Highly distorted apparent horizons and the hoop conjecture

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By analyzing the apparent horizon (AH) formation in the collision of two *pp* waves with rectangular sources in four dimensions, we study to what extent the AH can be distorted without violating the energy conditions. It is shown that the highly distorted AH can form in this system although it cannot be arbitrarily long. The hoop conjecture is examined for the formation of such highly distorted AHs, and our result gives a strong support to the hoop conjecture. We also point out the possible relation between the AH topology theorem and the hoop conjecture.

In D-dimensional spacetimes, the apparent horizon (AH) is defined as a $(D-2)$-dimensional surface whose outgoing null geodesic congruence has zero expansion. The formation of the AH implies the existence of the event horizon (EH) outside of it if the null energy condition is satisfied [1]. Although the black hole is usually defined by the EH, the AH is also of interest since the AH is a good indicator for the black hole formation.

The purpose of this paper is to examine to what extent the AH can be distorted in four-dimensional spacetimes. In higher-dimensional spacetimes, it is known that the AH of the spherical topology can be highly distorted. For example, in the system of the spindle-shaped matter in a four-dimensional conformally flat initial data, the AH can be arbitrarily long in some direction [2]. The distortion of four-dimensional static black holes was also studied [3]. Recently, it was shown that in the presence of a negative cosmological constant, the static black hole can be highly distorted in four dimensions [4]. In this paper, using a specific example, we study whether the highly distorted AHs can form in dynamical situations in four-dimensional spacetimes without violating the energy conditions. We give the system in which highly distorted AHs can actually form, and discuss whether the formation of such highly distorted AHs is consistent with the hoop conjecture.

The hoop conjecture was proposed as an attempt to give the necessary and sufficient condition for the black hole formation. *Black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $\mathcal{C} \lesssim 4\pi GM$* [5]. The value $4\pi GM$ comes from the circumference $2\pi r_h(M)$ of the Schwarzschild black hole of mass M, where $r_h(M) := 2GM$ is the Schwarzschild radius. This statement is often rephrased like "the concentration of a mass in every direction is necessary and sufficient for the black hole formation." The hoop conjecture has been tested using several systems, and no clear couter-example was found so far. Since this conjecture is loosely formulated, there are several discussions on the definitions of the black hole (EH or AH), the circumference $\mathcal{C}$, and the mass M [6–8] (see also [9] for the recent trial to reformulate the hoop conjecture).

The meaning of "$\lesssim$" is also unclear. When the hoop conjecture is tested using some systems, the authors usually choose the plausible scale of the system as $\mathcal{C}$ and the ADM mass or some quasilocal mass as M. Then they discuss whether $\mathcal{C}/2\pi r_h(M)$ becomes a parameter which indicates the AH formation [7,8,10–17]. In this paper, we follow this direction and use the total energy of the system as the definition of the mass M.

Figure 1 shows the system that we study in this paper. Two shock waves propagate in $\pm z$ directions at the speed of light and collide in a four-dimensional spacetime. Each wave has the energy p and there is a rectangular source at the center. The sizes of the rectangle in x and y directions are L and W, respectively. The positions of the two sources coincide when the shocks collide. Since the waves are infinitely contracted, the energy of the system is infinitely concentrated in the z direction at the instant of the collision. The values of L and W indicate the degree of the
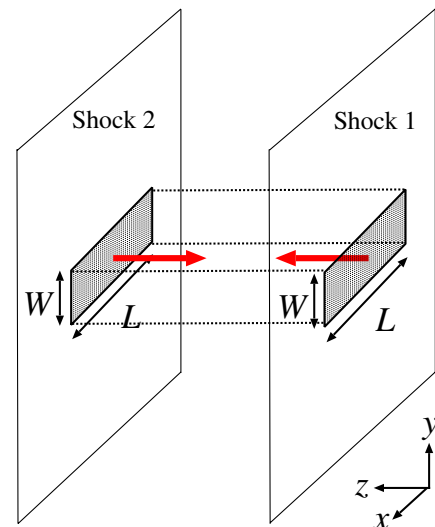

FIG. 1 (color online).   Setup of the system. Two shock waves with rectangular sources collide in four dimensions. The positions of the rectangles coincide at the instant of the collision. The sizes of the rectangles in x and y directions are L and W, respectively.

concentration of the energy in the x and y directions, respectively. We will show that the AH can be highly distorted for some values of W and L. However, it will be shown that the hoop conjecture holds also for the formation of such highly distorted AHs since the AH cannot be arbitrarily long in some direction. In the following, we adopt the gravitational radius of the system energy $r_h(2p)$ as the length unit, i.e. $r(2p) = 4Gp = 1$.

