# Vector modes generated by primordial density fluctuations

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While vector modes are usually ignored in cosmology since they are not produced during inflation they are inevitably produced from the interaction of density fluctuations of differing wavelengths. This effect may be calculated via a second-order perturbative expansion. We investigate this effect during the radiation era. We discuss the generation mechanism by investigating two scalar modes interacting, and we calculate the power of vector modes generated by a power-law spectrum of density perturbations on all scales.

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## I. INTRODUCTION

High-precision data from observations of the cosmic microwave background (CMB) [1] and large scale structure (LSS) [2] provide strong evidence for a nearly spatially flat universe with a primordial spectrum of adiabatic, Gaussian and nearly scale-invariant density perturbations. The standard cosmological model provides a remarkable theoretical basis for these and other observed features of our universe. Perturbations generated from inflation give a nearly scale-invariant spectrum of scalar (density) perturbations, and tensor (gravitational wave) modes, with amplitudes which are typically within a few orders-of-magnitude of each other. However, within this paradigm the amplitude of any corresponding vector modes is zero since a scalar field cannot support vector modes at linear order [3]; even if they were generated during inflation, vectors decay rapidly after they leave the Hubble radius during inflation, whereas scalars and tensors are typically frozen on super-Hubble scales, and only decay when they reenter the horizon. A generic prediction of inflation is therefore no vector modes.

There is an important caveat to this argument. Vector modes are generated via the nonlinear interaction of scalar (and tensor) perturbations of differing wavelength, and therefore inflation must generically predict a spectrum of vector modes, but at second-order in a perturbative expansion. Indeed, observations of the scalar spectrum requires this to be so independently of whether inflation is the correct model of the early universe or not. We shall consider the generation of vectors from scalars in some detail. The analogous process of gravitational wave generation by scalar-scalar interaction has been investigated [4–19], and the work presented here is closely related to these studies.

We shall principally expand on work of Mollerach *et al.* [20], who considered the effect of secondary vectors on the CMB. We shall investigate the generation method of vectors from two scalar modes (we show that unlike gravitational waves, vectors cannot be generated by a single mode), and we shall calculate the power spectra of the vector part of the metric in the radiation era. Our aim here is to principally discuss how vectors are generated, and get an overall estimate as to the magnitude and distribution of vectors at the end of the radiation era. How the results here relate to observables in the CMB, and the spectrum of vectors today, is left for future work.

Other people have discussed second-order vector modes before. Most recently, Mena *et al.* [21] considered second-order vector modes in a collapsing universe. Matarrese *et al.* [22] discussed the generation of primordial magnetic fields from density perturbations (although see [23]). Various other work has discussed vector mode generation on a more formal level [11,12,24–28].

There are a variety of other mechanisms which predict vector modes, all of which must happen after inflation, and usually predict a spectrum of modes on small scales. Such sources include cosmic strings [29], topological defects [30], fine-tuned anisotropies in collisionless neutrinos [31] and the presence of an primordial magnetic field [32–35]. The generated vector modes are highly non-Gaussian. Second-order inflationary vector modes considered here are also non-Gaussian and have a $\chi^2$-distribution.

Vector modes are likely to play a more prominent role in cosmology in the coming years, through their contributions to the CMB [24–26,36–40], which will impact on the B-mode polarization and could be the dominant contribution when compared with the second-order gravitational waves [20] (assuming one is able to subtract the lensing signal). They are also crucial in magnetogenesis [22,32–35,41–49] as the magnetic field vector has a dominant vector part. It is therefore appropriate to now consider in more detail the spectrum of vector modes which we know must exist by virtue of second-order effects.

The paper is organized as follows. In Sec. II we consider the formalism for investigating the generation of vectors from scalars. Then, in Sec. III we discuss the power spectra of vectors in the radiation era for power-law scalar modes. We also investigate the generation mechanism of vectors

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by considering the interaction of two distinct scalar modes. Finally, we conclude in Sec. IV.

## II. GENERATION OF VECTOR MODES FROM DENSITY PERTURBATIONS

We shall consider perturbations of a flat Robertson-Walker background up to second order. The metric is decomposed as

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \delta^{(2)} g_{\mu\nu}, \tag{1}$$

where Greek indices run from $0, \ldots, 3$ and Latin indices run from $1, \ldots, 3$. We are only investigating the second-order vector modes sourced by the first-order scalar perturbation; therefore, we have scalar degrees of freedom at first order, $\delta^{(1)} g_{\mu\nu}$, while the second-order perturbations, $\delta^{(2)} g_{\mu\nu}$, are pure vector modes, and second-order equations are therefore meaningful when projected out accordingly. Our metric in terms of conformal time with a longitudinal gauge chosen, is given as

$$\begin{aligned} \bar{g}_{00} &= -a^2(1 + 2\Phi^{(1)}), \qquad \bar{g}_{0i} = -\tfrac{1}{2} a^2 S_i^{(2)}, \\ \bar{g}_{ij} &= a^2(1 - 2\Phi^{(1)})\gamma_{ij}, \end{aligned} \tag{2}$$

where $\Phi^{(1)}$ is the first-order Bardeen potential, and $S_i^{(2)}$ describes the gauge-invariant [21] second-order vector modes, so that $\partial^i S_i^{(2)} = 0$. As there is no ambiguity in what follows we shall drop the order superscripts and just write $\Phi$ and $S_i$.

