr}\frac{q^2}{\mu p^*}\sqrt{\frac{M}{p^*}}. \tag{3.8}$$

The final form of the evolution equations is

$$\mathsf{p}' = -2\epsilon_{rr}\frac{1+ec}{\mathsf{p}^{1/2}}, \tag{3.9}$$

$$e' = \epsilon_c \frac{s(1+ec)}{\mathsf{p}} + \epsilon_{rr}\frac{(1+ec)(e-2c-3ec^2)}{\mathsf{p}^{3/2}}, \tag{3.10}$$

$$\omega' = -\epsilon_c \frac{c(1+ec)}{e\mathsf{p}} - \epsilon_{rr}\frac{(1+ec)s(2+3ec)}{e\mathsf{p}^{3/2}}, \tag{3.11}$$

$$\mathsf{t}' = \frac{\mathsf{p}^{3/2}}{(1+ec)^2}, \tag{3.12}$$

where $c = \cos(\phi - \omega)$ and $s = \sin(\phi - \omega)$. Integration proceeds from the initial values $\mathsf{p}(\phi = 0) = 1$, $e(\phi = 0) = e^*$, $\omega(\phi = 0) = 0$, and $\mathsf{t}(\phi = 0) = 0$. We shall assume that ϵ_c and ϵ_{rr} are small throughout the evolution. In spite of the fact that ϵ_{rr} is smaller than ϵ_c by a factor of order $\sqrt{M/p^*} \ll 1$, we shall formally treat them as being of the same order of magnitude.

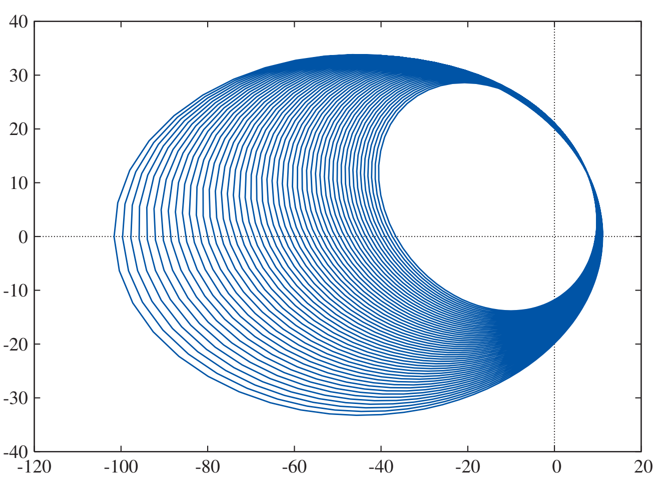

FIG. 1 (color online). Orbital evolution under the action of the electromagnetic self-force. We set $M = 1$, $q^2/\mu = 0.05$, and the orbital elements are integrated from the initial conditions $p^* = 20$, $e^* = 0.8$, and $\omega^* = 0$. With these choices we have $\epsilon_c = 2.500 \times 10^{-3}$ and $\epsilon_{rr} = 3.727 \times 10^{-4}$. The motion proceeds in the counterclockwise direction and is followed for 50 orbital cycles ($0 \le \phi < 100\pi$). At the end of the integration $\epsilon_c \phi = 0.7854$. The orbit is displayed in the x–y plane, with $x = r\cos\phi$ and $y = r\sin\phi$. In the course of its evolution the orbit becomes smaller, more circular, and its periapsis regresses.

Equations (3.9), (3.10), (3.11), and (3.12) can easily be integrated numerically, and the orbital motion reconstructed by inserting the solutions within Eq. (3.1). The result of such a numerical integration is presented in Fig. 1. Our goal, however, is to obtain as much analytical information as possible, and for this purpose we shall construct *approximate solutions* to these equations, taking advantage of the fact that ϵ_c and ϵ_{rr} are small. This will be carried out in the following subsections. The approximation we shall construct is distinct from the secular and radiative approximations considered in Sec. I; we shall refer to it as the *multiscale approximation*. We shall demonstrate that our multiscale approximation faithfully reproduces the numerical results at all points on the orbit, up to a time at which terms of second order in ϵ_c and ϵ_{rr} become important. From the multiscale approximation we shall be able to construct secular and radiative approximations, and we shall be able to ascertain their accuracy.

B. Multiscale approximation: orbital elements

The action of the electromagnetic self-force causes the orbital elements p, e, and ω to acquire a ϕ dependence governed by Eqs. (3.9), (3.10), (3.11), and (3.12). The corrections to the unperturbed solutions $\mathsf{p} = 1$, $e = e^*$, and $\omega = 0$ can be separated into two classes: *secular terms* that grow monotonically with ϕ and *nonsecular terms* that oscillate and average to zero over a complete orbital cycle ($0 \le \phi < 2\pi$). To capture these different behaviors, we need an approximation method that has the capability of producing a solution that stays accurate over a long interval $0 \le \phi < \phi_{max}$, with ϕ_{max} of the order of ϵ^{-1}, where ϵ is the overall smallness parameter of the problem; and in this interval, the difference between the exact and approximate solutions must be uniformly of order ϵ^2. (Because we have introduced two such parameters, we shall write $\epsilon_c = e_c\epsilon$, $\epsilon_{rr} = e_{rr}\epsilon$ and consider e_c, e_{rr} to be quantities of order unity.) These requirements rule out a simple-minded expansion in powers of ϵ, because this method would give rise to a solution that is accurate only for $\epsilon\phi \ll 1$. We adopt instead a multiscale analysis (see, for example, Chapter 11 of Ref. [21]).

In a multiscale expansion one introduces a dependence on a "long-scale" variable $z := \epsilon\phi$ in addition to the dependence on the "short-scale" variable ϕ. We write

$$\mathsf{p} = \mathsf{p}_0(z) + \epsilon \mathsf{p}_1(z, \phi) + \cdots, \tag{3.13}$$

$$e = e_0(z) + \epsilon e_1(z, \phi) + \cdots, \tag{3.14}$$

$$\omega = \omega_0(z) + \epsilon\omega_1(z, \phi) + \cdots, \tag{3.15}$$

and we seek to isolate all secular changes within the zeroth-order quantities, and to make all first-order quantities purely oscillatory. We use the chain rule

$$f' = \frac{\partial f}{\partial \phi} + \epsilon \frac{\partial f}{\partial z}$$

to evaluate the total derivative with respect to ϕ of a function $f(z, \phi)$.

To proceed we substitute Eqs. (3.13), (3.14), and (3.15) into Eqs. (3.9), (3.10), and (3.11) and obtain, to first order in ϵ,

$$\frac{d\mathsf{p}_0}{dz} + \frac{\partial \mathsf{p}_1}{\partial \phi} = -\frac{2e_{\rm rr}}{\mathsf{p}_0^{1/2}}\Big[1 + e_0 \cos v\Big], \tag{3.16}$$

$$\begin{aligned}\frac{de_0}{dz} + \frac{\partial e_1}{\partial \phi} = {}& \frac{e_{\rm c}}{\mathsf{p}_0}\Big[\sin v + \frac{1}{2}e_0 \sin 2v\Big] \\ & - \frac{e_{\rm rr}}{\mathsf{p}_0^{3/2}}\Big[\frac{3}{2}e_0 + \frac{1}{4}(8 + 5e_0^2)\cos v \\ & + \frac{5}{2}e_0 \cos 2v + \frac{3}{4}e_0^2 \cos 3v\Big],\end{aligned} \tag{3.17}$$

$$\begin{aligned}\frac{d\omega_0}{dz} + \frac{\partial \omega_1}{\partial \phi} = {}& -\frac{e_{\rm c}}{2\mathsf{p}_0}\Big[1 + \frac{2}{e_0}\cos v + \cos 2v\Big] \\ & - \frac{e_{\rm rr}}{\mathsf{p}_0^{3/2}}\Big[\frac{8 + 3e_0^2}{4e_0}\sin v + \frac{5}{2}\sin 2v \\ & + \frac{3}{4}e_0 \sin 3v\Big],\end{aligned} \tag{3.18}$$

where $v := \phi - \omega_0$. It is easy to recognize the terms on the right-hand sides that drive the secular changes in the orbital elements. We isolate these changes by setting

$$\frac{d\mathsf{p}_0}{dz} = -\frac{2e_{\rm rr}}{\mathsf{p}_0^{1/2}}, \qquad \frac{de_0}{dz} = -\frac{3e_{\rm rr}e_0}{2\mathsf{p}_0^{3/2}}, \qquad \frac{d\omega_0}{dz} = -\frac{e_{\rm c}}{2\mathsf{p}_0}. \tag{3.19}$$