In order to set up a shock wave with a rectangular source, we adopt the pp-wave metric

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{x}^2 + d\bar{y}^2 + \Phi(\bar{x}, \bar{y})\delta(\bar{u})d\bar{u}^2, \quad (1)$$

where $\delta(\bar{u})$ denotes the delta function. The energy-momentum tensor of this spacetime has the only nonzero component $T_{\bar{u}\bar{u}} = \hat{\rho}(\bar{x}, \bar{y})\delta(\bar{u})$. Here, $\hat{\rho}(\bar{x}, \bar{y})$ and $\Phi(\bar{x}, \bar{y})$ are related as

$$\bar{\nabla}^2\Phi = -16\pi G\hat{\rho} \quad (2)$$

through the Einstein equation, where $\bar{\nabla}^2$ is the flat space Laplacian of the $(\bar{x}, \bar{y})$-plane. We give the shock energy density $\hat{\rho}$ as

$$\hat{\rho} = p\vartheta_L(\bar{x})\vartheta_W(\bar{y}). \quad (3)$$

Here, $\vartheta_L(\bar{x})$ is defined by

$$\vartheta_L(\bar{x}) := \frac{1}{L}[\theta(\bar{x} + L/2) - \theta(\bar{x} - L/2)], \quad (4)$$

using the Heaviside step function $\theta(\bar{x})$. Then, the shock potential Φ is given by

$$\Phi = -\frac{1}{LW}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \log[(\bar{x} - \bar{x}')^2 + (\bar{y} - \bar{y}')^2]d\bar{x}'d\bar{y}', \quad (5)$$

in the unit $r_h(2p) = 1$. This is integrated as

$$\Phi = 3 - \frac{1}{LW}\sum_{\sigma_1,\sigma_2=\pm 1}\sigma_1\sigma_2\left[x_{\sigma_1}y_{\sigma_2}\log(\bar{x}_{\sigma_1}^2 + \bar{y}_{\sigma_2}^2)\right.$$
$$\left. + \bar{x}_{\sigma_1}^2\arctan\frac{\bar{y}_{\sigma_2}}{\bar{x}_{\sigma_1}} + \bar{y}_{\sigma_2}^2\arctan\frac{\bar{x}_{\sigma_1}}{\bar{y}_{\sigma_2}}\right], \quad (6)$$

where $\bar{x}_\sigma := \bar{x} + \sigma L/2$ and $\bar{y}_\sigma := \bar{y} + \sigma W/2$. This metric reduces to that of the Aichelburg-Sexl particle [18] in the limit $W \to 0$ and $L \to 0$.

Since the coordinates $(\bar{u}, \bar{v}, \bar{x}, \bar{y})$ are discontinuous at $\bar{u} = 0$, we need to introduce the continuous and smooth coordinates (u, v, x, y). Setting $(\bar{x}_1, \bar{x}_2) = (\bar{x}, \bar{y})$, such coordinates are introduced by

$$\bar{u} = u; \qquad \bar{v} = v + \theta(u)\Phi + \frac{1}{4}u\theta(u)(\nabla\Phi)^2;$$
$$\bar{x}_i = x_i + \frac{1}{2}u\theta(u)\nabla_i\Phi. \quad (7)$$