### A. Density perturbations at linear order

We shall consider the generation of vectors from scalars during the radiation era below, but for now we shall consider the situation where we have a linear constant equation of state, $c_s^2 = w$ where $w = p/\rho$ and $c_s$ is the sound speed. We shall assume that the first-order matter perturbations are adiabatic, i.e., that pressure perturbations obey $\delta^{(1)} p = c_s^2 \delta^{(1)} \rho$. Then the first-order equation of motion for the Bardeen potential in Fourier space is [50]

$$\Phi''(\boldsymbol{k}, \eta) + 3\mathcal{H}(1 + c_s^2)\Phi'(\boldsymbol{k}, \eta) + c_s^2 k^2 \Phi(\boldsymbol{k}, \eta) = 0, \tag{3}$$

where a prime denotes differentiation with respect to conformal time $\eta$.

In the radiation era, the scale factor, the conformal Hubble rate and energy density evolve as $a(\eta) = a_0 \frac{\eta}{\eta_0}$, $\mathcal{H} = \frac{a'}{a} = \frac{1}{\eta}$ and $\rho \propto \eta^{-4}$, and the general solution to (3), with $c_s^2 = \frac{1}{3}$ is

$$\begin{aligned} \Phi_r(\boldsymbol{k}, \eta) = {}& \frac{A_r(\boldsymbol{k})}{(k\eta)^3}\left[\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}}\cos\left(\frac{k\eta}{\sqrt{3}}\right)\right] \\ & + \frac{B_r(\boldsymbol{k})}{(k\eta)^3}\left[\frac{k\eta}{\sqrt{3}}\sin\left(\frac{k\eta}{\sqrt{3}}\right) + \cos\left(\frac{k\eta}{\sqrt{3}}\right)\right]. \end{aligned} \tag{4}$$

We shall ignore the decaying mode—that is terms with a $B_r(\boldsymbol{k})$ coefficient.

Assuming that the fluctuations are Gaussian, we may introduce Gaussian random variables, $\hat{E}$, with unit variance and the property

$$\langle \hat{E}^*(\boldsymbol{k}_1)\hat{E}(\boldsymbol{k}_2)\rangle = \delta^3(\boldsymbol{k}_1 - \boldsymbol{k}_2). \tag{5}$$

We can then separate the length and directional dependence of functions of $\boldsymbol{k}$ and write $\Phi(\boldsymbol{k}, \eta) = \Phi(k, \eta)\hat{E}(\boldsymbol{k})$ and $A_r(\boldsymbol{k}) = A_r(k)\hat{E}(\boldsymbol{k})$.

The power spectrum for the first-order scalar perturbation can be defined through

$$\langle \Phi^*(\boldsymbol{k}_1, \eta)\Phi(\boldsymbol{k}_2, \eta)\rangle = \frac{2\pi^2}{k^3}\delta^3(\boldsymbol{k}_1 - \boldsymbol{k}_2)\mathcal{P}_\Phi(k, \eta). \tag{6}$$

At early times during the radiation era the power spectrum becomes

$$\mathcal{P}_\Phi(k) \simeq A_r(k)^2 \frac{k^3}{486\pi^2}. \tag{7}$$

Relating the Bardeen potential to the comoving curvature perturbation at early times gives us

$$A_r(k)^2 \approx \frac{216\pi^2}{k^3}\Delta_{\mathcal{R}}^2(k), \tag{8}$$

where $\Delta_{\mathcal{R}}^2$ is primordial power spectrum for the curvature perturbation $\mathcal{R}$. Current observations show $\Delta_{\mathcal{R}}^2 \approx 2.4 \times 10^{-9}$ at a scale $k_{\rm CMB} = 0.002\ {\rm Mpc}^{-1}$, and is almost independent of wave number on these scales [1].

### B. Second-order vector modes

The vector perturbations at linear order satisfy an evolution equation and a momentum constraint equation which can be found by calculating the $i - j$ and $0 - i$ parts of the Einstein field equations (EFE's) respectively [21]. In the case of a perfect fluid (at first order only) there is no source in the vector evolution equation and it admits solutions proportional to $1/a^2$. The momentum constraint equation relates the vector perturbation to the 3-velocity perturbation, which in the perfect fluid case would be zero. However, the respective equations at second order differ significantly. First, as we will see, the evolution equation is sourced, allowing for the generation of vector modes. Second, the momentum constraint no longer excludes the existence of vector perturbations, provided we are only considering a perfect fluid up to first order (see [21]).

We calculate the evolution equation for $S_i$ in the usual manner, by expanding the EFE's up to second order, keeping terms quadratic in the first order quantities. We start with the trace reversed EFE's

$$\bar{R}_{\alpha\beta} = 8\pi G(\bar{T}_{\alpha\beta} - \tfrac{1}{2}\bar{g}_{\alpha\beta}\bar{T}) = 8\pi G\bar{\Upsilon}_{\alpha\beta}. \tag{9}$$

The second-order space-space part of the Ricci tensor can be written as

$$\delta^{(2)}R_{ij} = \delta^{(2)}R_{ij}^{\mathcal{V}} + \delta^{(2)}R_{ij}^{(S,S)}, \qquad (10)$$

where we have

$$\delta^{(2)}R_{ij}^{\mathcal{V}} = \tfrac{1}{2}\partial_{(i}S'_{j)} + \mathcal{H}\partial_{(i}S_{j)}, \qquad (11)$$

which contains only the second-order term; and

$$\begin{aligned}\delta^{(2)}R_{ij}^{(S,S)} = \Big[8\Big(\frac{a''}{a} + \mathcal{H}^2\Big)\Phi^2 + 16\mathcal{H}\Phi\Phi' + 2\Phi'^2 \\ + 2\Phi\Phi'' + 2\Phi\nabla^2\Phi + 2(\partial_m\Phi)(\partial^m\Phi)\Big]\gamma_{ij} \\ + 4\Phi(\partial_i\partial_j\Phi) + 2(\partial_i\Phi)(\partial_j\Phi), \qquad (12)\end{aligned}$$

which are the quadratic first-order scalar perturbations.

