PHYSICAL REVIEW D **77,** 034006 (2008)

Two chiral nonet model with massless quarks

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(Received 5 July 2007; published 4 February 2008)

We present a detailed study of a linear sigma model containing one chiral nonet transforming under $U(1)_A$ as a quark-antiquark composite and another chiral nonet transforming as a diquark-antidiquark composite (or, equivalently from a symmetry point of view, as a two meson molecule). The model provides an intuitive explanation of a current puzzle in low energy QCD: Recent work has suggested the existence of a lighter than 1 GeV nonet of scalar mesons which behave like four quark composites. On the other hand, the validity of a spontaneously broken chiral symmetric description would suggest that these states be chiral partners of the light pseudoscalar mesons, which are two quark composites. The model solves the problem by starting with the two chiral nonets mentioned and allowing them to mix with each other. The input of physical masses in the SU(3) invariant limit for two scalar octets and an excited pion octet results in a mixing pattern wherein the light scalars have a large four quark content while the light pseudoscalars have a large two quark content. One light isosinglet scalar is exceptionally light. In addition, the pion pion scattering is also studied and the current algebra theorem is verified for massless pions which contain some four quark admixture.

DOI: 10.1103/PhysRevD.77.034006 PACS numbers: 13.75.Lb, 11.15.Pg, 11.80.Et, 12.39.Fe

I. INTRODUCTION

The topic of anomalously light scalar mesons in QCD has become a subject of increasing interest in the last 15 years or so [1–39]. Of course light scalars, especially the light isoscalar called first sigma and now $f_0(600)$, have been discussed for at least 3 times as long, although without general agreement on their actual existence. The difficulty people had previously in accepting the light scalars was largely due to the great success of the simple quark model, in which the lightest scalars are expected to be p-wave quark-antiquark composite states and hence to be in the 1 to 1.5 GeV range, like the other p-wave states. It seems that physicists now believe more in their existence because there have been an increasing number of investigations, using a variety of techniques and models, which suggest that they do exist. Common features in many of these approaches have been the use of unitarity (which no one denies) and some input at low energy from chiral dynamics (which is also considered reasonable).

Of course, the strongly interacting gauge theory QCD has not been "solved" and any possible new features in the low energy region where the effective coupling constant is especially strong raise the hope of improving one's understanding of this basic theory. Perhaps the most fascinating possibility is that the very light scalars contain two quarks and two antiquarks. Variants based on a diquark-antidiquark picture [40] or a meson-meson "molecule" picture [41] have been discussed. In our approach, which is just based on the chiral symmetry properties of the underlying fields, these two possibilities can not be distinguished from each other. Either one may be distinguished from the quark-antiquark field by their $U(1)_A$ transformation property.

A lot of attention has been given to the question of a possible nonet grouping for the light (less than 1 GeV) scalars. The candidates are the already mentioned $f_0(600)$, the Kappa(800–900) (not conclusively established according to [1]), the established $a_0(980)$ and the established $f_0(980)$. It has been pointed out (See for examples [17,18,40]) that a characteristic signature of a four quark content would be an inverted mass ordering, with an almost degenerate $I = 0$, $I = 1$ pair being the heaviest rather than the lightest states when the light quark masses are " turned on." This seems to be the case.

Associating the four quark states with the lightest scalars naturally raises the question of where are the p-wave quark, antiquark scalars. The candidates for the nonzero isospin states are the established $a_0(1450)$ and the established $K_0^*(1430)$. For the $I = 0$ states the established candidates are the $f_0(1370)$, $f_0(1500)$ and the $f_0(1710)$, one of which may be a glueball. There is a slight puzzle since the nonstrange $a_0(1450)$ with a listed mass of 1474 MeV is heavier than the strange $K_0^*(1430)$ with a listed mass of 1414 MeV. In addition some branching ratios are not well predicted by SU(3) invariance. A possible way to overcome this problem [29,33] is to allow mixing between the lighter 4 quark and heavier 2 quark scalar nonets. This feature is incorporated as a basic part of the present paper.

While it is rather difficult to treat low energy QCD dynamically, great success has been obtained at low energies using the underlying chiral symmetry of QCD. We also incorporate this as an aid in getting more information about the system. This feature will be implemented by

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using linear rather than the more usual nonlinear representations for the pseudoscalar and scalar fields. If both four quark and two quark scalars are present, this means that four quark pseudoscalars should also appear in the model. Experimental candidates for the nonzero isospin, higher mass pseudoscalars are the $\pi(1300)$ and the two not yet conclusively established strange states $K(1460)$ and $K(1830)$. The candidates for the higher mass isoscalar pseudoscalars are the $\eta(1295)$, $\eta(1405)$, $\eta(1475)$ and the not conclusively established $\eta(1760)$. It is possible that one or more of these experimental candidates also contain glueball and radial excitation admixtures. At first glance it might seem puzzling that the picture seems to be: light mass two quark and heavy mass four quark states for the pseudoscalars at the same time as light mass four quark and heavy mass two quark states for their "chiral partners," the scalars. We shall make no initial assumption on this matter but let the experimental particle spectrum together with the mixing inherent in the model tell us the answer.

At a technical level it is amusing to note that the $U(1)_A$ transformation properties distinguish the two quark from the four quark fields. Since it is known that the $U(1)_A$ symmetry is badly broken in QCD, this means that we have to model the breaking in some detail. For this purpose we will use an extra term in addition to the usual one. We adopt a counting scheme for selecting the most important terms, out of the many possible ones. We assign a number N equal to the number of underlying quark plus antiquark lines associated with each effective term. Then it seems reasonable to pick up the terms with smallest N values. On this basis the extra term for saturating the $U(1)_A$ anomaly has the same justification as the conventional one.

Clearly, with so many scalar and pseudoscalar fields present, the model is fairly complicated to analyze. At the same time it is widely believed that massless (i.e. zero mass light quarks) QCD is an excellent qualitative approximation. Except for the pseudoscalar Nambu-Goldstone bosons of the theory, the masses of the physical particles made from light quarks are largely due to the spontaneous breakdown of chiral symmetry. We will employ this limit of the theory in the present paper and note that it very much simplifies the analysis. Especially, the characteristic mixing matrix pattern of the two quark and four quark states becomes very clear. The puzzle of opposite two quark vs four quark structures of the scalars and pseudoscalars seems to be neatly solved by the mixing mechanism.

Even though the nonlinear chiral model is more convenient for systematically studying the loop corrections at very low energies, the linear sigma models have a long history of elucidating key features of the strong interactions. Roughly speaking the use of the nonlinear model amounts to integrating out the scalars (although it is technically somewhat more general than that). Certainly for learning about the scalars themselves it is rather convenient to have them present in the Lagrangian to begin with. The most famous example of the linear model is of course the Higgs potential of the standard model. One of the classical triumphs of the nonlinear model is the derivation of the "current algebra" formula for low energy pion scattering. This can be obtained, though in a more complicated way, also in the linear model. We verify this in detail in the present paper. One might wonder, since the pion in the present model has a small (but non negligible) four quark content, whether the current algebra result strictly does hold in the present model. Our result shows that it does hold for the zero pion mass case we are considering here.

The two chiral nonet model was introduced in [31] as a convenient way to study the possibility of mixing between quark- antiquark ($q\bar{q}$) spin zero mesons and two quark- two antiquark ($qq\bar{q}\,\bar{q}$) spin zero mesons. Altogether there are two pseudoscalar and two scalar nonets contained in the model. It was found that, in the zero quark mass limit with just a few explicit chiral invariant terms contained, there was a possibility of a situation in which the lightest pseudoscalars could have zero mass (i.e. be Nambu-Goldstone bosons) and be primarily $q\bar{q}$ type while the next heaviest mesons could be scalars, primarily of $qq\bar{q}\,\bar{q}$ type. Furthermore, the next heaviest mesons could be pseudoscalars of mainly four quark type while the heaviest could be scalars, mainly of two quark type. A treatment [42] of the model with similar chiral invariant terms and several different quark mass terms also found that light scalars with relatively large admixtures of $qq\bar{q}\,\bar{q}$ type states are favored. Actually, the model can be rather complicated since there are twenty-one renormalizable chiral invariant terms which can be made as well as a similar number of renormalizable quark mass type terms which transform as the $(3, 3^*) + (3^*, 3)$ representation of chiral $SU(3)_L \times SU(3)_R$. In [43], the present authors studied the more general version of the model in which all possible chiral invariant, even non renormalizable, terms were included together with the single usual realization of the quark mass term. The same overall picture was found. However, because the method relied on the symmetry properties of the Lagrangian, only the properties of the pseudoscalar states and the strange scalar states could be studied. In the present paper, we shall initiate a much more systematic investigation. We first study precisely how the general results get constrained when a specific choice of invariant interaction terms is made. We introduce a scheme for ordering all the nonderivative terms of the Lagrangian according to their likely importance. This enables us to select a limited number of leading order terms in a meaningful way as well as to provide the framework for possible higher order extensions. At leading order, and with the extra simplification of zero quark masses, all our results were determined analytically, without any need for a numerical fitting procedure. There are essentially only four main input parameters and only one of them has a non negligible experimental error. We do our calculations for all allowable values of this parameter ($m[\pi(1300)]$) and also take into account the small experimental error on another of the three parameters. The results obtained dramatically predict the existence of a very low mass scalar isosinglet state. Especially, the puzzle concerning the coexistence of lighter (mainly) two quark pseudoscalars with lighter (largely) four quark scalars is clearly seen to be solved.

A brief review of the model and the relevant notation is presented in Sec. II. Section III shows the great simplifications obtained by going to massless QCD and also gives our notations appropriate to the flavor SU(3) invariant situation in this limit. General results, valid for any choice of terms in the invariant potential, are also presented in this section. Section IV gives a systematic procedure for deciding which terms are most important in the model. It mainly contains the worked out model using the leading terms in this scheme. A numerical analysis is presented and the masses of the two SU(3) singlet scalar states of the model are predicted. The two and four quark contents for each state of the model are displayed. Sections V, VI, and VII, are devoted to proving, for any choice of invariant potential, the current algebra theorem for the scattering of massless pions. Discussion and conclusions are given in Sec. VIII.

