Relation between light cone distribution amplitudes and shape function in *B* mesons

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The Bakamjian-Thomas relativistic quark model provides a Poincaré representation of bound states with a fixed number of constituents and, in the heavy quark limit, form factors of currents satisfy covariance and Isgur-Wise scaling. We compute the light cone distribution amplitudes (LCDA) of *B* mesons $\varphi_\pm^B(\omega)$ as well as the shape function $S(\omega)$, that enters in the decay $B \to X_s \gamma$, that are also covariant in this class of models. The LCDA and the shape function are related through the quark model wave function. The former satisfy, in the limit of vanishing constituent light quark mass, the integral relation given by QCD in the valence sector of Fock space. Using a Gaussian wave function, the obtained $S(\omega)$ is identical to the so-called roman shape function. From the parameters for the latter that fit the $B \to X_s \gamma$ spectrum we predict the behavior of $\varphi_\pm^B(\omega)$. We discuss the important role played by the constituent light quark mass. In particular, although $\varphi_-^B(0) \neq 0$ for vanishing light quark mass, a nonvanishing mass implies the unfamiliar result $\varphi_-^B(0) = 0$. Moreover, we incorporate the short distance behavior of QCD to $\varphi_+^B(\omega)$, which has sizeable effects at large ω. We obtain the values for the parameters $\bar{\Lambda} \cong 0.35$ GeV and $\lambda_B^{-1} \cong 1.43$ GeV^{-1}. We compare with other theoretical approaches and illustrate the great variety of models found in the literature for the functions $\varphi_\pm^B(\omega)$; hence the necessity of imposing further constraints as in the present paper. We briefly review also the different phenomena that are sensitive to the LCDA. The value that we find for λ_B^{-1} fulfills the upper bound recently measured by *BABAR*.

I. INTRODUCTION

The light cone distribution amplitudes (LCDA) of heavy-light mesons $\Phi_+^B(\xi)$, $\Phi_-^B(\xi)$ [1] or, in the heavy quark limit $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ [2], are fundamental functions that enter in the large energy limit of amplitudes of semileptonic decays [2] and in nonleptonic decays of *B* mesons [3], in the determination of the form factor $F_+^{B \to \pi}(0)$ [4] and, more directly, in the decay $B^- \to \gamma \ell \bar{\nu}_\ell$ [5–8].

On the theoretical side, these functions have been studied in the perturbative regime at large ω [7,9], and a number of very varied Ansätze have been proposed for the dominant part of them at low ξ or ω, where the function is peaked at $\xi \sim \frac{\Lambda_{\rm QCD}}{m_B}$ or $\omega \sim \Lambda_{\rm QCD}$ [2,7,8].

On the other hand, one can obtain model-independent information on these functions from the measurement of the spectrum in the decay $B^- \to \gamma \ell \bar{\nu}_\ell$, that is directly related to one of the LCDA [5–8].

On the theoretical side, although rigorous results are known in the perturbative regime $\omega \gg \Lambda_{\rm QCD}$, the guesses advanced for the main nonperturbative part of the LCDA amplitudes come essentially from QCD sum rules, imposing continuity between the perturbative and long distance regimes [7,9].

The motivation of the present work is to compute the LCDA in a class of relativistic quark models, namely, the Bakamjian-Thomas (BT) quark models [10–14]. This is a class of models with a fixed number of constituents where

the states are covariant under the Poincaré group. The model relies on an appropriate Lorentz boost of eigenfunctions of a Hamiltonian describing the spectrum at rest. On the other hand, one has demonstrated that the matrix elements of currents between hadrons are covariant in the heavy quark limit and exhibit Isgur-Wise scaling [15] in this limit [14]. Given a Hamiltonian describing the spectrum, the model provides an unambiguous result for the elastic Isgur-Wise function $\xi(w)$ [14,16]. On the other hand, the sum rules (SR) in the heavy quark limit of QCD, like Bjorken or Uraltsev SR are satisfied in the model [17,18], as well as SR involving higher derivatives of $\xi(w)$ at zero recoil [19].

The interest of computing the LCDA functions $\varphi_\pm^B(\omega)$ in this framework is to directly relate them, in an unambiguous way, to the shape function $S(\omega)$ [20–26] in $B \to X_s \gamma$, that can also be computed in the BT class of models.

One could calculate the LCDA within the BT scheme using a Hamiltonian describing the spectrum, like the Godfrey-Isgur (GI) model [27], that has been used elsewhere to compute the elastic Isgur-Wise function $\xi(w)$ and the inelastic ones $\tau_{1/2}(w)$, $\tau_{3/2}(w)$ [16]. However, we have tested the GI model to compute the shape function $S(\omega)$ and have realized that this model does not fit the $B \to X_s \gamma$ spectrum. This is the reason why we have decided on another phenomenological approach, namely, to relate the light-cone distribution amplitudes $\varphi_\pm^B(\omega)$ to the shape function $S(\omega)$, a relation that is provided by the BT scheme. Using this relation, and a SF $S(\omega)$ that fits the $B \to X_s \gamma$ spectrum, one can predict the LCDA. The problems for the SF and also a discussion of heavy-to-light form

factors within the BT scheme for the GI spectroscopic model will be given in detail elsewhere.

The paper is organized as follows. In Sec. II we give the master formulas defining the theoretical framework of BT quark models. In Sec. III we review the definitions of the LCDA at finite mass and in the heavy quark limit of QCD and in Sec. IV we set a main natural hypothesis to compute the LCDA within quark models. In Sec. V, to introduce the technicalities of the BT model, we review the calculation of the heavy meson decay constant. In Sec. VI we obtain our main results, namely, the expressions for the LCDA in the BT quark models. In Sec. VII we compute the SF $S(\omega)$ in BT models and show that, in the case of the harmonic oscillator, it is identical with the so-called roman shape function, used to fit the $B \to X_s \gamma$ spectrum [20–22,26]. In Sec. VIII we use the parameters of the latter to predict the LCDA functions $\varphi^B_\pm(\omega)$ and their moments. In Sec. IX, following Braun, Ivanov, and Korchemsky [7] and Lee and Neubert [9], we introduce the radiative tail of $\varphi^B_+(\omega)$. In Sec. X we compare our results with proposals for the LCDA in other theoretical schemes. In Sec. XI we review the different phenomena that are sensitive to the LCDA, and in Sec. XII we conclude.

II. THE BAKAMJIAN-THOMAS RELATIVISTIC QUARK MODEL

As explained in [14], the construction of the BT wave function in motion involves a *unitary* transformation that relates the wave function $\Psi^{(P)}_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ in terms of one-particle variables, the spin $\mathbf{S}_i$ and momenta $\mathbf{p}_i$ to the so-called *internal* wave function $\Psi^{\text{int}}_{s_1, \cdots, s_n}(\mathbf{P}, \mathbf{k}_2, \cdots, \mathbf{k}_n)$ given in terms of another set of variables, the total momentum $\mathbf{P}$ and the internal momenta $\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n$ ($\sum_i \mathbf{k}_i = 0$). This property ensures that, starting from an orthonormal set of internal wave functions, one gets an orthonormal set of wave functions in any frame. The base $\Psi^{(P)}_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ is useful to compute one-particle matrix elements like current one-quark matrix elements, while the second $\Psi^{\text{int}}_{s_1, \cdots, s_n}(\mathbf{P}, \mathbf{k}_2, \cdots, \mathbf{k}_n)$ allows to exhibit Poincaré covariance. In order to satisfy the Poincaré commutators, the unique requirement is that the mass operator M, i.e. the Hamiltonian describing the spectrum at rest, should depend only on the internal variables and be rotational invariant, i.e. M must commute with $\mathbf{P}$, $\frac{\partial}{\partial \mathbf{P}}$ and $\mathbf{S}$. The internal wave function at rest $(2\pi)^3 \delta(\mathbf{P}) \varphi_{s_1, \cdots, s_n}(\mathbf{k}_2, \cdots, \mathbf{k}_n)$ is an eigenstate of M, $\mathbf{P}$ (with $\mathbf{P} = 0$), $\mathbf{S}^2$ and $\mathbf{S}_z$, while the wave function in motion of momentum $\mathbf{P}$ is obtained by applying the boost $\mathbf{B}_P$, where $P^0 = \sqrt{\mathbf{P}^2 + M^2}$ involves the dynamical operator M.

The final output of the formalism that gives the total wave function in motion $\Psi^{(P)}_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ in terms of the internal wave function at rest $\varphi_{s_1, \cdots, s_n}(\mathbf{k}_2, \cdots, \mathbf{k}_n)$ is the formula

$$\Psi^{(P)}_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n) = (2\pi)^3 \delta\left(\sum_i \mathbf{p}_i - \mathbf{P}\right) \sqrt{\frac{\sum_i p_i^0}{M_0}} \left(\prod_i \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}}\right) \sum_{s_1', \cdots, s_n'} [D_i(\mathbf{R}_i)]_{s_i, s_i'} \varphi_{s_1', \cdots, s_n'}(\mathbf{k}_2, \cdots, \mathbf{k}_n) \tag{1}$$

where $p_i^0 = \sqrt{\mathbf{p}_i^2 + m_i^2}$ and M_0 is the free mass operator is given by $M_0 = \sqrt{(\sum_i p_i)^2}$.

The internal momenta of the hadron at rest are given in terms of the momenta of the hadron in motion by the *free boost* $k_i = \mathbf{B}^{-1}_{\sum_i p_i} p_i$ where the operator $\mathbf{B}_p$ is the boost $(\sqrt{p^2}, 0) \to p$, the Wigner rotations $\mathbf{R}_i$ in the preceding expression $\mathbf{R}_i = \mathbf{B}^{-1}_{p_i} \mathbf{B}^{-1}_{\sum_i p_i} \mathbf{B}_{k_i}$ and the states are normalized by $\langle \mathbf{P}', S'_z | \mathbf{P}, S_z \rangle = (2\pi)^3 \delta(\mathbf{P}' - \mathbf{P}) \delta_{S_z, S'_z}$.

