Helicity amplitudes for charmonium production in hadron-hadron and photon-hadron collisions

M. M. Meijer*

Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

J. Smith†

C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794-3840 USA

W. L. van Neerven‡

Lorentz Institute, University of Leiden, Postbus 9502, 2300 RA Leiden, The Netherlands

(Received 24 October 2007; published 20 February 2008)

We present the gluon-gluon and photon-gluon helicity amplitudes for color singlet and octet charmonium production in polarized and unpolarized hadron-hadron and photon-hadron collisions.

DOI: 10.1103/PhysRevD.77.034014 PACS numbers: 12.38.Bx, 13.60.Le, 13.85.Ni, 14.40.Gx

I. INTRODUCTION

The amplitudes for the production of charmonia states in hadron-hadron and photon-hadron collisions are usually calculated within the framework of nonrelativistic quantum chromodynamics [1]. There is a large amount literature on this subject and references to recent experimental and theoretical papers can be found in [2–4]. Several perturbative QCD reactions are required among them $g + g \to g +$ charmonia and $\gamma + g \to g +$ charmonia, where g represents a gluon. The latter can be in either color singlet or color octet states. Specific results have been presented in [5–9], among others. However a close examination of these papers reveals inconsistencies between the published results. Also while the individual helicity amplitudes are available in the color singlet case we could not find the corresponding results for the color octet case. These helicity amplitudes can be used to calculate the charmonia production cross sections by unpolarized gluons, by longitudinally polarized gluons and by transversely polarized gluons. Therefore we have calculated the helicity amplitudes and present our results below. For the benefit of the reader we also give some details of the calculation.

We used the helicity method described in the book by Gastmans and Wu [10] (see also [5]) to calculate processes where three gluons or two gluons and a photon form charmonium. Like Gastmans and Wu we projected out the lowest angular momenta states of the heavy quark pair, namely ${}^1S_0$, ${}^3S_1$, ${}^1P_1$, ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$, using appropriate projection operators (see [11]). We then flipped one of the gluons from incoming to outgoing and with these squared matrix elements calculated the unpolarized and longitudinally polarized differential cross sections.

*m.meijer@hef.ru.nl
†smith@max2.physics.sunysb.edu
‡Deceased.

II. GLUON-GLUON REACTION

Gastmans and Wu have presented results for the differential cross section for the production of a color singlet heavy quark pair in angular momentum states ${}^{2S+1}L_J$. They begin with the reaction with three incoming gluons where the momenta and colors of the particles are labeled as

$$g(k_1, a) + g(k_2, b) + g(k_3, c) \to q(p/2 + q) + \bar{q}(p/2 - q). \tag{1}$$

There are six Feynman diagrams where the three gluons couple directly to the heavy quark line and six diagrams where two gluons couple to the heavy quark line.

There are eight helicity matrix elements which are labeled by assigning either a + or a − to each gluon and which are related by CP conjugation and crossing. All eight can be derived from two, called $|M(+,+,+)|^2$ and $|M(+,+,-)|^2$. We will list them below.

The gluon helicities for the ${}^{2S+1}L_J$ $(+,+,+)$ combination are

$$\not{\epsilon}_1^+ = N[\not{k}_1\not{k}_2\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_2\not{k}_1(1+\gamma_5)] \tag{2a}$$
$$\not{\epsilon}_2^+ = N[\not{k}_2\not{k}_3\not{k}_1(1-\gamma_5) + \not{k}_1\not{k}_3\not{k}_2(1+\gamma_5)] \tag{2b}$$
$$\not{\epsilon}_3^+ = N[\not{k}_3\not{k}_1\not{k}_2(1-\gamma_5) + \not{k}_2\not{k}_1\not{k}_3(1+\gamma_5)], \tag{2c}$$

while those for the ${}^{2S+1}L_J$ $(+,+,-)$ combination are

$$\not{\epsilon}_1^+ = N[\not{k}_1\not{k}_2\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_2\not{k}_1(1+\gamma_5)] \tag{3a}$$
$$\not{\epsilon}_2^+ = -N[\not{k}_2\not{k}_1\not{k}_3(1-\gamma_5) + \not{k}_3\not{k}_1\not{k}_2(1+\gamma_5)] \tag{3b}$$
$$\not{\epsilon}_3^- = N[\not{k}_3\not{k}_1\not{k}_2(1+\gamma_5) + \not{k}_2\not{k}_1\not{k}_3(1-\gamma_5)], \tag{3c}$$

where $N = [(k_1 \cdot k_2)(k_2 \cdot k_3)(k_3 \cdot k_1)]^{-1/2}/4$ is a normalization factor. In principle there is an extra term of the form $(k_2 \cdot k_3)\not{k}_1$ in (2a) and corresponding terms in (2b), (2c), and (3a)–(3c). However the gluon-quark coupling is vectorlike and the vector current is conserved so these terms do not contribute to the helicity amplitudes. The helicity amplitudes are functions of the invariants $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_3 + k_1)^2$ and the mass of the pair $M \approx 2m$ where m is the heavy quark mass. Note that $s + t + u = M^2$, $N = (2stu)^{-1/2}$ and the color singlet projection operator is $\delta_{ij}/\sqrt{3}$. Also they depend on two parameters R_0 and R_1' which are the S-state wave function and the derivative of the P-state wave function evaluated at the origin. The former is defined in terms of the leptonic decay width

$$R_0^2 = M^2\Gamma(^3S_1 \rightarrow e^+e^-)/4\alpha^2 Q_f^2, \tag{4}$$

with $\alpha \approx 1/137$ the fine structure constant and Q_f is the fractional charge of the quarks. R_1' is determined from a fit to the charmonium potential and has the value

$$R_1'^2/M_\chi^2 \approx 0.006\ (\mathrm{GeV})^3. \tag{5}$$

R_0^2 and $R_1'^2$ have dimensions of mass^3 and mass^5, respectively. We will calculate the differential cross section for the reaction

$$g(k_1, a) + g(k_2, b) \rightarrow q(p/2 + q) + \bar{q}(p/2 - q) + g(k_3, c), \tag{6}$$

where the invariants are now $s = (k_1 + k_2)^2$, $t = (k_2 - k_3)^2$, $u = (k_1 - k_3)^2$.

The squares of the matrix elements for the reaction (6) follow from those in reaction (1) by crossing $k_3 \rightarrow -k_3$ and flipping the helicity of the third gluon. They are denoted by $M|(+,+;+)|^2$, $|M(+,+;-)|^2$, $|M(+,-;-)|^2$ and $|M(-,+;-)|^2$. Note that these are equal to $M|(-,-;-)|^2$, $|M(-,-;+)|^2$, $|M(-,+;+)|^2$ and $|M(+,-;+)|^2$, respectively, by CP conjugation. However the kinematic variables require permutations to reflect the crossing of gluon number three. These relations are

$$|M(+,+;+)|^2 = |M(+,+,-)|^2 \tag{7a}$$

$$|M(+,+;-)|^2 = |M(+,+,+)|^2 \tag{7b}$$

$$|M(+,-;-)|^2 = |M(+,+,-)|^2|_{s\leftrightarrow u} \tag{7c}$$

$$|M(-,+;-)|^2 = |M(+,+,-)|^2|_{s\leftrightarrow t}. \tag{7d}$$

It is very convenient to use completely symmetric variables which are then invariant under any crossing transformations. Hence we express several results in terms of the variables $M^2 = s + t + u$, $P = st + tu + us$ and $Q = stu$, which are invariant under $s \leftrightarrow t$ and $s \leftrightarrow u$. The denominators of the helicity amplitudes are written in these variables while the numerators contain terms in s. Therefore the crossing simply involves changing $s \rightarrow t$ and $s \rightarrow u$ in the numerators of our expressions.