II. BRIEF REVIEW OF MODEL

The fields of our "toy" model consist of a 3×3 matrix chiral nonet field, M which represents $q\bar{q}$ type states as well as a 3×3 matrix chiral nonet field, M' which represents $qq\bar{q}\,\bar{q}$ type states. They have the decompositions into scalar and pseudoscalar pieces:

$$M = S + i\phi, \qquad M' = S' + i\phi'. \tag{1}$$

They behave under "left handed" and "right handed" unitary unimodular (i.e. $SU(3)_L \times SU(3)_R$) transformations as

$$M \rightarrow U_L M U_R^\dagger, \qquad M' \rightarrow U_L M' U_R^\dagger. \tag{2}$$

However, under the $U(1)_A$ transformation which acts at the quark level as $q_{aL} \rightarrow e^{i\nu} q_{aL}$, $q_{aR} \rightarrow e^{-i\nu} q_{aR}$, the two fields behave differently:

$$M \rightarrow e^{2i\nu} M, \qquad M' \rightarrow e^{-4i\nu} M'. \tag{3}$$

We will be interested in the situation where nonzero vacuum values of the diagonal components of S and S' may exist. These will be denoted by

$$\langle S_a^b \rangle = \alpha_a \delta_a^b, \qquad \langle S_a'^b \rangle = \beta_a \delta_a^b. \tag{4}$$

In the isospin invariant limit, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ while in the SU(3) invariant limit, $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2 = \beta_3$. The general Lagrangian density which defines our model is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\partial_\mu M \partial_\mu M^\dagger) - \frac{1}{2}\mathrm{Tr}(\partial_\mu M' \partial_\mu M'^\dagger) - V_0(M, M') - V_{SB}, \tag{5}$$

where $V_0(M, M')$ stands for a general function made from $SU(3)_L \times SU(3)_R$ (but not necessarily $U(1)_A$) invariants formed out of M and M'. The last term, V_{SB}, stands for chiral symmetry breaking terms which transform in the same way as the quark mass terms in the fundamental QCD Lagrangian. In the present paper we shall, in later sections, specialize to the zero quark mass limit by taking $V_{SB} = 0$. Not only does this make the formalism much simpler but it is well known that, due to the spontaneous breakdown of chiral symmetry, the main mechanism of physical hadron mass generation is already accounted for. This is convenient for disentangling the general properties of each multiplet from the uncertainty as to which of the many possible mass type terms in the effective Lagrangian to include. We record the behaviors of the fields under infinitesimal transformations. Let us write the infinitesimal vector $(L + R)$ and axial vector $(L - R)$ transformations of ϕ and S as

$$\delta_V \phi = [E_V, \phi], \qquad \delta_A \phi = -i[E_A, S]_+, \qquad \delta_V S = [E_V, S], \qquad \delta_A S = i[E_A, \phi]_+. \tag{6}$$

Here, unitarity demands that the infinitesimal matrices obey

$$E_V^\dagger = -E_V, \qquad E_A^\dagger = -E_A. \tag{7}$$

If we demand that the transformations be unimodular, so that the $U(1)_A$ transformation is not included (the $U(1)_V$ transformation is trivial for mesons), we should also impose $\mathrm{Tr}(E_A) = 0$. However we will not do this so the effects of $U(1)_A$ will also be included. The transformation properties of the $qq\bar{q}\,\bar{q}$ type fields are:

$$\delta_V \phi' = [E_V, \phi'], \qquad \delta_A \phi' = -i[E_A, S']_+ + 2iS'\,\mathrm{Tr}(E_A), \qquad \delta_V S' = [E_V, S'], \qquad \delta_A S' = i[E_A, \phi']_+ - 2i\phi'\,\mathrm{Tr}(E_A). \tag{8}$$

The extra terms for the axial transformations reflect the different $U(1)_A$ transformation properties of M and M'.

We will employ two complementary approaches to make predictions. One approach will be to study generating equations for tree level vertices. These are like Ward identities and follow for any choice of $V_0(M, M')$ in Eq. (5). These predictions are consistent with but will not give all possible predictions which would arise if one considered, as a second approach, making a specific choice of terms in $V_0(M, M')$.

The method of treatment, as used earlier [44] to discuss the model containing only the field M, is based on two generating equations which reflect the invariance of V_0 under vector and axial vector transformations. Differentiating them once, relates two point vertices (masses) with one point vertices. Differentiating them twice relates three point vertices (trilinear couplings) with masses and so on. Under the infinitesimal vector and axial vector transformations we have

$$\begin{aligned}\delta_V V_0 &= \mathrm{Tr}\left(\frac{\partial V_0}{\partial \phi}\delta_V \phi + \frac{\partial V_0}{\partial S}\delta_V S\right) + (\phi, S) \to (\phi', S') \\ &= 0, \\ \delta_A V_0 &= \mathrm{Tr}\left(\frac{\partial V_0}{\partial \phi}\delta_A \phi + \frac{\partial V_0}{\partial S}\delta_A S\right) + (\phi, S) \to (\phi', S') \\ &= -\mathcal{L}_{\eta},\end{aligned} \tag{9}$$

wherein the nonzero value of the axial variation equation reflects the presence in V_0 of any terms which are not invariant under $\mathrm{U}(1)_\mathrm{A}$; these terms will provide mass to the $\eta'(958)$ meson. In [44], terms of this type were represented by any function of the chiral SU(3), but not $\mathrm{U}(1)_\mathrm{A}$, invariant det(M) plus its Hermitian conjugate. After QCD, 't Hooft found [45] that such a form would arise from instanton effects. If one wishes to model the $\mathrm{U}(1)_\mathrm{A}$ anomaly equation of QCD in the single M model the suggested form [46] is:

$$\mathcal{L}_{\eta} = -c_3\left[\ln\left(\frac{\det M}{\det M^\dagger}\right)\right]^2, \tag{10}$$

where c_3 is a numerical parameter. In the present $M - M'$ model this form is not unique and the most plausible modification [47] is to replace $\ln(\frac{\det M}{\det M^\dagger})$ by

$$\gamma_1 \ln\left(\frac{\det(M)}{\det(M^\dagger)}\right) + (1-\gamma_1)\ln\left(\frac{\mathrm{Tr}(MM'^\dagger)}{\mathrm{Tr}(M'M^\dagger)}\right), \tag{11}$$

where γ_1 is a dimensionless parameter. Using Eqs. (6) and (8) as well as the arbitrariness of the variations E_V and E_A yields the matrix generating equations,

$$\begin{aligned}&\left[\phi, \frac{\partial V_0}{\partial \phi}\right] + \left[S, \frac{\partial V_0}{\partial S}\right] + (\phi, S) \to (\phi', S') = 0, \\ &\left[\phi, \frac{\partial V_0}{\partial S}\right]_+ - \left[S, \frac{\partial V_0}{\partial \phi}\right]_+ + (\phi, S) \to (\phi', S') = 1\left[2\,\mathrm{Tr}\left(\phi'\frac{\partial V_0}{\partial S'} - S'\frac{\partial V_0}{\partial \phi'}\right) - 8c_3 i \ln\left(\frac{\det M}{\det M^\dagger}\right)\right],\end{aligned} \tag{12}$$

where the form of Eq. (10) was used. In addition, the replacement, Eq. (11) should be borne in mind. To get constraints on the particle masses we will differentiate these equations once with respect to each of the four matrix fields: ϕ, ϕ', S, S' and evaluate the equations in the ground state. Thus we also need the "minimum" condition,

$$\left\langle\frac{\partial V_0}{\partial S}\right\rangle + \left\langle\frac{\partial V_{SB}}{\partial S}\right\rangle = 0, \qquad \left\langle\frac{\partial V_0}{\partial S'}\right\rangle + \left\langle\frac{\partial V_{SB}}{\partial S'}\right\rangle = 0. \tag{13}$$

In Ref. [43] we considered the canonical term, $V_{SB} = -2\,\mathrm{Tr}(AS)$ as an effective representation of the fundamental quark mass terms; A is a diagonal matrix with entries proportional to the three quark masses. Next, let us differentiate successively the axial vector generating equation with respect to ϕ and to ϕ'. It is neater to write the results first for the case when fields with different upper and lower tensor indices are involved:

$$\begin{aligned}&(\alpha_a + \alpha_b)\left\langle\frac{\partial^2 V_0}{\partial \phi_b^a \partial \phi_a^b}\right\rangle + (\beta_a + \beta_b)\left\langle\frac{\partial^2 V_0}{\partial \phi'^a{}_b \partial \phi_a^b}\right\rangle = 2(A_a + A_b), \\ &(\alpha_a + \alpha_b)\left\langle\frac{\partial^2 V_0}{\partial \phi'^a{}_b \partial \phi_a^b}\right\rangle + (\beta_a + \beta_b)\left\langle\frac{\partial^2 V_0}{\partial \phi'^a_b \partial \phi'^b{}_a}\right\rangle = 0\end{aligned} \tag{14}$$

Next, let us write the corresponding equations for the case when the upper and lower tensor indices on each field are the same.

$$\begin{aligned}&\alpha_b\left\langle\frac{\partial^2 V_0}{\partial \phi_a^a \partial \phi_b^b}\right\rangle + \beta_b\left\langle\frac{\partial^2 V_0}{\partial \phi_a^a \partial \phi'^b{}_b}\right\rangle \\ &\quad = \sum_g \beta_g\left\langle\frac{\partial^2 V_0}{\partial \phi_a^a \partial \phi'^g{}_g}\right\rangle - \frac{8c_3}{\alpha_a}, \\ &\alpha_b\left\langle\frac{\partial^2 V_0}{\partial \phi'^a_a \partial \phi_b^b}\right\rangle + \beta_b\left\langle\frac{\partial^2 V_0}{\partial \phi'^a{}_a \partial \phi'^b{}_b}\right\rangle \\ &\quad = \sum_g \beta_g\left\langle\frac{\partial^2 V_0}{\partial \phi'^a_a \partial \phi'^g{}_g}\right\rangle.\end{aligned} \tag{15}$$

Note that the axial generating equation provides information on the masses of all the pseudoscalars. Further differentiations will relate a large number of trilinear and quadrilinear coupling constants to the meson masses and to the quark mass coefficients, A_a.

To fully characterize the system we will also require some knowledge of the axial vector and vector currents [44] obtained by Noether's method:

$$\begin{aligned}(J_\mu^{\mathrm{axial}})_a^b &= (\alpha_a + \alpha_b)\partial_\mu \phi_a^b + (\beta_a + \beta_b)\partial_\mu \phi'^b{}_a + \cdots, \\ (J_\mu^{\mathrm{vector}})_a^b &= i(\alpha_a - \alpha_b)\partial_\mu S_a^b + i(\beta_a - \beta_b)\partial_\mu S'^b{}_a + \cdots,\end{aligned} \tag{16}$$

where the dots stand for terms bilinear in the fields.

It will be helpful to briefly review the treatment of the $\pi - \pi'$ system as given in section 4 of Ref. [43]. Introduce the abbreviations

$$x_\pi = \frac{2A_1}{\alpha_1}, \qquad y_\pi = \left\langle \frac{\partial^2 V}{\partial \phi'^1{}_2 \partial \phi'^2{}_1} \right\rangle, \qquad z_\pi = \frac{\beta_1}{\alpha_1}. \tag{17}$$

Here we have introduced the total potential $V = V_0 + V_{SB}$. Substituting $a = 1$, $b = 2$ into both of Eqs. (14) enables us to write the (nondiagonal) matrix of squared π and π' masses as

$$(M_\pi^2) = \begin{bmatrix} x_\pi + z_\pi^2 y_\pi & -z_\pi y_\pi \\ -z_\pi y_\pi & y_\pi \end{bmatrix}. \tag{18}$$

We see that x_π would be the squared pion mass in the single M model and y_π represents the squared mass of the "bare" π'. The transformation between the diagonal fields (say π^+ and π'^+) and the original pion fields is defined as

$$\begin{bmatrix} \pi^+ \\ \pi'^+ \end{bmatrix} = R_\pi^{-1} \begin{bmatrix} \phi_1^2 \\ \phi_1'^2 \end{bmatrix} = \begin{bmatrix} \cos\theta_\pi & -\sin\theta_\pi \\ \sin\theta_\pi & \cos\theta_\pi \end{bmatrix} \begin{bmatrix} \phi_1^2 \\ \phi'^2{}_1 \end{bmatrix}, \tag{19}$$

which also defines the transformation matrix, R. The explicit diagonalization gives an expression for the mixing angle θ_π:

$$\tan(2\theta_\pi) = \frac{-2y_\pi z_\pi}{y_\pi(1 - z_\pi^2) - x_\pi}. \tag{20}$$

The mixing angle, θ_π can be connected to the experimentally known value of the pion decay constant. Substituting the expressions from Eq. (19) for ϕ_1^2 and $\phi_1'^2$ in terms of the physical fields π^+ and π'^+ into Eq. (16) yields

$$\begin{aligned} (J_\mu^{\text{axial}})_1^2 &= F_\pi \partial_\mu \pi^+ + F_{\pi'} \partial_\mu \pi'^+ + \cdots, \\ F_\pi &= (\alpha_1 + \alpha_2)\cos\theta_\pi - (\beta_1 + \beta_2)\sin\theta_\pi, \\ F_{\pi'} &= (\alpha_1 + \alpha_2)\sin\theta_\pi + (\beta_1 + \beta_2)\cos\theta_\pi. \end{aligned} \tag{21}$$

III. SIMPLIFICATION FOR ZERO QUARK MASSES

The zero quark mass limit is gotten by taking $V_{SB} = 0$. We assume that the original $SU(3)_L \times SU(3)_R$ symmetry is spontaneously broken to $SU(3)_V$ rather than some smaller subgroup. The vacuum expectation values of the scalar fields simplify to

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha, \qquad \beta_1 = \beta_2 = \beta_3 = \beta. \tag{22}$$

The mass spectrum also simplifies a lot. When quark masses are included in the isotopic spin invariant approximation there are 16 different masses. However in the zero quark mass limit there are only 8 different masses. These describe the four systems of degenerate SU(3) octet or SU(3) singlet fields:

$$(\hat{\phi}, \hat{\phi}'), \qquad (\phi_0, \phi_0'), \qquad (\hat{S}, \hat{S}'), \qquad (S_0, S_0'). \tag{23}$$

Here the hat stands for the eight members of the appropriate octets. The fields of each system can mix with each other but not with the fields of any other system. In addition to 8 different masses there will be four different mixing angles describing four orthogonal 2×2 matrices. The conventions are the same as in Eq. (19) so that θ_π now describes the mixings of the two pseudoscalar octets. Note that if only isotopic spin invariance were present, the isotopic spin zero fields of each parity would be characterized by a 4×4 mixing matrix with 6 angle parameters (See Eq. (64) of [43] for example). Notice also that the $\pi - \pi'$, $K - K'$ and $\eta_8 - \eta_8'$ mixings, for example, are all described by the same mixing parameter θ_π.