The nonsecular (oscillatory) corrections are then obtained by integrating

$$\frac{\partial \mathsf{p}_1}{\partial \phi} = -\frac{2e_{\rm rr}e_0}{\mathsf{p}_0^{1/2}}\cos v, \tag{3.20}$$

$$\begin{aligned}\frac{\partial e_1}{\partial \phi} = {}& \frac{e_{\rm c}}{\mathsf{p}_0}\Big[\sin v + \frac{1}{2}e_0 \sin 2v\Big] \\ & - \frac{e_{\rm rr}}{\mathsf{p}_0^{3/2}}\Big[\frac{1}{4}(8 + 5e_0^2)\cos v + \frac{5}{2}e_0 \cos 2v \\ & + \frac{3}{4}e_0^2 \cos 3v\Big],\end{aligned} \tag{3.21}$$

$$\begin{aligned}\frac{\partial \omega_1}{\partial \phi} = {}& -\frac{e_{\rm c}}{2\mathsf{p}_0}\Big[\frac{2}{e_0}\cos v + \cos 2v\Big] \\ & - \frac{e_{\rm rr}}{\mathsf{p}_0^{3/2}}\Big[\frac{8 + 3e_0^2}{4e_0}\sin v + \frac{5}{2}\sin 2v + \frac{3}{4}e_0 \sin 3v\Big].\end{aligned} \tag{3.22}$$

We must impose the initial conditions $\mathsf{p}_0 + \epsilon \mathsf{p}_1 + \cdots = 1$, $e_0 + \epsilon e_1 + \cdots = e^*$, and $\omega_0 + \epsilon \omega_1 + \cdots = 0$ when $\phi = 0$.

The general solutions to Eqs. (3.19) are

$$\mathsf{p}_0 = a(1 - 3e_{\rm rr}z/a^{3/2})^{2/3}, \tag{3.23}$$

$$e_0 = b(1 - 3e_{\rm rr}z/a^{3/2})^{1/2}, \tag{3.24}$$

$$\omega_0 = c - \frac{e_{\rm c}a^{1/2}}{2e_{\rm rr}}\Big[1 - (1 - 3e_{\rm rr}z/a^{3/2})^{1/3}\Big], \tag{3.25}$$

where a, b, and c are constants of integration that will be determined.

The purely oscillatory solutions to Eqs. (3.20), (3.21), and (3.22) are

$$\epsilon \mathsf{p}_1 = -\frac{2\epsilon_{\rm rr}e_0}{\mathsf{p}_0^{1/2}}\sin v, \tag{3.26}$$

$$\begin{aligned}\epsilon e_1 = {}& -\frac{\epsilon_{\rm c}}{\mathsf{p}_0}\Big[\cos v + \frac{1}{4}e_0 \cos 2v\Big] \\ & - \frac{\epsilon_{\rm rr}}{4\mathsf{p}_0^{3/2}}\Big[(8 + 5e_0^2)\sin v + 5e_0 \sin 2v + e_0^2 \sin 3v\Big],\end{aligned} \tag{3.27}$$

$$\begin{aligned}\epsilon \omega_1 = {}& -\frac{\epsilon_{\rm c}}{\mathsf{p}_0}\Big[\frac{1}{e_0}\sin v + \frac{1}{4}\sin 2v\Big] \\ & + \frac{\epsilon_{\rm rr}}{4\mathsf{p}_0^{3/2}}\Big[\frac{8 + 3e_0^2}{e_0}\cos v + 5\cos 2v + e_0 \cos 3v\Big].\end{aligned} \tag{3.28}$$

We recall that $v = \phi - \omega_0$.

To relate the constants a, b, and c to the initial conditions we note that $\mathsf{p}_0(0) = a$ and $\epsilon \mathsf{p}_1(0) = 0$; we therefore have $a = \mathsf{p}^* = 1$. Similarly, we note that $e_0(0) = b$ and $\epsilon e_1(0) = -\frac{1}{4}\epsilon_{\rm c}(4 + b)$; we therefore have $b = e^* + \frac{1}{4}\epsilon_{\rm c}(4 + e^*) + O(\epsilon^2)$. Finally, we note that $\omega_0(0) = c$ and $\epsilon \omega_1(0) = \epsilon_{\rm rr}(8 + 5b + 4b^2)/(4b)$; we therefore have $c = -\epsilon_{\rm rr}(8 + 5e^* + 4e^{*2})/(4e^*) + O(\epsilon^2)$. Making these substitutions in Eqs. (3.22), (3.23), and (3.24) gives

$$\mathsf{p}_0 = (1 - 3\epsilon_{\rm rr}\phi)^{2/3}, \tag{3.29}$$

$$e_0 = e^*\Big[1 + \epsilon_{\rm c}\frac{4 + e^*}{4e^*}\Big](1 - 3\epsilon_{\rm rr}\phi)^{1/2}, \tag{3.30}$$

$$\omega_0 = -\epsilon_{\rm rr}\frac{8 + 5e^* + 4e^{*2}}{4e^*} - \frac{\epsilon_{\rm c}}{2\epsilon_{\rm rr}}\Big[1 - (1 - 3\epsilon_{\rm rr}\phi)^{1/3}\Big]. \tag{3.31}$$

These expressions describe the secular changes in the orbital elements. Eqs. (3.26), (3.27), and (3.28), on the other hand, describe the nonsecular (oscillatory) changes. All together, these results give us the desired multiscale

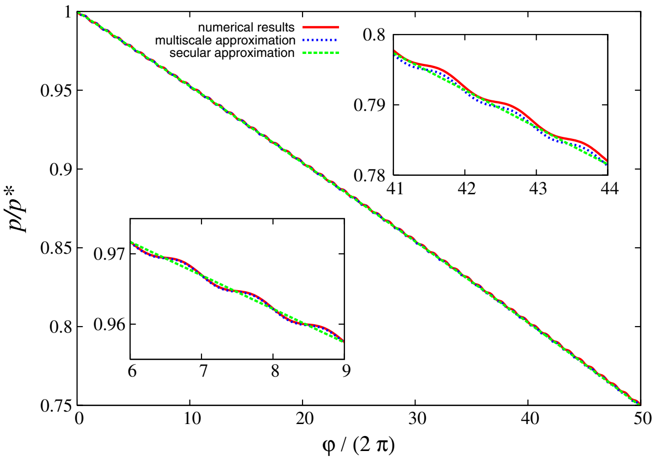

FIG. 2 (color online). Multiscale approximation for $p(\phi/2\pi)$ compared with exact numerical results. The numerical conditions are the same as in Fig. 1. The solid curve in red shows the exact evolution as computed numerically. The dotted curve in blue shows the evolution as predicted by the multiscale approximation, which includes a secular term as well as oscillations. The dashed curve in green shows the secular piece of the multiscale approximation. The large panel shows the entire evolution from $\phi = 0$ to $\phi = 100\pi$. The first inset (bottom left) shows the evolution in the small interval $6 < \phi/(2\pi) < 9$; early in the evolution the multiscale approximation is extremely accurate. The second inset (top right) shows the evolution in the small interval $41 < \phi/(2\pi) < 44$; here the multiscale approximation is less accurate, because $\epsilon\phi$ has become comparable to unity.

approximation for the orbital elements.[1] We compare the approximations with exact numerical results in Figs. 2–4.

C. Multiscale approximation: time

We now wish to construct a multiscale approximation for the time function $\mathsf{t}(\phi)$. To begin we recall that Eq. (3.12) must be integrated numerically even in the unperturbed situation, when p, e, and ω are all constant; the approximation, therefore, will also involve a numerical integration. A lazy option presents itself: The time function could be obtained simply by inserting our multiscale approximations for $\mathsf{p}(\phi)$, $e(\phi)$, and $\omega(\phi)$ within Eq. (3.12) and performing the integration numerically. In an effort to obtain maximum analytical insight, however, we choose to proceed differently.