The current one-quark matrix element acting on quark 1 between two hadrons is then given by the expression

$$\langle \Psi'(\mathbf{P}', S'_z) | J^{(1)} | \Psi(\mathbf{P}, S_z) \rangle = \int \frac{d\mathbf{p}'_1}{(2\pi)^3} \frac{d\mathbf{p}_1}{(2\pi)^3} \left(\prod_{i=2}^n \frac{d\mathbf{p}_i}{(2\pi)^3}\right)$$
$$\Psi^{P'}_{s_1', \cdots, s_n}(\mathbf{p}'_1, \cdots, \mathbf{p}_n)^* \langle \mathbf{p}'_1, s_1' | J^{(1)} | \mathbf{p}_1, s_1 \rangle \Psi^P_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n) \tag{2}$$

where $\Psi^P_{s_1, \cdots, s_n}(\mathbf{p}_1, \cdots, \mathbf{p}_n)$ is given in terms of the internal wave function by (1) and $\langle \mathbf{p}'_1, s_1' | J^{(1)} | \mathbf{p}_1, s_1 \rangle$ is the one-quark current matrix element.

As demonstrated in [14,28], in this formalism, in the heavy quark limit, current matrix elements are covariant and exhibit Isgur-Wise scaling, and one can compute Isgur-Wise functions like $\xi(w)$, $\tau_{1/2}(w)$, $\tau_{3/2}(w)$ [16].

In the present paper, as far as the LCDA are concerned, we are dealing with current matrix elements between one meson and the vacuum, i.e. $\langle 0 | J | \Psi \rangle$. In [29] such matrix elements were considered and it was demonstrated that the decay constants of heavy-light mesons are covariant— independent of the frame—in the heavy quark limit, and exhibit heavy quark scaling, i.e. $f_B \sqrt{m_B}$ is a constant in this

limit. This quantity was calculated for various Hamiltonians describing the meson spectrum.

We want now to go beyond and compute the LCDA $\varphi_{\pm}^{B}(\omega)$ starting from the meson-to-vacuum matrix elements $\langle 0|J|\Psi\rangle$. We can here advance our main result, that is parallel to the one obtained for meson-to-meson current matrix elements. The LCDA are covariant in the heavy quark limit, and can therefore be computed without any arbitrary parameter once the Hamiltonian giving the internal wave function is given. The same statement holds for the shape function $S(\omega)$ in $B \rightarrow X_s\gamma$, that can be expressed as a meson-to-meson matrix element $\langle\Psi|O|\Psi\rangle$. For the latter we will use the harmonic oscillator that gives the roman shape function [20–22,26]. We will then use the phenomenological parameters of this function, that fit the $B \rightarrow X_s\gamma$ spectrum, to predict the LCDA.

III. B MESON LIGHT-CONE DISTRIBUTION AMPLITUDES

Let us define the LCDA $\Phi_{+}^{B}(\xi)$, $\Phi_{-}^{B}(\xi)$ [1]

$$\langle 0|\bar{q}(z)S_{n_-}(z,0)\Gamma b(0)|\bar{B}_d(P)\rangle|_{z_+=z_\perp}$$
$$= -\frac{f_B}{4}\int_0^1 d\xi e^{-i\xi P_+ z_-}$$
$$\times \mathrm{Tr}\left\{\Gamma(\not{P}+m_B)\gamma_5\left[\Phi_{+}^{B}(\xi) - \frac{\not{n}_-}{v \cdot n_-}\frac{\Phi_{+}^{B}(\xi)-\Phi_{-}^{B}(\xi)}{2}\right]\right\}$$
(3)

where $S_{n_-}(z,0)$ is the Wilson line following the lightlike four vector $n_- = (1,0,0,-1)$ ($n_-^2 = 0$), v is the B four velocity $v = \frac{P}{m_B}$, Γ is an arbitrary Dirac matrix, and the center-of-mass motion is along the Oz axis. P_+, z_- are light-cone variables defined for any four vector p by $p_+ = \frac{p^0+p^z}{\sqrt{2}}$, $p_- = \frac{p^0-p^z}{\sqrt{2}}$. Sometimes one takes $v \cdot n_- = 1$, i.e. the B rest frame, but to exhibit covariance we have adopted a general value for $v \cdot n_-$. The LCDA $\Phi_{B_1}(\xi)$, $\Phi_{B_2}(\xi)$ satisfy the normalization conditions

$$\int_0^1 d\xi \Phi_{+}^{B}(\xi) = \int_0^1 d\xi \Phi_{-}^{B}(\xi) = 1. \tag{4}$$

In the heavy quark limit $m_b \rightarrow \infty$ it is useful to use a new variable

$$\omega = m_b\xi \tag{5}$$

keeping ω fixed, that yields the definition of the LCDA

[2,4,30],

$$\langle 0|\bar{q}(z)S_{n_-}(z,0)\Gamma h_v(0)|\bar{B}_d(v)\rangle|_{z_+=z_\perp}$$
$$= -\frac{f_B m_B}{4}\int_0^\infty d\omega e^{-i\omega v_+ z_-}$$
$$\times \mathrm{Tr}\left\{\Gamma(1+\not{v})\gamma_5\left[\varphi_{+}^{B}(\omega) - \frac{\varphi_{+}^{B}(\omega)-\varphi_{-}^{B}(\omega)}{2}\frac{\not{n}_-}{v \cdot n_-}\right]\right\}.$$
(6)

The relation between $\Phi_{\pm}^{B}(\xi)$ and $\varphi_{\pm}^{B}(\omega)$ is

$$\varphi_{\pm}^{B}(\omega) = \frac{1}{m_B}\Phi_{\pm}^{B}\left(\frac{\omega}{m_B}\right) \tag{7}$$

and $\varphi_{\pm}^{B}(\omega)$ satisfy the normalization conditions

$$\int_0^\infty d\omega \varphi_{\pm}^{B}(\omega) = 1. \tag{8}$$

IV. LCDA IN QUARK MODELS

In what follows, we will use the preceding relations to obtain the expression of $\varphi_{\pm}^{B}(\omega)$ in BT quark models. In the class of BT quark models, gluon exchange is included in the potential. Therefore, in a way, the gluon field is integrated out, but one loses the explicit gauge invariance that is ensured by the Wilson line of the preceding expressions.

In a quark model, what we can consider is the matrix element involving constituent quarks, in particular, constituent light quarks with a dynamical mass. Our ansatz will be to identify the QCD matrix element with the Wilson line with a matrix element involving the constituent quark field, or in more rigorous terms, one would say that one works in the light-cone gauge,

$$A_+ = 0 \qquad S_{n_-}(z,0) = 1 \tag{9}$$

and set

$$\langle 0|\bar{q}(z)S_{n_-}(z,0)\Gamma b(0)|\bar{B}_d(v)\rangle|_{z_+=z_\perp=0}$$
$$= \langle 0|\bar{q}_{\mathrm{constituent}}(z)\Gamma b(0)|\bar{B}_d(v)\rangle|_{z_+=z_\perp=0}. \tag{10}$$

This is our main hypothesis and the starting point of the quark model calculation, that then follows in a straightforward way. From now on the constituent light quark field $q_{\mathrm{constituent}}$ will be denoted by q. Of course, the condition (9) can hold only in field theory, and our BT scheme is just a model. Defining $\Phi_{\pm}(\xi)$ by the quark model expression:

$$\langle 0|\bar{q}(z)\Gamma b(0)|\bar{B}_d(P)\rangle|_{z_+=z_\perp} = -\frac{f_B}{4}\int_0^1 d\xi e^{-i\xi P_+ z_-}\mathrm{Tr}\left\{\Gamma(\not{P}+m_B)\gamma_5\left[\Phi_{+}^{B}(\xi) - \frac{\not{n}_-}{v \cdot n_-}\frac{\Phi_{+}^{B}(\xi)-\Phi_{-}^{B}(\xi)}{2}\right]\right\} \tag{11}$$

where $n_- = (1,0,0,-1)$, one obtains

$$\Phi_+^B(\xi) = \frac{1}{f_B m_B} \frac{P_+}{2\pi} \int dz_- e^{i\xi P_+ z_-} \langle 0|\bar{q}(z) \frac{\not{n}_-}{v \cdot n_-} \gamma_5 b(0)|\bar{B}_d(P)\rangle|_{z_+=z_\perp=0}$$

$$\Phi_-^B(\xi) = \frac{1}{f_B m_B} \frac{P_+}{2\pi} \int dz_- e^{i\xi P_+ z_-} \langle 0|\bar{q}(z)\left(2\not{v} - \frac{\not{n}_-}{v \cdot n_-}\right)\gamma_5 b(0)|\bar{B}_d(P)\rangle|_{z_+=z_\perp=0}.$$

(12)

Calling p_2 the four-momentum of the light quark, using translational invariance and integrating over z_-, these expressions write

$$\Phi_+^B(\xi) = \frac{1}{f_B m_B} \langle 0|\bar{q}(0)\delta\left(\xi - \frac{p_{2+}}{P_+}\right)\frac{\not{n}_-}{v \cdot n_-} \gamma_5 b(0)|\bar{B}_d(P)\rangle$$

$$\Phi_-^B(\xi) = \frac{1}{f_B m_B} \langle 0|\bar{q}(0)\delta\left(\xi - \frac{p_{2+}}{P_+}\right)\left(2\not{v} - \frac{\not{n}_-}{v \cdot n_-}\right)\gamma_5 b(0)|\bar{B}_d(P)\rangle.$$

(13)

These will be the starting formulas to compute these functions in the BT quark model, from which we will deduce their heavy quark limit $\varphi_\pm^B(\omega)$. But let us first compute the heavy meson decay constant f_B in the BT quark models, that will provide the desired normalization for the LCDA.