We next discuss the notations for resolving the nonets into SU(3) octets and singlets. Matrix notation is sometimes convenient; we use the convention $\phi_a^b \to \phi_{ab}$. The properly normalized singlet states are

$$\begin{aligned} \phi_0 &= \frac{1}{\sqrt{3}} \mathrm{Tr}(\phi), \qquad \phi_0' = \frac{1}{\sqrt{3}} \mathrm{Tr}(\phi'), \\ S_0 &= \frac{1}{\sqrt{3}} \mathrm{Tr}(S), \qquad S_0' = \frac{1}{\sqrt{3}} \mathrm{Tr}(S'). \end{aligned} \tag{24}$$

Then we have the matrix decompositions:

$$\begin{aligned} \phi &= \hat{\phi} + \frac{1}{\sqrt{3}} \phi_0 1, \qquad \phi' = \hat{\phi}' + \frac{1}{\sqrt{3}} \phi_0' 1, \\ S &= \hat{S} + \frac{1}{\sqrt{3}} S_0 1, \qquad S' = \hat{S}' + \frac{1}{\sqrt{3}} S_0' 1, \end{aligned} \tag{25}$$

wherein $\hat{\phi}$, $\hat{\phi}'$, $\hat{S}$, and $\hat{S}'$ are all 3×3 traceless matrices. The singlet scalar fields may be further decomposed as

$$S_0 = \sqrt{3}\alpha + \tilde{S}_0, \qquad S_0' = \sqrt{3}\beta + \tilde{S}_0'. \tag{26}$$

Here $\tilde{S}_0$ and $\tilde{S}_0'$ are the fluctuation fields around the true ground state of the model.

Setting $x_\pi = 0$, corresponding to zero quark masses, simplifies Eq. (20) for the $\pi - \pi'$ mixing angle to

$$\tan 2\theta_\pi = \frac{-2z_\pi}{1 - z_\pi^2} \equiv \frac{2\tan\theta_\pi}{1 - \tan^2\theta_\pi}. \tag{27}$$

This immediately yields

$$\tan\theta_\pi = -\frac{\beta}{\alpha}. \tag{28}$$

Substituting this into Eq. (21) yields the simple results

$$F_\pi = 2\sqrt{\alpha^2 + \beta^2}, \qquad F_{\pi'} = 0. \tag{29}$$

One may note, for comparison, from Table 4 in [48] that $F_{\pi'}$ and also $F_{K'}$ are not exactly zero in the presence of non zero quark masses, although they are very heavily suppressed. This feature suggests the essential reliability of the zero quark mass limit.

Next consider the pseudoscalar octets, $\hat{\phi}$ and $\hat{\phi}'$ in the model. Because of SU(3) symmetry it is sufficient to give just the two $I = I_3 = 1$ fields, ϕ_1^2 and $\phi_1'^2$. Their mixing matrix, Eq. (18) becomes in the limit of zero quark masses:

$$(M_\pi^2) = y_\pi \begin{bmatrix} z_\pi^2 & -z_\pi \\ -z_\pi & 1 \end{bmatrix} = \left\langle \frac{\partial^2 V_0}{\partial \phi'^2{}_1 \partial \phi'^1{}_2} \right\rangle \begin{bmatrix} \beta^2/\alpha^2 & -\beta/\alpha \\ -\beta/\alpha & 1 \end{bmatrix} \quad (30)$$

It is easy to see that this matrix has zero determinant and to identify the usual (but zero mass) pseudoscalar pion as

$$\pi^+ = \frac{2}{F_\pi}(\alpha \phi_1^2 + \beta \phi'^2{}_1), \quad (31)$$

where $F_\pi = 131$ MeV. The physical massive pion "excitation" is clearly $\pi'^+ = \frac{2}{F_\pi}(-\beta\phi_1^2 + \alpha\phi_1'^2)$ and has a squared mass, $m^2(\pi') = y_\pi(1 + \beta^2/\alpha^2)$. We notice that, just from our general treatment, the $\pi - \pi'$ system can be described by the three parameters α, β and y_π. However, there are only two physical quantities, F_π and $m^2(\pi')$, to compare with. Thus the mixing angle between the usual and the "excited" pseudoscalar octet states is not predicted in general. In order to predict this interesting quantity we have to specify our choice of chiral invariant terms in the potential, V. A similar situation will be seen to hold for trilinear and quadrilinear coupling constants involving the physical pseudoscalars. There are many constraints just from chiral symmetry but a complete (though clearly model dependent) description will depend on the particular choice of terms in the potential.

It is also amusing to look at the $\phi_0 - \phi_0'$ sector in the zero quark mass limit. There is a rather drastic simplification since the introduction of quark masses results in additional mixing with the isoscalar members of the corresponding octets. That requires a six parameter 4×4 transformation matrix rather than the single parameter 2×2 matrix we now will get. Using the formula,

$$\frac{\partial}{\partial \phi_0} = \frac{1}{\sqrt{3}}\left(\frac{\partial}{\partial \phi_1^1} + \frac{\partial}{\partial \phi_2^2} + \frac{\partial}{\partial \phi_3^3}\right), \quad (32)$$

in both of Eqs. (15), we end up with the prediagonal $\phi_0 - \phi_0'$ mass squared matrix:

$$(M_0^2) = \begin{bmatrix} z_0^2 y_0 - \frac{8c_3(2\gamma_1+1)^2}{3\alpha^2} & -z_0 y_0 + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} \\ -z_0 y_0 + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} & y_0 - \frac{8c_3(1-\gamma_1)^2}{3\beta^2} \end{bmatrix}. \quad (33)$$

Here $z_0 = -2\beta/\alpha$ and

$$y_0 = \left\langle \frac{\partial^2 V}{\partial \phi_0' \partial \phi'{}_0} \right\rangle. \quad (34)$$

The mixing angle, θ_0, is defined by the convention:

$$\begin{bmatrix} \phi_{0p} \\ \phi_{0p}' \end{bmatrix} = R_0^{-1} \begin{bmatrix} \phi_0 \\ \phi_0' \end{bmatrix} \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_0' \end{bmatrix}, \quad (35)$$

In the limit where c_3, defined in Eq. (10), vanishes it is seen that the determinant of the mass squared matrix in Eq. (33) vanishes. This is understandable since c_3 multiplies the terms which violate $\mathrm{U(1)_A}$ symmetry and a zero mass singlet pseudoscalar boson must exist since the symmetry is broken spontaneously. In this limit the mixing angle is related to the pseudoscalar octet one by $\tan\theta_0 = -2\tan\theta_\pi$. It should be remarked that the effect of non zero c_3 is actually quite large so the limit where it vanishes is mainly of academic interest.

Of course, the $\mathrm{U(1)_A}$ transformation is relevant in setting up this model since, as seen from Eq. (3), it distinguishes the two quark fields from the four quark fields. We shall consider here, models in which the terms multiplied by c_3 are the only ones which violate $\mathrm{U(1)_A}$ symmetry. In that case the divergence of the axial current in the model exactly mocks up the QCD axial anomaly at tree level. Alternatively, a term like $\det(M)$ + H.c. could be used with similar results; such a term does not however mock up the $\mathrm{U(1)_A}$ anomaly equation.

We have seen that quite a lot of information about the pseudoscalar particle masses and mixings follows just from the axial generating equations, reflecting the spontaneous breakdown of the octet axial symmetries. On the other hand, Eq. (31) of Ref. [43] shows that, in the case where spontaneous breakdown preserves the SU(3) invariance of the vacuum, there will be no such model independent information about the masses and mixings of the scalars. To find that information, one must make models with specific choices of the invariant terms. In preparation we give notations for the scalar mass and transformation matrices, analogous to those we adopted for the pseudoscalars, in the case where quark masses are absent and the vacuum is assumed to be $\mathrm{SU(3)_V}$ invariant. The prediagonal 2×2 matrix for the $I = I_3 = 1$ scalar meson squared masses is denoted (X_a^2) and the mass diagonal fields, a^+ and a'^+ are related to the nondiagonal ones by

$$\begin{bmatrix} a^+ \\ a'^+ \end{bmatrix} = L_a^{-1} \begin{bmatrix} S_1^2 \\ S_1'^2 \end{bmatrix} = \begin{bmatrix} \cos\psi_a & -\sin\psi_a \\ \sin\psi_a & \cos\psi_a \end{bmatrix} \begin{bmatrix} S_1^2 \\ S_1'^2 \end{bmatrix}. \quad (36)$$

This is sufficient to describe the mixing of all the scalar octet particles with corresponding SU(3) quantum numbers. For the $S_0 - S_0'$ mixing, we define the prediagonal squared mass matrix to be (X_0^2) while the mass diagonal fields S_{0p} and S_{0p}' are defined by

$$\begin{bmatrix} S_{0p} \\ S_{0p}' \end{bmatrix} = L_0^{-1} \begin{bmatrix} S_0 \\ S_0' \end{bmatrix} = \begin{bmatrix} \cos\psi_0 & -\sin\psi_0 \\ \sin\psi_0 & \cos\psi_0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_0' \end{bmatrix}. \quad (37)$$

IV. MODEL FOR MASSES AND MIXINGS

As just discussed, it is necessary to make a specific choice of terms in the $SU(3)_L \times SU(3)_R$ invariant potential V_0 in order to be able to predict all physical properties of the system. This is a non trivial issue since, for example, if we restrict V_0 to be renormalizable, there are twenty-one terms [43] with this symmetry. We will adopt two criteria for which terms to include. First we list the six $SU(3)_L \times SU(3)_R$ invariant terms which satisfy these criteria and seem the most reasonable for an initial treatment:

$$V_0 = -c_2 \mathrm{Tr}(MM^\dagger) + c_4^a \mathrm{Tr}(MM^\dagger MM^\dagger) + d_2 \mathrm{Tr}(M'M'^\dagger) + e_3^a(\epsilon_{abc}\epsilon^{def} M_d^a M_e^b M'^c{}_f + \mathrm{H.c.}) + c_3\left[\gamma_1 \ln\left(\frac{\det M}{\det M^\dagger}\right) + (1-\gamma_1)\ln\left(\frac{\mathrm{Tr}(MM'^\dagger)}{\mathrm{Tr}(M'M^\dagger)}\right)\right]^2. \tag{38}$$