We substitute Eqs. (3.13), (3.14), and (3.15) and the explicit expressions of Eqs. (3.26), (3.27), and (3.28) into Eq. (3.12), and we expand in powers of ϵ. Through first order we obtain

$$\mathsf{t}' = \frac{\mathsf{p}_0^{3/2}}{(1+e_0\cos v)^2} + \frac{1}{2}\epsilon_{\rm c}\mathsf{p}_0^{1/2}\frac{4+e_0\cos v}{(1+e_0\cos v)^3} - \frac{1}{2}\epsilon_{\rm rr}e_0\frac{\sin v + e_0\sin 2v}{(1+e_0\cos v)^3}, \tag{3.32}$$

[1] We take this opportunity to make an observation. We notice from Eq. (3.29) that when $\epsilon\phi$ is comparable to unity, the dissipative term in the self-force produces a change in p that is also of order unity. In this calculation the radiation-reaction force is linear in ϵ, and there are no corrections of order ϵ^2. If such corrections were present, however, they would produce an additional change of order ϵ in p. Next we notice from Eq. (3.30) that the conservative term in the self-force produces a change of order ϵ in e, in addition to the change of order unity that comes from the radiation-reaction force. We conclude that *second-order terms in the radiation-reaction force* would produce effects that scale with the same power of ϵ as those produced by the conservative force. As we shall see below, the conservative force must be included in the calculation when the evolution of the orbital phase is required to stay accurate to order ϵ^0 during a radiation-reaction time. In a context where the radiation-reaction force would contain a second-order term, the same accuracy would be achieved only after including this term as well in the calculation. We thank Tanja Hinderer and Éanna Flanagan for making this point clear to us.

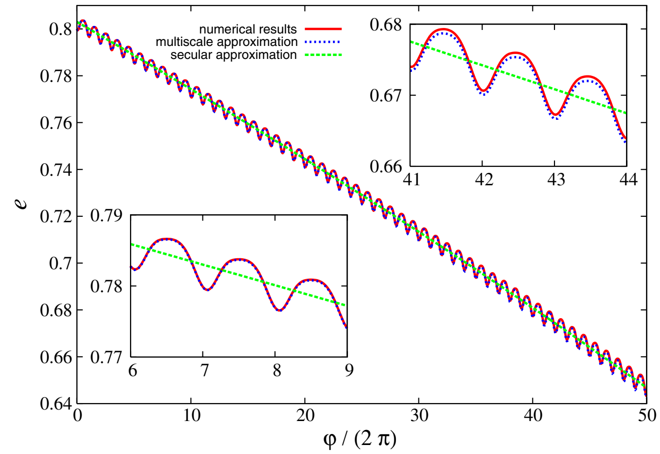

FIG. 3 (color online). Multiscale approximation for $e(\phi/2\pi)$ compared with exact numerical results. The numerical conditions are the same as in Fig. 1, and the caption of Fig. 2 provides the relevant details.

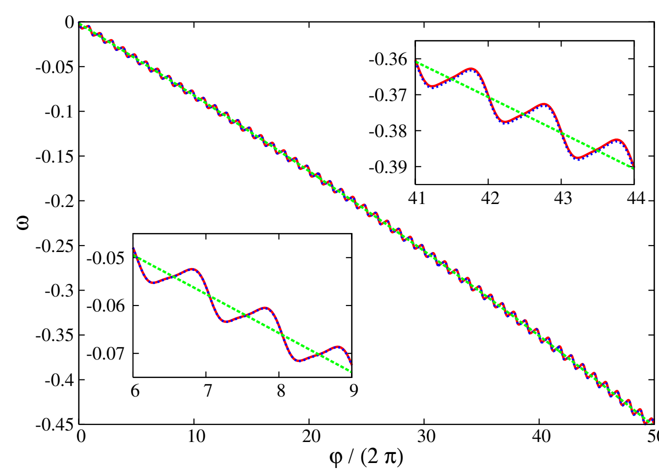

FIG. 4 (color online). Multiscale approximation for $\omega(\phi/2\pi)$ compared with exact numerical results. The numerical conditions are the same as in Fig. 1, and the caption of Fig. 2 provides the relevant details.

where $v = \phi - \omega_0$, and p_0, e_0, and ω_0 are the functions of $z := \epsilon_{\rm rr}\phi$ displayed in Eqs. (3.29), (3.30), and (3.31). The first term in Eq. (3.32) is obtained by inserting $\mathsf{p} = \mathsf{p}_0$, $e = e_0$, and $\omega = \omega_0$ within Eq. (3.12); the second and third terms are contributed by the oscillatory terms p_1, e_1, and ω_1.

We wish to find an approximate solution to Eq. (3.32), and once more we wish to distinguish between secular and nonsecular terms. Two sources of complications present themselves. First, while it was easy in Eq. (3.16), (3.17), and (3.18) to separate the secular terms from the oscillations, the factors of $1 + e_0 \cos v$ in the denominators of Eq. (3.32) make this separation more difficult. Second, while p, e, and ω, are simply constant at the unperturbed level, the Keplerian version of $\mathsf{t}(\phi)$ is already a complicated function of ϕ that contains secular and oscillating terms. The situation here is therefore more complicated, but we will, nevertheless, be able to express the solution to Eq. (3.32) in the form

$$\mathsf{t}(\phi) = \mathsf{t}_0(\phi) + \epsilon \mathsf{t}_1(\phi) + \cdots, \tag{3.33}$$

with $\mathsf{t}_0(\phi)$ incorporating the Keplerian behavior (including secular terms and oscillations) as well as the secular changes produced by the electromagnetic self-force, and with $\epsilon\mathsf{t}_1(\phi)$ being purely oscillatory.

To isolate the oscillatory terms in Eq. (3.32) we calculate the averages

$$\langle f\rangle_\phi(\phi) := \frac{1}{2\pi}\int_{\phi-\pi}^{\phi+\pi} f(\phi')d\phi'$$

of the various functions of ϕ that appear on its right-hand side; these averages are calculated while keeping p_0, e_0, and ω_0 constant over the integration domain. Defining $f_1 = (1 + e_0 \cos v)^{-2}$, $f_2 = (4 + e_0 \cos v) \times (1 + e_0 \cos v)^{-3}$, and $f_3 = (\sin v + e_0 \sin 2v) \times (1 + e_0 \cos v)^{-3}$, we find that $\langle f_1\rangle_\phi = (1 - e_0^2)^{-3/2}$, $\langle f_2\rangle_\phi = \frac{1}{2}(8 + e_0^2)(1 - e_0^2)^{-5/2}$, and $\langle f_3\rangle = 0$. The function multiplying $\epsilon_{\rm rr}$ in Eq. (3.32) is therefore purely oscillatory, but the function multiplying $\epsilon_{\rm c}$ contains a secular component. To isolate this we rewrite Eq. (3.32) into the equivalent form

$$\begin{aligned}\mathsf{t}' = {}& \frac{\mathsf{p}_0^{3/2}}{(1 + e_0\cos v)^2}\left[1 + \epsilon_{\rm c}\frac{8 + e_0^2}{4\mathsf{p}_0(1 - e_0^2)}\right] \\ & - \frac{1}{4}\epsilon_{\rm c}\frac{\mathsf{p}_0^{1/2}e_0}{1 - e_0^2}\frac{9e_0 + 3(2 + e_0^2)\cos v}{(1 + e_0\cos v)^3} \\ & - \frac{1}{2}\epsilon_{\rm rr}e_0\frac{\sin v + e_0\sin 2v}{(1 + e_0\cos v)^3},\end{aligned} \tag{3.34}$$

in which a term $\frac{1}{4}\epsilon_{\rm c}\mathsf{p}_0^{1/2}(8 + e_0^2)(1 - e_0^2)^{-1}(1 + e_0\cos v)^{-2}$ was removed from the second term in Eq. (3.32) and inserted within the first term. In Eq. (3.34), the functions that appear in the second and third lines are purely oscillatory.

It is important to notice that the term proportional to $\epsilon_{\rm c}$ in the first line of Eq. (3.34) is a secular correction to t' that originates with the oscillatory terms p_1, e_1, and ω_1 in the orbital elements. These oscillations combine in a nonlinear fashion, and they contribute an additional secular term beyond the one that comes from p_0, e_0, and ω_0. It would be a significant mistake to discard the oscillations in the orbital elements when constructing the time function.