V. DECAY CONSTANT IN BT MODELS

The calculation of the matrix elements to obtain the LCDA $\varphi_\pm^B(\omega)$ is just reminiscent of the one made to obtain the corresponding decay constant [29]. In this latter case one needs the matrix element

$$\langle 0|\bar{q}(0)\Gamma b(0)|\bar{B}_d(v)\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sqrt{\frac{\sum_i p_i^0}{M_0}} \sqrt{\frac{k_1^0 k_2^0}{p_1^0 p_2^0}} \sqrt{\frac{m_1 m_2}{p_1^0 p_2^0}} \frac{1}{\sqrt{2}} \varphi(\mathbf{k}_2) \text{Tr}\left[\Gamma\gamma^5 \mathbf{B}_u \mathbf{B}_{k_2} \frac{1+\gamma^0}{2} \mathbf{B}_u^{-1}\mathbf{B}_u \frac{1+\gamma^0}{2} \mathbf{B}_{k_1}^{-1}\mathbf{B}_u^{-1}\right]$$

(14)

where $\varphi(\mathbf{k}_2)$ is the internal wave function at rest, with the normalization

$$\int \frac{d\mathbf{k}_2}{(2\pi)^3} |\varphi(\mathbf{k}_2)|^2 = 1.$$

(15)

p_1 and p_2 (m_1 and m_2) are the quark four-momenta (masses) of, respectively, the heavy and light quarks, $\mathbf{B}_p$ is the 4×4 boost matrix associated with the four-vector p, and the four vector u, M_0 and the relation between k_i and p_i are given by

$$u = \frac{p_1 + p_2}{M_0} \qquad M_0 = \sqrt{(p_1 + p_2)^2} \qquad \mathbf{B}_u k_i = p_i \quad (i = 1, 2)$$

(16)

where k_1 and k_2 are the four-momenta of the quarks in the rest frame of the B meson and $\mathbf{B}_u$ is the boost associated with the four-vector u. The products of 4×4 matrices under the trace read

$$\mathbf{B}_u \mathbf{B}_{k_2} \frac{1+\gamma^0}{2} \mathbf{B}_u^{-1} = \frac{m_2 + \not{p}_2}{\sqrt{2m_2(k_2^0 + m_2)}} \frac{1+\not{u}}{2} \qquad \mathbf{B}_u \frac{1+\gamma^0}{2} \mathbf{B}_{k_1} \mathbf{B}_u^{-1} = \frac{1+\not{u}}{2} \frac{m_1 + \not{p}_1}{\sqrt{2m_1(k_1^0 + m_1)}}.$$

(17)

This yields the expression

$$\langle 0|\bar{q}(0)\Gamma b(0)|\bar{B}_d(P)\rangle = -\frac{\sqrt{N_c}}{8} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sqrt{2} \frac{\sqrt{u^0}}{p_1^0 p_2^0} \sqrt{\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}} \text{Tr}[\gamma^5 \Gamma(m_2 + \not{p}_2)(1 + \not{u})(m_1 + \not{p}_1)]\varphi(\mathbf{k}_2).$$

(18)

Using the current $\Gamma = \gamma_\mu \gamma_5$ we obtain, after some algebra and the definition of the four vector u (16),

$$\langle 0|\bar{q}(0)\gamma_\mu \gamma_5 b(0)|\bar{B}_d(P)\rangle = \frac{\sqrt{N_c}}{\sqrt{2}} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{\sqrt{u^0}}{p_1^0 p_2^0} \sqrt{\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}}\left(1 + \frac{m_1 + m_2}{M_0}\right)(p_{1\mu}m_2 + p_{2\mu}m_1)\varphi(\mathbf{k}_2).$$

(19)

Since the BT states are normalized according to

$$\langle \bar{B}_d(P')|\bar{B}_d(P)\rangle_{\mathrm{BT}} = (2\pi)^3 \delta(\mathbf{P} - \mathbf{P}') \tag{20}$$

while the covariant normalization is $\langle \bar{B}_d(P')|\bar{B}_d(P)\rangle = (2\pi)^3 2P^0 \delta(\mathbf{P} - \mathbf{P}')$, we have to identify the former matrix element with the definition of the decay constant

$$\langle 0|\bar{q}(0)\gamma_\mu \gamma_5 b(0)|\bar{B}_d(P)\rangle = \frac{f_B P_\mu}{\sqrt{2P^0}} \tag{21}$$

one obtains

$$f_B\sqrt{m_B} = \sqrt{N_c}\sqrt{v^0} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{\sqrt{u^0}}{p_1^0 p_2^0} \sqrt{\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}} \left(1 + \frac{m_1 + m_2}{M_0}\right) [m_1(p_2 \cdot v) + m_2(p_1 \cdot v)]\varphi(\mathbf{k}_2). \tag{22}$$

This expression is not covariant, but becomes covariant in the heavy quark limit. For $m_1 \to \infty$ one has

$$m_0 \to m_1 \to m_B, \qquad u \to v, \qquad \frac{p_1}{m_1} \to v, \qquad \frac{k_1^0}{m_1} \to 1, \qquad k_2^0 \to p_2 \cdot v \tag{23}$$

where v is the B meson four-velocity, and one gets the expression of the B decay constant in the BT model in the heavy quark limit [29],

$$f_B\sqrt{m_B} = \sqrt{2}\sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \sqrt{(p_2 \cdot v)(p_2 \cdot v + m_2)}\varphi(\sqrt{(p_2 \cdot v)^2 - m_2^2}). \tag{24}$$

This expression is covariant, satisfies heavy quark scaling, and gives the decay constant in terms of the internal wave function. In the B rest frame one gets

$$f_B\sqrt{m_B} = \frac{\sqrt{N_c}}{\sqrt{2}} \frac{1}{\pi^2} \int_0^\infty dp\,p^2 \left(\frac{m_2 + \sqrt{p^2 + m_2^2}}{\sqrt{p^2 + m_2^2}}\right)^{1/2} \varphi(p). \tag{25}$$

We have checked that these expressions for the decay constant hold exactly in the equivalent light-front approach of Cardarelli *at al.* [13] in the heavy mass limit.

VI. LCDA IN THE HEAVY QUARK LIMIT IN BT MODELS

According to (13), we have to compute a generic matrix element

$$\Phi_B(\xi) = \frac{1}{f_B m_B}\langle 0|\bar{q}(0)\delta\left(\xi - \frac{p_{2+}}{P_+}\right)\Gamma b(0)|\bar{B}_d(P)\rangle|_{z_+ = z_\perp = 0} \tag{26}$$

where Γ is a Dirac matrix. In the BT quark model, this expression writes, taking into account the appropriate normalizations,

$$\Phi(\xi) = -\sqrt{2P^0}\frac{1}{f_B m_B}\frac{\sqrt{N_c}}{8}\int \frac{d\mathbf{p}_2}{(2\pi)^3}\frac{1}{p_2^0}\delta\left(\xi - \frac{p_{2+}}{P_+}\right)$$

$$= \sqrt{2}\frac{\sqrt{u^0}}{p_1^0}\sqrt{\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}}\mathrm{Tr}[\gamma^5\Gamma(m_2 + \not{p}_2)(1 + \not{u})(m_1 + \not{p}_1)]\varphi(\mathbf{k}_2). \tag{27}$$

Changing the measure and integrating relatively to p_{2+} and p_{2-}, one obtains

$$\Phi(\xi) = -\sqrt{2P^0}\frac{1}{f_B m_B}\frac{\sqrt{N_c}}{8}\frac{1}{\xi}\frac{1}{2\pi}\left[\int \frac{d\mathbf{p}_{2\perp}}{(2\pi)^2}\sqrt{2}\frac{\sqrt{u^0}}{p_1^0}\sqrt{\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}}\right.$$

$$\times \mathrm{Tr}[\gamma^5\Gamma(m_2 + \not{p}_2)(1 + \not{u})(m_1 + \not{p}_1)]\varphi(\mathbf{k}_2)]_{p_{2+} = \xi P_+, p_{2-} = [(\mathbf{p}_{2\perp})^2 + m_2^2]/(2P_+\xi)}. \tag{28}$$

Making use of the definition of the four-vector u (16) and computing the trace particularizing, respectively, to $\Gamma = \frac{\not{n}_-}{v \cdot n_-}\gamma_5$

and $\Gamma = (2\rlap{/}{p} - \frac{\rlap{/}{n}_-}{v \cdot n_-})\gamma_5$, one gets

$$
\Phi_+^B(\xi) = \sqrt{2P^0}\frac{1}{f_B m_B}\frac{\sqrt{N_c}}{2}\frac{1}{\xi}\frac{1}{2\pi}\left\{\int\frac{d\mathbf{p}_{2\perp}}{(2\pi)^2}\sqrt{2}\frac{\sqrt{u^0}}{p_1^0}\sqrt{\frac{k_1^0 k_2^0}{(k_1^0+m_1)(k_2^0+m_2)}}\left(1+\frac{m_1+m_2}{M_0}\right)\right.
$$

$$
\left.\times\left[\frac{m_1(p_2\cdot n_-)+m_2(p_1\cdot n_-)}{v\cdot n_-}\right]\varphi(\mathbf{k}_2)\right\}_{p_{2+}=\xi P_+, p_{2-}=((\mathbf{p}_{2\perp})^2+m_2^2)/(2P_+\xi)}
$$

$$
\Phi_-^B(\xi) = \sqrt{2P^0}\frac{1}{f_B m_B}\frac{\sqrt{N_c}}{2}\frac{1}{\xi}\frac{1}{2\pi}\left[\int\frac{d\mathbf{p}_{2\perp}}{(2\pi)^2}\sqrt{2}\frac{\sqrt{u^0}}{p_1^0}\sqrt{\frac{k_1^0 k_2^0}{(k_1^0+m_1)(k_2^0+m_2)}}\left(1+\frac{m_1+m_2}{M_0}\right)\right.
$$

$$
\left.\times\left\{2[m_1(p_2\cdot v)+m_2(p_1\cdot v)]-\frac{m_1(p_2\cdot n_-)+m_2(p_1\cdot n_-)}{v\cdot n_-}\right\}\varphi(\mathbf{k}_2)\right]_{p_{2+}=\xi P_+, p_{2-}=((\mathbf{p}_{2\perp})^2+m_2^2)/(2P_+\xi)}.
$$

(29)

A. Heavy quark limit

At finite mass the expressions (29) are not covariant. In the heavy quark limit, using now the variable $\omega = m_1\xi$,

$$
u\to v_1\to v \qquad k_1^0\to m_1, \qquad k_2^0\to p_2\cdot v \qquad M_0\to m_1\to m_B \qquad m_1\xi\to\omega \tag{30}
$$

one obtains, denoting $\mathbf{p}_{2\perp} = \mathbf{p}_\perp$

$$
\varphi_+^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}}\sqrt{2}\frac{1}{8\pi^2}\int d\mathbf{p}_\perp^2\sqrt{\frac{\mathbf{p}_\perp^2+m_2^2+\omega^2}{\mathbf{p}_\perp^2+(\omega+m_2)^2}}\frac{\omega+m_2}{\omega}\varphi\left(\frac{\sqrt{(\mathbf{p}_\perp^2+\omega^2+m_2^2)^2-4\omega^2 m_2^2}}{2\omega}\right)
$$

$$
\varphi_-^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}}\sqrt{2}\frac{1}{8\pi^2}\int d\mathbf{p}_\perp^2\sqrt{\frac{\mathbf{p}_\perp^2+\omega^2+m_2^2}{\mathbf{p}_\perp^2+(\omega+m_2)^2}}\frac{\mathbf{p}_\perp^2+m_2(\omega+m_2)}{\omega^2}\varphi\left(\frac{\sqrt{(\mathbf{p}_\perp^2+\omega^2+m_2^2)^2-4\omega^2 m_2^2}}{2\omega}\right).
$$