All the terms except the last two have been chosen to also possess the $U(1)_A$ invariance. Those terms are clearly nonrenormalizable and violate $U(1)_A$ invariance in a special way. They have, as previously discussed [see Eq. (11)], the correct $U(1)_A$ property so that the resulting Lagrangian can exactly mock up the $U(1)_A$ anomaly of QCD. Of course, we are using the effective Lagrangian at tree level and renormalizability is not an issue at this level. Renormalizable terms of the instanton determinant type and the type $\mathrm{Tr}(MM'^\dagger)$ + H.c. could be used instead with not much change in the result. However, the role that the $U(1)_A$ transformation is playing in distinguishing "four quark" from "two quark" effective fields suggests that we try to reproduce as much as possible of the behavior of QCD under axial $U(1)_A$. The ln terms chosen also have the convenient feature that they confine the $U(1)_A$ violating effects to the SU(3) singlet pseudoscalar sector of the model. The first four terms were chosen from the 12 renormalizable and $U(1)_A$ invariant ones in the formula, Eq. (A1) of [43] (please see also Appendix A of the present paper) by imposing the criterion that effective vertices describing the smallest numbers of quarks plus antiquarks be retained. This quantity, representing the total number of fermion lines at each effective vertex can be written as,

$$N = 2n + 4n', \tag{39}$$

where n is the number of times M or $M^\dagger$ appears in each term while n' is the number of times M' or $M'^\dagger$ appears in each term. Thus, the c_2 term has $N = 4$ while the c_4^a, d_2, and e_3^a terms each have $N = 8$. For simplicity, we have neglected the $N = 8$ term, $c_4^b[\mathrm{Tr}(MM^\dagger)]^2$ which is suppressed, in the single M model, by the quark line rule. It may be noted that the quantities $\det(M)$ and $\mathrm{Tr}(MM'^\dagger)$ which enter into those two terms which saturate the $U(1)_A$ anomaly have $N = 6$. On the other hand, the terms in

$$e_4^a \mathrm{Tr}(MM^\dagger M'M'^\dagger) + e_4^b \mathrm{Tr}(MM'^\dagger M'M^\dagger) \tag{40}$$

each represent 12 quarks plus antiquarks at the same vertex and will not be included at the present stage. Similarly, the term $d_4^a \mathrm{Tr}(M'M'^\dagger M'M'^\dagger)$ representing 16 quarks and antiquarks will not be included. In the future, $U(1)_A$ invariant terms with higher values of N may be used to systematically improve the approximation as well as $U(1)_A$ violating operators with higher values of N which may be inserted into an obvious generalization of Eq. (11). The minimum equations for this potential are:

$$\left\langle \frac{\partial V_0}{\partial S_a^a} \right\rangle = 2\alpha(-c_2 + 2c_4^a\alpha^2 + 4e_3^a\beta) = 0, \tag{41}$$

$$\left\langle \frac{\partial V_0}{\partial S_a'^a} \right\rangle = 2(d_2\beta + 2e_3^a\alpha^2) = 0. \tag{42}$$

Notice that α is an overall factor in Eq. (41) so that, in addition to the physical spontaneous breakdown solution where $\alpha \neq 0$ there is a solution with $\alpha = 0$. On the other hand, β is not an overall factor of Eq. (42) and it is easy to see that β is necessarily nonzero in the physical situation where α is nonzero. The minimum equations clearly eliminate two parameters from the model.

Next, we shall give the matrix elements of the four squared mass mixing matrices based on the use of the specific potential of Eq. (38). First consider the matrix describing any of the eight degenerate 0^- quark-antiquark fields mixing with their corresponding four quark partners. Without using the minimum equations, one obtains:

$$(M_\pi^2) = \begin{bmatrix} 2(-c_2 + 2c_4^a\alpha^2 + 2e_3^a\beta) & 4e_3^a\alpha \\ 4e_3^a\alpha & 2d_2 \end{bmatrix}. \tag{43}$$

This corresponds to the general form given in Eq. (33) when we identify, $y_\pi = 2d_2$ and $z_\pi = -2\alpha e_3^a/d_2$. Note that $z_\pi \equiv \beta/\alpha$.

The matrix describing the mixing of the two pseudoscalar singlets is similarly written as

$$(M_0^2) = \begin{bmatrix} -2(c_2 - 2c_4^a\alpha^2 + 4e_3^a\beta) - \frac{8c_3(2\gamma_1+1)^2}{3\alpha^2} & -8e_3^a\alpha + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} \\ -8e_3^a\alpha + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} & 2d_2 - \frac{8c_3(1-\gamma_1)^2}{3\beta^2} \end{bmatrix}. \tag{44}$$

This corresponds to the general form given in Eq. (30) when we identify, $y_0 = 2d_2$ and $z_0 = -2\beta/\alpha = 4e_3^a\alpha/d_2$.

For the mixing matrix of the octet scalars, the specific potential of Eq. (38) directly gives

$$(X_a^2) = \begin{bmatrix} 2(-c_2 + 6c_4^a\alpha^2 - 2e_3^a\beta) & -4\alpha e_3^a \\ -4\alpha e_3^a & 2d_2 \end{bmatrix}. \quad (45)$$

Finally the squared mass mixing matrix for the singlet scalars is similarly obtained as

$$(X_0^2) = \begin{bmatrix} 2(-c_2 + 6c_4^a\alpha^2 + 4e_3^a\beta) & 8\alpha e_3^a \\ 8\alpha e_3^a & 2d_2 \end{bmatrix}. \quad (46)$$

Now let us consider the comparison of this model with experiment. To start with there are 8 parameters (α, β, c_2, d_2, c_4^a, e_3^a, c_3 and γ_1. These can be reduced to six by use of the two minimum equations just given. We note that the parameters c_3 and γ_1), associated with modeling the $U(1)_A$ anomaly, do not contribute to either the minimum equations or to the mass matrices of the particles which are not 0^- singlets. Thus it is convenient to first determine the other four independent parameters. As the corresponding four experimental inputs [1] we take the nonstrange quantities:

$$\begin{aligned} m(0^+\text{octet}) &= m[a_0(980)] = 984.7 \pm 1.2\ \text{MeV} \\ m(0^+\text{octet}') &= m[a_0(1450)] = 1474 \pm 19\ \text{MeV} \\ m(0^-\text{octet}') &= m[\pi(1300)] = 1300 \pm 100\ \text{MeV} \\ F_\pi &= 131\ \text{MeV} \end{aligned} \quad (47)$$

Evidently, a large experimental uncertainty appears in the mass of $\pi(1300)$; we shall initially take the other masses as fixed at their central values and vary this mass in the indicated range. As shown in Eq. (B1) in Appendix B, it is straightforward to determine the four independent parameters in terms of these masses. There is a complication which must be taken into account; from studying the predicted masses of the 0^+ SU(3) singlet states one finds that the positivity of the eigenvalues of their squared mass matrix, Eq. (46) is only satisfied when

$$m[\pi(1300)] < 1302\ \text{MeV}. \quad (48)$$

Further restrictions on the allowed range of $m[\pi(1300)]$ will arise when we calculate the masses of the 0^- SU(3) singlet states. Before that we mention the two predicted masses for the 0^+ SU(3) singlet states; as $m[\pi(1300)]$ varies from 1200 to 1300 MeV,

$$\begin{aligned} m(0^+\text{singlet}) &= 510 \rightarrow 28(410)\ \text{MeV}, \\ m(0^+\text{singlet}') &= 1506 \rightarrow 1555(1520)\ \text{MeV}. \end{aligned} \quad (49)$$

The predictions in parentheses correspond to the likely additional constraints from the positivity of the 0^- SU(3) singlet states. Plots are shown in Fig. 1.

Clearly, the most dramatic feature is the very low mass of the lighter SU(3) singlet scalar meson. Of course, one expects the addition of quark mass type terms to modify the details somewhat. On the other hand, there are a number of allowed different quark mass terms so it is notable that the characteristic very light mass scalar exists apart from the ambiguity in choice of the quark mass terms.

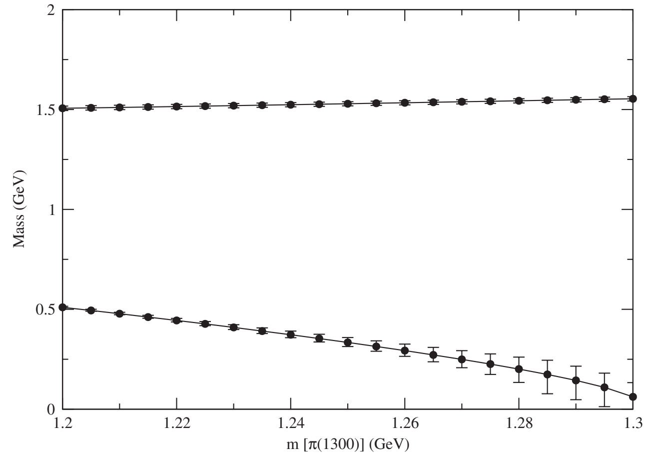

FIG. 1. The predictions for the masses of the two SU(3) singlet scalars vs $m[\pi(1300)]$. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

The four independent parameters which appear in the Lagrangian (c_2, d_2, c_4^a, e_3^a) are shown, as functions of $m[\pi(1300)]$, in Fig. 2. The vacuum expectation values α and β of the two and four quark scalar fields are similarly shown in Fig. 3. It is seen that β and α are each insensitive to varying $m[\pi(1300)]$ and their ratio is about 0.40.

To calculate the masses of the SU(3) singlet pseudoscalars we must diagonalize Eq. (33) with the specific choices of parameters $y_0 = 2d_2$ and $z_0 = 4e_3^a\alpha/d_2$ corresponding to the potential of Eq. (38). This enables us to fit in principle, for any choice of $m[\pi(1300)]$, the two parameters c_3 and γ_1 in terms of the experimental masses of $\eta(958)$ and one of the candidates $\eta(1295)$, $\eta(1405)$, $\eta(1475)$ and $\eta(1760)$. The specific formulas are given as Eqs. (B2) and (B3) in Appendix B. However, as mentioned above, the positivity of the eigenvalues of the matrix (M_0^2) imposes additional constraints on the choice of $m[\pi(1300)]$ in Eq. (48). This appears in solving for γ_1 using the quadratic Eq. (B2) and requiring its discriminant to be positive. In Fig. 4, the discriminants are shown as functions of $m[\pi(1300)]$ for each of the four possible candidates for the heavier 0^- SU(3) singlet. This clearly shows that the two lowest mass candidates have negative discriminants and can be ruled out according to our criterion. The perhaps most likely candidate $\eta(1475)$ [this case will be denoted scenario 1] has a positive discriminant for $m[\pi(1300)]$ less than about 1.23 GeV. This leads to the modified allowed ranges for the 0^+ singlet states, shown in parentheses in Eq. (49). There is no restriction on the heaviest candidate, $\eta(1760)$ [this case will be denoted scenario 2].