The second-order trace reversed space-space part of the energy momentum tensor is

$$\begin{aligned}\delta^{(2)}\Upsilon_{ij} = a^2[-(1-w)\Phi\delta^{(1)}\rho\gamma_{ij} \\ + (1+w)\rho(\partial_i v_{(1)})(\partial_j v_{(1)})]. \qquad (13)\end{aligned}$$

It is obvious that we require the following zeroth and first-order equations,

$$\mathcal{H}^2 = \tfrac{8}{3}\pi G a^2\rho,$$

and

$$\partial_i v_{(1)} = -\frac{1}{4\pi G a^2\rho(1+w)}[\partial_i\Phi' + \mathcal{H}(\partial_i\Phi)].$$

Note that the terms with $\gamma_{ij}$ as a coefficient will not play a role since the $\gamma_{ij}$ terms are eliminated by the operator $\hat{\mathcal{V}}_i^{lm}$ defined below. Because of the limited quantities we keep in our metric the tensorial equations we calculate are only valid for vector modes, and so these must be projected out.

We define the Fourier transform of the vector perturbation as

$$\begin{aligned}S_i(\boldsymbol{x},\eta) = \frac{1}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}[S(\boldsymbol{k},\eta)e_i(\boldsymbol{k}) \\ + \bar{S}(\boldsymbol{k},\eta)\bar{e}_i(\boldsymbol{k})]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad (14)\end{aligned}$$

where the two orthonormal basis vectors $\boldsymbol{e}$ and $\bar{\boldsymbol{e}}$ are orthogonal to $\boldsymbol{k}$. We shall use the operator $\hat{\mathcal{V}}_i^{lm}$ to extract out the divergenceless vector from a rank-2 tensor

$$\begin{aligned}\hat{\mathcal{V}}_i^{lm} = -\frac{2i}{(2\pi)^3}\int d^3\boldsymbol{k}'k'^{-2}\int d^3\boldsymbol{x}'k'^l[e_i(\boldsymbol{k}')e^m(\boldsymbol{k}') \\ + \bar{e}_i(\boldsymbol{k}')\bar{e}^m(\boldsymbol{k}')]e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \qquad (15)\end{aligned}$$

This operator will remove any rank-2 tensor which is constructed from derivatives of a scalar potential. Specifically, any second-order scalar perturbations will be removed and therefore such perturbations have been neglected in our analysis. The action of this operator is to produce a rank-1 vector which is a pure vector mode. Further details concerning the operator can be found in the appendix.

We can then obtain, from the $i-j$ component of the EFE's, the evolution equation for the second-order vector perturbations

$$\hat{\mathcal{V}}_i^{lm}(\partial_{(l}S'_{m)} + 2\mathcal{H}\partial_{(l}S_{m)}) = 2\hat{\mathcal{V}}_i^{lm}\Sigma_{lm}, \qquad (16)$$

where the source term is given by

$$\begin{aligned}\Sigma_{lm} = -4\Phi(\partial_l\partial_m\Phi) - \frac{2(1+3w)}{3(1+w)}(\partial_l\Phi)(\partial_m\Phi) \\ + \frac{4}{3\mathcal{H}^2(1+w)}[(\partial_l\Phi')(\partial_m\Phi') + 2\mathcal{H}(\partial_l\Phi) \\ \times(\partial_m\Phi')]. \qquad (17)\end{aligned}$$

For either polarization, the evolution equation in Fourier space for the vector mode becomes

$$S'(\boldsymbol{k},\eta) + 2\mathcal{H}S(\boldsymbol{k},\eta) = \Sigma(\boldsymbol{k},\eta), \qquad (18)$$

where the source term $\Sigma(\boldsymbol{k},\eta)$ is an appropriate convolution over the quadratic first-order quantities,

$$\begin{aligned}\Sigma(\boldsymbol{k},\eta) = -\frac{4i}{k^2(2\pi)^{3/2}}k^i e^j(\boldsymbol{k})\int d^3\boldsymbol{k}'(k'_ik'_j) \\ \times\Big[\frac{10+6w}{3(1+w)}\Phi(\boldsymbol{k}',\eta)\Phi(\boldsymbol{k}-\boldsymbol{k}',\eta) \\ + \frac{4}{3(1+w)\mathcal{H}^2(\eta)}\Phi'(\boldsymbol{k}',\eta)\Phi'(\boldsymbol{k}-\boldsymbol{k}',\eta) \\ + \frac{8}{3(1+w)\mathcal{H}(\eta)}\Phi(\boldsymbol{k}',\eta)\Phi'(\boldsymbol{k}-\boldsymbol{k}',\eta)\Big]. \qquad (19)\end{aligned}$$

The general solution for $S(\boldsymbol{k},\eta)$ can be written as

$$S(\boldsymbol{k},\eta) = \frac{1}{a^2(\eta)}\int_{\eta_0}^{\eta}d\tilde{\eta}a^2(\tilde{\eta})\Sigma(\boldsymbol{k},\tilde{\eta}). \qquad (20)$$

We have set the initial conditions for the vector mode to zero at $\eta = \eta_0$.