(31)

In the heavy quark limit, the LCDA $\varphi_\pm^B(\omega)$ are covariant, since the boost is along Oz, and the variable ω can be written in the covariant form,

$$
\omega = m_B\xi = m_B\frac{p_{2+}}{P_+} = m_B\frac{p_2\cdot n_-}{P\cdot n_-} = \frac{p_2\cdot v_-}{v\cdot n_-}. \tag{32}
$$

Performing the change of variables

$$
p = \frac{\sqrt{(\mathbf{p}_\perp^2+\omega^2+m_2^2)^2-4\omega^2 m_2^2}}{2\omega} \tag{33}
$$

one obtains

$$
\varphi_+^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}}\sqrt{2}\frac{1}{4\pi^2}(\omega+m_2)\int_{p_0(\omega)}^\infty dp\frac{p}{(p^2+m_2^2)^{1/4}(\sqrt{p^2+m_2^2}+m_2)^{1/2}}\varphi(p)
$$

$$
\varphi_-^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}}\sqrt{2}\frac{1}{4\pi^2}\int_{p_0(\omega)}^\infty dp\frac{p(2\sqrt{p^2+m_2^2}-\omega+m_2)}{(p^2+m_2^2)^{1/4}(\sqrt{p^2+m_2^2}+m_2)^{1/2}}\varphi(p)
$$

(34)

with

$$
p_0(\omega) = \frac{|\omega-m_2|(\omega+m_2)}{2\omega}. \tag{35}
$$

One can check the normalization (8) by changing the order of the integrals

$$
\omega > 0, \quad p > p_0(\omega) \quad \Leftrightarrow \quad p > 0,
$$
$$
\omega_-(p) < \omega < \omega_+(p) \tag{36}
$$

with

$$
\omega_\pm(p) = \sqrt{p^2+m_2^2}\pm p. \tag{37}
$$

The integrals over ω are trivial

$$
\int_{\omega_-(p)}^{\omega_+(p)}d\omega(m_2+\omega) = \int_{\omega_-(p)}^{\omega_+(p)}d\omega(2\sqrt{p^2+m_2^2}-\omega+m_2)
$$

$$
= 2p(\sqrt{p^2+m_2^2}+m_2) \tag{38}
$$

and the normalization (8) follows.

B. Limit of vanishing light quark mass

In the limit of vanishing light quark mass, one can immediately demonstrate from expressions (34) that $\varphi_\pm^B(\omega)$ satisfy the differential equation, the so-called Wandzura-Wilczek approximation [4] [31]

$$\frac{d\varphi_-^B(\omega)}{d\omega} = -\frac{\varphi_+^B(\omega)}{\omega} \tag{39}$$

or, equivalently, the integral relation

$$\varphi_-^B(\omega) = \int_\omega^\infty d\rho \, \frac{\varphi_+^B(\rho)}{\rho}. \tag{40}$$

Moreover, for vanishing light quark mass, one gets from (34), for small ω,

$$\varphi_+^B(\omega) \sim \omega \qquad \varphi_-^B(\omega) \sim \text{Constant} \qquad (m_2 = 0). \tag{41}$$

The relation (39) or (40) holds in QCD if one restricts to the valence quark sector of Fock space [4,31]. It is reassuring that this relation holds also in the BT quark model, since it is obviously formulated in the valence quark approximation. For this relation to be satisfied one needs nevertheless the dynamical light quark mass to vanish. However, it will be interesting to grasp the significance of the corrections due to a nonvanishing m_2 and check its numerical effect on $\varphi_\pm^B(\omega)$, as we will do below.

C. Behavior of $\varphi_\pm^B(\omega)$ for nonvanishing light quark mass

A specific trend of our results for the LCDA $\varphi_\pm^B(\omega)$ is the important role played by the light quark mass m_2, as will show the numerical results of Sec. VIII. The light quark mass has dramatic consequences, namely, that the first derivative of $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ vanish, as we can see by inspection of formulas (34) and (35):

$$\varphi_+^{B\prime}(0) = 0 \qquad \varphi_-^B(0) = 0. \tag{42}$$

This is apparently at odds with the general belief, based on QCDSR, that predicts the behavior (41) [2,6–8]. The relation (40) is strongly violated in the presence of a constituent light quark mass, since the l.h.s. of (40) vanishes: $\varphi_-^B(\omega)$ vanishes for $\omega \to 0$, since the lower limit (35) of the integral behaves like

$$p_0(\omega) \sim \frac{m_2^2}{\omega} \to \infty \quad \text{for } \omega \to 0. \tag{43}$$

In the QCDSR approach one has an expression giving the correction to the behavior (41) from the $\langle \bar{q}q \rangle$ condensate, i.e. the first order correction due to the *dynamical* light quark mass, the gap induced by dynamical chiral symmetry breaking. This is the constituent mass involved in our quark model calculation, proportional to the condensate. The corrections at first order in $\langle \bar{q}q \rangle$ have been given by Grozin and Neubert [2] that write, in a simplified notation,

$$\varphi_+^B(\omega) = \varphi_+^{B(0)}(\omega) - \langle \bar{q}q \rangle \tilde{f}(\omega)$$
$$\varphi_-^B(\omega) = \varphi_-^{B(0)}(\omega) - \langle \bar{q}q \rangle \tilde{f}(\omega) \tag{44}$$

where $\varphi_\pm^B(\omega)$ are LCDA at leading order, independent of $\langle \bar{q}q \rangle$, satisfying (39), and the function $\tilde{f}(\omega)$ has the behavior

$$\tilde{f}(0) = \tilde{f}(\infty) = 0. \tag{45}$$

Notice moreover that the correction dependent of $\langle \bar{q}q \rangle$ is the same for $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$.

In our scheme, the first order correction in m_2 to $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ are also equal, but our calculations contain higher orders in the constituent light quark mass, as we observe from (31), (34), and (35), giving the much stronger behavior (42).

D. Moments of $\varphi_\pm^B(\omega)$

Defining the moments

$$M_+^{(N)} = \int_0^\infty d\omega \, \omega^N \varphi_+^B(\omega) \qquad M_-^{(N)} = \int_0^\infty d\omega \, \omega^N \varphi_-^B(\omega) \tag{46}$$

one finds for $m_2 \neq 0$ that $M_+^{(0)} = M_-^{(0)} = 1$, i.e. the normalization condition (8). Moreover, one finds for any moment with $N \geq 0$, for vanishing light quark mass $m_2 = 0$,

$$M_+^{(N)} = (N+1)M_-^{(N)} \tag{47}$$

that holds obviously in the BT class of quark models since it follows from the QCD relation (39). Of interest are the moments $M_+^{(1)}$, $M_-^{(1)}$ that are given in the valence sector of QCD by [2]

$$M_+^{(1)} = 2M_-^{(1)} = \frac{4\bar{\Lambda}}{3} \tag{48}$$

and should allow us to compute $\bar{\Lambda}$ in the BT models.

The moment $M_+^{(-1)} = \lambda_B^{-1}$, satisfies, due to the positivity condition in the absence of a radiative tail $\varphi_+(\omega) \geq 0$ [5],

$$\lambda_B^{-1} = M_+^{(-1)} \geq \frac{1}{M_+^{(1)}} = \frac{3}{4\bar{\Lambda}}. \tag{49}$$

VII. THE SHAPE FUNCTION $S(\omega)$ IN THE BT APPROACH

A. Definitions of the shape function

The shape function $S(\omega)$ that enters in the description of the decay $B \to X_s \gamma$ [22,23] is defined by the expression

$$S(\omega) \frac{1}{2} \text{Tr}\left(\Gamma \frac{1 + \not{v}}{2}\right) = \frac{\langle \bar{B}(p_B)|\bar{h}_v \Gamma \delta(\omega - iD_+)h_v|\bar{B}(p_B)\rangle}{2m_B} \tag{50}$$

where $p_+ = p^0 + p^z$ and the support of $S(\omega)$ is

$$-\infty < \omega \le \bar{\Lambda}. \tag{51}$$

One uses also another definition [24,25]

$$\hat{S}(\hat{\omega}) = S(\bar{\Lambda} - \hat{\omega}) \tag{52}$$

with the support

$$0 \le \hat{\omega} < \infty. \tag{53}$$

The functions $S(\omega)$ or $\hat{S}(\hat{\omega})$ can also be computed in the BT model.

B. Calculation of the shape function in the BT model

Since the field h_v annihilates the b quark within the $\bar{B}$ meson, one can introduce a complete set of intermediate states of the spectator quark $|p_2, s_2\rangle\langle p_2, s_2|$ in the preceding expression

$$\langle \bar{B}(p_B)|\bar{h}_v(x)\delta(\omega - iD_+)h_v(x)|\bar{B}(p_B)\rangle$$
$$= \sum_{p_2, s_2} \langle \bar{B}(p_B)|\bar{h}_v(x)|p_2, s_2\rangle$$
$$\times \langle p_2, s_2|\delta(\omega - iD_+)h_v(x)|\bar{B}(p_B)\rangle \tag{54}$$

with

$$\langle p_2, s_2|h_v(x)|\bar{B}(p_B)\rangle = e^{i(p_2 - p_B + m_b v)\cdot x}\langle p_2, s_2|h_v(0)|\bar{B}(p_B)\rangle$$
$$\langle \bar{B}(p_B)|\bar{h}_v(x)|p_2, s_2\rangle = e^{-i(p_2 - p_B + m_b v)\cdot x}\langle \bar{B}(p_B)|\bar{h}_v(0)|p_2, s_2\rangle. \tag{55}$$