Since Eq. (B2) is a quadratic equation for γ_1, one expects that there may be two physical solutions for γ_1. This turns out to be the case. In Fig. 5 we show plots of γ_1

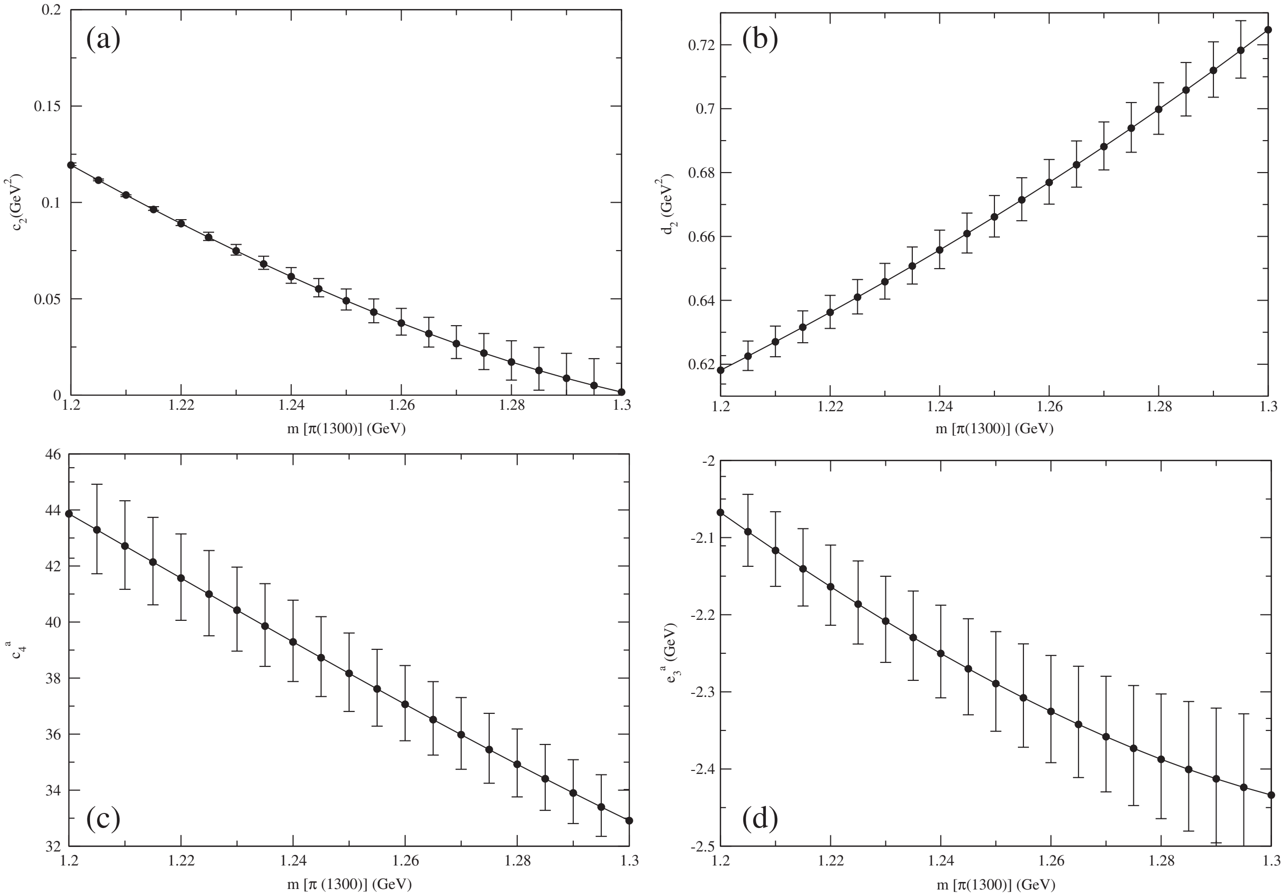

FIG. 2. Starting from the upper left and proceeding clockwise: c_2 vs $m[\pi(1300)]$, d_2 vs $m[\pi(1300)]$, e_3^a vs $m[\pi(1300)]$ and c_4^a vs $m[\pi(1300)]$. The range of $m[\pi(1300)]$ corresponds to the restrictions imposed by the positivity of the scalar SU(3) singlet masses. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

as a function of $m[\pi(1300)]$ for each of the scenarios mentioned above. The quantity c_3 is given in Eq. (B3) and is seen in Fig. 6 to be single valued in its dependence on $m[\pi(1300)]$.

It is very interesting to see what the model has to say about the four quark percentages of the particles it describes. The percentages for the pion, the lighter 0^+ singlet and the $a_0(980)$ are displayed in Fig. 7 as functions of the

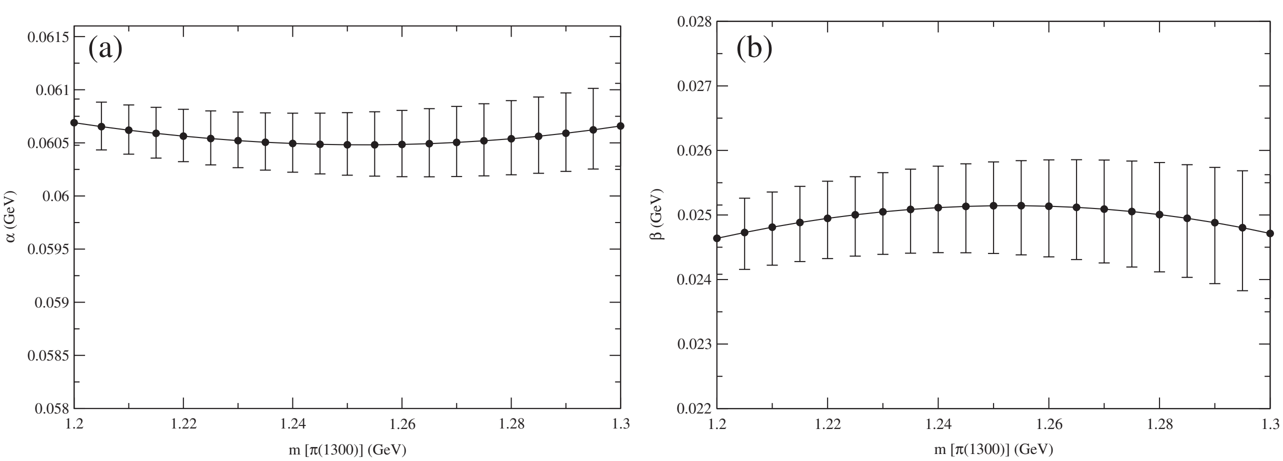

FIG. 3. Dependences of the two quark vacuum value α (left) and the four quark vacuum value β (right) on the choice of $m[\pi(1300)]$. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

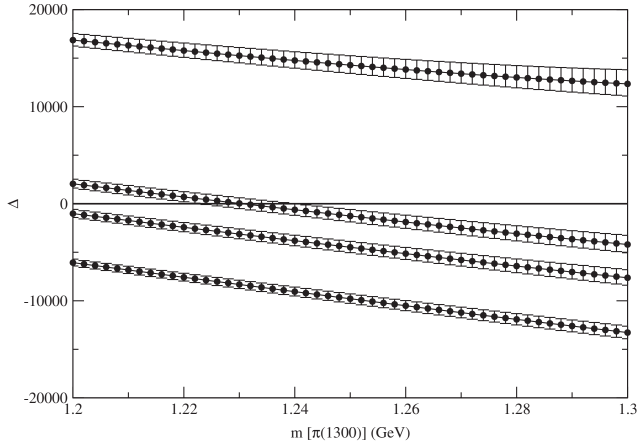

FIG. 4. The discriminant for Eq. (B2) vs $m[\pi(1300)]$. The curves from bottom to top, respectively, represent the choices for the heavier 0^- SU(3) singlet to be $\eta(1295)$, $\eta(1405)$, $\eta(1475)$, and $\eta(1760)$. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

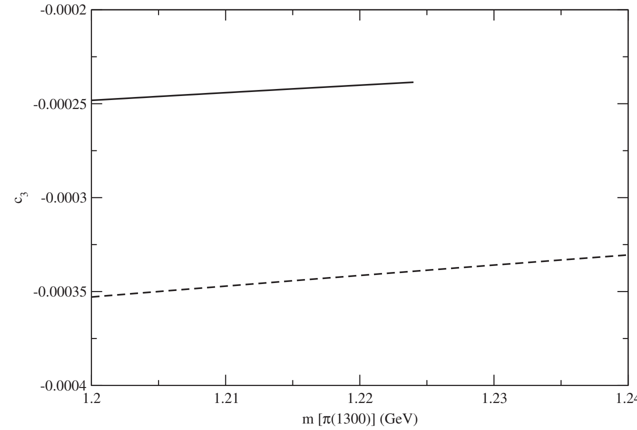

FIG. 6. c_3 in units of GeV^4 vs $m[\pi(1300)]$. The upper curve corresponds to the scenario where the heavier 0^- SU(3) singlet is identified with the $\eta(1475)$ while the lower curve corresponds to $\eta(1760)$ as the heavier 0^- SU(3) singlet.

precise value of the input parameter $m[\pi(1300)]$. The pion four quark content (equal to $100\sin^2\theta_\pi$) is seen to be about 17%. Of course the heavier pion would have about an 83% four quark content. On the other hand, the octet scalar states present a reversed picture: the $a_0(980)$ has a large four quark content while the $a_0(1450)$ has a smaller four quark content. The very light and the rather heavy 0^+ singlets are about maximally mixed, having roughly equal contributions from the 4 quark and 2 quark components.

In Fig. 8 the four quark percentages of the 0^- SU(3) singlets are shown for both scenarios. The perhaps more plausible scenario takes $\eta(1475)$ as the heavy 0^- singlet state. In this case we see that for the solution with smaller γ_1, the four quark content of the familiar $\eta(958)$ is about 25% while for the solution with larger γ_1, the four quark content of $\eta(958)$ is about 55%. Thus the smaller γ_1 solution seems more plausible physically. In the case where the $\eta(1760)$ is identified as the heavier partner of the $\eta(958)$ the smaller γ_1 solution yields an $\eta(958)$ with a four quark content of about 7% while the larger γ_1 solution yields an $\eta(958)$ with a four quark content of about 82%.

Values of all the model parameters as well as numerical values of the mixing matrices, for a typical choice of $m[\pi(1300)]$, are listed at the end of Appendix B.

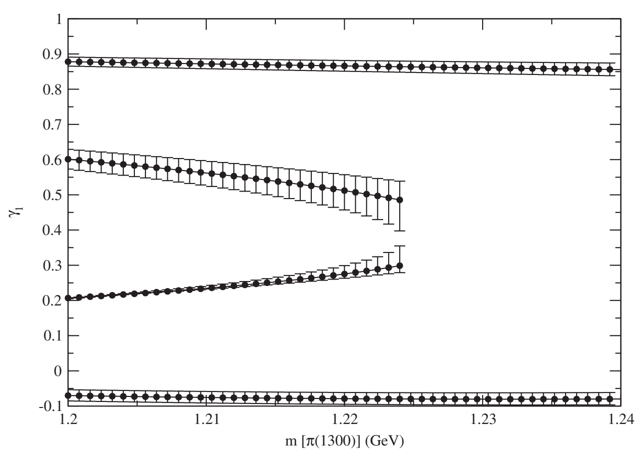

FIG. 5. γ_1 vs $m[\pi(1300)]$. The top and bottom curves correspond to choosing the $\eta(1760)$ as the heavier 0^- SU(3) singlet while the middle two curves correspond to choosing the $\eta(1475)$ as the heavier 0^- SU(3) singlet. Note that for each scenario, the two curves are associated with different solutions of the quadratic Eq. (B2) for γ_1. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

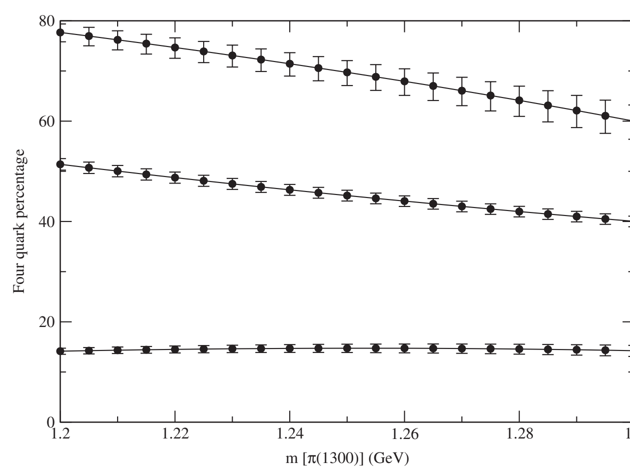

FIG. 7. Plot of the four quark percentages of various particles in the model as functions of the undetermined input parameter, $m[\pi(1300)]$. Starting from the bottom and going up, the curves, respectively, show the four quark percentages of the pion, the 0^+ singlet, and the $a_0(980)$. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