The solution to Eq. (3.34) is

$$\begin{aligned}\mathsf{t} = {}& \int_0^\phi \frac{\mathsf{p}_0^{3/2}}{(1 + e_0\cos v')^2}\left[1 + \epsilon_{\rm c}\frac{8 + e_0^2}{4\mathsf{p}_0(1 - e_0^2)}\right]d\phi' \\ & - \frac{1}{4}\epsilon_{\rm c}\int_0^\phi \frac{\mathsf{p}_0^{1/2}e_0}{1 - e_0^2}\frac{9e_0 + 3(2 + e_0^2)\cos v'}{(1 + e_0\cos v')^3}d\phi' \\ & - \frac{1}{2}\epsilon_{\rm rr}\int_0^\phi e_0\frac{\sin v' + e_0\sin 2v'}{(1 + e_0\cos v')^3}d\phi',\end{aligned} \tag{3.35}$$

where $v' := \phi' - \omega_0(\phi')$. We must leave the first integral alone, but we shall manage to evaluate the second and third integrals. Because the changes in p_0, e_0, and ω_0 are of order ϵ, because the second and third integrals already come with a factor of ϵ in front, because the integrands are purely oscillatory functions, and because the calculation of $\mathsf{t}(\phi)$ is carried out consistently to first order in ϵ, we are permitted to treat p_0, e_0, and ω_0 as constants when evaluating the integrals. We thus obtain

$$\begin{aligned}\mathsf{t} = {}& \int_0^\phi \frac{\mathsf{p}_0^{3/2}}{(1 + e_0\cos v')^2}\left[1 + \epsilon_{\rm c}\frac{8 + e_0^2}{4\mathsf{p}_0(1 - e_0^2)}\right]d\phi' \\ & - \frac{1}{2}\epsilon_{\rm c}\frac{\mathsf{p}_0^{1/2}e_0}{1 - e_0^2}\frac{3\sin v + \frac{3}{4}e_0\sin 2v}{(1 + e_0\cos v)^2} \\ & - \frac{1}{4}\epsilon_{\rm rr}\left[\frac{3 + 4e_0\cos v}{(1 + e_0\cos v)^2} - \frac{3 + 4e_0}{(1 + e_0)^2}\right].\end{aligned} \tag{3.36}$$

We have not yet achieved the form of Eq. (3.33). The reason is that while the function in the second line of Eq. (3.36) is purely oscillatory (it has a zero average), this is not true of the function in the third line. Defining $f_4 = (3 + 4e_0\cos v)(1 + e\cos v)^{-2}$, we find that $\langle f_4\rangle_\phi = (3 - 4e_0^2)(1 - e_0^2)^{-3/2}$ and this combines with the second term on the third line to contribute secular terms. Removing these from the third line and inserting them within the first line, we finally arrive at the desired expression for the time function $\mathsf{t}(\phi)$.

Our final result is that $\mathsf{t}(\phi)$ can be expressed as in Eq. (3.33), with

$$\begin{aligned}\mathsf{t}_0 = {}& \int_0^\phi \frac{\mathsf{p}_0^{3/2}}{(1 + e_0\cos v')^2}\left[1 + \epsilon_{\rm c}\frac{8 + e_0^2}{4\mathsf{p}_0(1 - e_0^2)}\right]d\phi' \\ & + \frac{1}{4}\epsilon_{\rm rr}\left[\frac{3 + 4e_0}{(1 + e_0)^2} - \frac{3 - 4e_0^2}{(1 - e_0^2)^{3/2}}\right]\end{aligned} \tag{3.37}$$

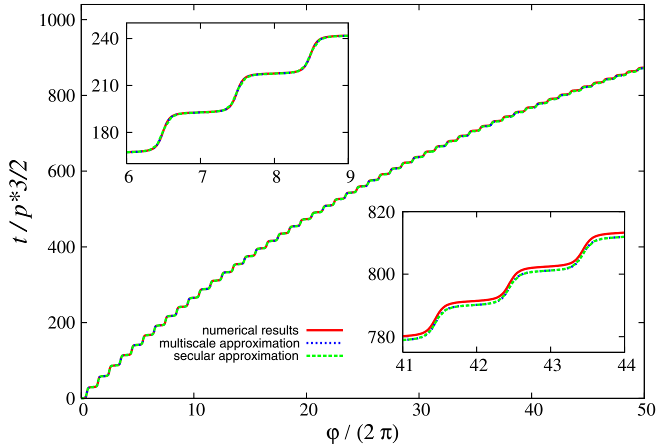

FIG. 5 (color online). Multiscale approximation for $t(\phi/2\pi)$ compared with exact numerical results. The numerical conditions are the same as in Fig. 1. The solid curve in red shows the exact evolution as computed numerically. The dotted curve in blue shows the evolution as predicted by the multiscale approximation $t_0 + \epsilon t_1$, where t_0 incorporates the Keplerian behavior in addition to the secular changes produced by the perturbing force, and where $\epsilon t_1(\phi)$ is purely oscillatory. The dashed curve in green is a plot of t_0 only. The large panel shows the entire evolution from $\phi = 0$ to $\phi = 100\pi$. The first inset (top left) shows the evolution in the small interval $6 < \phi/(2\pi) < 9$. The second inset (bottom right) shows the evolution in the interval $41 < \phi/(2\pi) < 44$.

and

$$\epsilon t_1 = -\frac{1}{2}\epsilon_c \frac{p_0^{1/2} e_0}{1-e_0^2}\frac{3\sin v + \frac{3}{4}e_0\sin 2v}{(1+e_0\cos v')^2} - \frac{1}{4}\epsilon_{rr}\left[\frac{3+4e_0\cos v}{(1+e_0\cos v)^2} - \frac{3-4e_0^2}{(1-e_0^2)^{3/2}}\right], \quad (3.38)$$

where $v = \phi - \omega_0$, and p_0, e_0, and ω_0 are the functions of $z := \epsilon_{rr}\phi$ displayed in Eqs. (3.29), (3.30), and (3.31). By design, $t_0(\phi)$ incorporates the Keplerian behavior (including all Keplerian oscillations) in addition to the secular changes produced by the perturbing force; the function $\epsilon t_1(\phi)$ is purely oscillatory, in the sense that its ϕ average is zero. A comparison between the exact time function $t(\phi)$ and the multiscale approximation is presented in Fig. 5.

IV. DISCUSSION

With the technical details out of the way, we may now return to the themes introduced in Sec. I. To launch the discussion we summarize the main results obtained in Sec. III.

A. Summary of our results

In the method of osculating orbital elements, the motion of a charged particle subjected to the electromagnetic self-force of Eq. (1.2) is at all times described by

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \omega)}, \quad (4.1)$$

the Keplerian relation of Eq. (3.1). The orbital elements p, e, ω, however, acquire a ϕ dependence that accounts for the perturbation created by the self-force. With the definitions of Eq. (3.8), these quantities evolve according to Eqs. (3.9), (3.10), and (3.11), and integrating Eq. (3.12) produces $t(\phi)$. The motion is then fully determined.

The evolution equations can be integrated numerically, or they can be integrated analytically via a multiscale analysis that produces a faithful approximation over the long interval $0 \le \phi \lesssim \epsilon^{-1}$. Moreover, the multiscale analysis produces a clean separation of the solutions into secular and oscillatory pieces. The secular changes in the orbital elements are described by the functions p_0, e_0, and ω_0 displayed in Eqs. (3.29), (3.30), and (3.31). We copy them here for convenience:

$$p_0 = p^*(1 - 3\epsilon_{rr}\phi)^{2/3}, \quad (4.2)$$

$$e_0 = e^*\left[1 + \epsilon_c \frac{4+e^*}{4e^*}\right](1 - 3\epsilon_{rr}\phi)^{1/2}, \quad (4.3)$$

$$\omega_0 = -\epsilon_{rr}\frac{8 + 5e^* + 4e^{*2}}{4e^*} - \frac{\epsilon_c}{2\epsilon_{rr}}\left[1 - (1 - 3\epsilon_{rr}\phi)^{1/3}\right], \quad (4.4)$$

where

$$\epsilon_c := \lambda_c \frac{q^2}{\mu p^*}, \quad (4.5)$$

$$\epsilon_{rr} := \frac{2}{3}\lambda_{rr}\frac{q^2}{\mu p^*}\sqrt{\frac{M}{p^*}}, \quad (4.6)$$

and where $p^* := p(\phi = 0)$ and $e^* := e(\phi = 0)$; we have set $\omega(\phi = 0) = 0$. The oscillatory changes in the orbital elements are given by ϵp_1, ϵe_1, and $\epsilon\omega_1$ displayed in Eqs. (3.26), (3.27), and (3.28).