By this coordinate transformation, the metric becomes

$$ds^2 = -dudv + H_{ik}H_{jk}dx^i dx^j, \quad (8)$$

$$H_{ij} = \delta_{ij} + \frac{1}{2}u\theta(u)\nabla_i\nabla_j\Phi. \quad (9)$$

Using the continuous and smooth coordinates, we can set up the system of two pp waves by just combining the two metrics

$$ds^2 = -dudv + [H_{ik}^{(1)}H_{jk}^{(1)} + H_{ik}^{(2)}H_{jk}^{(2)} - \delta_{ij}]dx^i dx^j, \quad (10)$$

$$H_{ij}^{(1)} = \delta_{ij} + \frac{1}{2}u\theta(u)\nabla_i\nabla_j\Phi; \quad (11)$$

$$H_{ij}^{(2)} = \delta_{ij} + \frac{1}{2}v\theta(v)\nabla_i\nabla_j\Phi, \quad (12)$$

since the incoming waves do not interact before the collision. This metric can be applied except at the interaction region $u > 0$, $v > 0$.

We study the AH on the slice $v \le 0 = u$ and $u \le 0 = v$. On this slice, the AH is a union of two surfaces S_1: $v = -\Psi(x, y)$ in $v \le 0 = u$ and S_2: $u = -\Psi(x, y)$ in $u \le 0 = v$. The surfaces S_1 and S_2 are connected on a common boundary B in $u = v = 0$. The equation and the boundary conditions for the AH on this slice were derived in [19]. The AH equation is $\nabla^2(\Psi - \Phi) = 0$, and the boundary conditions are $\Psi = 0$ and $(\nabla\Psi)^2 = 4$ on B. The numerical method for solving this problem was established in [20]. We used the grid numbers (50×50) for radial and angular coordinates in most cases. Since the numerical error grows up as the AH becomes distorted, we increased the grid numbers up to (400×400) appropriately.

Now we show the numerical results. In order to discuss the degree of distortion of the AH, we introduce the spherical-polar coordinates (r, θ) in the (x, y)-plane. In these coordinates, the boundary B is given as $r = f(\theta)$. Then, the parameter $\zeta := r_{max}/r_{min}$ gives a good indicator for the degree of distortion, where r_{max} and r_{min} are the maximum and minimum values of $f(\theta)$, respectively. Let us first look at the case $L = W$, where the source has the shape of a regular square. In this case, the boundary B becomes small and a little bit distorted as the value of $L(= W)$ is increased. We could not find the AH for $L = W > 1.637$. Figure 2 shows the shape of B for $L = W = 1.637$. For this value of $L(= W)$, it is found that $r_{min} = f(0) = f(\pi/2)$ and $r_{max} = f(\pi/4)$, and the distortion parameter is $\zeta \simeq 1.05$.

Next, we fix W and increase L. Let us choose $W = 0.5$ as an example. The shapes of the boundary B for $L = 1.00$, 2.00, 2.60 are shown in Fig. 3. We could not find the solution of the AH for $L \ge 2.61$. For these values of W and L, $r_{min} = f(\pi/2)$ and $r_{max} = f(0)$. The distortion parameter ζ becomes larger as L is increased, and it is $\zeta \simeq 2.34$ for $L = 2.60$.

Let us look at the case $W = 0.0$, where we can find the highly distorted AH. Figure 4 shows the shapes of the boundary B for $L = 1.00$, 2.00, 3.00, 3.13. We could not find the AH for $L \ge 3.14$. As we can see, the AH is highly
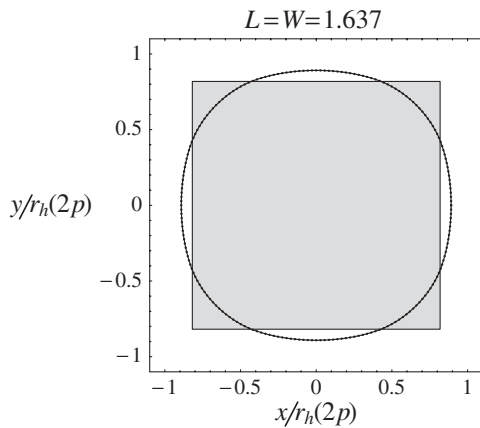
FIG. 2. The shape of the boundary B (i.e., the cross section of the AH and $u = v = 0$) for $L = W = 1.637$. The shape of the source is indicated by the gray region. For $L = W > 1.637$, we could not find the AH. The distortion parameter $\zeta := r_{max}/r_{min}$ is 1.05.

distorted for $L = 3.13$. The distortion parameter ζ has the tendency to become larger as W is decreased and L is increased for $L > W$, and it takes the maximum value for $W = 0.0$ and $L = 3.13$. This maximum value of ζ is more than 15 and much larger than unity. Therefore, the highly distorted AHs can form in this system. But we point out that the value of the maximum radius r_{max} of the boundary B is restricted from above as $r_{max} \lesssim 1.57$. Therefore, the AH cannot become arbitrarily long in some direction.