### 1. Power spectrum

The power spectrum of the induced vector mode is defined as

$$\langle S^*(\boldsymbol{k}_1,\eta)S(\boldsymbol{k}_2,\eta)\rangle = \frac{2\pi^2}{k^3}\delta^3(\boldsymbol{k}_1-\boldsymbol{k}_2)\mathcal{P}_{\mathcal{V}}(k,\eta). \qquad (21)$$

Substituting the solution (20) into (21) we find

$$\langle S^*(\boldsymbol{k}_1,\eta)S(\boldsymbol{k}_2,\eta)\rangle = \frac{1}{a^4(\eta)}\int_{\eta_0}^{\eta} d\tilde{\eta}_2\int_{\eta_0}^{\eta} d\tilde{\eta}_1 a^2(\tilde{\eta}_1)a^2(\tilde{\eta}_2)\langle \Sigma^*(\boldsymbol{k}_1,\tilde{\eta}_1)\Sigma(\boldsymbol{k}_2,\tilde{\eta}_2)\rangle$$
$$= \frac{16}{(2\pi)^3 a^4(\eta)}\frac{[k_1^m e^n(\boldsymbol{k}_1)][k_2^i e^j(\boldsymbol{k}_2)]}{k_1^2k_2^2}\int_{\eta_0}^{\eta} d\tilde{\eta}_2\int_{\eta_0}^{\eta} d\tilde{\eta}_1 a^2(\tilde{\eta}_1)a^2(\tilde{\eta}_2)\int d^3k_1'\int d^3k_2'(k'_{1m}k'_{1n})$$
$$\times (k'_{2i}k'_{2j})\Xi(k_1', |\boldsymbol{k}_1-\boldsymbol{k}_1'|,\tilde{\eta}_1)\Xi(k_2', |\boldsymbol{k}_2-\boldsymbol{k}_2'|,\tilde{\eta}_2)\langle \hat{E}^*(\boldsymbol{k}_1')\hat{E}^*(\boldsymbol{k}_1-\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2')\hat{E}(\boldsymbol{k}_2-\boldsymbol{k}_2')\rangle \quad (22)$$

where

$$\Xi(\mathcal{K}_1,\mathcal{K}_2,\eta) = \frac{10+6w}{3(1+w)}\Phi(\mathcal{K}_1,\eta)\Phi(\mathcal{K}_2,\eta) + \frac{4}{3(1+w)\mathcal{H}^2(\tilde{\eta})}\Phi'(\mathcal{K}_1,\eta)\Phi'(\mathcal{K}_2,\eta) + \frac{8}{3(1+w)\mathcal{H}(\tilde{\eta})}\Phi(\mathcal{K}_1,\eta)\Phi'(\mathcal{K}_2,\eta).$$

However, Wick's theorem tells us that

$$\langle \hat{E}^*(\boldsymbol{k}_1')\hat{E}^*(\boldsymbol{k}_1-\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2')\hat{E}(\boldsymbol{k}_2-\boldsymbol{k}_2')\rangle = \langle \hat{E}^*(\boldsymbol{k}_1')\hat{E}^*(\boldsymbol{k}_1-\boldsymbol{k}_1')\rangle\langle\hat{E}(\boldsymbol{k}_2')\hat{E}(\boldsymbol{k}_2-\boldsymbol{k}_2')\rangle + \langle \hat{E}^*(\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2')\rangle\langle\hat{E}^*(\boldsymbol{k}_1-\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2-\boldsymbol{k}_2')\rangle + \langle \hat{E}^*(\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2-\boldsymbol{k}_2')\rangle\langle\hat{E}^*(\boldsymbol{k}_1-\boldsymbol{k}_1')\hat{E}(\boldsymbol{k}_2')\rangle. \quad (23)$$

Therefore, the power spectrum of the induced vector mode is

$$\mathcal{P}_{\mathcal{V}}(k,\eta) = \frac{1}{k\pi^5 a^4(\eta)}\int_{\eta_0}^{\eta} d\tilde{\eta}_2\int_{\eta_0}^{\eta} d\tilde{\eta}_1 a^2(\tilde{\eta}_1)a^2(\tilde{\eta}_2) \times \int d^3k'(k^a k'_a)[e^b(\boldsymbol{k})k'_b][e^j(\boldsymbol{k})k'_j] \times \Xi(k', |\boldsymbol{k}-\boldsymbol{k}'|,\tilde{\eta}_1)[(k^i k'_i)\Xi(k', |\boldsymbol{k}-\boldsymbol{k}'|,\tilde{\eta}_2) + (k^i k'_i - k^2)\Xi(|\boldsymbol{k}-\boldsymbol{k}'|, k',\tilde{\eta}_2)]. \quad (24)$$

In order to compute the integrals over Fourier space, we first introduce the dimensionless variables $u$ and $v$, where

$$v = \frac{k'}{k} \quad \text{and} \quad u = \sqrt{1+v^2-2v\cos\theta}.$$

If we rewrite Eq. (24) using spherical coordinates in Fourier space, we can carry out the azimuthal integral trivially. Using the two new variables, the power spectrum then becomes

$$\mathcal{P}_{\mathcal{V}}(k,\eta) = \frac{k^8}{16\pi^4 a^4(\eta)}\int_{\eta_0}^{\eta} d\tilde{\eta}_2\int_{\eta_0}^{\eta} d\tilde{\eta}_1 a^2(\tilde{\eta}_1)a^2(\tilde{\eta}_2) \times \int_0^\infty dv\int_{|v-1|}^{v+1} du(uv)(v^2+1-u^2) \times [(u^2-1-v^2)^2-4v^2] \times \Xi(kv, ku, \tilde{\eta}_1)\{(u^2-1-v^2)\Xi(kv, ku,\tilde{\eta}_2) + (u^2+1-v^2)\Xi(ku, kv, \tilde{\eta}_2)\}. \quad (25)$$

The power spectrum can now be calculated once the power spectra (initial conditions) for the scalar modes are chosen.