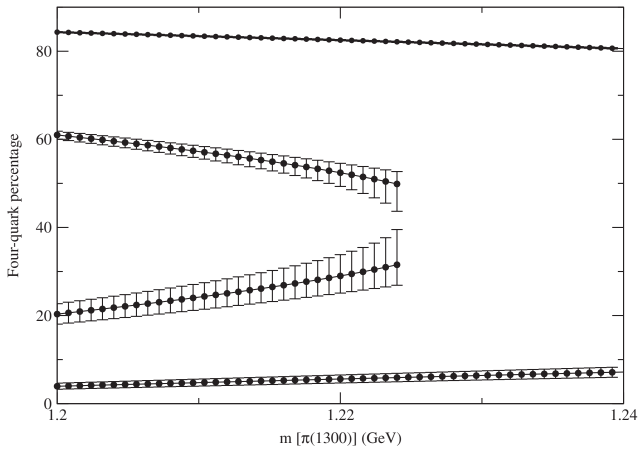

FIG. 8. Plot of the four quark percentages of the $\eta(958)$ as functions of the undetermined input parameter, $m[\pi(1300)]$ for two scenarios. The top and bottom curves correspond to choosing the $\eta(1760)$ as the heavier 0^- SU(3) singlet while the middle two curves correspond to choosing the $\eta(1475)$ as the heavier 0^- SU(3) singlet. Note that for each scenario, the two curves are associated with different solutions of the quadratic equation (B2) for γ_1. The error bars give the effect of the uncertainty in the $a_0(1450)$ mass.

V. THREE POINT VERTICES

The three point vertices are useful for calculating the widths of the various mesons and also for the calculation of meson-meson scattering. These can be calculated for a specific model, like the one with the choice of terms given in the previous section, by straightforward differentiation. However, one may also obtain model independent (in the sense of being independent of the choice of invariant terms in V_0) information about these from the generating equation. We shall do that here, specializing to the scalar-pseudoscalar-pseudoscalar vertices needed for pion pion scattering. These are obtained by successively differentiating the two equations in Eq. (12) with respect to one scalar field and one pseudoscalar field. First we introduce the notations:

$$\begin{aligned}
r_1 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1^2 \partial S_0} \right\rangle & q_1 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1^2 \partial S_8} \right\rangle \\
r_2 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1^2 \partial S_0'} \right\rangle & q_2 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1^2 \partial S'_8} \right\rangle \\
r_3 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1'^2 \partial S_0} \right\rangle & q_3 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1'^2 \partial S_8} \right\rangle \\
r_4 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1'^2 \partial S'_0} \right\rangle & q_4 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2^1 \partial\phi_1'^2 \partial S'_8} \right\rangle \\
r_5 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2'^1 \partial\phi'^2{}_1 \partial S_0} \right\rangle & q_5 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2'^1 \partial\phi'^2{}_1 \partial S_8} \right\rangle \\
r_6 &= \left\langle \frac{\partial^3 V_0}{\partial\phi_2'^1 \partial\phi'^2{}_1 \partial S'_0} \right\rangle & q_6 &= \left\langle \frac{\partial^3 V_0}{\partial\phi'^1{}_2 \partial\phi'^2{}_1 \partial S'_8} \right\rangle.
\end{aligned} \tag{50}$$

Note that S_0 was defined in Eq. (24) while S_8, for example, is the isoscalar member of the SU(3) octet defined as

$$S_8 = \frac{1}{\sqrt{6}}(S_1^1 + S_2^2 - 2S_3^3). \tag{51}$$

Now using the generating equations as just discussed, we obtain the following relations connecting the trilinear coupling constants r_i with corresponding mass squared matrices for the $S_0 - S_0'$ and the $\pi - \pi'$ systems.

$$\begin{aligned}
\alpha r_1 + \beta r_3 &= \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial S_0^2} \right\rangle - \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2^1 \partial\phi_1^2} \right\rangle \\
\alpha r_2 + \beta r_4 &= \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial S_0 \partial S'_0} \right\rangle - \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2^1 \partial\phi_1'^2} \right\rangle \\
\alpha r_3 + \beta r_5 &= \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial S_0 \partial S'_0} \right\rangle - \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2^1 \partial\phi_1'^2} \right\rangle \\
\alpha r_4 + \beta r_6 &= \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial S'^2{}_0} \right\rangle - \frac{1}{\sqrt{3}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2'^1 \partial\phi'^2{}_1} \right\rangle
\end{aligned} \tag{52}$$

Similar equations are obtained for the q_i trilinear couplings and the mass squared matrices for the $S_8 - S_8'$ systems:

$$\begin{aligned}
\alpha q_1 + \beta q_3 &= \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial S_8^2} \right\rangle - \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2 \partial\phi_1^2} \right\rangle \\
\alpha q_2 + \beta q_4 &= \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial S_8 \partial S'_8} \right\rangle - \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2^1 \partial\phi'^2{}_1} \right\rangle \\
\alpha q_3 + \beta q_5 &= \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial S_8 \partial S'_8} \right\rangle - \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2^1 \partial\phi'^2{}_1} \right\rangle \\
\alpha q_4 + \beta q_6 &= \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial S'^2{}_8} \right\rangle - \frac{1}{\sqrt{6}}\left\langle \frac{\partial^2 V_0}{\partial\phi_2'^1 \partial\phi'^2{}_1} \right\rangle
\end{aligned} \tag{53}$$

Equations (52) and (53) relate eight different linear combinations of the three point vertices to two point vertices for the fields of pure $q\bar{q}$ and pure $qq\bar{q}\bar{q}$ types. Since there are 12 *a priori* unknown three point vertices according to Eq. (50), it is clear that there is, in general, not enough information available to determine all the three point vertices in terms of the two point ones. However, we will see that the available relations are sufficient to prove the desired low energy theorem. To relate the quantities in Eqs. (52) and (53) to quantities pertaining to mass eigenstates we introduce an index notation to distinguish unprimed from primed fields; for example:

$$\phi_1^2 = (\phi_1^2)_1, \qquad \phi_1'^2 = (\phi_1^2)_2. \tag{54}$$

With this notation, which we apply to all fields of the model, the coupling constant of the Goldstone boson pions to the mass diagonal SU(3) singlet scalars may be compactly written as

$$g_{0D} = \left\langle \frac{\partial^3 V}{\partial \pi^+ \partial \pi^- \partial (S_{0p})_D} \right\rangle = \sum_{A,B,C} (R_\pi)_{A1}(R_\pi)_{B1}(L_0)_{CD} \left\langle \frac{\partial^3 V}{\partial (\phi_1^2)_A \partial (\phi_2^1)_B \partial (S_0)_C} \right\rangle. \tag{55}$$

The transformation matrix elements, $(R_\pi)_{AB}$ and $(L_0)_{AB}$ may be read from Eqs. (19) and (37). Note that the capital Latin subscripts take on the values 1 and 2 as in Eq. (54) above. There is a similar equation involving the $S_8 - S_8'$ scalars which yields the physical coupling constant of two Goldstone pions with S_8, g_{8D}:

$$g_{8D} = \left\langle \frac{\partial^3 V}{\partial \pi^+ \partial \pi^- \partial (S_{8p})_D} \right\rangle = \sum_{A,B,C} (R_\pi)_{A1}(R_\pi)_{B1}(L_a)_{CD} \left\langle \frac{\partial^3 V}{\partial (\phi_1^2)_A \partial (\phi_2^1)_B \partial (S_8)_C} \right\rangle. \tag{56}$$

Here, L is the transformation matrix defined in Eq. (36). Using the compact form of Eq. (55), one may compactly express the comparison of Eq. (52) with Eq. (50) as:

$$\frac{\sqrt{3}F_\pi}{2} \sum_B (R_\pi^{-1})_{1B} \left\langle \frac{\partial^3 V_0}{\partial (\phi_1^2)_A \partial (\phi_2^1)_B \partial (S_0)_H} \right\rangle = (X_0^2)_{AH} - (M_\pi^2)_{AH}. \tag{57}$$

(M_π^2) is given in Eq. (30) and (X_0^2) is the model independent version of Eq. (46). Similarly,

$$\frac{\sqrt{6}F_\pi}{2} \sum_B (R_\pi^{-1})_{1B} \left\langle \frac{\partial^3 V_0}{\partial (\phi_1^2)_A \partial (\phi_2^1)_B \partial (S_8)_H} \right\rangle = (X_a^2)_{AH} - (M_\pi^2)_{AH}. \tag{58}$$

Here (X_a^2) is the model independent version of Eq. (45). Note that according to our conventions the nondiagonal and diagonal (hatted) squared mass matrices are related as

$$\begin{aligned} \sum_{B,C} (R_\pi^{-1})_{AB}(M_\pi^2)_{BC}(R_\pi)_{CD} &= (\hat{M}_\pi^2)_{AD}, \\ \sum_{B,C} (R_0^{-1})_{AB}(M_0^2)_{BC}(R_0)_{CD} &= (\hat{M}_0^2)_{AD}, \\ \sum_{B,C} (L_a^{-1})_{AB}(X_a^2)_{BC}(L_a)_{CD} &= (\hat{X}_a^2)_{AD}, \\ \sum_{B,C} (L_0^{-1})_{AB}(X_0^2)_{BC}(L_0)_{CD} &= (\hat{X}_0^2)_{AD}, \end{aligned} \tag{59}$$

VI. LOW ENERGY PION SCATTERING

There are two reasons for next discussing the pi-pi scattering in this model. First, since the isosinglet scalar resonances above are being considered at tree level, one expects, as can be seen in the single M model also discussed in [31] and at the two flavor level in [6], that unitarity corrections for the scattering amplitudes will alter their masses and widths. Second, since the pion looks unconventional in this model (having a non-neglegible four quark component) one might worry that the fairly precise "current algebra" formula for the near to threshold scattering amplitude might acquire unacceptably large corrections.

Of course, for computing the near threshold pion pion scattering, it is well known that the use of a nonlinear sigma model is more convenient. However, we are also interested in unitarizing the model in the resonance region where the nonlinear model, which can be obtained by integrating out the resonances, is clearly not applicable.