We test the hoop conjecture using our result. For this purpose, it is convenient to introduce a parameter

$$H := \frac{\mathcal{C}}{2\pi r_h(M)}. \tag{13}$$

In this paper, we adopt the circumference $2(W + L)$ of the source as the definition of $\mathcal{C}$ and the total energy $2p$ of the system as the definition of M. The gray region in Fig. 5 shows the region of the AH formation in the (L, W)-plane. The contours of $H = 0.97$ and $H = 1.05$ are also shown.
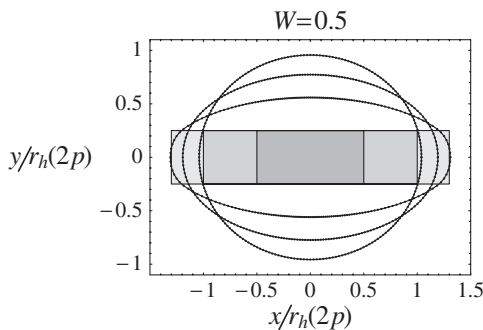

FIG. 3. The shapes of the boundary B for $W = 0.5$ and $L = 1.00, 2.00, 2.60$. The shapes of the source are indicated by the gray regions. For $L \geq 2.61$, we could not find the AH. For $L = 2.60$, the distortion parameter is $\zeta = 2.34$.
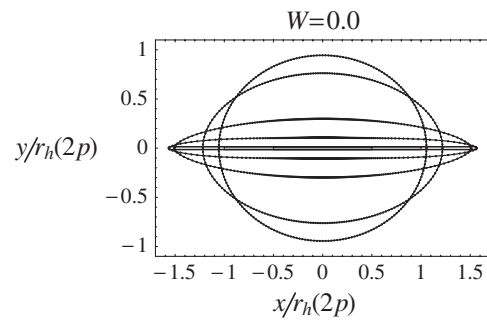

FIG. 4. The shapes of the boundary B for $W = 0.0$ and $L = 1.00, 2.00, 3.00, 3.13$. The shapes of the source are indicated by the gray regions. For $L \geq 3.14$, we could not find the AH. For $L = 3.13$, the distortion parameter is $\zeta \simeq 15$ and the AH is highly distorted.

From this figure, we see that the AH forms if $H \leq 0.97$ and the AH does not form if $H \geq 1.05$. Therefore, the parameter H gives a good indicator for the AH formation in this system and our result is consistent with the hoop conjecture. The hoop conjecture holds well also for the formation of the highly distorted AHs. This is because the AH can be highly distorted, but cannot be arbitrarily long in some direction.

It is worth pointing out that the highly distorted AH can form also in the system of collapsing convex null-dust shell [13]. For a cylindrical shell with two hemisphere caps, the AH can form for very small radius of the cylinder $r \ll r_h(M)$, while the length of the cylinder can be as large as $4r_h(M)$ (see Fig. 1(b) in [13]). Therefore, it is expected that in many systems the highly distorted AHs can form but its size in the largest direction is restricted from above.