After we add the helicity amplitudes to form differential cross sections we can again use the completely symmetric variables above. We briefly describe how to do this. Completely symmetric functions in s, t and u are made out of combinations like $s^at^bu^c + s^at^cu^b + s^bt^au^c + s^bt^cu^a + s^ct^au^b + s^ct^bu^a \equiv A(a,b,c)$, where we are free to choose $a \le b \le c$. Therefore take

$$A(a,b,c) = Q^cA(a',b',0) \equiv Q^cB(a',b'), \tag{8}$$

where $a' = a - c$ and $b' = b - c$. The function $B(a,b)$ then has the following properties:

$$\begin{aligned} B(a,b) &= B(b,a) \\ PB(a,b) &= (st + tu + us)(s^at^b + s^au^b + s^bt^a + s^bu^a + t^au^b + t^bu^a) \\ &= B(a+1,b+1) + Q(B(a-1,b) + B(a,b-1)) \\ M^2B(a,b) &= (s + t + u)(s^at^b + s^au^b + s^bt^a + s^bu^a + t^au^b + t^bu^a) \\ &= B(a+1,b) + B(a,b+1) + QB(a-1,b-1). \end{aligned} \tag{9}$$

We can therefore calculate it recursively from

$$\begin{aligned} B(a,b) &= PB(a-1,b-1) - Q(B(a-2,b-1) + B(a-1,b-2)) \\ B(a,1) &= PB(a-1,0)/2 - QB(a-2,0)/2 \\ B(a,0) &= M^2B(a-1,0) - 2B(a-1,1). \end{aligned} \tag{10}$$

Starting from the initial values

$$B(0,0) = 6, \qquad B(1,0) = 2M^2, \qquad B(1,1) = 2P,$$

we find

$$\begin{aligned} B(2,0) &= 2M^4 - 4P \\ B(3,0) &= 2M^6 - 6M^2P + 6Q \\ B(2,1) &= M^2P - 3Q \\ B(4,0) &= 2M^8 - 8M^4P + 8QM^2 + 4P^2 \\ B(3,1) &= M^4P - 2P^2 - QM^2 \\ B(2,2) &= 2P^2 - 4QM^2 \\ B(5,0) &= 2M^{10} - 10M^6P + 10M^4Q + 10M^2P^2 - 10PQ \\ B(4,1) &= M^6P - 3M^2P^2 + 5PQ - M^4Q \\ B(3,2) &= M^2P^2 - PQ - 2M^4Q. \end{aligned} \tag{11}$$

This procedure was continued up to high enough powers to express the numerators of our answers for unpolarized differential cross sections in terms of M^2, P and Q.

We have also calculated the corresponding amplitudes for the production of a color octet heavy quark pair which requires four additional Feynman diagrams for processes where only one gluon couples to the heavy quark pair. These contain three gluon and four gluon couplings. The color octet projection operator is required so the factor

$\delta_{ij}/\sqrt{3}$ in the color singlet case is replaced by $\sqrt{2}T^a_{ij}$. This follows from the fact that any 3×3 matrix M can be decomposed into a trace part $\mathrm{Tr}M/3$ and eight parts proportional to the color matrices $NT^a\,\mathrm{Tr}(MT^a)$ where $N = 2$. The normalization is determined by the identity $T^b = NT^a\,\mathrm{Tr}(T^bT^a) = NT^2\delta^{ab}/2 = NT^b/2$. Hence the normalization factors of $1\sqrt{3}$ and $\sqrt{2}$ are required in the color singlet and color octet projections, respectively. The color octet amplitudes cannot be determined from decay processes so they are fit to quarkonium production differential cross sections in proton-proton, proton-antiproton and photon-hadron collisions.

The differential cross section for unpolarized reactions such as $P + P \to c\bar{c} + X$ contains the sum of the squares of the helicity amplitudes, $|M(+,+;+)|^2 + |M(+,+;-)|^2 + |M(+,-;-)|^2 + |M(-,+;-)|^2$. The color singlet case results are given by [5], which we refer to as GW, and [10]. The color octet results are available in the Appendix of Cho and Leibovich [6], which we refer to as CL. The differential cross sections for longitudinally polarized collisions contain the differences $|M(+,+;+)|^2 + |M(+,+;-)|^2 - |M(+,-;-)|^2 - |M(-,+;-)|^2$, and are listed for both the color singlet and the color octet cases in the paper of Klasen, Kniehl, Mihaila and Steinhauser [9], which we refer to as KKMS. We compare our results with those in these (and other) papers. Note that all these helicity amplitudes can also be used to construct differential cross sections for charmonium production by transversely polarized gluons but we leave this to another publication.

A. Matrix elements squared

We now list the results for the squares of the color singlet matrix elements when the heavy quark pair (with mass M) is in the appropriate angular momentum state. It is convenient to rename $R_0^2 = \langle R[^1S_0^{(1)}]\rangle = \langle R[^3S_1^{(1)}]\rangle$ and $R'^2 = \langle R[^1P_1^{(1)}]\rangle = \langle R[^3P_0^{(1)}]\rangle = \langle R[^3P_1^{(1)}]\rangle = \langle R[^3P_2^{(1)}]\rangle$, where the final superscript indicates the color singlet.

We begin with the color singlet case. When the heavy quark pair is in the $^1S_0$ state we find

$$|M(+,+,+)|^2 = \frac{16g^6\langle R[^1S_0^{(1)}]\rangle}{\pi M}\frac{M^8P^2}{Q(Q-M^2P)^2} \tag{12a}$$

$$|M(+,+,-)|^2 = \frac{16g^6\langle R[^1S_0^{(1)}]\rangle}{\pi M}\frac{s^4P^2}{Q(Q-M^2P)^2}, \tag{12b}$$

after the color states of the gluons have been summed over. These results agree with the squares of (8.29) and (8.40) in GW.

When the heavy quark pair is in the $^3S_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{13a}$$

$$|M(+,+,-)|^2 = \frac{160g^6\langle R[^3S_1^{(1)}]\rangle}{9\pi M}\frac{M^2s^2(s-M^2)^2}{(Q-M^2P)^2}, \tag{13b}$$

after the color states of the gluons have been summed over. The polarization of the spin one charmonium state has also been summed over. The second result agrees with the (8.50) in GW after correcting an obvious typographical error that the $(t - M^2)$ should read $(t - M^2)^2$.

When the heavy quark pair is in the $^1P_1$ state we find

$$|M(+,+,+)|^2 = \frac{640g^6\langle R[^1P_1^{(1)}]\rangle}{3\pi M^3}\frac{M^{10}(-M^2P+5Q)}{(Q-M^2P)^3} \tag{14a}$$

$$|M(+,+,-)|^2 = \frac{640g^6\langle R[^1P_1^{(1)}]\rangle}{3\pi M^3}\frac{M^2s^2}{(Q-M^2P)^3} \times [3M^4Q - M^6P + 2Qs^2], \tag{14b}$$

after summing over gluon colors and final state polarizations, Here we agree with the results (8.55) and (8.57) in GW.

When the heavy quark pair is in the $^3P_0$ state we find

$$|M(+,+,+)|^2 = \frac{64g^6\langle R[^3P_0^{(1)}]\rangle}{\pi M^3}\frac{9M^8P^2(Q-M^2P)^2}{Q(Q-M^2P)^4} \tag{15a}$$

$$|M(+,+,-)|^2 = \frac{64g^6\langle R[^3P_0^{(1)}]\rangle}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4} \times [Q^2 - s^2Q(s-3M^2) + 3PM^2s^3]^2, \tag{15b}$$

after summing over gluon colors and final state polarizations. Here we agree with the results in (8.59) in GW.

When the heavy quark pair is in the $^3P_1$ we find

$$|M(+,+,+)|^2 = 0 \tag{16a}$$

$$|M(+,+,-)|^2 = \frac{192g^6\langle R[^3P_1^{(1)}]\rangle}{\pi M^3}\frac{(s-M^2)^2s^2}{(Q-M^2P)^4}[2Q(5M^4P - M^8 + P^2 - (4P - 2sM^2 + 4s^2 - M^4)(s-M^2)^2) - Q^2(15M^2 - 8s) - 4M^2P^3 + M^6P^2], \tag{16b}$$

after summing over the gluon colors and final state polarizations. These results agree with (8.63) in GW.