## III. VECTOR MODE POWER SPECTRA

We shall now investigate the power spectrum of the induced vector modes during the radiation era.

After substituting for the first-order solution for $\Phi$ for the radiation era, the power spectrum then becomes

$$\mathcal{P}_{\mathcal{V}}(k,\eta) = \frac{(243)^2}{4(k\eta)^4}\int_0^\infty dv\int_{|v-1|}^{v+1} du\mathcal{P}_\Phi(ku) \times \mathcal{P}_\Phi(kv)\mathcal{F}(u,v,x), \quad (26)$$

where

$$\mathcal{F}(u,v,x) = \frac{1}{(uv)^8}(v^2+1-u^2)[(u^2-1-v^2)^2-4v^2] \times \int_{x_0}^{x} d\tilde{x}_1 \mathcal{I}_1(\tilde{x}_1)\Big[(u^2-1-v^2) \times \int_{x_0}^{x} d\tilde{x}_2\mathcal{I}_1(\tilde{x}_2) + (u^2+1-v^2) \times \int_{x_0}^{x} d\tilde{x}_2\mathcal{I}_2(\tilde{x}_2)\Big], \quad (27)$$

and $x$ is another dimensionless variable defined as $x = k\eta$, and $x_0 = k\eta_0$. We have defined the functions

$$\mathcal{I}_j(x) = \sum_{m=1}^{5}\sum_{n=1}^{4}\sin(\alpha_n x + \phi_n)\frac{M_{nm}^j}{x^{m-1}}, \quad (28)$$

with the coefficients $\alpha_n$, $\phi_n$, and $M_{nm}^j$ defined as

$$\phi_n = \frac{\pi}{2}\begin{pmatrix}\mathbf{1}\\ \mathbf{0}\end{pmatrix} \quad \text{and} \quad \alpha_n = \frac{1}{\sqrt{3}}\begin{pmatrix}-u\mathbf{1}+v\boldsymbol{b}\\ u\mathbf{1}+v\boldsymbol{a}\end{pmatrix}, \quad (29)$$

$$M^1_{nm} = \begin{pmatrix} \frac{u^2v^2}{18}\boldsymbol{b} & \boldsymbol{0} & (\frac{1}{6}u^2 + \frac{1}{2}v^2)\boldsymbol{a} + uv\boldsymbol{1} & \boldsymbol{0} & 3\boldsymbol{b} \\ \boldsymbol{0} & \frac{u^2v}{6\sqrt{3}}\boldsymbol{1} + \frac{uv^2}{2\sqrt{3}}\boldsymbol{a} & \boldsymbol{0} & -\sqrt{3}v\boldsymbol{1} + \sqrt{3}u\boldsymbol{b} & \boldsymbol{0} \end{pmatrix}, \tag{30}$$

$$M^2_{nm} = \begin{pmatrix} \frac{u^2v^2}{18}\boldsymbol{b} & \boldsymbol{0} & (\frac{1}{2}u^2 + \frac{1}{6}v^2)\boldsymbol{a} + uv\boldsymbol{1} & \boldsymbol{0} & 3\boldsymbol{b} \\ \boldsymbol{0} & \frac{u^2v}{2\sqrt{3}}\boldsymbol{1} + \frac{uv^2}{6\sqrt{3}}\boldsymbol{a} & \boldsymbol{0} & -\sqrt{3}v\boldsymbol{1} + \sqrt{3}u\boldsymbol{b} & \boldsymbol{0} \end{pmatrix}. \tag{31}$$

Here we have defined the matrices $\boldsymbol{1}$, $\boldsymbol{0}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$\boldsymbol{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{32}$$

As we have four integrals to carry out it is useful to calculate them analytically where possible. We can do this for the $x$-integrals to get

$$\int_{x_0}^{x} d\tilde{x} I_j(\tilde{x}) = \sum_{m=1}^{5}\sum_{n=1}^{4} M^j_{nm}\bigg\{\bigg[\sum_{k=1}^{m-3}\frac{(m-k-3)!}{(m-2)!}\alpha_n^k \sin\bigg(\alpha_n\tilde{x} + \phi_n + \frac{k+2}{2}\pi\bigg)\tilde{x}^{(2+k-m)}\bigg]_{x_0}^{x} - \frac{\alpha_n^{(m-2)}}{(m-2)!}\bigg[\mathrm{Si}(\alpha_n\tilde{x})\cos\bigg(\phi_n + \frac{m}{2}\pi\bigg) + \mathrm{Ci}(\alpha_n\tilde{x})\sin\bigg(\phi_n + \frac{m}{2}\pi\bigg)\bigg]_{x_0}^{x}\bigg\}. \tag{33}$$

For the radiation era we assume that all modes are well outside the horizon when the interaction begins and therefore can set $x_0 = 0$.