The piece of the time function that incorporates Keplerian behavior and secular changes produced by the electromagnetic self-force is t_0, and this is displayed in Eq. (3.37). We copy it here for convenience:

$$t_0 = \sqrt{\frac{p^{*3}}{M}}\left\{\int_0^\phi \frac{(p_0/p^*)^{3/2}}{(1+e_0\cos v')^2}\left[1 + \epsilon_c\frac{(8+e_0^2)p^*}{4p_0(1-e_0^2)}\right]d\phi' + \frac{1}{4}\epsilon_{rr}\left[\frac{3+4e_0}{(1+e_0)^2} - \frac{3-4e_0^2}{(1-e_0^2)^{3/2}}\right]\right\}, \quad (4.7)$$

where $v' = \phi' - \omega_0(\phi')$; this, like the Keplerian time function, is expressed in terms of an integral that must be evaluated numerically. The oscillatory piece of the time function is ϵt_1, and this is given by Eq. (3.38). We recall that the term proportional to ϵ_c inside the integral is produced by the oscillatory pieces p_1, e_1, and ω_1 of the orbital elements; the oscillations combine to produce a secular correction to the time function.

B. Conservative and dissipative terms in the self-force

The effect of each term in Eq. (1.2) can easily be identified if we focus our attention on the secular changes in the orbital elements and time function described by Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7). Because these accumulate in the long run, while the oscillations that are not contained in those equations average to zero, it is clear that it is the secular pieces of $p(\phi)$, $e(\phi)$, $\omega(\phi)$, and $t(\phi)$ that are the most important to capture.

The effects of the conservative piece of the self-force are identified by selecting the terms in ϵ_c; the effects of the radiation-reaction piece are identified by ϵ_{rr}. An examination of Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7) reveals that the radiation-reaction force drives secular changes in the principal orbital elements p and e, but that it affects ω only indirectly, and only if there is a conservative force. In addition, the radiation-reaction force affects the time function indirectly through the changes in the principal elements, and also directly as can be seen in the second line of Eq. (4.7). On the other hand, the conservative force drives secular changes in the positional element ω, but it affects e only through the factor that comes in front of $(1 - 3\epsilon_{rr}\phi)^{1/2}$. In addition, the conservative force affects the time function directly, as can be seen from the correction term inside the integral.

The combined effects of the conservative and radiation-reaction forces on the time function can be neatly summarized by computing $P := \int_{\phi}^{\phi+2\pi} t(\phi')d\phi'$, the period of an orbital cycle. Ignoring the changes in the orbital elements while performing the integration, we obtain

$$P = 2\pi\sqrt{\frac{p_0^3}{M(1-e_0^2)^3}}\left(1 + \frac{1}{4}\epsilon_c\frac{p^*}{p_0}\frac{8+e_0^2}{1-e_0^2}\right). \tag{4.8}$$

The factor in front of the large brackets is the Keplerian period expressed in terms of the changing orbital elements; these changes, we recall, are driven by the radiation-reaction force. The second term gives the correction contributed by the conservative force. It may be recalled that this correction originates from oscillations in p, e, and ω, and it may be noted that it becomes large when $e_0 \to 1$.

C. Limitations of the radiative approximation

As we have defined it in Sec. I, the radiative approximation is obtained by setting $\epsilon_c = 0$ in our results. This preserves the secular changes in p and e, but it completely turns off the secular evolution of ω. In addition, the radiative approximation discards an important correction term in the time function, the one proportional to ϵ_c in Eq. (4.7). This term, in fact, dominates over the radiation-reaction corrections, because ϵ_c is numerically larger than ϵ_{rr} by a factor of order $\sqrt{p^*/M} \gg 1$, as can be seen from Eqs. (4.5) and (4.6). In addition, we have noted that the correction becomes increasingly large as e_0 increases toward unity. As a result, the radiative approximation provides a poor estimation of the orbital period, as can be seen from Eq. (4.8).

The combined effect of omitting the secular changes in ω and missing an important correction in the time function can be seen most clearly by examining the radial phase variable $\Phi := \phi - \omega$ expressed as a function of time; this is the function that appears in $r(t) = p/(1 + e\cos\Phi)$, as can be seen from Eq. (4.1). In Fig. 6 we compare the results of two numerical computations, one carried out with the full electromagnetic self-force (including conservative and radiation-reaction pieces), the other carried out in the radiative approximation, with only the radiation-reaction piece of the self-force. The figure, and the quantitative analysis presented in the caption, reveal very clearly that the radiative approximation gives a rather poor representation of the phase function; in our simulation, the mismatch after 50 orbits is nearly three full radial cycles. The origin of the discrepancy is also clearly identified in the caption: It is the missing conservative correction to the time function $t(\phi)$ that is mostly responsible for the phase mismatch.

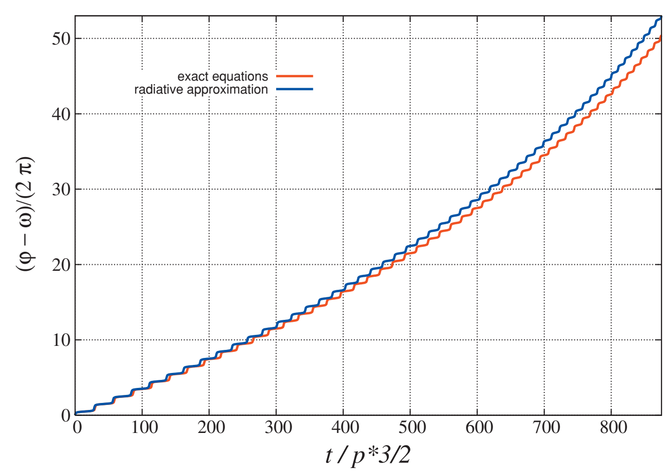

FIG. 6 (color online). Effect of the radiative approximation on the radial phase function $\Phi(t)$, where $\Phi = \phi - \omega(\phi)$. Plotted are $\Phi/(2\pi)$ versus $\mathsf{t} = \sqrt{M/p^{*3}}t$, with the same numerical conditions as in Fig. 1. The lower curve in red is the exact evolution as computed numerically with the help of the full self-force, which includes conservative and radiation-reaction pieces. The upper curve in blue is the evolution obtained in the radiative approximation, in which the conservative force is switched off. At the end of the integration, for $\mathsf{t} = 875$, we have $\Phi/(2\pi) = 50.40$ under the action of the full self-force, with a contribution $\omega/(2\pi) = -0.073$ coming from the periapsis advance. The radiative approximation gives instead $\Phi/(2\pi) = 53.06$, with a contribution $\omega/(2\pi) = 0.000\,16$ coming from the shift in periapsis. The total phase mismatch is $\Delta\Phi = 2.66(2\pi)$, nearly three radial cycles out of 50 orbits. Because the contribution from ω is small in both cases, we conclude that the main source of error is contained in the missing conservative terms in $t(\phi)$.

D. Secular approximation: ϕ average

The multiscale approximation method of Sec. III was adopted precisely because it produces a clean separation between secular and nonsecular terms in the expressions for the orbital elements and the time function. The zeroth-order quantities displayed earlier in this section were constructed so as to represent the secular changes, and the oscillatory corrections were carefully designed to average to zero. It is clear, therefore, that Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7) achieve the goals of a secular approximation, and we would be justified to write

$$p_{\rm sec}(\phi) = p_0(\phi), \qquad e_{\rm sec}(\phi) = e_0(\phi), \qquad \omega_{\rm sec}(\phi) = \omega_0(\phi), \qquad t_{\rm sec}(\phi) = t_0(\phi). \tag{4.9}$$

This, in the language introduced in Sec. II, consists of defining the secular approximation by averaging the exact solution (as represented by the multiscale approximation, which has been demonstrated to be faithful to the exact numerical results) over the orbital parameter ϕ.