When the heavy quark pair is in the $^3P_2$ we find

$$|M(+,+,+)|^2 = 0 \tag{17a}$$

$$\begin{aligned}|M(+,+,-)|^2 &= \frac{64g^6\langle R[^3P_2^{(1)}]\rangle}{\pi M^3}\frac{1}{Q(Q-M^2P)^4}[12M^8P^4(3s-M^2)(s-M^2) - 12M^4P^5 s(s-3M^2)\\ &\quad + 2P^2Q^3(s-11M^2) - 3M^6P^3Q(s-M^2)(25s-8M^2) + 12M^2P^4Q(s^2-4M^2s-3M^4)\\ &\quad + M^4P^2Q^2(8s^2+9M^2s-15M^4) - 2P^3Q^2(s^2-5M^2s-30M^4) + M^2PQ^3(29s^2-51M^2s+18M^4)\\ &\quad - M^2Q^4(9s-11M^2)],\end{aligned} \tag{17b}$$

after summing over gluon colors and final state polarizations. These results agree with (8.70) in GW.

Now we present the corresponding results for the color octet projections. These results do not seem to be available in the literature. We have only found expressions for the differential cross sections which we will compare to ours later on. We give these results since we need the differences between the helicity combinations to check the octet longitudinally polarized differential cross sections. The constants from the wave functions are now simply renamed as $R^2 \to \langle R[^1S_0^{(8)}]\rangle$ etc., since there are other definitions in the literature. We will present the relations between the definitions later on.

When the heavy quark pair is in the $^1S_0$ state we find

$$|M(+,+,+)|^2 = \frac{40g^6\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{M^8(P^2-M^2Q)}{Q(Q-M^2P)^2} \tag{18a}$$

$$|M(+,+,-)|^2 = \frac{40g^6\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{s^3}{Q(Q-M^2P)^2}[PQ + s^3(s-M^2)^2 - s^2Q], \tag{18b}$$

after the color states of the gluons have been summed over.

When the heavy quark pair is in the $^3S_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{19a}$$

$$|M(+,+,-)|^2 = \frac{16g^6\langle R[^3S_1^{(8)}]\rangle}{3\pi M}\frac{s^2(s-M^2)^2}{M^2(Q-M^2P)^2}(19M^4-27P), \tag{19b}$$

after the color states of the gluons have been summed over.

When the heavy quark pair is in the $^1P_1$ state we find

$$|M(+,+,+)|^2 = \frac{32g^6\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{M^6}{Q(Q-M^2P)^3}[217M^4Q^2 - 54PQ^2 + 43M^6PQ - 27M^2P^2Q - 27M^4P^3] \tag{20a}$$

$$\begin{aligned}|M(+,+,-)|^2 &= \frac{32g^6\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{s^3}{Q(Q-M^2P)^3}[Qs^2(t+u)(-174u^2+26tu-174t^2)\\ &\quad + Qs(-98u^4 - 278tu^3 - 468t^2u^2 - 278t^3u - 98t^4) + Q(t+u)(-38u^4 - 82tu^3 - 169t^2u^2\\ &\quad - 82t^3u - 38t^4) + s^4(-27u^4 - 152tu^3 + 10t^2u^2 - 152t^3u - 27t^4)\\ &\quad + t^2u^2(t+u)(-38u^3 - 60tu^2 - 60t^2u - 38t^3) + s^5(t+u)(-27u^2 - 11tu - 27t^2)],\end{aligned} \tag{20b}$$

after summing over gluon colors and final state polarizations.

When the heavy quark pair is in the $^3P_0$ state we find

$$|M(+,+,+)|^2 = \frac{160g^6\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{9M^8(P^2-M^2Q)(Q-M^2P)^2}{Q(Q-M^2P)^4} \tag{21a}$$

$$\begin{aligned}|M(+,+,-)|^2 &= \frac{160g^6\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{(s-M^2)^2}{Q(Q-M^2P)^4}[Q^4 + 9s^8M^4(s-M^2)^2 + Qs^5M^2(6s^3 - 6M^6 + 33sM^4 - 42s^2M^2)\\ &\quad + Q^2s^2(44s^2M^4 + 4M^8 - 18s^3M^2 + s^4) + Q^3s(-2(s-M^2)^2 + 9sM^2)],\end{aligned} \tag{21b}$$

after summing over gluon colors and final state polarizations.

When the heavy quark pair is in the $^3P_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{22a}$$

$$|M(+,+,-)|^2 = \frac{960 g^6 \langle R[^3P_1^{(8)}]\rangle}{\pi M^3} \frac{(s-M^2)^2(P+s^2-sM^2)}{Q(Q-M^2P)^4}[Q^3(s-2M^2) + s^5M^2(M^8-4sM^6+7s^2M^4-6s^3M^2+2s^4) + Qs^3(M^8-4sM^6+11s^2M^4-10s^3M^2+s^4) + Q^2s(M^6+7s^2M^2-2s^3)], \tag{22b}$$

after summing over gluon colors and final state polarizations.

When the heavy quark pair is in the $^3P_2$ state we find

$$|M(+,+,+)|^2 = 0 \tag{23a}$$

$$|M(+,+,-)|^2 = \frac{320 g^6 \langle R[^3P_2^{(8)}]\rangle}{\pi M^3} \frac{(s-M^2)^2}{Qs^4(Q-M^2P)^4}[6s^8M^4(s-M^2)^6 + Qs^6M^2(18M^{12}-114sM^{10}+285s^2M^8-354s^3M^6+225s^4M^4-66s^5M^2+6s^6) + Q^2s^4(24M^{12}-132sM^{10}+313s^2M^8-336s^3M^6+161s^4M^4-30s^5M^2+s^6) + Q^3s^2(18M^{10}-78sM^8+141s^2M^6-110s^3M^4+25s^4M^2-2s^5) + Q^4(s^4-6M^2(s-M^2)^3 + 6sM^4(s-M^2))], \tag{23b}$$

after summing over gluon colors and final state polarizations.

B. Unpolarized differential cross sections

These follow from the sum of the squares of the helicity matrix elements $|M(+,+;+)|^2 + |M(+,+;-)|^2 + |M(+,-;-)|^2 + |M(-,+;-)|^2$ with the substitutions $s \to t$ and $s \to u$ as described above. However to sum over all polarization states we have to multiply by 2 to include the CP conjugates. Then one adds the average over the initial gluon colors and polarizations (1/256) and multiplies by an overall factor of $1/(16\pi s^2)$. We also rewrite the numerators of our answers in terms of M^2, P and Q. Finally we note that $Q = s^2 p_t^2$ where p_t is the transverse momentum of the heavy quark pair so it is easy to rewrite the results for $d\sigma/dt$ into those for $d\sigma/dp_t^2$.

We begin with the color singlet case, where our results can be compared with those in GW and KKMS. The latter authors give the differential cross sections as functions of polarization factors $\xi_a\xi_b$ in the form $a(s,t,u) + \xi_a\xi_b b(s,t,u)$. The unpolarized cross sections are obtained by setting $\xi_a\xi_b = 0$. We call these the first terms and the coefficients of $\xi_a\xi_b$, which yield the longitudinally polarized differential cross sections, the second terms. Note that, due to the differences in the definitions of the wave functions, our comments concern the polynomial dependence of $a(s,t,u)$ and $b(s,t,u)$ on the invariants. However we will also identify the prefactors. This is possible because their polarized differential cross sections agree with ours.

When the heavy quark pair is in the $^1S_0$ state we find

$$\frac{d\sigma}{dt} = \frac{\pi\alpha_s^3\langle R[^1S_0^{(1)}]\rangle}{Ms^2} \frac{P^2}{Q(Q-M^2P)^2}[(P-M^4)^2 + 2M^2Q], \tag{24}$$

which agrees with (8.46) in GW. However it does not agree with the first term in (A.16) in KKMS, who use the notation where $\langle R[^1S_0^{(1)}]\rangle = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

When the heavy quark pair is in the $^3S_1$ state we find

$$\frac{d\sigma}{dt} = \frac{10\pi\alpha_s^3\langle R[^3S_1^{(1)}]\rangle}{9Ms^2} \frac{M^2(P^2-M^2Q)}{(Q-M^2P)^2}, \tag{25}$$

which agrees with (8.52) in GW and also agrees with the first terms in (A.17) in KKMS, who use the notation where $\langle R[^3S_1^{(1)}]\rangle = 4\pi\langle O[^3S_1^{(1)}]\rangle/3$.