We use now for the operator $\delta(\omega - iD_+)$ the following representation

$$\delta(\omega - iD_+) = \frac{1}{2\pi}\int_{-\infty}^{\infty}ds\, e^{i(\omega - iD_+)s} \tag{56}$$

and make the identification in the quark model, that follows from the hypothesis (9)

$$D_+ \to \partial_+ = \frac{\partial}{\partial x^+}. \tag{57}$$

One obtains, for any function $f(x^+)$

$$e^{s\partial_+}f(x^+) = \sum_{n=0}^{\infty}\frac{1}{n!}(s\partial_+)^n f(x^+) = f(x^+ + s) \tag{58}$$

from $p \cdot x = p_+ x_- + p_- x_+ - \mathbf{p}_\perp \cdot \mathbf{x}_\perp$ with $p_\pm = p^0 \pm p^z$ and $\partial_+ = \frac{\partial}{\partial x^+} = \frac{\partial}{\partial x_-}$ one finds

$$e^{s\partial_+}e^{i(p_2 - p_B + m_b v)\cdot x} = e^{i\{(p_2 - p_B + m_b v)_+(x_- + s) + (p_2 - p_B + m_b v)_- x_+ - (\mathbf{p}_2 - \mathbf{p}_B + m_b \mathbf{v})_\perp \cdot \mathbf{x}_\perp\}} \tag{59}$$

and therefore

$$\delta(\omega - i\partial_+)e^{i(p_2 - p_B + m_b v)\cdot x} = \delta[\omega + (p_2 - p_B + m_b v)_+]e^{i(p_2 - p_B + m_b v)\cdot x} \tag{60}$$

hence, in the quark model

$$S(\omega)\frac{1}{2}\mathrm{Tr}\left(\Gamma\frac{1 + \slashed{v}}{2}\right) = \frac{\langle \bar{B}(p_B)|\bar{h}_v(x)\delta(\omega - i\partial_+)h_v(x)|\bar{B}(p_B)\rangle}{2m_B} = \frac{\langle \bar{B}(p_B)|\bar{h}_v(x)\delta[\omega + (p_2 - p_B + m_b v)_+]h_v(x)|\bar{B}(p_B)\rangle}{2m_B}. \tag{61}$$

In the $\bar{B}$ rest fame $m_b v_+ = m_b$, $p_B = (m_B, \mathbf{0})$ and therefore, from $\bar{\Lambda} = m_B - m_b$ one gets

$$S(\omega)\frac{1}{2}\mathrm{Tr}\left(\Gamma\frac{1 + \slashed{v}}{2}\right) = \frac{\langle \bar{B}(p_B)|\bar{h}_v(x)\Gamma\delta(\omega + p_{2+} - \bar{\Lambda})h_v(x)|\bar{B}(p_B)\rangle}{2m_B}. \tag{62}$$

One obtains for the shape function (52)

$$\hat{S}(\hat{\omega})\frac{1}{2}\mathrm{Tr}\left(\Gamma\frac{1 + \slashed{v}}{2}\right) = \frac{\langle \bar{B}(p_B)|\bar{h}_v(x)\Gamma\delta(\hat{\omega} - p_{2+})h_v(x)|B(p_B)\rangle}{2m_B}. \tag{63}$$

In these expressions p_{2+} is the component $p_{2+} = p_2^0 + p_2^z$ of the spectator light quark.

Let us now compute explicitly this last expression in the BT model, proceeding along the same lines as we have done in Sec. VI for the LCDA. One needs to compute the matrix element of the operator $\Gamma\delta(\hat{\omega} - p_{2+})$ in the forward direction

$$\langle \bar{B}(\mathbf{P})|\Gamma\delta(\hat{\omega} - p_{2+})|\bar{B}(\mathbf{P})\rangle = \int\frac{d\mathbf{p}_2}{(2\pi)^3}\frac{1}{p_2^0}[\varphi(\mathbf{k}_2)]^2\delta(\omega - p_{2+})\frac{u^0}{(p_1^0)}\frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}\frac{1}{16}$$
$$\times \mathrm{Tr}[\Gamma(m_1 + \slashed{p}_1)(1 + \slashed{u})(m_2 + \slashed{p}_2)(1 + \slashed{u})(m_1 + \slashed{p}_1)] \tag{64}$$

where k_i are related to p_i through the boost (16), $\mathbf{B}_u k_i = p_i$. Changing the measure, and taking into account that now $p_{2+} = p_2^0 + p_2^z$, one can write

$$\langle \bar{B}(\mathbf{P})|\Gamma\delta(\hat{\omega} - p_{2+}|\bar{B}(\mathbf{P})\rangle = \int \frac{dp_{2+}\,dp_{2-}}{p_{2+}} \frac{1}{2\pi} \delta\left(p_{2-} - \frac{\mathbf{p}_\perp^2 + m_2^2}{p_{2+}}\right) \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{u^0}{(p_1^0)} \frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)}$$

$$\times [\varphi(\mathbf{k}_2)]^2 \delta(\hat{\omega} - p_{2+}) \frac{1}{16} \text{Tr}[\Gamma(m_1 + \not{p}_1)(1 + \not{u})(m_2 + \not{p}_2)(1 + \not{u})(m_1 + \not{p}_1)]. \quad (65)$$

Integrating relatively to p_{2+} and p_{2-}

$$\langle \bar{B}(\mathbf{P}|\Gamma\delta(\hat{\omega} - p_{2+})|\bar{B}(\mathbf{P})\rangle = \frac{1}{16} \frac{1}{2\pi} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{1}{\hat{\omega}} \left\{ [\varphi(\mathbf{k}_2)]^2 \frac{u^0}{(p_1^0)^2} \frac{k_1^0 k_2^0}{(k_1^0 + m_1)(k_2^0 + m_2)} \right.$$

$$\left. \times \text{Tr}[\Gamma(m_1 + \not{p}_1)(1 + \not{u})(m_2 + \not{p}_2)(1 + \not{u})(m_1 + \not{p}_1)] \right\}_{p_{2+}=\hat{\omega},\, p_{2-}=(\mathbf{p}_\perp^2 + m_2^2)/\hat{\omega}}. \quad (66)$$

C. Heavy quark limit

In the heavy mass limit one has $u \to v_1 \to v$, $k_2^0 \to p_2 \cdot v$ and $M_0 \to m_1 \to m_B$ and therefore, after some algebra,

$$\langle \bar{B}(\mathbf{P}|\Gamma\delta(\hat{\omega} - p_{2+})|\bar{B}(\mathbf{P})\rangle = \frac{1}{8} \frac{1}{2\pi} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{1}{\hat{\omega}} \left\{ [\varphi(\sqrt{(p_2 \cdot v)^2 - m_2^2})]^2 \frac{1}{v^0} \frac{p_2 \cdot v}{p_2 \cdot v + m_2} \right.$$

$$\left. \times \text{Tr}[\Gamma(1 + \not{v})(m_2 + \not{p}_2)(1 + \not{v})] \right\}_{p_{2+}=\hat{\omega},\, p_{2-}=(\mathbf{p}_\perp^2 + m_2^2)/\hat{\omega}} \quad (67)$$

and from

$$\text{Tr}[\Gamma(1 + \not{v})(m_2 + \not{p}_2)(1 + \not{v})] = 8[m_2 + (p_2 \cdot v)] \frac{1}{2} \text{Tr}\left(\Gamma \frac{1 + \not{v}}{2}\right) \quad (68)$$

one gets finally

$$\langle \bar{B}(\mathbf{P}|\Gamma\delta(\hat{\omega} - p_{2+})|\bar{B}(\mathbf{P})\rangle = \frac{1}{2} \text{Tr}\left(\Gamma \frac{1 + \not{v}}{2}\right) \frac{1}{2\pi} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{1}{\hat{\omega}} \left\{ [\varphi(\sqrt{(p_2 \cdot v)^2 - m_2^2})]^2 \frac{1}{v^0}(p_2 \cdot v) \right\}_{p_{2+}=\hat{\omega},\, p_{2-}=(\mathbf{p}_\perp^2 + m_2^2)/\hat{\omega}}. \quad (69)$$

Let us identify with the definition (52). With the normalization of the BT model (20) we have to identify the matrix element (69) with

$$\frac{\langle \bar{B}(\mathbf{P})|\Gamma\delta(\hat{\omega} - p_{2+})|\bar{B}(\mathbf{P})\rangle}{2M_B} = \frac{1}{2} \text{Tr}\left(\Gamma \frac{1 + \not{v}}{2}\right) \frac{\hat{S}(\hat{\omega})}{2P^0}. \quad (70)$$

Therefore one gets

$$\hat{S}(\hat{\omega}) = \frac{1}{2} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{1}{\omega} \{(p_2 \cdot v)$$

$$\times [\varphi(\sqrt{(p_2 \cdot v)^2 - m_2^2})]^2\}_{p_{2+}=\hat{\omega},\, p_{2-}=(\mathbf{p}_\perp^2 + m_2^2)/\hat{\omega}} \quad (71)$$

and from

$$p_2 \cdot v = \frac{\mathbf{p}_\perp^2 + m_2^2 + \hat{\omega}^2}{2\hat{\omega}} \quad (72)$$

one obtains finally

$$\hat{S}(\hat{\omega}) = \frac{1}{2} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \frac{\mathbf{p}_\perp^2 + m_2^2 + \hat{\omega}^2}{2\hat{\omega}}$$

$$\times \left[\varphi\left(\frac{\sqrt{(\mathbf{p}_\perp^2 + \hat{\omega}^2 + m_2^2)^2 - 4\hat{\omega}^2 m_2^2}}{2\hat{\omega}} \right) \right]^2. \quad (73)$$

Again, similarly to the LCDA, one can obtain a more compact form of $\hat{S}(\hat{\omega})$ by performing the change of variables

$$k = \frac{\sqrt{(\mathbf{p}_\perp^2 + \hat{\omega}^2 + m_2^2)^2 - 4\hat{\omega}^2 m_2^2}}{2\hat{\omega}} \quad (74)$$

and one obtains the simple final result for the shape function in the BT model

$$\hat{S}(\hat{\omega}) = \frac{1}{4\pi^2} \int_{k_0(\hat{\omega})}^\infty dk k [\varphi(k)]^2 \qquad (0 \le \hat{\omega} < \infty) \quad (75)$$

with

$$k_0(\hat{\omega}) = \frac{|\hat{\omega} - m_2|(\hat{\omega} + m_2)}{2\hat{\omega}} \quad (76)$$

that gives the value $\hat{\omega}_{\text{max}}$ for which $\hat{S}(\hat{\omega})$ attains its maximum value $\hat{S}_{\text{max}} = \hat{S}(\hat{\omega}_{\text{max}})$,

$$\hat{\omega}_{\text{max}} = m_2 \qquad \hat{S}_{\text{max}} = \frac{1}{4\pi^2} \int_0^\infty dk k [\varphi(k)]^2. \quad (77)$$

For the other definition of the shape function $S(\omega)$ (50) one reads, from (52),

$$S(\omega) = \frac{1}{4\pi^2} \int_{\kappa_0(\omega)}^\infty dk k [\varphi(k)]^2 \quad (78)$$

with

$$\kappa_0(\omega) = \frac{|\bar{\Lambda} - \omega - m_2|(\bar{\Lambda} - \omega + m_2)}{2(\bar{\Lambda} - \omega)} \quad (79)$$

$$(-\infty < \omega \le \bar{\Lambda}).$$

To check the normalization condition

$$\int_0^\infty d\hat{\omega} \, \hat{S}(\hat{\omega}) = 1 \quad (80)$$

we use (78) and exchange the order of the integrals over $\hat{\omega}$ and k,

$$\hat{\omega} > 0, \quad k > k_0(\hat{\omega}) \quad \Leftrightarrow \quad k > 0,$$
$$\hat{\omega}_-(k) < \hat{\omega} < \hat{\omega}_+(k) \quad (81)$$

where

$$\hat{\omega}_\pm(k) = \sqrt{k^2 + m_2^2} \pm k. \quad (82)$$

One obtains

$$\int_0^\infty d\hat{\omega} \, \hat{S}(\hat{\omega}) = \frac{1}{4\pi^2} \int_0^\infty dk k [\varphi(k)]^2 \int_{\hat{\omega}_-(k)}^{\hat{\omega}_+(k)} d\hat{\omega}$$
$$= \frac{1}{2\pi^2} \int_0^\infty dk k^2 [\varphi(k)]^2 = 1 \quad (83)$$

from the normalization of the internal wave function (15).