The question we wish to explore here is whether the secular approximation of Eq. (4.9) could be formulated directly, without the help of the exact solution. The dynamical equations that govern the secular approximation can be obtained by differentiating Eqs. (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7) with respect to ϕ. We obtain

$$p'_{\rm sec} = -2(\epsilon_{\rm rr} p^{*3/2}) p_{\rm sec}^{-1/2}, \tag{4.10}$$

$$e'_{\rm sec} = -\frac{3}{2}(\epsilon_{\rm rr} p^{*3/2}) e_{\rm sec} p_{\rm sec}^{-3/2}, \tag{4.11}$$

$$\omega'_{\rm sec} = -\frac{1}{2}(\epsilon_{\rm c} p^*) p_{\rm sec}^{-1}, \tag{4.12}$$

$$t'_{\rm sec} = \frac{\sqrt{p_{\rm sec}^3/M}}{(1 + e_{\rm sec}\cos v)^2}\left[1 + \frac{(\epsilon_{\rm c} p^*)(8 + e_{\rm sec}^2)}{4(1 - e_{\rm sec}^2) p_{\rm sec}}\right], \tag{4.13}$$

in which $v = \phi - \omega_{\rm sec}(\phi)$ and where we have ignored (as we should) terms of order ϵ^2. These equations must come with the initial conditions

$$p_{\rm sec}(\phi = 0) = p^*, \tag{4.14}$$

$$e_{\rm sec}(\phi = 0) = e^*\left[1 + \epsilon_{\rm c}\frac{4 + e^*}{4e^*}\right], \tag{4.15}$$

$$\omega_{\rm sec}(\phi = 0) = -\epsilon_{\rm rr}\frac{8 + 5e^* + 4e^{*2}}{4e^*}, \tag{4.16}$$

$$t_{\rm sec}(\phi = 0) = \frac{1}{4}\epsilon_{\rm rr}\sqrt{\frac{p^{*3}}{M}}\left[\frac{3 + 4e^*}{(1 + e^*)^2} - \frac{3 - 4e^{*2}}{(1 - e^{*2})^{3/2}}\right] \tag{4.17}$$

in order to reproduce precisely the secular evolution predicted by the multiscale approximation.

The differential equations for $p_{\rm sec}$, $e_{\rm sec}$, and $\omega_{\rm sec}$ are easy to motivate: Eqs. (4.10), (4.11), and (4.12) are the same as Eq. (3.19), and they can be obtained directly by submitting the exact Eqs. (3.9), (3.10), and (3.11) to an averaging procedure. The differential equation for $t_{\rm sec}$, however, is not so easy to justify. It is not reproduced by averaging Eq. (3.12) over ϕ, which would fail to account for the important conservative correction proportional to $\epsilon_{\rm c}$; the averaging would also remove the Keplerian oscillations of the time function. And the initial values of Eqs. (4.14), (4.15), (4.16), and (4.17) cannot be justified at all without knowledge of the oscillatory terms in the multiscale approximation. To integrate the differential equations with the approximate initial conditions $p_{\rm sec}(0) = p^*$, $e_{\rm sec}(0) = e^*$, $\omega_{\rm sec}(0) = 0$, and $t_{\rm sec}(0) = 0$ would produce solutions that are offset from the exact solutions by quantities of order ϵ. It is noteworthy that while the corrections to $p_{\rm sec}(0)$, $\omega_{\rm sec}(0)$, and $t_{\rm sec}(0)$ come as additive terms that become increasingly irrelevant as ϕ increases, the correction to $e_{\rm sec}(0)$ comes as a *multiplicative factor*; this correction never becomes irrelevant.

Our main conclusion is this: The secular approximation defined by the system of Eqs. (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17) would be very difficult to formulate without prior knowledge of the exact solution, as represented by the faithful multiscale approximation. This conclusion reflects the lesson learned from the illustrative example described in Sec. II.

E. Secular approximation: t average

The secular approximation considered in the preceding subsection is obtained by removing the oscillations in ϕ from the exact expressions for the orbital elements. Because we are ultimately interested in the time behavior of the elements, it is perhaps more meaningful to define the secular approximation by averaging with respect to t instead of ϕ. In this alternative secular approximation, we write

$$p_{\rm sec}(\phi) = \langle p\rangle_t(\phi), \qquad e_{\rm sec}(\phi) = \langle e\rangle_t(\phi), \qquad \omega_{\rm sec}(\phi) = \langle \omega\rangle_t(\phi), \qquad t_{\rm sec}(\phi) = \langle t\rangle_t(\phi) \tag{4.18}$$

in place of Eq. (4.9), where the time average is defined as in Eq. (2.9),

$$\langle q\rangle_t := \frac{\int_{\phi-\pi}^{\phi+\pi} q(\phi')(dt/d\phi')d\phi'}{\int_{\phi-\pi}^{\phi+\pi}(dt/d\phi')d\phi'}. \tag{4.19}$$

The oscillations contained in $dt/d\phi$, as revealed in Eq. (3.32), ensure that this version of the secular approximation is quite distinct from the version examined in the preceding subsection.

Performing the calculations produces

$$p_{\rm sec} = p_0, \tag{4.20}$$

$$e_{\rm sec} = e_0 + \frac{1}{4}(\epsilon_{\rm c} p^*) \frac{2 + e_0^2 - 2(1 - e_0^2)^{3/2}}{e_0 p_0}, \tag{4.21}$$

$$\omega_{\rm sec} = \omega_0 - \frac{1}{4}(\epsilon_{\rm rr} p^{*3/2}) \frac{2 + 5e_0^2 - 2(1 - e_0^2)^{3/2}}{e_0^2 p_0^{3/2}}. \tag{4.22}$$

It is clear that these expressions do not agree with those of the preceding subsection, in which we made the assignments $p_{\rm sec} = p_0$, $e_{\rm sec} = e_0$, and $\omega_{\rm sec} = \omega_0$.

In view of the complexity involved, we shall not attempt to find an explicit expression for $t_{\rm sec}$. Nor shall we prolong the discussion by writing down dynamical equations and initial conditions for $p_{\rm sec}$, $e_{\rm sec}$, $\omega_{\rm sec}$, and $t_{\rm sec}$. We can simply jump to the main conclusion, which is the same as in the preceding subsection: The dynamical equations and initial conditions associated with this version of the secular approximation would be very difficult to formulate without prior knowledge of the exact solution. Once more this conclusion reflects the lesson learned from the illustrative example described in Sec. II.

V. CONCLUSIONS

We examined the motion of a charged particle in a weak gravitational field. In addition to the Newtonian gravity $\boldsymbol{g}$ exerted by a large body of mass M, the particle is subjected to the electromagnetic self-force described by Eq. (1.2). As we have argued in Sec. I, this toy problem shares many of the features of the gravitational self-force problem, and yet it is sufficiently simple that it can be solved completely with simple numerical methods, and virtually completely with simple analytical methods.

After subjecting the equations of motion to a multiscale analysis in Sec. III, we summarized our main results in Sec. IV and investigated the main themes introduced in Sec. I. We first examined the roles of the conservative and radiation-reaction pieces of the self-force. We showed that the radiation-reaction force drives secular changes in the principal orbital elements p and e, while the conservative force drives secular changes in the positional element ω as well as in the time function $t(\phi)$.

This led us to our first conclusion: The radiative approximation to the true self-force does not account for the secular changes in all the orbital elements; this gives rise to an important phase mismatch between an orbital evolution driven by the radiation-reaction force, and one driven by the true self-force. The radiative approximation does not achieve the goals of a secular approximation.

This was also the conclusion of our previous work (paper I: Ref. [17]), but we believe that we have established these statements more firmly in this work. In addition, the source of the phase mismatch was correctly identified here, while it was attributed incorrectly in paper I: it is the conservative correction in the time function that is mostly responsible for the dephasing, and not the secular change in ω.

We next considered the issue of formulating secular approximations to the dynamical equations that govern the evolution of the orbital elements. Having access to a faithful, analytical representation of this evolution, as provided by the multiscale approximation, it was an easy task to perform averages and to obtain expressions that capture the secular changes in the orbital elements (and the time function). And having access to those expressions, it was again an easy task to identify the differential equations that govern their behavior, as well as the appropriate initial conditions. The issue, of course, is whether the simplified dynamical equations, those that would govern the purely secular changes in the orbital elements, could be obtained directly in a context in which the exact solutions are not known.

Our answer is in the negative, and this led us to our second conclusion: A secular approximation to the exact differential equations and initial conditions, designed to capture the secular changes in the orbital elements and to discard the oscillations, would be very difficult to formulate without prior knowledge of the exact solution. While some of the approximate differential equations can be obtained by submitting the exact equations to an averaging procedure, other equations cannot be obtained so simply. And even if the correct differential equations can be identified, their integration must proceed from initial conditions that differ from the exact initial conditions; the difference is determined by the oscillations, and those must be known before the approximate initial conditions can be prescribed.

In addition to these issues, the formulation of a secular approximation must resolve a fundamental ambiguity: Which oscillations are to be removed? In our analysis we had to distinguish carefully between taking a ϕ average to remove oscillations in ϕ, and taking a t average to remove oscillations in t. Different choices lead to different secular approximations, different dynamical equations, and a different prescription for initial conditions.