When the heavy quark pair is in the $^1P_1$ state we find

$$\frac{d\sigma}{dt} = \frac{40\pi\alpha_s^3\langle R[^1P_1^{(1)}]\rangle}{3M^3s^2} \frac{M^2}{(Q-M^2P)^3} \times [-M^{10}P + M^6P^2 + Q(5M^8 - 7M^4P + 2P^2) + 4M^2Q^2], \tag{26}$$

which agrees with (8.58) in GW. It also agrees with the first terms in (A.18) in KKMS, who use the notation where $\langle R[^1P_1^{(1)}]\rangle = 4\pi\langle O[^1P_1^{(1)}]\rangle/9$.

When the heavy quark pair is in the $^3P_0$ state we find

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha_s^3\langle R[^3P_0^{(1)}]\rangle}{M^3s^2} \frac{1}{Q(Q-M^2P)^4} \times [-2M^8P^2Q^2 + 6M^6P^3Q(3P-M^4) - 2M^2PQ^3(P-M^4) + P^2(3PM^2-Q)^2(P-M^4)^2 + 6M^4Q^4], \tag{27}$$

which agrees with (8.60) in GW. It does not agree with the first terms in (A.19) in KKMS, who use the notation where $\langle R[^3P_0^{(1)}]\rangle = 4\pi\langle O[^3P_0^{(1)}]\rangle/3$.

When the heavy quark pair is in the $^3P_1$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{12\pi\alpha_s^3\langle R[^3P_1^{(1)}]\rangle}{M^3s^2}\frac{P^2}{(Q-M^2P)^4} \times [-15M^2Q^2 + M^2P^2(M^4-4P) - 2Q(M^8-5M^4P-P^2)], \quad (28)$$

which agrees with (8.64) in GW. It does not agree with the first terms in (A.20) in KKMS, who use the notation where $\langle R[^3P_1^{(1)}]\rangle = 4\pi\langle O[^3P_1^{(1)}]\rangle/9$.

When the heavy quark pair is in the $^3P_2$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3\langle R[^3P_2^{(1)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4} \times [12M^4P^4(P-M^4)^2 + M^2PQ^3(16M^4-61P) - 3M^2P^3Q(8M^8-M^4P+4P^2) + 12M^4Q^4 - 2P^2Q^2(7M^8-43M^4P-P^2)], \quad (29)$$

which agrees with (8.71) in GW after correcting a typographical error. They have $(8M^8 - M^4P + P^2)$ which should read $(8M^8 - M^4P + 4P^2)$. The expression is given correctly in their published paper [5]. Also it does not agree with the first terms in (A.21) in KKMS, who use the notation where $\langle R[^3P_2^{(1)}]\rangle = 4\pi\langle O[^3P_2^{(1)}]\rangle/15$.

In view of these differences we contacted the authors of the KKMS paper. They calculated their results with projection operators for the sums over the gluon polarization states, which required the calculation of additional ghost diagrams. However they inadvertently presented the formulas (A.16), (A.19), (A.20) and (A.21) without the contributions from these ghost terms. They claim that the correct formulas are included in their FORTRAN programs and that their numerical results are therefore correct.

Now we give the color octet results which can be compared with the results for the squares of the matrix elements in the appendix of CL and with the first parts of the expressions in Appendix A of KKMS.

When the heavy quark pair is in the $^1S_0$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{5\pi\alpha_s^3\langle R[^1S_0^{(8)}]\rangle}{2Ms^2}\frac{P^2-M^2Q}{Q(Q-M^2P)^2} \times [(P-M^4)^2 + 2M^2Q], \quad (30)$$

which agrees with (A5a) in CL. It does not agree with the first part of (A.22) in KKMS, who use the notation where $\langle R[^1S_0^{(8)}]\rangle = \pi\langle O[^1S_0^{(8)}]\rangle/2$.

When the heavy quark pair is in the $^3S_1$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R[^3S_1^{(8)}]\rangle}{3Ms^2}\frac{(P^2-M^2Q)(19M^4-27P)}{M^2(Q-M^2P)^2}, \quad (31)$$

which agrees with the sum of (A5b) plus (A5c) in CL. It does not agree with the first terms in (A.23) in KKMS, who use the notation where $\langle R[^3S_1^{(8)}]\rangle = \pi\langle O[^3S_1^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^1P_1$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{2\pi\alpha_s^3\langle R[^1P_1^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^3} \times [179M^4Q^3 + 217M^{10}Q^2 - 27M^2P^5 + 54M^6P^4 - 27M^{10}P^3 + 135PQ^3 + 103M^2P^2Q^2 - 212M^6PQ^2 - 124M^8P^2Q + 43M^{12}PQ + 27P^4Q], \quad (32)$$

which is not given in CL. It does not agree with the first terms in (A.24) in KKMS, who use the notation where $\langle R[^1P_1^{(8)}]\rangle = \pi\langle O[^1P_1^{(8)}]\rangle/18$.

When the heavy quark pair is in the $^3P_0$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R[^3P_0^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4} \times [9M^4P^4(P-M^4)^2 + 3M^{10}P^3Q - 6M^2P^5Q + 27M^6P^4Q + 18M^{12}PQ^2 - 32M^8P^2Q^2 - 9M^{14}P^2Q - 4M^4P^3Q^2 + 5M^4Q^4 + P^4Q^2 + 11M^6PQ^3 - M^2P^2Q^3 - 13M^{10}Q^3], \quad (33)$$

which agrees with (A5d) in CL. It does not agree with the first terms in (A.25) in KKMS, who use the notation where $\langle R[^3P_0^{(8)}]\rangle = \pi\langle O[^3P_0^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^3P_1$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{60\pi\alpha_s^3\langle R[^3P_1^{(8)}]\rangle}{M^3s^2}\frac{1}{(Q-M^2P)^4} \times [P^4Q + M^{10}Q^2 + M^6P^4 - 2M^2P^5 - 2M^8P^2Q + 7M^4P^3Q - 3M^6PQ^2 - 9M^2P^2Q^2 + 6M^4Q^3], \quad (34)$$

which agrees with the sum of (A5e) and (A5f) in CL. It does not agree with the first terms in (A.26) in KKMS, who use the notation where $\langle R[^3P_1^{(8)}]\rangle = \pi\langle O[^3P_1^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^3P_2$ state we find

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{20\pi\alpha_s^3\langle R[^3P_2^{(8)}]\rangle}{M^3s^2}\frac{1}{Q(Q-M^2P)^4} \times [6M^{12}P^4 - 12M^8P^5 + 6M^4P^6 + 11M^4Q^4 + Q(-6M^{14}P^2 - 3M^{10}P^3 + 3M^6P^4 - 6M^2P^5) + Q^2(24M^{12}P - 29M^8P^2 + 41M^4P^3 + P^4) + Q^3(-19M^{10} + 14M^6P - 31M^2P^2)], \quad (35)$$

which agrees with the sum of (A5g) plus (A5h) plus (A5i) in CL, after correcting an obvious typographical error that the term $-M\hat{s}^2$, which multiplies the second line in (A5i), should read $-M^2\hat{s}$. It does not agree with the first terms in (A.27) in KKMS, who use the notation where $\langle R[^3P_2^{(8)}]\rangle = \pi\langle O[^3P_2^{(8)}]\rangle/30$. The explanation for the difference between our results and (A.22)–(A.27) in KKMS is again that they inadvertently neglected to include ghost contributions to their amplitudes. However they claim that they did so in their computer programs so their numerical results are correct.