### A. Interaction of scalar modes

Before calculating the power spectrum for the case of power-law scalar modes, it is useful to investigate how the vector modes are generated from individual scalar modes. It has been shown that a single scalar mode with an isotropic distribution will induce second-order gravitational waves [14]. This is not the case with vector modes: scalar modes of differing wavelengths need to interact to generate vector modes, as we shall see. To investigate this we choose then a scalar power spectrum of the form

$$\mathcal{P}_\Phi(k) = \tfrac{4}{9}\mathcal{A}^2\Delta^2_{\mathcal{R}}(k_{\mathrm{CMB}})\{\delta[\ln(k_1/k)] + \delta[\ln(k_2/k)]\}, \tag{34}$$

where $\mathcal{A}$ is the mean amplitude of each wave number, $k_i$, relative to the observed amplitude of the primordial power spectrum, $\Delta^2_{\mathcal{R}}(k_{\mathrm{CMB}})$, at wave number $k_{\mathrm{CMB}} \gg k_i$. We assume for simplicity that they both have the same amplitude. Carrying out the $u$ and $v$ integrals, we then find that the vector power spectra becomes, in terms of $v_i = k_i/k$,

$$\mathcal{P}_{\mathcal{V}}(k,\eta) = 26\,244\mathcal{A}^4\Delta^4_{\mathcal{R}}\frac{k_1k_2}{x^4k^2}[\mathcal{F}(v_1,v_2,x) + \mathcal{F}(v_2,v_1,x)] \tag{35}$$

provided $v_1 + 1 > v_2 > |v_1 - 1|$, and is zero otherwise. Therefore modes are induced for all wave numbers $k$ such that $k_1 + k_2 > k > |k_1 - k_2|$, and are scattered into angles such that $u_1 = v_2$, and $u_2 = v_1$ (a further requirement from carrying out the integration). In the case where only one input mode is present, so $v_2 = v_1$, this inequality becomes $k_1 > k/2$ while we also have $v_1 = u_1$, as in [14]; from Eq. (27) with $u = v$, however, we see that $\mathcal{F}$ vanishes in this case. Therefore, we can see that vector modes cannot be induced by a single scalar degree of freedom. The physical reason for this is because vector modes are associated with rotational degrees of freedom. A consequence of $k_1 = k_2$ is that $\theta_1 = \pm\theta_2$, i.e., the input modes only have momentum along the same axis in Fourier space. Consequently, there is no angular momentum generated, and hence no vectors.

Provided that $k_1 \neq k_2$, we can have vectors induced over the appropriate range of wavelengths. Closely separated scalar modes will produce a much broader spectrum of vector modes while modes of vastly differing wavelengths will produce a very narrow range of vectors, with wave numbers close to the largest input wave number. Note that as the generated wave numbers are restricted from above and below, we cannot expect any noise on large scales, as is the case for gravitational waves. This is also evidenced by the fact that one input mode cannot produce any vectors—there would be nothing to set the long wavelength cutoff in that case.

In Fig. 1 we show the induced vector modes as a function of $x$ for various $v_1$ with $v_2 = 1$ (so we require $0 < v_1 < 2$), i.e., the evolution of modes of wave number $k = k_2$. While the generated mode is outside the Hubble radius, there is power-law growth, with $\mathcal{P}_{\mathcal{V}} \sim \eta^2$. When $k_2$ enters the Hubble radius, the principle generation of vector modes stops shortly thereafter, and the induced modes start to decay as $\eta^{-4}$. This continues until the longer wavelength mode enters the Hubble radius at $k_1\eta = 1 \Rightarrow x \sim 1/v_1$. This then generates a further burst of vector modes, which we can see by progressively more pronounced knees, as $v_1 \to 0$, in the curves at late times. For the case when $v_1 = 1.5$, on the other hand, we see some confusion as the modes enter the horizon more-or-less together, before decaying as normal.

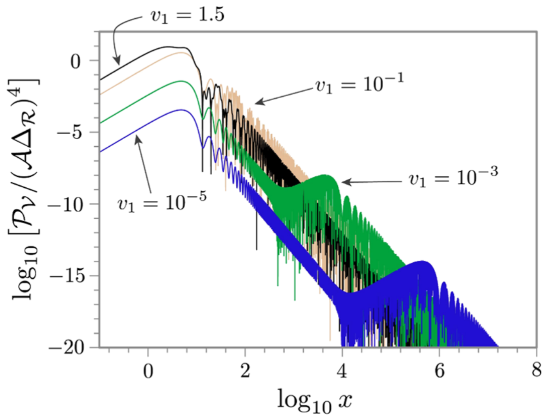


FIG. 1 (color online). The power spectrum of vector modes induced by two interacting scalar modes. Although maximum power is generated in the scenario $k_1/k_2 \sim 1$ shortly after the Hubble radius is crossed, at late times scalar interactions with vastly differing wavelengths produce more power once the long-wavelength mode enters the Hubble radius.

Thus we see that vector modes are only efficiently generated when at least one of the scalar modes is entering its Hubble radius—provided another scalar mode exists to help seed the vector mode. This explains why we have knees in the evolution of the generated vector modes, since two interacting scalars enter the Hubble radius at different times. The power generated into vectors as each mode enters depends on the relative ratio $k_1/k_2$. Modes of similar wavelength generate more overall power, because they are entering at the same time; modes that are widely separated in wave number do not generate as much overall power but produce more pronounced knees instead.

### B. Power-law scalar modes

Let us now investigate the spectrum of vector modes from power-law scalar modes. To do this, we assume that the input power spectrum is

$$\mathcal{P}_\Phi(k) = \frac{4}{9}\Delta_\mathcal{R}^2\left(\frac{k}{k_{\rm CMB}}\right)^{n_s-1}, \qquad (36)$$

where the index $n_s$ tells us the tilt of the spectrum relative to scale-invariance, $n_s = 1$, and $k_{\rm CMB}$ is a pivot scale for the power spectrum [1]. The induced vector modes are then given by

$$\mathcal{P}_\mathcal{V}(k,\eta) = \frac{729\Delta_\mathcal{R}^4}{(k\eta)^4}\left(\frac{k}{k_{\rm CMB}}\right)^{2(n_s-1)}\mathcal{F}_{n_s}(x), \qquad (37)$$

where $\mathcal{F}_{n_s}(x)$ is defined as

$$\mathcal{F}_{n_s}(x) = \int_0^\infty dv \int_{|v-1|}^{v+1} du (uv)^{n_s-1}\mathcal{F}(u,v,x). \qquad (38)$$

We integrate this numerically, and show the results in Fig. 2 for the case $n_s = 1$. The tilt of the scalar power spectrum tends to affect the amplitude of second-order modes on large scales at the level of a few percent [14,20]. Viewing $x$ as time for constant $k$, we see that the modes grow as $\eta$, peak when inside the Hubble radius and decay as $\eta^{-4}$. While the modes are decaying there are faint oscillations as shown in the top panel of the figure. This Figure can also be interpreted as the power spectrum at fixed time, showing the usual features.