The work presented here leaves a number of questions to be examined. The most important one is this: Do the conclusions of this paper have any relevance to the gravitational self-force? While our analysis of the electromagnetic self-force leaves no room for controversy, the question of how our results will transfer to the more interesting case of the gravitational self-force might be cause for debate. We believe that the analogy between the electromagnetic and gravitational self-forces is close, we believe that our general conclusions do carry over to this case, and we believe that our work serves as a useful cautionary tale for the gravitational self-force. But we admit that the analogy relies on the usual formulation of the gravitational self-force in the Lorenz gauge, and that the analogy may be lost in alternative formulations—the gravitational self-force is not gauge invariant, and its effect on the description of orbital evolutions will depend on the choice of gauge. For example, Mino [7,22] has proposed a formulation of the gravitational self-force in a "radiation-reaction gauge" in which the full self-force is equal (for a radiation-reaction time) to the radiative self-force. In Mino's proposed formulation, the radiative approximation is exact over a radiation-reaction time, and the issues raised here may not at all be relevant. How our conclusions might apply to the gravitational case is indeed a controversial topic, but we consider its discussion to be beyond the scope of this work. Indeed, this paper is concerned with the electromagnetic self-force, and the case of the gravitational self-force is considered separately in a companion paper [23]. In our companion work we argue that the Lorenz-gauge formulation of the gravitational self-force is physically meaningful, that the Lorenz gauge is most likely to keep quantities other than the self-force (such as the gravitational potentials) under control, and that the conclusions of this paper do carry over to the gravitational case.

ACKNOWLEDGMENTS

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APPENDIX A: KEPLERIAN MOTION

In this Appendix we provide a complete description of Kepler's problem. This material is well-known, and can be found in any textbook on celestial mechanics (see, for example, Ref. [24]), but we include it here for completeness and as a way of defining our notation.

Two bodies of masses m_1 and m_2 move under their mutual gravitational attraction. The equation of motion for the relative position $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2$ is

$$\boldsymbol{a} = \boldsymbol{g}, \tag{A1}$$

where $\boldsymbol{a} := d^2\boldsymbol{r}/dt^2$ is the relative acceleration vector, and $\boldsymbol{g} = -M\boldsymbol{r}/r^3$ is the gravitational field. Here $M = m_1 + m_2$ is the total mass, and $r = |\boldsymbol{r}|$ is the distance between the two bodies. We set $G = 1$.

Conservation of angular momentum implies that the motion takes place within a fixed plane. We use polar coordinates (r, ϕ) in this plane, and we resolve all vectors in the associated basis $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}})$. The relation with the Cartesian description is $x = r\cos\phi$, $y = r\sin\phi$, $\hat{\boldsymbol{r}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$. The position vector is $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, the velocity vector is $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$, and the acceleration vector is

$$\boldsymbol{a} = (\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\boldsymbol{\phi}}. \tag{A2}$$

An overdot indicates differentiation with respect to t.

Eqs. (A1) and (A2) imply

$$r^2\dot{\phi} = \text{const} =: \sqrt{Mp}, \tag{A3}$$

which defines the semilatus rectum p. We also have

$$\ddot{r} + \frac{M}{r^2} - \frac{Mp}{r^3} = 0, \tag{A4}$$

which integrates to

$$\frac{1}{2}\dot{r}^2 - \frac{M}{r} + \frac{Mp}{2r^2} = \text{const} =: -\frac{M}{2p}(1 - e^2). \tag{A5}$$

The constant is the system's conserved energy per unit reduced mass, and the last equation defines the eccentricity e.

Eliminating time from Eqs. (A3) and (A4) produces a differential equation for $r(\phi)$ which integrates to

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \omega)}, \tag{A6}$$

where ω is an additional constant of the motion. This equation describes an off-centered ellipse of semimajor axis

$$a = \frac{p}{1 - e^2} \tag{A7}$$

and eccentricity e. The constant ω, known as *longitude of periapsis*, determines the orientation of the ellipse in the plane. The orbit is at periapsis $r = p/(1 + e)$ whenever $\cos(\phi - \omega) = 1$, and is at apoapsis $r = p/(1 - e)$ whenever $\cos(\phi - \omega) = -1$.

Equations (A3) and (A6) imply

$$\dot{r} = e\sqrt{\frac{M}{p}}\sin(\phi - \omega) \tag{A8}$$

and

$$\dot{\phi} = \sqrt{\frac{M}{p^3}}[1 + e\cos(\phi - \omega)]^2. \tag{A9}$$

This last equation integrates to

$$t(\phi) = t_{\text{peri}} + \sqrt{\frac{p^3}{M}}\int_\omega^\phi \frac{d\phi'}{[1 + e\cos(\phi' - \omega)]^2} \tag{A10}$$

and determines the time. The fourth (and final) constant of integration t_{peri} is *time at periapsis*, and is such that $t(\phi = \omega) = t_{\text{peri}}$. According to Eq. (A10) the orbital period is

$$P = \frac{2\pi}{n}, \qquad n := \sqrt{\frac{M}{a^3}}, \tag{A11}$$

where n is known as the *mean motion*.

APPENDIX B: OSCULATING ORBITAL ELEMENTS

In this Appendix we develop a method of osculating orbital elements for the integration of the equations of motion associated with a perturbed Keplerian orbit. The general idea is very old, and many variations of this method can be found in the literature (see, for example, Ref. [24]). But we find that the version presented here is perhaps a little unusual, while being especially convenient and well suited to our purposes. For these reasons we judge it worthwhile to develop it in full here.

We consider the equations of motion

$$\boldsymbol{a} = \boldsymbol{g} + \boldsymbol{f}, \tag{B1}$$

in which $\boldsymbol{f}$ is a perturbing force (divided by the system's reduced mass) that depends on the relative position vector $\boldsymbol{r}$ and (possibly) the relative velocity vector $\boldsymbol{v}$. (The notation is introduced in Appendix A.) We seek to integrate Eq. (B1) for $\boldsymbol{r}(t)$ using a *method of osculating orbital elements*. We assume, for simplicity, that the perturbing force can be decomposed as

$$\boldsymbol{f} = R\hat{\boldsymbol{r}} + S\hat{\boldsymbol{\phi}}, \tag{B2}$$

so that it lies within the orbital plane. The perturbed orbit, therefore, will stay within the same plane.

1. First formulation

Let

$$I^A := \{p, e, \omega, t_{\rm peri}\} \tag{B3}$$

collectively stand for the Keplerian orbital elements introduced in Appendix A, let

$$\boldsymbol{r}_{\rm K}(I^A, t) \tag{B4}$$

stand for the position vector of a Keplerian orbit, and let

$$\boldsymbol{v}_{\rm K}(I^A, t) \tag{B5}$$

be the Keplerian velocity vector. The method of osculating elements states that the perturbed motion is described at all times by Eqs. (B4) and (B5), but that the orbital elements acquire a time dependence. In mathematical terms, the position vector of the perturbed orbit is

$$\boldsymbol{r} = \boldsymbol{r}_{\rm K}(I^A(t), t) \tag{B6}$$

and its velocity vector is

$$\boldsymbol{v} = \boldsymbol{v}_{\rm K}(I^A(t), t). \tag{B7}$$

Differentiating Eq. (B6) with respect to time yields

$$\boldsymbol{v} = \frac{\partial \boldsymbol{r}_{\rm K}}{\partial I^A}\frac{dI^A}{dt} + \frac{\partial \boldsymbol{r}_{\rm K}}{\partial t}.$$

The second term, in which $\boldsymbol{r}_{\rm K}$ is differentiated while keeping I^A constant, is recognized as $\boldsymbol{v}_{\rm K}$, the Keplerian velocity vector. Comparing with Eq. (B7) gives

$$\frac{\partial \boldsymbol{r}_{\rm K}}{\partial I^A}\dot{I}^A = 0. \tag{B8}$$

Differentiating Eq. (B7) with respect to time yields

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}_{\rm K}}{\partial I^A}\frac{dI^A}{dt} + \frac{\partial \boldsymbol{v}_{\rm K}}{\partial t}.$$

The second term gives $\boldsymbol{g}$, and comparing with Eq. (B1) gives

$$\frac{\partial \boldsymbol{v}_{\rm K}}{\partial I^A}\dot{I}^A = \boldsymbol{f}. \tag{B9}$$

Eqs. (B8) and (B9) can be solved for $\dot{I}^A$ in terms of the perturbing force. The equations of motion have become a system of first-order differential equations for the orbital elements. The method of osculating orbital elements therefore transforms the original phase space spanned by $(\boldsymbol{r}, \boldsymbol{v})$ into a new phase space spanned by the coordinates I^A. In the planar context considered here, the original phase space is spanned by $(r, \phi, \dot{r}, \dot{\phi})$ while the new phase space is spanned by $(p, e, \omega, t_{\rm peri})$.