From (52) and (83) it follows, for the function $S(\omega)$,

$$\int_{-\infty}^{\bar{\Lambda}} d\omega S(\omega) = 1. \quad (84)$$

D. Gaussian wave function

In principle, one could calculate the wave function $\varphi(k)$ entering in $\varphi_\pm^B(\omega)$ (34) and $S(\omega)$ (78) from the quark potential. However, as argued in the Introduction, we prefer here to adopt simply a Gaussian wave function, that has been used elsewhere to obtain $S(\omega)$ and to fit the $B \to X_s \gamma$ spectrum [20–22,26].

Let us compute the shape function $S(\omega)$ using the harmonic oscillator potential, with the internal wave function at rest $\varphi(\mathbf{k}_2)$ is given by

$$\varphi_{\text{HO}}(\mathbf{k}_2) = (2\pi)^{3/2} \left(\frac{R^2}{\pi}\right)^{3/4} \exp\left(-\frac{R^2 \mathbf{k}_2^2}{2}\right). \quad (85)$$

We obtain

$$S(\omega) = \frac{R}{\sqrt{\pi}} \exp\left\{-\frac{1}{4} \frac{[(\bar{\Lambda} - \omega)^2 - m_2^2]^2}{R^2(\bar{\Lambda} - \omega)^2}\right\}. \quad (86)$$

With the notation

$$R = \frac{1}{p_F} \qquad \rho = \frac{m_2^2}{p_F^2} \qquad x = \frac{\omega}{\bar{\Lambda}} \quad (87)$$

one gets

$$S(\omega) = \frac{1}{\sqrt{\pi}} \frac{1}{p_F} \exp\left\{-\frac{1}{4}\left[\frac{p_F}{\bar{\Lambda}} \frac{\rho}{1 - x} - \frac{\bar{\Lambda}}{p_F}(1 - x)\right]^2\right\}$$
$$\left(x = \frac{\omega}{\bar{\Lambda}}\right) \quad (88)$$

that is correctly normalized to 1.

We obtain therefore in the BT model with harmonic oscillator potential the so-called "roman" shape function [20–22,26].

VIII. PREDICTIONS FOR THE LCDA $\varphi_\pm^B(\omega)$

A. Parameters of the roman shape function

Limosani and Nozaki [26] have made a recent fit to the Belle $B \to X_s \gamma$ data [32], in order to extract the b-quark shape function parameters $\bar{\Lambda}_{\text{SF}}$ and λ_1^{SF} ($\lambda_1^{\text{SF}} = -\mu_\pi^2$) in a number of models for the shape function, among them the roman shape function. Writing (87) in their notation

$$\omega = k_+ \qquad F(k_+) = S(\omega) \quad (89)$$

$$F(k_+) = N \frac{\kappa}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left[\frac{1}{\kappa} \frac{\rho}{1 - x} - \kappa(1 - x)\right]^2\right\} \quad (90)$$

with

$$x = \frac{k_+}{\bar{\Lambda}_{\text{SF}}} \qquad \kappa = \frac{\bar{\Lambda}_{\text{SF}}}{p_F} \qquad \rho = \frac{m_2^2}{p_F^2} \quad (91)$$

amounts to replace in (88) $\bar{\Lambda}$ by $\bar{\Lambda}_{\text{SF}}$ of the shape function renormalization scheme [33]. Of course, the BT quark model is not field theory, and this scheme is too rough to distinguish between $\bar{\Lambda}$ and $\bar{\Lambda}_{\text{SF}}$, that differ by QCD radiative corrections. Therefore, we can make the replacement of $\bar{\Lambda}$ by $\bar{\Lambda}_{\text{SF}}$ in the formulas of the precedent section, and this will be our final model for the shape function.

The first moments of $F(k_+)$

$$A_n = \int_{-\infty}^{\bar{\Lambda}_{\text{SF}}} dk_+ k_+^n F(k_+) \quad (92)$$

are given in QCD by

$$A_0 = 1 \qquad A_1 = 0 \qquad A_2 = -\frac{\lambda_1^{\text{SF}}}{3}. \quad (93)$$

Computing them explicitly using the shape function (90) one obtains

$$A_n = N(\bar{\Lambda}_{SF})^{n+1} \frac{\kappa}{\sqrt{\pi}} e^{\rho/2} \sum_k (-1)^k \binom{n}{k}$$

$$\times \left(\frac{\rho}{\kappa^2}\right)^{(k+1)/2} K_{(k+1)/2}\left(\frac{\rho}{2}\right) \quad (94)$$

and the conditions (93) yield respectively

$$N = \frac{1}{\bar{\Lambda}_{SF}} \qquad \kappa = \frac{\rho}{\sqrt{\pi}} e^{\rho/2} K_1\left(\frac{\rho}{2}\right)$$
$$\lambda_1^{SF} = -3\left[\frac{2}{\kappa^2}\left(\frac{1+\rho}{2}\right) - 1\right](\bar{\Lambda}_{SF})^2. \quad (95)$$

The best fit of Limosani and Nozaki [26] gives

$$\bar{\Lambda}_{SF} = 0.66 \text{ GeV} \qquad \lambda_1^{SF} = -0.39 \text{ GeV}^2 \quad (96)$$

that corresponds, using (95), to the values

$$\rho = 0.776 \qquad \kappa = 1.462 \quad (97)$$

or, from (87) and (91), to the quark model parameters

$$R = \frac{1}{p_F} = 2.216 \text{ GeV}^{-1} \qquad m_2 = 0.398 \text{ GeV}. \quad (98)$$

On can also compute the value $(k_+)_{max}$ of k_+, for which $F(k_+)$ becomes maximum, $F_{max} = F[(k_+)_{max}]$,

$$(k_+)_{max} = \left(1 - \frac{\sqrt{\rho}}{\kappa}\right)\bar{\Lambda}_{SF} \qquad F_{max} = N\frac{\kappa}{\sqrt{\pi}} \quad (99)$$

or numerically,

$$(k_+)_{max} = 0.262 \qquad F_{max} = 1.250 \text{ GeV}^{-1}. \quad (100)$$

The other definition of the roman shape function $\hat{S}(\hat{\omega})$ (52) has a very simple expression in the BT model

$$\hat{S}(\hat{\omega}) = \frac{R}{\sqrt{\pi}} \exp\left[-\frac{R^2}{4}\left(\frac{m_2^2}{\hat{\omega}} - \hat{\omega}\right)^2\right] \quad (101)$$

that is identical to $\hat{S}(\hat{\omega})_{roman}$ with

$$\rho = R^2 m_2^2 \qquad \bar{\Lambda}_{SF} = \frac{R^2 m_2^2}{\sqrt{\pi}} \exp\left(\frac{R^2 m_2^2}{2}\right) K_1\left(\frac{R^2 m_2^2}{2}\right)$$
$$\lambda_1^{SF} = -3\left[\frac{2}{R^2} + m_2^2 - (\bar{\Lambda}_{SF})^2\right] \quad (102)$$

and

$$\hat{\omega}_{max} = 0.398 \text{ GeV} \qquad \hat{S}_{max} = 1.250 \text{ GeV}^{-1}. \quad (103)$$

B. Predictions for $\varphi_\pm^B(\omega)$

Using the harmonic oscillator wave function (85), the parameters (98) and expressions (31) or (34) we can predict the LCDA $\varphi_\pm^B(\omega)$ within the BT model.

In the limit of vanishing light quark mass $m_2 = 0$, in which the relation (39) between $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ holds, one obtains simple analytic expressions in terms of the error function $\Phi(x)$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}. \quad (104)$$

For the decay constant one gets

$$f_B\sqrt{m_B} = \sqrt{N_c} \frac{1}{\pi^{3/4}} \sqrt{\frac{R}{2}} \int_0^\infty d\omega\,\omega\left[1 - \Phi\left(\frac{R\omega}{2\sqrt{2}}\right)\right] \quad (105)$$

and for the LCDA,

$$\varphi_+^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}} \frac{1}{\pi^{3/4}} \sqrt{\frac{R}{2}} \omega\left[1 - \Phi\left(\frac{R\omega}{2\sqrt{2}}\right)\right]$$

$$\varphi_-^B(\omega) = \frac{\sqrt{N_c}}{f_B\sqrt{m_B}} \frac{1}{\pi^{3/4}} \sqrt{\frac{2}{R}}\left\{\sqrt{\frac{2}{\pi}} \exp\left(-\frac{R^2\omega^2}{8}\right)\right.$$
$$\left. - \frac{R\omega}{2}\left[1 - \Phi\left(\frac{R\omega}{2\sqrt{2}}\right)\right]\right\}. \quad (106)$$

To illustrate numerical results, and give a feeling of the dependence on the light quark mass, we chose two sets of parameters:

(1) The realistic case of the parameters of the quark wave function (98) $R = 2.216 \text{ GeV}^{-1}$ and $m_2 = 0.398 \text{ GeV}$. From these parameters fitted on the $B \to X_s\gamma$ spectrum we predict

$$f_B\sqrt{m_B} = 0.388 \text{ GeV}^{3/2} \quad (107)$$

and the functions $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ that are plotted, respectively, in Figs. 1 and 2. The heavy quark limit value (107) gives, using the physical B mass, $f_B \cong 170$ MeV, a little smaller than the popular values for this quantity.