The invariant pion pion scattering amplitude for $\pi_i(p_1) + \pi_j(p_2) \to \pi_k(p_3) + \pi_l(p_4)$ is decomposed as

$$\delta_{ij}\delta_{kl}A(s,t,u) + \delta_{ik}\delta_{jl}A(t,s,u) + \delta_{il}\delta_{jk}A(u,t,s), \tag{60}$$

where s, t, and u are the usual Mandelstam variables. Note that the phase of the above amplitude simply corresponds to taking the matrix element of the Lagrangian density for a four pion contact interaction. The $I = 0$, $I = 1$, and $I = 2$ amplitudes correspond to the projections:

$$\begin{aligned} T^0(s,t,u) &= 3A(s,t,u) + A(t,s,u) + A(u,t,s), \\ T^1(s,t,u) &= A(t,s,u) - A(u,t,s), \\ T^2(s,t,u) &= A(t,s,u) + A(u,t,s). \end{aligned} \tag{61}$$

It is straightforward to calculate $A(s,t,u)$ using the three point vertices for two massless pions coupling to a physical scalar [See Eqs. (55) and (56)] as well as the four point coupling constant, g for four massless pions:

$$g = \left\langle \frac{\partial^4 V_0}{\partial \pi^+ \partial \pi^- \partial \pi^+ \partial \pi^-} \right\rangle. \tag{62}$$

The result is simply

$$A(s,t,u) = -\frac{g}{2} + \sum_D \left(\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD} - s} + \frac{g_{0D}^2}{(\hat{X}_0^2)_{DD} - s} \right). \tag{63}$$

Note that the sum goes over the two SU(3) singlet scalars as well as the two isosinglet scalars belonging to SU(3) octets. We are presently interested in the threshold region (near $s = 0$ for massless pions) so we expand this formula to first order in s:

$$A(s,t,u) \approx -\frac{g}{2} + \left(\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD}} + \frac{g_{0D}^2}{(\hat{X}_0^2)_{DD}} \right) + s \left(\frac{g_{8D}^2}{[(\hat{X}_a^2)_{DD}]^2} + \frac{g_{0D}^2}{[(\hat{X}_0^2)_{DD}]^2} \right). \tag{64}$$

In this equation the summation over D has not been explicitly written and the summation over repeated indices is to be assumed; note that the quantity $(\hat{X}_a^2)_{DD}$, for example, is a single number indexed by D. Observe that the four point vertex does not contribute to the terms linear in s. Let us then evaluate the s term first. Begin by substituting Eq. (57) into Eq. (55) and noticing that the term $(M_\pi^2)_{AH}$ makes zero contribution since that piece can be manipulated, using Eq. (59), to be proportional to the zero masses of the physical Goldstone bosons. The physical trilinear coupling constant is next obtained as $g_{0D} = \frac{2}{\sqrt{3}F_\pi}(R_\pi)_{A1} \times (X_0^2)_{AH}(L_0)_{HD}$. Then the quantity appearing in Eq. (64) can be evaluated as

$$\begin{aligned}\frac{g_{0D}^2}{[(\hat{X}_0^2)_{DD}]^2} &= \frac{4}{3F_\pi^2}(R_\pi)_{A1}(X_0^2)_{AH}(L_0)_{HD}\frac{1}{[(\hat{X}_0^2)_{DD}]^2}(R_\pi)_{C1}(X_0^2)_{CK}(L_0)_{KD}\\ &= \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1G}(L_0)_{GE}(L_0^{-1})_{EA}(X_0^2)_{AH}(L_0)_{HD}\frac{1}{[(\hat{X}_0^2)_{DD}]^2}(L_0^{-1})_{DK}(X_0^2)_{KC}(L_0)_{CF}(L_0^{-1})_{FJ}(R_\pi)_{J1}\\ &= \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1G}(L_0)_{GE}(\hat{X}_0^2)_{ED}\frac{1}{[(\hat{X}_0^2)_{DD}]^2}(\hat{X}_0^2)_{DF}(L_0^{-1})_{FJ}(R_\pi)_{J1} = \frac{4}{3F_\pi^2}.\end{aligned} \tag{65}$$

Similarly,

$$\frac{g_{8D}^2}{[(\hat{X}_a^2)_{DD}]^2} = \frac{4}{6F_\pi^2}. \tag{66}$$

The s dependent part of the scattering amplitude near threshold finally takes the simple form:

$$A(s, t, u) = \frac{2s}{F_\pi^2}. \tag{67}$$

This may be recognized as the usual current algebra formula [49] in the case where the pion mass is set to zero. We will complete its derivation in the next section, where it will be shown that the s independent terms in Eq. (64) cancel each other. It should be remarked that the present derivation holds for any choice of chiral invariant terms in V_0, not necessarily just for the leading terms in Eq. (38).

Of course, the current algebra result is just the first term in an expansion in powers of s. The higher terms will have the structure of a geometric series:

$$A(s, t, u) = s\left[\frac{2}{F_\pi^2} + s\sum_i \frac{g_i^2}{m_i^6} + s^2\sum_i \frac{g_i^2}{m_i^8} + \cdots\right], \tag{68}$$

wherein we have amalgamated all four scalars as the m_i and their corresponding coupling constants to two pions as the g_i. It may be noted that the entire amplitude is proportional to s. The zero of the amplitude at $s = 0$ is referred to as the Adler zero. Notice also that the higher terms involve the scalar masses and hence will vanish as the $m_i \to \infty$. In the case of the linear-in-s current algebra term, the non zero result arose because the g_i's increase as m_i^2. Taking the scalar masses to infinity is the same as integrating them out of the Lagrangian which results, as pointed out in the original paper [50] by Gell-Mann and Levy, in a nonlinear sigma model. The magic cancellations in that case are very easy to see. Clearly they are more intricate in the present case.

From the starting equation (63) it is seen that the radius of convergence of the series in s is equal to the squared mass of the lightest scalar meson. To go beyond this point, in principle one should calculate all loop diagrams. A simple approximation is to identify the partial wave corresponding to the tree term with the K matrix amplitude. This gives results for amplitudes spanning a considerable range in s in reasonable agreement with present experimental indications. This was carried out for the SU(2) single M linear sigma model in [6] and for the SU(3) single M linear sigma model in [31].

VII. FOUR POINT VERTICES

We start by establishing the notations for the quadrilinear coupling constants involving the prediagonal fields:

$$\begin{aligned} &p_1 = \left\langle\frac{\partial^4 V_0}{\partial\phi_2^1\partial\phi_1^2\partial\phi_2^1\partial\phi_1^2}\right\rangle \qquad p_2 = \left\langle\frac{\partial^4 V_0}{\partial\phi'^1{}_2\partial\phi_1^2\partial\phi_2^1\partial\phi_1^2}\right\rangle \qquad p_{31} = \left\langle\frac{\partial^4 V_0}{\partial\phi'^1{}_2\partial\phi'^2{}_1\partial\phi_2^1\partial\phi_1^2}\right\rangle\\ &p_{32} = \left\langle\frac{\partial^4 V_0}{\partial\phi'^1{}_2\partial\phi_1^2\partial\phi'^1{}_2\partial\phi_1^2}\right\rangle \qquad p_4 = \left\langle\frac{\partial^4 V_0}{\partial\phi'^1{}_2\partial\phi'^2{}_1\partial\phi'^1{}_2\partial\phi_1^2}\right\rangle \qquad p_5 = \left\langle\frac{\partial^4 V_0}{\partial\phi'^1{}_2\partial\phi'^2{}_1\partial\phi'^1_2\partial\phi'^2{}_1}\right\rangle.\end{aligned} \tag{69}$$

We find the following equations relating these quadrilinear coupling constants to the trilinear coupling constants in Eq. (50) by differentiating the second generating equation in Eq. (12) three times with respect to pseudoscalar fields:

$$
\begin{aligned}
p_1 &= \frac{\beta^2}{\alpha^2} p_{31} - \frac{\beta}{\sqrt{2}\alpha^2}\left(\frac{1}{\sqrt{3}}q_2 + \frac{2}{\sqrt{6}}r_2 + \frac{1}{\sqrt{3}}q_3 + \frac{2}{\sqrt{6}}r_3\right) + \frac{\sqrt{2}}{\alpha}\left(\frac{1}{\sqrt{3}}q_1 + \frac{2}{\sqrt{6}}r_1\right) \\
p_2 &= -\frac{\beta}{\alpha} p_{31} + \frac{1}{\sqrt{2}\alpha}\left(\frac{1}{\sqrt{3}}q_2 + \frac{2}{\sqrt{6}}r_2 + \frac{1}{\sqrt{3}}q_3 + \frac{2}{\sqrt{6}}r_3\right) \\
p_{32} &= p_{31} + \frac{1}{\beta\sqrt{2}}\left(\frac{1}{\sqrt{3}}(q_3 - q_2) + \frac{2}{\sqrt{6}}(r_3 - r_2)\right) \\
p_4 &= -\frac{\alpha}{\beta} p_{31} + \frac{1}{\sqrt{2}\beta}\left(\frac{1}{\sqrt{3}}q_4 + \frac{2}{\sqrt{6}}r_4 + \frac{1}{\sqrt{3}}q_5 + \frac{2}{\sqrt{6}}r_5\right) \\
p_5 &= \frac{\alpha^2}{\beta^2} p_{31} - \frac{\alpha}{\sqrt{2}\beta^2}\left(\frac{1}{\sqrt{3}}q_2 + \frac{2}{\sqrt{6}}r_2 + \frac{1}{\sqrt{3}}q_3 + \frac{2}{\sqrt{6}}r_3\right) + \frac{\sqrt{2}}{\beta}\left(\frac{1}{\sqrt{3}}q_6 + \frac{2}{\sqrt{6}}r_6\right).
\end{aligned}
\tag{70}
$$

Notice that the above equations were obtained by expressing five out of the six quantities in Eq. (69) in terms of trilinear coupling constants as well as the sixth quadrilinear, p_{31}. This shows that all the quadrilinear coupling constants cannot be obtained in terms of the trilinear ones. Nevertheless, as we will now see, the physical quadrilinear coupling constant, g can be completely expressed in terms of the bilinear coupling constants. Using the definition in Eq. (62), we express the physical four point coupling constant in terms of the bare four point coupling constants as,

$$
g = (R_\pi)_{A1}(R_\pi)_{B1}(R_\pi)_{C1}(R_\pi)_{D1} \times \left\langle \frac{\partial^4 V_0}{\partial(\phi_1^2)_A \partial(\phi_2^1)_B \partial(\phi_1^2)_C \partial(\phi_2^1)_D} \right\rangle, \tag{71}
$$

which may be explicitly written as

$$
\begin{aligned}
g = {} & \cos^4\theta_\pi p_1 - 4\cos^3\theta_\pi \sin\theta_\pi p_2 \\
& + \cos^2\theta_\pi \sin^2\theta_\pi (4p_{31} + 2p_{32}) \\
& - 4\cos\theta_\pi \sin^3\theta_\pi p_4 + \sin^4\theta_\pi p_5.
\end{aligned}
\tag{72}
$$

Substituting Eq. (70) into Eq. (72) and then using Eqs. (52) and (53) gives the formula for the quadrilinear coupling constant:

$$
g = \frac{1}{(\alpha^2 + \beta^2)^2}\left[\frac{2}{3}(\alpha^2 (X_0^2)_{11} + 2\alpha\beta (X_0^2)_{12} + \beta^2 (X_0^2)_{22}) + \frac{1}{3}(\alpha^2 (X_a^2)_{11} + 2\alpha\beta (X_a^2)_{12} + \beta^2 (X_a^2)_{22})\right] \tag{73}
$$

Noting $\alpha = (R_\pi^{-1})_{11} F_\pi/2$ and $\beta = (R_\pi^{-1})_{12} F_\pi/2$ we rewrite Eq. (73) as,

$$
g = \frac{8}{F_\pi^2}\left(\frac{1}{3}(R_\pi^{-1})_{1D}(X_0^2)_{DJ}(R_\pi)_{J1} + \frac{1}{6}(R_\pi^{-1})_{1D}(X_0^2)_{DJ}(R_\pi)_{J1}\right). \tag{74}
$$

In order to verify the cancellation of the s independent terms in Eq. (64) we should subtract half of Eq. (74) from the sum of the following two expressions:

$$
\begin{aligned}
\frac{g_{0D}^2}{(\hat{X}_0^2)_{DD}} &= \frac{4}{3F_\pi^2}(R_\pi)_{A1}(X_0^2)_{AH}(L_0)_{HG}\frac{1}{(\hat{X}_0^2)_{GG}}(R_\pi)_{C1}(X_0^2)_{CK}(L_0)_{KG} \\
&= \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1D}(L_0)_{DE}(L_0^{-1})_{EA}(X_0^2)_{AH}(L_0)_{HG}\frac{1}{(\hat{X}_0^2)_{GG}}(L_0^{-1})_{GK}(X_0^2)_{KC}(L_0)_{CF}(L_0^{-1})_{FJ}(R_\pi)_{J1} \\
&= \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1D}(L_0)_{DE}(\hat{X}_0^2)_{EG}\frac{1}{(\hat{X}_0^2)_{GG}}(\hat{X}_0^2)_{GF}(L_0^{-1})_{FJ}(R_\pi)_{J1} = \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1D}(X_0^2)_{DJ}(R_\pi)_{J1}
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD}} &= \frac{4}{6F_\pi^2}(R_\pi)_{A1}(X_a^2)_{AH}(L_0)_{HG}\frac{1}{(\hat{X}_a^2)_{GG}}(R_\pi)_{C1}(X_a^2)_{CK}(L_0)_{KG} \\
&= \frac{4}{6F_\pi^2}(R_\pi^{-1})_{1D}(L_0)_{DE}(L_0^{-1})_{EA}(X_a^2)_{AH}(L_0)_{HG}\frac{1}{(\hat{X}_a^2)_{GG}}(L_0^{-1})_{GK}(X_a^2)_{KC}(L_0)_{CF}(L_0^{-1})_{FJ}(R_\pi)_{J1} \\
&= \frac{4}{6F_\pi^2}(R_\pi^{-1})_{1D}(L_0)_{DE}(\hat{X}_a^2)_{EG}\frac{1}{(\hat{X}_a^2)_{GG}}(\hat{X}_a^2)_{GF}(L_0^{-1})_{FJ}(R_\pi)_{J1} = \frac{4}{6F_\pi^2}(R_\pi^{-1})_{1D}(X_a^2)_{DJ}(R_\pi)_{J1}
\end{aligned}
\tag{76}
$$

It has thus been shown that the simple formula Eq. (67) holds near threshold in the case of massless pions for an arbitrary potential, V_0.