Concretely the equations of motion are

$$\ddot{r} - r\dot{\phi}^2 + \frac{M}{r^2} = R, \qquad \frac{d}{dt}(r^2\dot{\phi}) = rS. \tag{B10}$$

By virtue of Eq. (A3) and the osculating conditions of Eqs. (B6) and (B7), $r^2\dot{\phi} = \sqrt{Mp}$ and the second of Eqs. (B10) implies $rS = \frac{1}{2}\sqrt{M/p}\,\dot{p}$. Inserting Eq. (A6) yields

$$\dot{p} = 2\sqrt{\frac{p^3}{M}}\frac{1}{1 + e\cos(\phi - \omega)}S, \tag{B11}$$

the new equation of motion for $p(t)$.

To work out the remaining equations we substitute Eq. (A8) into the first of Eq. (B10). This gives

$$R = -\dot{p}\frac{e}{2}\sqrt{\frac{M}{p^3}}\sin(\phi - \omega) + \dot{e}\sqrt{\frac{M}{p}}\sin(\phi - \omega) - \dot{\omega}e\sqrt{\frac{M}{p}}\cos(\phi - \omega), \tag{B12}$$

after canceling out all Keplerian terms. An additional equation is obtained by differentiating Eq. (A6) with respect to time and demanding that the result be compatible with Eq. (A8). After some algebra we obtain

$$0 = \dot{p} - \frac{p\cos(\phi - \omega)}{1 + e\cos(\phi - \omega)}\dot{e} - \frac{ep\sin(\phi - \omega)}{1 + e\cos(\phi - \omega)}\dot{\omega}. \tag{B13}$$

Equations (B11) and (B13) imply

$$\dot{e} = \sqrt{\frac{p}{M}}\left[\sin(\phi - \omega)R + \frac{e + 2\cos(\phi - \omega) + e\cos^2(\phi - \omega)}{1 + e\cos(\phi - \omega)}S\right] \tag{B14}$$

and

$$e\dot{\omega} = \sqrt{\frac{p}{M}}\left[-\cos(\phi - \omega)R + \frac{\sin(\phi - \omega)[2 + e\cos(\phi - \omega)]}{1 + e\cos(\phi - \omega)}S\right]. \tag{B15}$$

In these equations, ϕ is a function of time that must be obtained by integrating Eq. (A9),

$$\dot{\phi} = \sqrt{\frac{M}{p^3}}[1 + e\cos(\phi - \omega)]^2,$$

in which p, e, and ω are now time-varying orbital elements.

Our system of equations currently leaves out $t_{\rm peri}$, the fourth orbital element. An equation for $\dot{t}_{\rm peri}$, however, will not be required.

2. Second formulation

The preceding system of equations achieves a cleaner structure if we change the independent variable from t to ϕ via Eq. (A9). Writing, for example, $p' := dp/d\phi = \dot{p}/\dot{\phi}$, we obtain

$$p' = \frac{2p^3}{M}\frac{1}{(1 + ec)^3}S, \tag{B16}$$

$$e' = \frac{p^2}{M}\left[\frac{s}{(1 + ec)^2}R + \frac{e + 2c + ec^2}{(1 + ec)^3}S\right], \tag{B17}$$

$$e\omega' = \frac{p^2}{M}\left[-\frac{c}{(1 + ec)^2}R + \frac{s(2 + ec)}{(1 + ec)^3}S\right], \tag{B18}$$

$$t' = \sqrt{\frac{p^3}{M}}\frac{1}{(1 + ec)^2}, \tag{B19}$$

where

$$c := \cos(\phi - \omega), \qquad s := \sin(\phi - \omega). \tag{B20}$$

The first three equations for $p(\phi)$, $e(\phi)$, and $\omega(\phi)$ constitute a closed system that can be solved independently of the fourth equation, which determines $t(\phi)$. These equations are exact, they are convenient to deal with, and they can easily be implemented numerically. (The equations are ill behaved when $e \to 0$; a transformation to new variables $\alpha = e\cos\omega$, $\beta = e\sin\omega$ eliminates this pathology.) It is understood that the system of equations is accompanied by the Keplerian representation of the motion, that is, equations such as $r(\phi) = p/[1 + e\cos(\phi - \omega)]$ and $r' = ep\sin(\phi - \omega)/[1 + e\cos(\phi - \omega)]^2$.

The second formulation of the method can be understood as follows. Let

$$I^A := \{p, e, \omega\} \tag{B21}$$

collectively stand for the relevant orbital elements, let

$$\boldsymbol{r}_{\rm K}(I^A, \phi), \qquad t_{\rm K}(I^A, \phi) \tag{B22}$$

stand for the position vector of a Keplerian orbit, parametrized by longitude ϕ, and let

$$\boldsymbol{r}'_{\rm K}(I^A, \phi) := \frac{\partial \boldsymbol{r}_{\rm K}}{\partial \phi}, \qquad t'_{\rm K}(I^A, \phi) := \frac{\partial t_{\rm K}}{\partial \phi}. \tag{B23}$$

The Keplerian velocity vector can then be expressed as $\boldsymbol{v}_{\rm K} = \boldsymbol{r}'_{\rm K}/t'_{\rm K}$.

The method of osculating elements states that the perturbed motion continues to be described by Eqs. (B22) and (B23), but that the orbital elements acquire a ϕ dependence. In mathematical terms, the position vector of the perturbed orbit is

$$\boldsymbol{r} = \boldsymbol{r}_{\rm K}(I^A(\phi), \phi), \qquad t = t_{\rm K}(I^A(\phi), \phi) \tag{B24}$$

and we impose also

$$\boldsymbol{r}' = \boldsymbol{r}'_{\rm K}(I^A(\phi), \phi), \qquad t' = t'_{\rm K}(I^A(\phi), \phi). \tag{B25}$$

The first two equations are equivalent to Eq. (B6), and the last two equations are equivalent to Eq. (B7). The second of Eqs. (B25) is the same as Eq. (B19).

Differentiating Eq. (B24) with respect to ϕ yields

$$\boldsymbol{r}' = \frac{\partial \boldsymbol{r}_{\rm K}}{\partial I^A}\frac{dI^A}{d\phi} + \frac{\partial \boldsymbol{r}_{\rm K}}{\partial \phi}.$$

Comparing with Eq. (B25) gives

$$\frac{\partial \boldsymbol{r}_{\rm K}}{\partial I^A}I'_A = 0. \tag{B26}$$

Differentiating Eq. (B7) with respect to ϕ and dividing by t' from Eq. (B25) yields

$$\boldsymbol{a} = \frac{\boldsymbol{v}'}{t'} = \frac{1}{t'_{\rm K}}\left[\frac{\partial \boldsymbol{v}_{\rm K}}{\partial I^A}\frac{dI^A}{d\phi} + \frac{\partial \boldsymbol{v}_{\rm K}}{\partial \phi}\right].$$

The second term gives $\boldsymbol{g}$, and comparing with Eq. (B1) gives

$$\frac{1}{t'_{\rm K}}\frac{\partial \boldsymbol{v}_{\rm K}}{\partial I^A}I'_A = \boldsymbol{f}. \tag{B27}$$

Equations (B26) and (B27) can be solved for I'_A in terms of the perturbing force, and the end result is the system of Eqs. (B16)–(B18). In this formulation the method of osculating orbital elements transforms the original phase space spanned by $r(t)$, $\phi(t)$, $\dot{r}(t)$, and $\dot{\phi}(t)$ into a new phase space spanned by $p(\phi)$, $e(\phi)$, $\omega(\phi)$, and $t(\phi)$.

The second formulation of the method of osculating elements is distinguished by the facts that it involves ϕ as a running orbital parameter, and it removes t_{peri} from the list of phase-space variables. This formulation leads to the important advantages that Eqs. (B16)–(B18) form a closed set of equations; these equations can be integrated first, and $t(\phi)$ can be recovered at a later stage by solving Eq. (B19).

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