(2) For comparison we adopt the vanishing light quark mass case, taking the same radius $R = 2.216 \text{ GeV}^{-1}$ but $m_2 = 0$. We find $f_B\sqrt{m_B} = 0.315 \text{ GeV}^{3/2}$ and the functions $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ that are also plotted in Figs. 1 and 2.
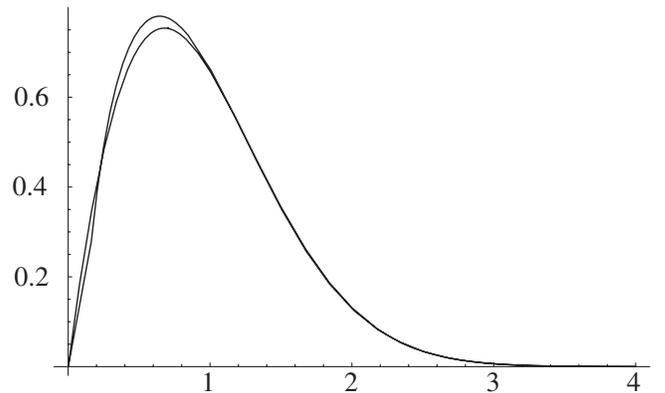
FIG. 1. The function $\varphi_+^B(\omega)$ for a Gaussian wave function for the sets of parameters (98) obtained from the roman shape function [26] $(R, m_2) = (2.216 \text{ GeV}^{-1}, 0.398 \text{ GeV})$ (higher curve) and $(R, m_2) = (2.216 \text{ GeV}^{-1}, 0)$ (lower curve). For nonvanishing m_2 the first derivative vanishes at $\omega = 0$.
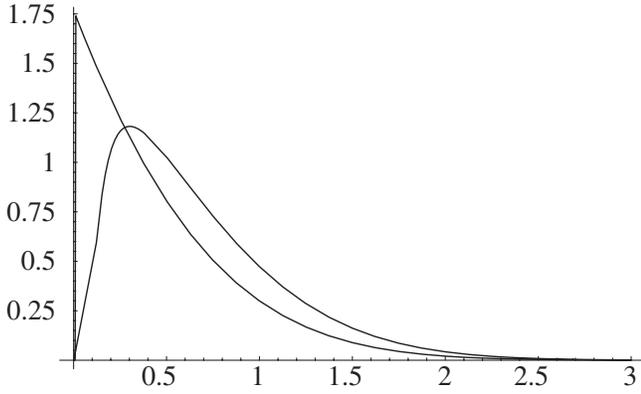
FIG. 2. The function $\varphi^B_-(\omega)$ for a Gaussian wave function for the sets of parameters obtained from the roman shape function [26] $(R, m_2) = (2.216\ \text{GeV}^{-1}, 0.398\ \text{GeV})$ and $(R, m_2) = (2.216\ \text{GeV}^{-1}, 0)$. For nonvanishing m_2 the function vanishes at $\omega = 0$.

We observe that the nonvanishing light quark mass gives $\varphi^B_-(0) = 0$ and $\varphi^{B'}_+(0) = 0$, while for vanishing light quark mass one obtains $\varphi^B_-(0) \neq 0$ and also $\varphi^{B'}_+(0) \neq 0$ like in QCD sum rules. Since the vanishing $\varphi^B_-(0) = 0$ for $m_2 \neq 0$ is an unfamiliar feature, we plot in Fig. 3 the evolution of $\varphi^B_-(\omega)$ with increasing values of $m_2 = 0, 0.1, 0.2, 0.3$ GeV.

C. Moments of $\varphi^B_+(\omega)$

Let us now compute the moments of $\varphi^B_\pm(\omega)$ using the harmonic oscillator wave function (85) that gives the roman shape function (90) and (91) with the parameters (98). We obtain for the first moments $M^{(1)}_\pm$

$$M^{(1)\text{BT}}_+ = 2M^{(1)\text{BT}}_- \cong \frac{4\bar{\Lambda}_{\text{SF}}}{3} = 0.964\ \text{GeV} \qquad (108)$$

that gives $\bar{\Lambda}_{\text{SF}} \cong 0.723$ GeV, in qualitative agreement at the 10% level with the value (96) obtained from the fit to $B \to X_s\gamma$, and for the moment $M^{(-1)\text{BT}}_+$, called also λ^{-1}_B
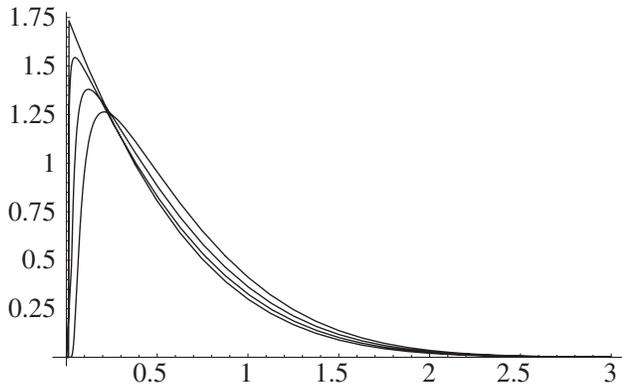
FIG. 3. Evolution of $\varphi^B_-(\omega)$ with the constituent light quark mass for $R = 2.216\ \text{GeV}^{-1}$ and $m_2 = 0, 0.1, 0.2, 0.3$ GeV.

$$(\lambda^{-1}_B)^{\text{BT}} = M^{(-1)\text{BT}}_+ = 1.521\ \text{GeV}^{-1} \qquad (109)$$

IX. RADIATIVE CORRECTIONS

The importance of the radiative tail has been underlined by Braun, Ivanov and Korchemsky [7]. When taking into account one-loop QCD corrections, the moments for $N \geq 0$ of $\varphi^B_+(\omega)$ are divergent, as already pointed out by Grozin and Neubert [2] and by Descotes-Genon and Sachrajda [6].

Braun *et al.* give a parametrization of $\varphi^B_+(\omega, \mu)$ based on QCD sum rules plus the QCD behavior $-\frac{1}{\omega}\log(\frac{\omega}{\mu})$ at large ω,

$$\varphi^B_+(\omega, \mu) = \frac{4\lambda^{-1}_B}{\pi}\frac{\omega\mu}{\omega^2 + \mu^2}$$
$$\times \left[\frac{\mu^2}{\omega^2 + \mu^2} - \frac{2(\sigma_B - 1)}{\pi^2}\log\left(\frac{\omega}{\mu}\right)\right] \qquad (110)$$

where the parameters λ^{-1}_B and σ_B are defined in terms of the integrals

$$\lambda^{-1}_B = \int_0^\infty d\omega\,\frac{\varphi^B_+(\omega, \mu)}{\omega}$$
$$\sigma_B\lambda^{-1}_B = -\int_0^\infty d\omega\,\frac{\varphi^B_+(\omega, \mu)}{\omega}\log\left(\frac{\omega}{\mu}\right) \qquad (111)$$

We realize that, due to the logarithm in the radiative tail, the moments $M^{(N)}_+$ for $N \geq 0$ are divergent.

As explained below in Sec. 10.11, the parametrization (110) is a simplified and approximated form of a full QCD sum rules calculation that includes radiative corrections.

A thorough study of the radiative corrections has been done by Lee and Neubert [9]. The lowest positive moments $M^{(0)}_+$ and $M^{(1)}_+$ are given as a power expansion in the ultraviolet cutoff Λ_{UV}

$$M^{(0)}_+(\Lambda_{UV}) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-2\log^2\left(\frac{\Lambda_{UV}}{\mu}\right) + 2\log\left(\frac{\Lambda_{UV}}{\mu}\right)\right.$$
$$\left. - \frac{\pi^2}{12}\right] + \frac{16\bar{\Lambda}}{3\Lambda_{UV}}\frac{C_F\alpha_s}{4\pi}\left[\log\left(\frac{\Lambda_{UV}}{\mu}\right) - 1\right] \qquad (112)$$

$$M^{(1)}_+(\Lambda_{UV}) = \Lambda_{UV}\frac{C_F\alpha_s}{4\pi}\left[-4\log\left(\frac{\Lambda_{UV}}{\mu}\right) + 6\right]$$
$$+ \frac{4\bar{\Lambda}}{3}\left\{1 + \frac{C_F\alpha_s}{4\pi}\left[-2\log^2\left(\frac{\Lambda_{UV}}{\mu}\right)\right.\right.$$
$$\left.\left. + 8\log\left(\frac{\Lambda_{UV}}{\mu}\right) - \frac{7}{4} - \frac{\pi^2}{12}\right]\right\}. \qquad (113)$$

In the limit $\alpha_s \to 0$, one recovers $M_0 = 1$ and $M_1 = \frac{4\bar{\Lambda}}{3}$. Lee and Neubert define the radiative tail of the function $\varphi^B_+(\omega, \mu)$ by the prescription

$$\varphi_+^{\mathrm{RAD}}(\omega) = \left[\frac{dM_0(\Lambda_{UV}, \mu)}{d\Lambda_{UV}}\right]_{\Lambda_{UV}=\omega} \qquad (114)$$

that gives

$$\varphi_+^{\mathrm{RAD}}(\omega) = \frac{C_F \alpha_s}{\pi\omega}\left\{\left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right)\right]\right.$$
$$\left. + \frac{4\bar{\Lambda}}{3}\frac{1}{\omega}\left[2 - \log\left(\frac{\omega}{\mu}\right)\right]\right\}. \qquad (115)$$

Notice that for large enough ω the radiative tail becomes negative, and that at lowest order it agrees with (110).

To include the radiative tail is not without ambiguity. To add it to our calculation of the long distance part of $\varphi_+^B(\omega)$, we follow two different models, that have different continuity properties, but that lead to almost identical results: first, a similar procedure to the one proposed by Lee *et al.* [9], although different in its details; second, a procedure close to the one followed by Braun *et al.* [7]. We now expose both methods, and compare the results at the end.

A. Model to add the radiative tail following Lee *et al.*

Our first model for the function $\varphi_+^B(\omega)$, including the radiative corrections follows essentially the prescription of [9]

$$\varphi_+^B(\omega) = N\varphi_+^{\mathrm{BT}}(\omega) + \theta(\omega - \omega_t)\varphi_+^{\mathrm{RAD}}(\omega) \qquad (116)$$

where $\varphi_+^{\mathrm{BT}}(\omega)$ is the function (31) or (34) with the harmonic oscillator wave function (85) and the parameters of the roman shape function (98).