Let us discuss whether the formation of a highly distorted AH leads to an interesting gravitational phenomena. In the higher-dimensional cases, one would expect that the
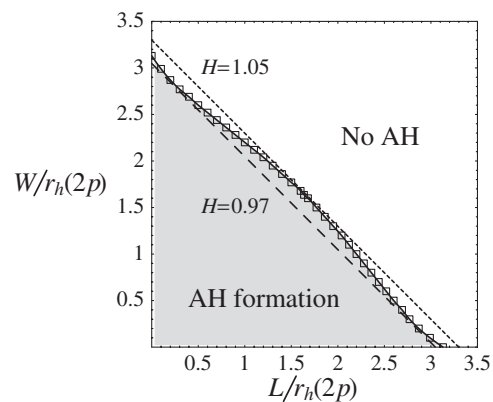

FIG. 5. The region of the AH formation in the (L, W)-plane (the gray region). The border of this region is shown by a solid line, on which the numerical data is shown by squares ($\square$). The contours of $H = 0.97$ and $H = 1.05$ are shown by a dashed line and a dotted line, respectively. The AH forms if $H \leq 0.97$ and does not form if $H \geq 1.05$.

Gregory-Laflamme instability [21] occurs when a highly distorted AH forms. However, such an instability is not known in the four-dimensional case, since there is no black string solution. Although there is a four-dimensional cylindrical black hole solution in the presence of a negative cosmological constant Λ, it turns out to be stable [22]. This is in contrast to the fact that the uniform AdS black strings in higher dimensions are unstable for sufficiently small $|\Lambda|$ [23]. Therefore, we cannot expect the interesting phenomena such as the pinch-off of the AH. A highly distorted AH could form in a dynamical situation, and it will become less distorted in the temporal evolution. In our system, the final state is expected to be a Schwarzschild black hole.

The mass of the final Schwarzschild black hole M_{BH} is determined by the amount of gravitational radiation. Using the area theorem, we can evaluate the lower bound on the mass of the final state as

$$M_{\mathrm{AH}} = \frac{1}{G}\sqrt{\frac{A_{\mathrm{AH}}}{16\pi}}, \qquad (14)$$

with the AH area A_{AH}. In our system, there is a tendency that M_{AH} becomes smaller as the AH becomes more distorted. In the case $W = 0.0$ and $L = 3.13$, M_{AH} is less than 30% of the total system energy $2p$. Although M_{AH} is just the lower bound on M_{BH}, it is natural to expect that there is some correlation between M_{AH} and M_{BH}. Therefore, if a highly distorted AH forms in some system, a lot of gravitational wave could be radiated.

Finally, we point out the possible relation between the theorem on the AH topology [24] and the hoop conjecture. For this purpose, let us recall the study of [2] on the AH formation in several systems in the momentarily static conformally flat four-dimensional space (i.e., the initial data of the five-dimensional spacetime). They showed that (i) the arbitrarily long AH of the spherical topology S^3 forms for the spindle-shaped matter distribution and (ii) the AH of the ring topology $S^1 \times S^2$ forms for the ring-shaped matter distribution, if the ring radius is sufficiently large. The result of (ii) is naturally expected from the result of (i), since the ring-shaped matter distribution is achieved by bending the spindle-shaped matter distribution. Therefore, it is indicated that if a long AH of the spherical topology can form, an AH of the ring topology also can form. Conversely, it is expected that if an AH of the ring topology cannot form, a long AH of the spherical topology cannot form. Since an AH of the ring topology (or equivalently the torus topology $S^1 \times S^1$) is forbidden in four dimensions [24], the formation of a long AH is also expected to be prohibited. This statement is consistent with the ''only when'' part of the hoop conjecture.

The above discussion gives one plausible interpretation for the reason why the ''only when'' part of the hoop conjecture holds in four dimensions. We also hope that this observation could give a hint for studies which attempt to prove the only when part of the hoop conjecture, since at least physically the AH topology theorem is related to the hoop conjecture. Here, we would like to note that currently there is no theorem which corresponds to the only when part of the hoop conjecture, though there is a strong theorem by Schoen and Yau [25] which corresponds to the ''when'' part of the hoop conjecture.

To summarize, we studied the collision of shock waves with rectangular sources, and found that the highly distorted AH can form in this system although it cannot be arbitrarily long. The hoop conjecture remarkably holds also for the formation of such highly distorted AHs, and our result gives a strong support to the hoop conjecture. The higher-dimensional generalization of the study in this paper, especially in the case $W = 0$, is interesting in the context of the black hole production at accelerators in TeV gravity scenarios. It will be reported in our forthcoming paper.

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