In view of the differences in the above results and before contacting KKMS we recalculated the differential cross sections by summing over the physical polarizations of the external gluons using the covariant expression

$$\sum_{\alpha=+,-} \epsilon^{\mu}(k,\alpha)\epsilon^{\nu}(k,\alpha) = P^{\mu\nu}(n,k), \tag{36}$$

with

$$P^{\mu\nu}(n,k) = -g_{\mu\nu} + (n_\mu k_\nu + k_\mu n_\nu)/n\cdot k, \tag{37}$$

where n_μ satisfies $n_\mu P^{\mu\nu} = P^{\mu\nu} n_\nu = 0$ and $n^2 = 0$. One uses this sum for each external gluon and the answer for the square of the matrix elements should be independent of n_μ. This method does not require any ghosts and yielded the same answers we obtained above for the differential cross sections.

C. Polarized differential cross sections

Now we calculate the expressions $|M(+,+;+)|^2 + |M(+,+;-)|^2 - |M(+,-;-)|^2 - |M(-,+;-)|^2$, which yield the longitudinally polarized differential cross sections.

We begin with the color singlet expressions. These are available in KKMS as the second terms, i.e., $b(s,t,u)$, those terms proportional to $\xi_a\xi_b$. The prefactors are identified as in the unpolarized differential cross sections given previously. We repeat them here for convenience.

When the heavy quark pair is in the $^1S_0$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R[^1S_0^{(1)}]\rangle}{s^2M}\frac{P^2}{Qs^2(Q-M^2P)^2} \times [s^6 - 2Q^2 + 4Qs(s-M^2)^2 + s^2M^8 - s^2(s-M^2)^4]. \tag{38}$$

This is in agreement with the second terms in (A.16) in KKMS, when we make the replacement $\langle R[^1S_0^{(1)}]\rangle = 4\pi\langle O[^1S_0^{(1)}]\rangle$.

When the heavy quark pair is in the $^3S_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R[^3S_1^{(1)}]\rangle}{9Ms^2}\frac{M^2Q(s^2-P)}{s(Q-M^2P)^2}. \tag{39}$$

This is in agreement with the second terms in (A.17) in KKMS, when we make the replacement $\langle R[^3S_1^{(1)}]\rangle = 4\pi\langle O[^3S_1^{(1)}]\rangle/3$.

When the heavy quark pair is in the $^1P_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{40\pi\alpha_s^3\langle R[^1P_1^{(1)}]\rangle}{3M^3s^2}\frac{M^2}{(Q-M^2P)^3}[stu(2u^4+4tu^3+6t^2u^2+4t^3u+2t^4) + s^2(5tu^4+7t^2u^3+7t^3u^2+5t^4u + t^5+u^5) + s^3(10tu^3+10t^2u^2+10t^3u+4t^4+4u^4) + s^4(10tu^2+10t^2u+6t^3+6u^3) + s^5(4tu+4t^2+4u^2) + s^6(t+u) + t^2u^5 + 3t^3u^4 + 3t^4u^3 + t^5u^2]. \tag{40}$$

This is in agreement with the second terms in (A.18) in KKMS, when we make the replacement $\langle R[^1P_1^{(1)}]\rangle = 4\pi\langle O[^1P_1^{(1)}]\rangle/9$.

When the heavy quark pair is in the $^3P_0$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3\langle R[^3P_0^{(1)}]\rangle}{M^3s^2}\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4}[18s^9M^4(s-M^2)^6 + 9s^{10}M^6(s-M^2)^4 + Qs^7M^2(66M^{12} - 327sM^{10} + 684s^2M^8 - 762s^3M^6 + 462s^4M^4 - 135s^5M^2 + 12s^6) + Q^3s^3(66M^{10} - 260sM^8 + 374s^2M^6 - 5s^5 - 237s^3M^4 + 62s^4M^2) + Q^5(6sM^2 - s^2 - 9M^4) + Q^4s(18M^8 - 75sM^6 + 80s^2M^4 - 31s^3M^2 + 4s^4) + Q^2s^5(96M^{12} - 422sM^{10} + 750s^2M^8 - 663s^3M^6 + 286s^4M^4 - 49s^5M^2 + 2s^6)]. \tag{41}$$

This is in agreement with the second terms in (A.19) in KKMS, when we make the replacement $\langle R[^3P_0^{(1)}]\rangle = 4\pi\langle O[^3P_0^{(1)}]\rangle/3$.

When the heavy quark pair is in the $^3P_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{12\pi\alpha_s^3\langle R[^3P_1^{(1)}]\rangle}{M^3s^2}\frac{Q(Q+s^2M^2)}{s^5(Q-M^2P)^4}[2s^6(-M^8 + 5sM^6 - 9s^2M^4 + 7s^3M^2 - 2s^4) + Qs^3(M^8 - 4sM^6 + 11s^2M^4 - 18s^3M^2 + 10s^4) + Q^2s(M^6 - 6sM^4 + 11s^2M^2 - 8s^3) + 2Q^3(s - 2M^2)]. \tag{42}$$

This is in agreement with the second terms in (A.20) in KKMS, when we make the replacement $\langle R[^3P_1^{(1)}]\rangle = 4\pi\langle O[^3P_1^{(1)}]\rangle/9$.

When the heavy quark pair is in the $^3P_2$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{4\pi\alpha_s^3\langle R[^3P_2^{(1)}]\rangle}{M^3s^2}\frac{Q+s^2M^2}{Qs^6(Q-M^2P)^4}[-24s^9M^4(s-M^2)^6 - 12s^{10}M^6(s-M^2)^4 + Qs^7M^4(-48M^{10}+276sM^8-648s^2M^6+768s^3M^4-456s^4M^2+108s^5) + Q^4s(24M^8-63sM^6+34s^2M^4-5s^3M^2+8s^4) + Q^3s^3(15s^3M^4-79sM^8+28s^2M^6-2s^4M^2-10s^5 + 48M^{10}) + Q^5(-12M^4+12sM^2-2s^2) + Q^2s^6(-330sM^8+306s^2M^6-82s^3M^4-14s^4M^2+4s^5+116M^{10})]. \tag{43}$$

This is in agreement with the second terms in (A.21) in KKMS, when we make the replacement $\langle R[^3P_2^{(1)}]\rangle = 4\pi\langle O[^3P_2^{(1)}]\rangle/15$. Our polarized differential cross sections agree with those in KKMS because their method of calculation does not require ghost contributions.

We now switch to the color octet results. They can be compared with those in the appendix of KKMS.

When the heavy quark pair is in the $^1S_0$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{5\pi\alpha_s^3\langle R[^1S_0^{(8)}]\rangle}{2s^2M}\frac{1}{s^4Q(Q-M^2P)^2}[2s^7M^2(M^2-s)^4 + s^8M^4(M^2-s)^2 + Qs^5(4M^8-15sM^6+20s^2M^4-12s^3M^2+2s^4) + Q^2s^3(4M^6-12sM^4+14s^2M^2-5s^3) + Q^3s(2M^4-5sM^2+4s^2) - Q^4]. \tag{44}$$

This is in agreement with the second terms in (A.22) in KKMS, if we make the replacement $\langle R[^1S_0^{(8)}]\rangle = \pi\langle O[^1S_0^{(8)}]\rangle/2$.

When the heavy quark pair is in the $^3S_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{\pi\alpha_s^3\langle R[^3S_1^{(8)}]\rangle}{3Ms^2}\frac{Q(19M^4-27P)(s^2-P)}{M^2s(Q-M^2P)^2}, \tag{45}$$

which agrees with the second terms in (A.23) in KKMS, if we make the replacement $\langle R[^3S_1^{(8)}]\rangle = \pi\langle O[^3S_1^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^1P_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{2\pi\alpha_s^3\langle R[^1P_1^{(8)}]\rangle}{M^3s^2}\frac{M^2-s}{Qs^6(Q-M^2P)^4}[(27(M^2-s)^3M^6s^{11}(2M^4-3sM^2+2s^2) + Qs^9(M^2-s)M^4(-621sM^6+864s^2M^4-567s^3M^2+108s^4+173M^8) + Q^2s^7M^2(-1395sM^8+2307s^2M^6-1988s^3M^4+621s^4M^2-54s^5+249M^{10}) + Q^3s^5(-1488sM^8+2314s^2M^6-1492s^3M^4+189s^4M^2+249M^{10}) + Q^4s^3(-779sM^6+1379s^2M^4-449s^3M^2+173M^8) + Q^5s(-162sM^4+373s^2M^2+27s^3+54M^6) - Q^6 27(M^2-s))]. \tag{46}$$

This is in agreement with the second terms in (A.24) in KKMS, if we make the replacement $\langle R[^1P_1^{(8)}]\rangle = \pi\langle O[^1P_1^{(8)}]\rangle/18$.