VIII. SUMMARY AND DISCUSSION

We have given a detailed treatment of a systematic approach to the study of a linear sigma model containing one chiral nonet transforming under $U(1)_A$ as a quark-antiquark composite and another chiral nonet transforming as a diquark-antidiquark composite (or, equivalently from a symmetry point of view, as a two meson molecule). Some highlights of this work have been presented elsewhere [51]. The model provides an intuitive explanation of a current puzzle in low energy QCD: Recent work has suggested the existence of a lighter than 1 GeV nonet of scalar mesons which behave like four quark composites. On the other hand, the validity of a spontaneously broken chiral symmetric description would suggest that these states be (perhaps somewhat distorted) chiral partners of the light pseudoscalar mesons which are two quark composites. The model solves the problem by starting with the two chiral nonets mentioned and allowing them to mix with each other. Working with the SU(3) invariant version of the model it is seen that the four experimental inputs given in Eq. (47) (note that the lighter 0^- nonet automatically has zero mass in the limit in which we are working) enforce a mixing whereby the light scalars have a large four quark content while the light pseudoscalars have a large two quark content. In addition, one light isosinglet scalar is exceptionally light [see Eq. (49)].

Of the four experimental inputs just mentioned, there is a large uncertainty associated only with the mass of the ''heavy'' pion, the $\pi(1300)$. It turns out that there is in fact some sensitivity to the precise choice of $m[\pi(1300)]$ so that this quantity is really being considered as a free parameter within the range of the quoted rather large experimental error. Thus the model parameters and predictions calculated in Sec. IV are all displayed as functions of $m[\pi(1300)]$. The effects of the not so large allowed variations in the mass of the $a_0(1450)$ are shown as error bars in these plots.

In our treatment there are two parameters, associated with the masses and mixings of the SU(3) singlet pseudoscalars, which describe the $U(1)_A$ anomaly in the effective Lagrangian. These parameters do not affect properties of the other particles and may be traded for the masses of the $\eta(958)$ and one of the heavier candidates $\eta(1295)$, $\eta(1405)$, $\eta(1475)$ or $\eta(1760)$. The positivity of the eigenvalues allows only the last two candidates. For either of these it is noted in Sec. IV that there are two solutions for the two quark vs four quark content of the $\eta(958)$. The presumably favored solution results in $\eta(958)$ with a mainly two quark content, while the less favored solution results in a mainly four quark content for the $\eta(958)$.

In Secs. V, VI, and VII we gave a detailed proof that the low energy theorem for pion pion scattering holds in the present model with massless pions, for any choice of chiral invariant potential. The proof made use of the ''generating equations,'' stated in Sec. II, to relate the four particle, three particle and two particle (i.e. mass term) vertices to each other. We carried out this somewhat lengthy calculation for two reasons. First, since the pion in the model has a non negligible, though small four quark content, one might wonder whether the theorem actually does hold. Second, it is expected to be useful to calculate the scattering amplitude, Eq. (63) in the resonance region, rather than close to threshold, as the theorem requires.

Clearly, there are a number of other interesting directions for further work. We plan to add mass terms in the same systematic scheme employed in Sec. IV for selecting the most important chiral invariant terms. Mixing with glueball states and possibly other chiral nonets is also an intriguing possibility. Of course, an important ingredient to be taken into account would be the changes in the model parameters which result from unitarizing the tree level scattering amplitudes and comparing with the unitarized amplitudes with experiment. This was carried out for the 2 flavor Gell-Mann-Levy model in [6] and for the 3 flavor single M model in [31].

ACKNOWLEDGMENTS

We are happy to thank A. Abdel-Rehim, D. Black, M. Harada, S. Moussa, S. Nasri, and F. Sannino for many helpful related discussions. The work of A. H. F. has been partially supported by a 2006-2007 Crouse Grant from the School of Arts and Sciences, SUNY Institute of Technology. The work of R. J. and J. S. is supported in part by the U. S. DOE under Contract no. DE-FG-02-85ER 40231.

APPENDIX A: SOME CORRECTIONS

We have found the following minor corrections to Ref. [43]:

(1) In Eq. (A1) the fifth term on the right-hand side should properly read, $d_2\,\mathrm{Tr}(M'M'^\dagger)$.

(2) In the sentence immediately following Eq. (A1), d_2 should be added to the list of coefficients which are $U(1)_A$ invariant.

(3) In Eq. (19), the denominator of the argument of the ''ln'' in the first term should read $\det M^\dagger$.

(4) In the last line of Eq. (58) the left-hand side should read β_3.

(5) In the last approximate equality in Eq. (60) the left-hand side should read β_3.

APPENDIX B: PARAMETER DETERMINATION

Given the inputs: the pion decay constant, F_π; the mass of the $a_0(980)$, m_a; the mass of the $a_0(1450)$, $m_{a'}$; the mass

TABLE I. Calculated Lagrangian parameters: c_2, d_2, e_3^a, c_4^a and vacuum values: α, β.

c_2 (GeV^2)	9.64×10^{-2}
d_2 (GeV^2)	6.32×10^{-1}
e_3^a (GeV)	-2.14
c_4^a	42.1
α (GeV)	6.06×10^{-2}
β (GeV)	2.49×10^{-2}

of the $\pi(1300)$, $m_{\pi'}$, the independent model parameters which do not involve the $U(1)_A$ violating terms can be successively determined (in the order given) by the equations:

$$\begin{aligned}
2d_2 &= \frac{m_a^2 m_{a'}^2}{m_a^2 + m_{a'}^2 - m_{\pi'}^2}\\
(\alpha e_3^a)^2 &= \frac{1}{64}((m_a^2 - m_{a'}^2)^2 - [4d_2 - (m_a^2 + m_{a'}^2)]^2)\\
4c_2 &= m_a^2 + m_{a'}^2 - 2d_2 - \frac{56(\alpha e_3^a)^2}{d_2}\\
\frac{\beta}{\alpha} &= \frac{-2(\alpha e_3^a)}{d_2}\\
\alpha^2 &= \frac{1}{4}\frac{F_\pi^2}{1 + (\beta/\alpha)^2}\\
c_4^a &= \frac{1}{2\alpha^2}\left(c_2 + \frac{8(\alpha e_3^a)^2}{d_2}\right).
\end{aligned} \tag{B1}$$

The first equation tells us that d_2 is positive for the experimental input masses. We take α and β to be positive. Then the fourth equation shows that e_3^a must be negative. Finally c_2 and c_4^a will be positive.

Once the above parameters are determined, the parameters γ_1 and c_3 of the $U(1)_A$ violating sector are obtained in terms of the mass of the $\eta(958)$, $m_{\eta 1}$ and the mass of a suitable heavier 0^- isosinglet, $m_{\eta 2}$ as follows. First, γ_1 is found as a solution of the quadratic equation:

$$\begin{gathered}
0 = S\gamma_1^2 + T\gamma_1 + U, \qquad S = r\left(4 + \frac{\alpha^2}{\beta^2}\right),\\
T = r\left(4 - 2\frac{\alpha^2}{\beta^2}\right), \qquad U = r\left(1 + \frac{\alpha^2}{\beta^2}\right) - 36,\\
r = \frac{4m_{\eta 1}^2 m_{\eta 2}^2}{y_0[m_{\eta 1}^2 + m_{\eta 2}^2 - y_0(1 + z_0^2)]}.
\end{gathered} \tag{B2}$$

In addition,

$$c_3 = -\frac{m_{\eta 1}^2 m_{\eta 2}^2 \alpha^2}{24 y_0}. \tag{B3}$$

Next we give the numerical values of the parameters for the central values of all the listed input masses except for $m[\pi(1300)]$ which instead will take the typical value allowed by both the data and by the model, 1215 MeV. Table I shows the results for the parameters which are not associated with the $U(1)_A$ violating part of the Lagrangian.

Table II shows the calculated Lagrangian parameters associated with the $U(1)_A$ violating terms. Two "scenarios" associated with different identifications of the heavy η which is the partner of the $\eta(958)$ are shown (I assumes $\eta(1475)$ to be chosen while II assumes $\eta(1760)$ to be chosen.) For each scenario, the two solutions (labeled 1 and 2) are shown.

Using these parameters we next list the mixing matrices for, respectively, the two 0^- octet states, the two 0^+ octet states and the two 0^+ singlet states:

$$\begin{aligned}
(R_\pi^{-1}) &= \begin{bmatrix} 0.925 & 0.380 \\ -0.380 & 0.925 \end{bmatrix},\\
(L_a^{-1}) &= \begin{bmatrix} -0.496 & 0.869 \\ 0.869 & 0.496 \end{bmatrix},\\
(L_0^{-1}) &= \begin{bmatrix} 0.711 & 0.703 \\ -0.703 & 0.711 \end{bmatrix}.
\end{aligned} \tag{B4}$$

Similarly, the mixing matrices for the two solutions for scenario I of the 0^- singlet states are

$$\begin{aligned}
I1\colon (R_0^{-1}) &= \begin{bmatrix} -0.671 & 0.742 \\ 0.742 & 0.671 \end{bmatrix},\\
I2\colon (R_0^{-1}) &= \begin{bmatrix} 0.858 & -0.514 \\ 0.514 & 0.858 \end{bmatrix}.
\end{aligned} \tag{B5}$$

Finally, the mixing matrices for the two solutions for scenario II of the 0^- singlet states are

$$\begin{aligned}
II1\colon (R_0^{-1}) &= \begin{bmatrix} -0.413 & 0.910 \\ 0.910 & 0.413 \end{bmatrix},\\
II2\colon (R_0^{-1}) &= \begin{bmatrix} 0.974 & -0.228 \\ 0.228 & 0.074 \end{bmatrix}.
\end{aligned} \tag{B6}$$

TABLE II. Calculated parameters: c_3 and γ_1.

	I1	I2	II1	II2
c_3 (GeV^4)	-2.42×10^{-4}	-2.42×10^{-4}	-3.44×10^{-4}	-3.44×10^{-4}
γ_1	5.38×10^{-1}	2.53×10^{-1}	8.69×10^{-1}	-7.76×10^{-2}

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