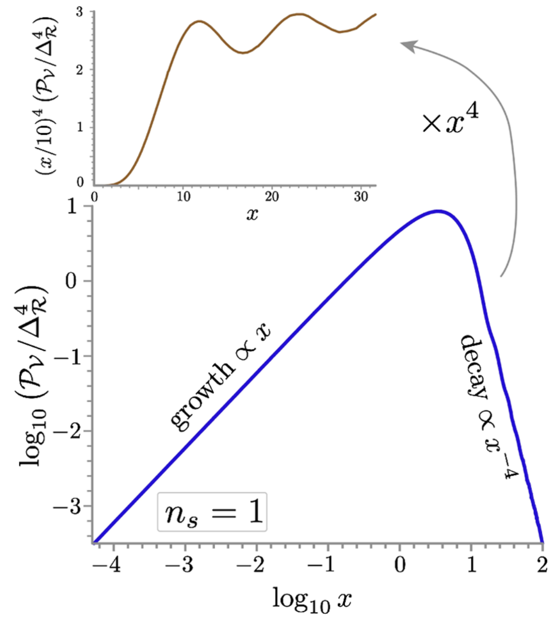


FIG. 2 (color online). The power spectrum of vector modes induced by scale-invariant scalar modes. Scalar modes outside the Hubble radius interact to give power-law growth at until the modes enter the Hubble radius. Thereafter the modes decay as normal vectors, with some gentle oscillatory features.

It is worth mentioning that we have taken the upper limit of the $k'$ integral to be infinity. In reality there is a cutoff from the end of inflation, at $\eta = \eta_*$, corresponding to modes which are inside the Hubble radius at that time, $k_* = 1/\eta_*$, so giving a finite upper limit to the $v$-integral, $v_* = k_*/k$ (although this is on very small scales in reality). This causes a break from linear scaling in $x$ in the power spectrum for $x \lesssim 1/v_*$, and we may analytically find the leading behavior of Eq. (26) is $\mathcal{P}_\mathcal{V} \sim \frac{32}{15}v_* x^2$. Why is this the case?

In the interacting delta function case we saw that modes are efficiently produced—and grow like $x^2$—when both modes are outside the Hubble radius; once one is inside and the other outside there is effectively no generation of vectors. In power-law case, then, the modes which are generating vectors are those outside the Hubble radius, providing an effective cutoff to the $v$-integral of $v \sim 1/x$, so giving us growth $\propto x$, when $v_* \gg 1/x$. When we have the cutoff $v_*$ on the other hand, for early times when $\eta < 1/k_*$, all relevant modes are outside the Hubble radius, and interact coherently giving us growth $\propto \eta^2$. For $\eta > 1/k_*$, modes which have entered the Hubble radius no longer contribute to the generation of modes outside the

Hubble radius giving weaker growth $\propto x$. Of course, we are not in a position here to analyze times before inflation ends, but this helps us understand why we have the $x$-scaling behavior we do.

## IV. CONCLUSIONS

An important feature of inflation is the lack of vector modes: if they were observed to have a similar spectrum and amplitude to the scalars then this could prove difficult for inflation, and lend favor to other theories of the early universe, e.g. Pre Big Bang scenarios and Ekpyrotic models [21,51,52]. However, there is a $\chi^2$-distribution of vectors produced by inflation as a consequence of the nonlinear interaction of scalar modes, which has received relatively little attention to date.

We have investigated the generation of vector modes induced by primordial density perturbations during the radiation dominated era. Performing a perturbative expansion to second order, we isolated the scalar terms which source the vector perturbations. We then calculated the power spectrum of the metric vector mode, and analyzed its form. In order to understand the generation of modes we investigated individual scalar modes generating vectors, and demonstrated that, contrary to the case of gravitational waves, vector modes cannot be generated by an isotropic distribution of scalars of a single wavelength, owing to the spin-1 nature of vector modes: rotational degrees of freedom must be generated by scattering of nonparallel input modes. We then demonstrated that vectors are generated by modes of differing wavelength whenever one of the two scalar modes is entering the Hubble radius. The amplitude of the generated modes depends on the ratio of input wave numbers; maximum power is generated when the modes are not too widely separated. After investigating the generation of modes, we then presented the power spectrum for scale-invariant scalar modes, displaying our results in terms of the variable $x = k\eta$: i.e., they may be interpreted the temporal evolution of a single scalar mode, or the power at a fixed time. Interestingly the maximum power generated is the same at all times, but the position of this peak changes with wavelength, such that $x \sim 1 \Rightarrow k \sim 1/\eta$. This is due to the fact that the modes are efficiently generated as they enter the Hubble radius, and are not generated significantly while outside.

There are some open questions raised by the study presented here. In particular, it is not clear how the power spectrum for $S_i$ we have calculated will be related to observable quantities. It is gauge invariant so must be observable, by the results presented in [53]; it also represents all possible degrees of freedom of vectors generated by scalars, under the conditions laid out here. Thus, although there may be a "better" variable, it must be related to $S_i$ by quadrature (plus some further scalar-squared contributions). The effects we have presented here will have interesting implications for a variety of phenomena such as the CMB; the issue of how significant is left for future work.