When the heavy quark pair is in the $^3P_0$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{10\pi\alpha_s^3\langle R[^3P_0^{(8)}]\rangle}{M^3 s^2}\frac{1}{Qs^6(Q - M^2P)^4}[9s^{11}M^6(2(M^2 - s)^6 + sM^2(M^2 - s)^4) + 3Qs^9M^4(23M^{12} - 126sM^{10} + 285s^2M^8 - 342s^3M^6 + 228s^4M^4 - 78s^5M^2 + 10s^6) + Q^2s^7M^2(117M^{12} - 612sM^{10} + 1285s^2M^8 - 1358s^3M^6 + 738s^4M^4 - 184s^5M^2 + 14s^6) + Q^3s^5(117M^{12} - 553sM^{10} + 1021s^2M^8 + 2s^6 - 881s^3M^6 + 352s^4M^4 - 54s^5M^2) + Q^5s(18M^8 - 81sM^6 + 88s^2M^4 - 33s^3M^2 + 4s^4) + Q^4s^3(69M^{10} - 292sM^8 + 439s^2M^6 - 275s^3M^4 + 68s^4M^2 - 5s^5) - Q^6(s - 3M^2)^2]. \tag{47}$$

This is in agreement with the second terms in (A.25) in KKMS, if we make the replacement $\langle R[^3P_0^{(8)}]\rangle = \pi\langle O[^3P_0^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^3P_1$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{60\pi\alpha_s^3\langle R[^3P_1^{(8)}]\rangle}{M^3 s^2}\frac{Q}{s^5(Q - M^2P)^4}[s^7M^2(-M^{10} + 5sM^8 - 10s^2M^6 + 11s^3M^4 + 2s^5 - 7s^4M^2) + Qs^5(-2M^{10} + 8sM^8 - 14s^2M^6 + 2s^5 + 17s^3M^4 - 12s^4M^2) + Q^4(-s + 2M^2) + Q^2s^3(-2M^8 + 4sM^6 - 8s^2M^4 + 10s^3M^2 - 5s^4) + Q^3s(-M^6 + 3sM^4 - 5s^2M^2 + 4s^3)]. \tag{48}$$

This is in agreement with the second terms in (A.26) in KKMS, if we make the replacement $\langle R[^3P_1^{(8)}]\rangle = \pi\langle O[^3P_1^{(8)}]\rangle/6$.

When the heavy quark pair is in the $^3P_2$ state we find

$$\frac{\mathrm{d}\Delta\sigma}{\mathrm{d}t} = \frac{20\pi\alpha_s^3\langle R[^3P_2^{(8)}]\rangle}{M^3 s^2}\frac{1}{Qs^6(Q - M^2P)^4}[-12s^{11}M^6(M^2 - s)^6 - 6s^{12}M^8(M^2 - s)^4 + Qs^9M^4(-27M^{12} + 177sM^{10} - 447s^2M^8 + 567s^3M^6 - 378s^4M^4 + 120s^5M^2 - 12s^6) + Q^2s^7M^2(-15M^{12} + 138sM^{10} - 398s^2M^8 + 481s^3M^6 - 255s^4M^4 + 47s^5M^2 + 2s^6) + Q^3s^5(15M^{12} + 5sM^{10} - 89s^2M^8 + 115s^3M^6 - 35s^4M^4 - 12s^5M^2 + 2s^6) + Q^5s(12M^8 - 39sM^6 + 25s^2M^4 - 3s^3M^2 + 4s^4) + Q^4s^3(27M^{10} - 55sM^8 + 37s^2M^6 - 5s^3M^4 + 2s^4M^2 - 5s^5) + Q^6(6sM^2 - s^2 - 6M^4)]. \tag{49}$$

This is in agreement with the second terms in (A.27) in KKMS, if we make the replacement $\langle R[^3P_2^{(8)}]\rangle = \pi\langle O[^3P_2^{(8)}]\rangle/30$.

Again we agree with the KKMS results because, in their method of calculation, the polarized differential cross sections do not require ghost contributions.

III. PHOTON-GLUON PRODUCTION

We also calculate the helicity matrix elements for the reaction

$$\gamma(k_1) + g(k_2, b) + g(k_3, c) \to q(p/2 + q) + \bar{q}(p/2 - q). \tag{50}$$

Here we can compare our results with those in KKMS as well as those in Yuan, Dong, Hao and Chao [8], which we refer to as YDHC and in Ko, Lee and Song [7], which we refer to as KLS.

In this case there is no $t \leftrightarrow u$ symmetry. Also we have to change our choice for the helicities. We use

$$\not\epsilon_1^{\pm} = N[\not k_1\not k_2\not k_3(1 \mp \gamma_5) - \not k_2\not k_3\not k_1(1 \pm \gamma_5) \pm 2k_2\cdot k_3\not k_1\gamma_5] \tag{51a}$$

$$\not\epsilon_2^{\pm} = N[\not k_3\not k_1\not k_2(1 \pm \gamma_5) + \not k_2\not k_1\not k_3(1 \mp \gamma_5) - 2k_1\cdot k_3\not k_2] \tag{51b}$$

$$\not\epsilon_3^{\pm} = N[\not k_1\not k_2\not k_3(1 \pm \gamma_5) + \not k_3\not k_2\not k_1(1 \mp \gamma_5) - 2k_1\cdot k_2\not k_3]. \tag{51c}$$

The squares of the matrix elements now no longer have all the symmetries as in the previous case, so we need the four helicity amplitudes $|M(+,+,+)|^2$, $|M(+,+,-)|^2$, $|M(+,-,+)|^2$ and $|M(-,+,+)|^2$.

When we calculate the differential cross section for

$$\gamma(k_1) + g(k_2, b) \to q(p/2 + q) + \bar{q}(p/2 - q) + g(k_3, b) \tag{52}$$

we have to cross gluon number three. For the unpolarized differential cross section we need the sum of the above terms and for the polarized one we need the difference similar to the three gluon case. Since the sum is over all the polarization states we have to multiply by 2 to include the CP conjugates. Then divide by 32 to average over the initial gluon colors and initial gluon and photon polarizations. Finally we have to divide by $16\pi s^2$.

We begin with the color singlet case. For the $^1S_0$, $^3P_0$, $^3P_1$ and $^3P_2$ states we find

$$|M(+,+,+)|^2 = |M(+,+,-)|^2 = |M(+,-,+)|^2 = |M(-,+,+)|^2 = 0, \tag{53}$$

after summing over gluon colors and final state polarizations.

When the heavy quark pair is in the $^3S_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{54a}$$

$$|M(+,+,-)|^2 = \frac{128g^4e^2\langle R[^3S_1^{(1)}]\rangle}{3\pi M}\frac{M^2s^2(t+u)^2}{(Q-M^2P)^2} \tag{54b}$$

$$|M(+,-,+)|^2 = \frac{128g^4e^2\langle R[^3S_1^{(1)}]\rangle}{3\pi M}\frac{M^2u^2(s+t)^2}{(Q-M^2P)^2} \tag{54c}$$

$$|M(-,+,+)|^2 = \frac{128g^4e^2\langle R[^3S_1^{(1)}]\rangle}{3\pi M}\frac{M^2t^2(u+s)^2}{(Q-M^2P)^2}, \tag{54d}$$

after summing over gluon colors. Our results agree with the upolarized and polarized results in (A.3) in KKMS if we make the replacement $\langle R[^3S_1^{(0)}]\rangle = 16\pi\langle O[^3S_1^{(0)}]\rangle/3$.

When the heavy quark pair is in the $^1P_1$ state we find

$$|M(+,+,+)|^2 = \frac{1024g^4e^2\langle R[^1P_1^{(1)}]\rangle}{\pi M^3}\frac{M^{10}(5Q-M^2P)}{(Q-M^2P)^3} \tag{55a}$$

$$|M(+,+,-)|^2 = \frac{1024g^4e^2\langle R[^1P_1^{(1)}]\rangle}{\pi M^3}\frac{M^2s^2}{(Q-M^2P)^3}[5M^4Q-4PQ-M^6P-2Q(t^2+u^2)] \tag{55b}$$

$$|M(+,-,+)|^2 = \frac{1024g^4e^2\langle R[^1P_1^{(1)}]\rangle}{\pi M^3}\frac{M^2u^2}{(Q-M^2P)^3}[5M^4Q-4PQ-M^6P-2Q(s^2+t^2)] \tag{55c}$$

$$|M(-,+,+)|^2 = \frac{1024g^4e^2\langle R[^1P_1^{(1)}]\rangle}{\pi M^3}\frac{M^2t^2}{(Q-M^2P)^3}[5M^4Q-4PQ-M^6P-2Q(s^2+u^2)], \tag{55d}$$

after summing over gluon colors and final state polarizations.

When we calculate the differential cross sections they agree with the unpolarized and polarized results in (A.4) in KKMS if we make the replacement $\langle R[^1P_1^{(0)}]\rangle = 8\pi\langle O[^1P_1^{(0)}]\rangle/9$.

We now switch to the color octet case. When the heavy quark pair is in the $^1S_0$ state we find

$$|M(+,+,+)|^2 = \frac{96g^4e^2\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{usM^8}{t(Q-M^2P)^2} \tag{56a}$$

$$|M(+,+,-)|^2 = \frac{96g^4e^2\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{uss^4}{t(Q-M^2P)^2} \tag{56b}$$

$$|M(+,-,+)|^2 = \frac{96g^4e^2\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{usu^4}{t(Q-M^2P)^2} \tag{56c}$$

$$|M(-,+,+)|^2 = \frac{96g^4e^2\langle R[^1S_0^{(8)}]\rangle}{\pi M}\frac{ust^4}{t(Q-M^2P)^2}, \tag{56d}$$

after summing over gluon colors. When we calculate the unpolarized differential cross section, it does not agree with the first terms in (A.5) in KKMS. However the polarized differential cross section agrees with the second terms in (A.5) in KKMS if we make the replacement $\langle R[^1S_0^{(8)}]\rangle = 2\pi\langle O[^1S_0^{(8)}]\rangle$. The reason for the different results is again caused by the fact that ghost contributions were not included in the KKMS analytic answers but are included in the KKMS FORTRAN programs. Both the sum and the difference agree with (A1) and (A2) in YDHC. The sum agrees with (A1) in KLS.

When the heavy quark pair is in the $^3S_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{57a}$$

$$|M(+,+,-)|^2 = \frac{320g^4e^2\langle R[^3S_1^{(8)}]\rangle}{3\pi M}\frac{s^2M^2(t+u)^2}{(Q-M^2P)^2} \tag{57b}$$

$$|M(+,-,+)|^2 = \frac{320g^4e^2\langle R[^3S_1^{(8)}]\rangle}{3\pi M}\frac{u^2M^2(t+s)^2}{(Q-M^2P)^2} \tag{57c}$$

$$|M(-,+,+)|^2 = \frac{320g^4e^2\langle R[^3S_1^{(8)}]\rangle}{3\pi M}\frac{t^2M^2(s+u)^2}{(Q-M^2P)^2}, \tag{57d}$$

after summing over gluon colors. The sum and the difference both agree with (A.6) in KKMS if we make the replacement $\langle R[^3S_1^{(8)}]\rangle = 2\pi\langle O[^3S_1^{(8)}]\rangle/3$. They also agree with (A4) and (A5) in YDHC.

When the heavy quark pair is in the $^1P_1$ state we find

$$|M(+,+,+)|^2 = \frac{2560g^4e^2\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{M^{10}(5Q - M^2P)}{(Q - M^2P)^3} \tag{58a}$$

$$|M(+,+,-)|^2 = \frac{2560g^4e^2\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{M^2s^2}{(Q - M^2P)^3}[3M^4Q - M^6P + 2Qs^2] \tag{58b}$$

$$|M(+,-,+)|^2 = \frac{2560g^4e^2\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{M^2u^2}{(Q - M^2P)^3}[3M^4Q - M^6P + 2Qu^2] \tag{58c}$$

$$|M(-,+,+)|^2 = \frac{2560g^4e^2\langle R[^1P_1^{(8)}]\rangle}{\pi M^3}\frac{M^2t^2}{(Q - M^2P)^3}[3M^4Q - M^6P + 2Qt^2], \tag{58d}$$

after summing over gluon colors and final state polarizations. The sum and the difference both agree with (A.7) in KKMS if we make the replacement $\langle R[^1P_1^{(8)}]\rangle = \pi\langle O[^1P_1^{(8)}]\rangle/9$.

When the heavy quark pair is in the $^3P_0$ state we find

$$|M(+,+,+)|^2 = \frac{128g^4e^2\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{9usM^8(M^2 - s)^2}{t(Q - M^2P)^4}[t^2u^2 + 2Q(M^2 - s) + s^2(M^2 - s)^2 + 2sQ + 2s^3(M^2 - s) + s^4] \tag{59a}$$

$$|M(+,+,-)|^2 = \frac{128g^4e^2\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{uss^2(M^2 - s)^2}{t(Q - M^2P)^4}[4t^2u^2(M^2 - s)^2 + 4Qtu(M^2 - s) + Q^2 - 12sQ(M^2 - s)^2 - 18s^2Q(M^2 - s) + 9s^6 + 9s^4(M^2 - s)^2 - 6s^3Q + 18s^5(M^2 - s)] \tag{59b}$$

$$|M(+,-,+)|^2 = \frac{128g^4e^2\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{usu^2(M^2 - u)^2}{t(Q - M^2P)^4}[4t^2s^2(M^2 - u)^2 + 4Qts(M^2 - u) + Q^2 - 12uQ(M^2 - u)^2 - 18u^2Q(M^2 - u) + 9u^6 + 9u^4(M^2 - u)^2 - 6u^3Q + 18u^5(M^2 - u)] \tag{59c}$$

$$|M(-,+,+)|^2 = \frac{128g^4e^2\langle R[^3P_0^{(8)}]\rangle}{\pi M^3}\frac{ust^4(M^2 - t)^2}{t(Q - M^2P)^4}[4(M^2 - t)^4 + 4su(M^2 - t)^2 + s^2u^2 + 28t(M^2 - t)^3 + 14Q(M^2 - t) + 25t^4 + 69t^2(M^2 - t)^2 + 10tQ + 70t^3(M^2 - t)], \tag{59d}$$

after summing over gluon colors and final state polarizations. Only the difference agrees with (A.8) in KKMS if we make the replacement $\langle R[^3P_0^{(8)}]\rangle = 2\pi\langle O[^3P_0^{(8)}]\rangle$. The sum and the difference agree with (A6) and (A7) in YDHC once a typographical error is corrected; the last term in these equations should have been $(t + u)^{-2}$ instead of $(t + s)^{-2}$. The sum agrees with (A2) in KLS.