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## APPENDIX: THE EXTRACTION OPERATOR

In this section we consider the extraction operator discussed in this paper. We start by defining the Fourier basis used for the purposes of harmonic decompositions. An arbitrary scalar in real space can be expressed as a Fourier integral

$$S(\boldsymbol{x}, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3\boldsymbol{k} S(\boldsymbol{k}, \eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A1}$$

A divergence free vector in real space can then be expressed as a Fourier integral

$$V_a(\boldsymbol{x}, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3\boldsymbol{k}[\bar{V}(\boldsymbol{k}, \eta)\bar{e}_a(\boldsymbol{k}) + V(\boldsymbol{k}, \eta) e_a(\boldsymbol{k})] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A2}$$

where $e_a(\boldsymbol{k})$ and $\bar{e}_a(\boldsymbol{k})$ are orthogonal parity vectors, which are also orthogonal to $\boldsymbol{k}$. Similarly, a transverse traceless tensor (a tensor mode) in real space can be expressed as

$$T_{ab}(\boldsymbol{x}, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3\boldsymbol{k}[T(\boldsymbol{k}, \eta) q_{ab}(\boldsymbol{k}) + \bar{T}(\boldsymbol{k}, \eta)\bar{q}_{ab}(\boldsymbol{k})] e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A3}$$

where the two polarization tensors $q_{ab}$ and $\bar{q}_{ab}$ can also be expressed in terms of the parity vectors $e_a$ and $\bar{e}_a$, and are orthogonal to $\boldsymbol{k}$. An arbitrary symmetric trace-free spatial tensor in real space

$$A_{ab}(\boldsymbol{x}, \eta) = [\partial_a \partial_b - \tfrac{1}{3}\gamma_{ab}\partial^c\partial_c] A^{(S)}(\boldsymbol{x}, \eta) + \partial_{(a} A^{(V)}_{b)}(\boldsymbol{x}, \eta) + A^{(T)}_{ab}(\boldsymbol{x}, \eta),$$

which has explicit scalar, vector and tensor contributions. We can also express $A_{ab}$ as a Fourier integral

$$A_{ab}(\boldsymbol{x}, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3\boldsymbol{k}\bigg\{-\bigg[k_a k_b - \frac{1}{3}\gamma_{ab}k^2\bigg] A^{(S)}(\boldsymbol{k}, \eta) + iA^{(V)}(\boldsymbol{k}, \eta) k_{(a}e_{b)} + i\bar{A}^{(V)}(\boldsymbol{k}, \eta) k_{(a}\bar{e}_{b)} + A^{(T)}(\boldsymbol{k}, \eta) q_{ab} + \bar{A}^{(T)}(\boldsymbol{k}, \eta)\bar{q}_{ab}\bigg\} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A4}$$

The even-parity vector contribution can then be extracted in Fourier space by applying the following operator

$$\hat{\mathcal{V}}_i^{lm} = -\frac{2i}{(2\pi)^3}\int d^3\boldsymbol{k}' k'^{-2}\int d^3\boldsymbol{x}' k'^l [e_i(\boldsymbol{k}')e^m(\boldsymbol{k}') + \bar{e}_i(\boldsymbol{k}')\bar{e}^m(\boldsymbol{k}')]e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \quad \text{(A5)}$$

Applying the operator $\hat{\mathcal{V}}_l^{ab}$ to Eq. (A4) gives

$$A_l^{(V)}(\boldsymbol{x},\eta) = \hat{\mathcal{V}}_l^{ab}A_{ab} = -\frac{2i}{(2\pi)^3}\int d^3\boldsymbol{k}' k'^{-2}\int d^3\boldsymbol{x}' k'^a [e_l(\boldsymbol{k}')e^b(\boldsymbol{k}') + \bar{e}_l(\boldsymbol{k}')\bar{e}^b(\boldsymbol{k}')]e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}-\boldsymbol{x}')}A_{ab}(\boldsymbol{x}',\eta). \quad \text{(A6)}$$

The extraction of the vector component is made possible by taking advantage of the various properties of both the parity vectors ($e_a$ and $\bar{e}_a$) and the polarization tensors ($q_{ab}$ and $\bar{q}_{ab}$). We now consider as an example one possible contribution from the first-order squared terms. For simplicity we start with contributions from a term made up of the product of first-order scalars and do not reconstruct in real space, looking only at the Fourier amplitudes. Consider a term of the type

$$\Phi\partial_a\partial_b\Phi = \frac{1}{(2\pi)^3}\left[\int d^3\boldsymbol{k}_1\Phi(\boldsymbol{k}_1,\eta)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}\right]\times\left[\int d^3\boldsymbol{k}_2\Phi(\boldsymbol{k}_2,\eta)(-k_{2a}k_{2b})e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}\right]. \quad \text{(A7)}$$

The Fourier amplitude of the vector part of this is then

$$[\Phi\partial_a\partial_b\Phi]^{(V)}(\boldsymbol{k}) = \frac{2i}{(2\pi)^{3/2}}\int d^3\boldsymbol{k}_2\Phi(\boldsymbol{k}-\boldsymbol{k}_2,\eta)\Phi(\boldsymbol{k}_2,\eta)\times\frac{k^a[e^b(\boldsymbol{k})+\bar{e}^b(\boldsymbol{k})]k_{2a}k_{2b}}{k^2}, \quad \text{(A8)}$$

where we have carried out a real space integral and a $k$-space integral.


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