When the heavy quark pair is in the $^3P_1$ state we find

$$|M(+,+,+)|^2 = 0 \tag{60a}$$

$$|M(+,+,-)|^2 = \frac{1152 g^4 e^2 \langle R[^3P_1^{(8)}]\rangle}{\pi M^3}\frac{s^2(M^2-s)^2}{(Q-M^2P)^4}[s^5(t+u)^2 + s(9t^2u^4 + 20t^3u^3 + 13t^4u^2 + 2t^5u) + s^2(tu^4 + 26t^2u^3 + 38t^3u^2 + 13t^4u + t^5 + u^5) + s^3(6tu^3 + 34t^2u^2 + 22t^3u + 3t^4 - u^4) + s^4(9tu^2 + 13t^2u + 3t^3 - u^3) + t^2u^2(t+u)^3] \tag{60b}$$

$$|M(+,-,+)|^2 = \frac{1152 g^4 e^2 \langle R[^3P_1^{(8)}]\rangle}{\pi M^3}\frac{u^2(M^2-u)^2}{(Q-M^2P)^4}[u^5(s+t)^2 + u(2st^5 + 13s^2t^4 + 20s^3t^3 + 9s^4t^2) + u^2(13st^4 + 38s^2t^3 + 26s^3t^2 + s^4t + s^5 + t^5) + u^3(22st^3 + 34s^2t^2 + 6s^3t - s^4 + 3t^4) + u^4(13st^2 + 9s^2t - s^3 + 3t^3) + s^2t^2(t+s)^3] \tag{60c}$$

$$|M(-,+,+)|^2 = \frac{1152 g^4 e^2 \langle R[^3P_1^{(8)}]\rangle}{\pi M^3}\frac{t^2(M^2-t)^2}{(Q-M^2P)^4}[ts^2u^2(5(u+s)^2 - 2su) - 4s^3u^3(u+s) + t^2((u+s)^5 + 8s^2u^2(u+s)) + s^2u^2(u+s)^3 + t^3(3(u+s)^4 - 2su(s^2+u^2)) + t^5(s^2+u^2) + t^4(3(u+s)^3 - 4su(u+s))], \tag{60d}$$

after summing over gluon colors and final state polarizations. Only the difference agrees with (A.9) in KKMS if we make the replacement $\langle R[^3P_1^{(8)}]\rangle = \pi\langle O[^3P_1^{(8)}]\rangle/4$. The sum and the difference agree with (A8) and (A9), respectively, in YDHC. The sum also agrees with (A3) in KLS once the factor $(s^2-u^2)^2$ is replaced by $(s^2-u^2)^4$.

For $^3P_2$ our results below are obtained after summing over gluon colors and final state polarizations.

$$|M(+,+,+)|^2 = 0 \tag{61a}$$

$$|M(+,+,-)|^2 = \frac{48 g^4 e^2 \langle R[^3P_2^{(8)}]\rangle}{\pi M^3}\frac{s^2u(M^2-s)^2}{Q(Q-M^2P)^4}[(12t^2u^7 + 48t^3u^6 + 72t^4u^5 + 48t^5u^4 + 12t^6u^3) + s(24tu^7 + 96t^2u^6 + 171t^3u^5 + 177t^4u^4 + 105t^5u^3 + 27t^6u^2) + s^2(72tu^6 + 140t^2u^5 + 187t^3u^4 + 200t^4u^3 + 111t^5u^2 + 18t^6u + 12u^7) + s^3(51tu^5 + 59t^2u^4 + 134t^3u^3 + 162t^4u^2 + 63t^5u + 3t^6 + 24u^6) + s^4(-3tu^4 + 26t^2u^3 + 102t^3u^2 + 78t^4u + 9t^5 + 12u^5) + s^5(-3tu^3 + 27t^2u^2 + 39t^3u + 9t^4) + s^6(3tu^2 + 6t^2u + 3t^3)] \tag{61b}$$

$$|M(+,-,+)|^2 = \frac{48 g^4 e^2 \langle R[^3P_2^{(8)}]\rangle}{\pi M^3}\frac{su^2(M^2-u)^2}{Q(Q-M^2P)^4}[(12s^3t^6 + 48s^4t^5 + 72s^5t^4 + 48s^6t^3 + 12s^7t^2) + u(27s^2t^6 + 105s^3t^5 + 177s^4t^4 + 171s^5t^3 + 96s^6t^2 + 24s^7t) + u^2(18st^6 + 111s^2t^5 + 200s^3t^4 + 187s^4t^3 + 140s^5t^2 + 72s^6t + 12s^7) + u^3(3t^6 + 63st^5 + 162s^2t^4 + 134s^3t^3 + 59s^4t^2 + 51s^5t + 24s^6) + u^4(9t^5 + 78st^4 + 102s^2t^3 + 26s^3t^2 - 3s^4t + 12s^5) + u^5(9t^4 + 39st^3 + 27s^2t^2 - 3s^3t) + u^6(3t^3 + 6st^2 + 3s^2t)] \tag{61c}$$

$$|M(-,+,+)|^2 = \frac{48 g^4 e^2 \langle R[^3P_2^{(8)}]\rangle}{\pi M^3}\frac{su(M^2-t)^2}{Q(Q-M^2P)^4}[24Qsu(u^5 + 5su^4 + 10s^2u^3 + 10s^3u^2 + 5s^4u + s^4) + 12Qt(u^6 + 10su^5 + 29s^2u^4 + 40s^3u^3 + 29s^4u^2 + 10s^5u + s^6) + 3Qt^2(16u^5 + 89su^4 + 183s^2u^3 + 183s^3u^2 + 89s^4u + 16s^5) + Qt^3(92u^4 + 367su^3 + 552s^2u^2 + 367s^3u + 92s^4) + t^5(3u^5 + 119su^4 + 334s^2u^3 + 334s^3u^2 + 119s^4u + 3s^5) + t^6(9u^4 + 102su^3 + 182s^2u^2 + 102s^3u + 9s^4) + t^7(9u^3 + 47su^2 + 47s^2u + 9s^3) + t^8(3u^2 + 8su + 3s^2) + 12s^3u^3(u^4 + 4su^3 + 6s^2u^2 + 4s^3u + s^4)]. \tag{61d}$$

We find that only the difference of these results agrees with (A.10) in KKMS if we make the replacement $\langle R[^3P_2^{(8)}]\rangle = 16\pi\langle O[^3P_2^{(8)}]\rangle/15$. The difference also agrees with (A10) and (A11) in YDHC. The sum does not agree with (A4) in KLS or with (A10) in YDHC. The incorrect results in (A.5), (A.8), (A.9) and (A.10) of the KKMS paper are again due to the inadvertent omission of ghost contributions. However the authors claim to have inserted the correct results in their FORTRAN programs.

IV. CONCLUSIONS

We have calculated the gluon-gluon and photon-gluon amplitudes for the production of color singlet and color octet charmonium production. These amplitudes are required for the QCD analysis of charmonium production in polarized and unpolarized hadron-hadron and photon-hadron collisions. Our calculations clarify several inconsistencies in previously published results.

ACKNOWLEDGMENTS

The work of J. S. was partially supported by the National Science Foundation Grant No. PHY-0354776. He would like to thank Professor P. van Baal of the Lorentz Institute, University of Leiden for hospitality and support. He would also like to thank Michael Klasen and Luminita Mihaila for correspondence regarding the formulas in the Appendix of the KKMS paper. The work of M. M. M. was supported by the Dutch Ministry of Education, Culture and Science.

[1] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995).
[2] M. Kraemer, Prog. Part. Nucl. Phys. **47**, 141 (2001).
[3] J-P. Landsberg, Int. J. Mod. Phys. A **21**, 3857 (2006).
[4] G. T. Bodwin, Int. J. Mod. Phys. A **21**, 785 (2006).
[5] R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. **B291**, 731 (1987).
[6] P. Cho and A. K. Leibovich, Phys. Rev. D **53**, 6203 (1996).
[7] P. Ko, J. Lee, and H. S. Song, Phys. Rev. D **54**, 4312 (1996); **60**, 119902(E) (1999).
[8] F. Yuan, H-S. Dong, L-K. Hao, and K. T. Chao, Phys. Rev. D **61**, 114013 (2000).
[9] M. Klasen, B. A. Kniehl, L. N. Mihaila, and M. Steinhauser, Phys. Rev. D **68**, 034017 (2003).
[10] R. Gastmans and T. T. Wu, *The Ubiquitous Photon, Helicity Methods for QED and QCD* (Oxford Science Publishers, Oxford, 1990).
[11] B. Gubernina, J. H. Kuhn, R. D. Peccei, and R. Ruckl, Nucl. Phys. **B174**, 317 (1980).