Without loss of generality at leading order in α_s, we take the radiative tail $\varphi_+^{\mathrm{RAD}}(\omega)$ as given by expression (115) with $\bar{\Lambda}$ replaced now by $\bar{\Lambda}_{\mathrm{SF}}(\mu)$, as determined from the fit to $B \to X_s \gamma$

$$\varphi_+^{\mathrm{RAD}}(\omega) = \frac{C_F \alpha_s}{\pi\omega}\left\{\left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right)\right]\right.$$
$$\left. + \frac{4\bar{\Lambda}_{\mathrm{SF}}(\mu)}{3}\frac{1}{\omega}\left[2 - \log\left(\frac{\omega}{\mu}\right)\right]\right\}. \qquad (117)$$

The relation between $\bar{\Lambda}_{\mathrm{SF}}(\mu) \equiv \bar{\Lambda}_{\mathrm{SF}}(\mu, \mu)$ and $\bar{\Lambda}$ is [25]

$$\bar{\Lambda} = \bar{\Lambda}_{\mathrm{SF}}(\mu) - \frac{C_F \alpha_s}{4\pi}4\mu. \qquad (118)$$

We choose from now on $\mu = 1.5$ GeV as an illustration,

$$\mu = 1.5 \text{ GeV} \qquad C_F \alpha_s = 0.470. \qquad (119)$$

For this value of μ one has, from $B \to X_s \gamma$ and $B \to X_u \ell \bar{\nu}_\ell$ [9,33],

$$\bar{\Lambda}_{\mathrm{SF}}(\mu) = (0.65 \pm 0.06) \text{ GeV} \qquad (\mu = 1.5 \text{ GeV}) \qquad (120)$$

that is in agreement with the determination (96) from [26].

The relation between ω_t, defined by the vanishing of the radiative tail to ensure continuity, and $\bar{\Lambda}_{\mathrm{SF}}(\mu)$ is given by

$$\frac{1}{2} - \log\left(\frac{\omega_t}{\mu}\right) + \frac{4\bar{\Lambda}_{\mathrm{SF}}(\mu)}{3}\frac{1}{\omega_t}\left[2 - \log\left(\frac{\omega_t}{\mu}\right)\right] = 0. \quad (121)$$

$\bar{\Lambda}_{\mathrm{SF}}(\mu)$ becomes a function of ω_t and N is a parameter to be determined by the matching with the QCD behavior (112) and (113).

In terms of $\bar{\Lambda}_{\mathrm{SF}}(\mu)$, at the lowest order in α_s one can rewrite the moments (112) and (113)

$$M_+^{(0)}(\Lambda_{UV}) = 1 + \frac{C_F \alpha_s}{4\pi}\left[-2\log^2\left(\frac{\Lambda_{UV}}{\mu}\right)\right.$$
$$\left. + 2\log\left(\frac{\Lambda_{UV}}{\mu}\right) - \frac{\pi^2}{12}\right] + \frac{16\bar{\Lambda}_{\mathrm{SF}}(\mu)}{3\Lambda_{UV}}\frac{C_F \alpha_s}{4\pi}$$
$$\times \left[\log\left(\frac{\Lambda_{UV}}{\mu}\right) - 1\right] \qquad (122)$$

$$M_+^{(1)}(\Lambda_{UV}) = \Lambda_{UV}\frac{C_F \alpha_s}{4\pi}\left[-4\log\left(\frac{\Lambda_{UV}}{\mu}\right) + 6\right]$$
$$+ \frac{4\bar{\Lambda}_{\mathrm{SF}}(\mu)}{3}\left\{1 + \frac{C_F \alpha_s}{4\pi}\left[-2\log^2\left(\frac{\Lambda_{UV}}{\mu}\right)\right.\right.$$
$$\left.\left. + 8\log\left(\frac{\Lambda_{UV}}{\mu}\right) - \frac{7}{4} - \frac{\pi^2}{12}\right]\right\} - \frac{C_F \alpha_s}{4\pi}\frac{16\mu}{3}.$$
$$(123)$$

Notice that a new term proportional to α_s appears in the first moment $M_+^{(1)}(\Lambda_{UV})$.

Using the LCDA $\varphi_+^B(\omega)$ from the BT model (34) with the harmonic oscillator wave function (85) and the parameters (98) we then compute the moments in the model

$$M_+^{(0)\mathrm{model}}(\Lambda_{UV}) = \int_0^{\Lambda_{UV}} d\omega \varphi_+^B(\omega)$$
$$M_+^{(1)\mathrm{model}}(\Lambda_{UV}) = \int_0^{\Lambda_{UV}} d\omega \omega \varphi_+^B(\omega) \qquad (124)$$

and match with the OPE expressions (122) and (123).

Making the approximation, that will be discussed below,

$$\int_0^{\Lambda_{UV}} d\omega \varphi_+^B(\omega) \cong \int_0^{\infty} d\omega \varphi_+^B(\omega) \qquad (125)$$

$$\int_0^{\Lambda_{UV}} d\omega \omega \varphi_+^B(\omega) \cong \int_0^{\infty} d\omega \omega \varphi_+^B(\omega) \qquad (126)$$

the matching implies

$$M_+^{(0)}(\Lambda_{UV}) = NM_+^{(0)\mathrm{BT}} + M_+^{(0)}(\Lambda_{UV}) - M_+^{(0)}(\omega_t)$$
$$M_+^{(1)}(\Lambda_{UV}) = NM_+^{(1)\mathrm{BT}} + M_+^{(1)}(\Lambda_{UV}) - M_+^{(1)}(\omega_t) \qquad (127)$$

that gives, since $M_+^{(0)\mathrm{BT}} = 1$,

$$N = M_+^{(0)}(\omega_t) \qquad (128)$$

$$M_+^{(1)\text{BT}} = \frac{M_+^{(1)}(\omega_t)}{M_+^{(0)}(\omega_t)} = \frac{M_+^{(1)}(\omega_t)}{N}. \tag{129}$$

Equation (128) gives N in terms of ω_t

$$N(\omega_t, \mu) = 1 + \frac{C_F \alpha_s}{4\pi} \left[-2\log^2\left(\frac{\omega_t}{\mu}\right) + 2\log\left(\frac{\omega_t}{\mu}\right) - \frac{\pi^2}{12} \right]$$
$$+ \frac{16\bar{\Lambda}_{\text{SF}}(\mu)}{3\omega_t} \frac{C_F \alpha_s}{4\pi} \left[\log\left(\frac{\omega_t}{\mu}\right) - 1 \right] \tag{130}$$

and Eq. (129) gives, expanding to first order in α_s,

$$M_+^{(1)\text{BT}} = \frac{4\bar{\Lambda}_{\text{SF}}(\mu)}{3} \left[1 + \frac{C_F \alpha_s}{4\pi} \left\{ -6\log\left(\frac{\omega_t}{\mu}\right) - \frac{16\bar{\Lambda}_{\text{SF}}(\mu)}{3\omega_t} \right. \right.$$
$$\left. \left. \times \left[\log\left(\frac{\omega_t}{\mu}\right) - 1 \right] \right\} \right] - \frac{C_F \alpha_s}{4\pi} \omega_t \left[4\log\left(\frac{\omega_t}{\mu}\right) - 6 \right]$$
$$- \frac{C_F \alpha_s}{4\pi} \frac{16\mu}{3}. \tag{131}$$

Using the result (108) of the model $M_+^{(1)\text{BT}} = 0.964$ GeV, this equation gives another relation, besides (121), relating ω_t and $\bar{\Lambda}_{\text{SF}}(\mu)$.

From (121) and (131) we can solve for ω_t and $\bar{\Lambda}_{\text{SF}}(\mu)$ for a given value of μ and see if the value obtained for $\bar{\Lambda}_{\text{SF}}(\mu)$ is consistent with the known value of $\bar{\Lambda}_{\text{SF}}(\mu)$ (120) from the fit to $B \to X_s \gamma$. Once ω_t and $\bar{\Lambda}_{\text{SF}}(\mu)$ are known one can compute N and $\bar{\Lambda}$ from (118) and (130).

From (121) and (131) we obtain, for $\mu = 1.5$ GeV,

$$\begin{aligned} \omega_t &= 3.288 \text{ GeV} & N &= 0.974 \\ \bar{\Lambda}_{\text{SF}}(\mu) &= 0.578 \text{ GeV} & \bar{\Lambda} &= 0.354 \text{ GeV.} \end{aligned} \tag{132}$$

These values for $\bar{\Lambda}_{\text{SF}}(\mu)$ and $\bar{\Lambda}$ that come from the OPE constraints are only about 10% lower from the values (120) coming from the fit to $B \to X_s \gamma$ and $B \to X_u \ell \bar{\nu}_\ell$. We conclude that the situation is good enough.

A different status from $\bar{\Lambda}$ has the parameter λ_B^{-1}, that enters in a number of processes that we examine blow. The value that we obtain including the radiative corrections is

$$\lambda_B^{-1} = 1.429 \text{ GeV}^{-1} \tag{133}$$

to be compared with the value (109) without the radiative tail. The correction is small. We find, for the parameter σ_B (111)

$$\sigma_B = 1.207. \tag{134}$$

The function (116) with its radiative tail is plotted in Fig. 4.
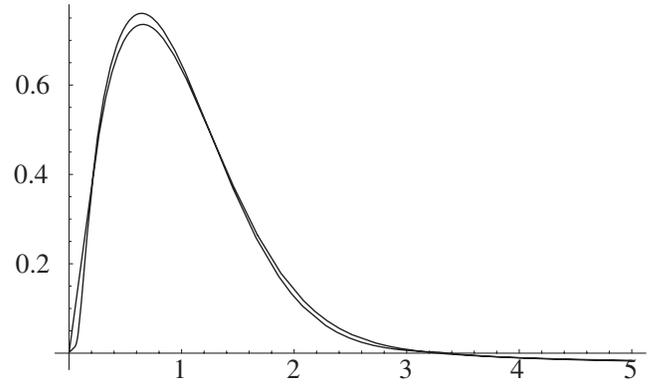
FIG. 4. The function $\varphi_+^B(\omega)$ in the BT approach with its radiative tail adopting the SF renormalization scheme for $\mu = 1.5$ GeV. The two curves show the two ways of gluing the radiative tail, following Lee *et al.* [9] (lower curve) or Braun *et al.* [7] (upper curve).

B. Model to add the radiative tail following Braun *et al.*

This model follows the regularization of Braun *et al.* as illustrated by Eq. (110), but using the full radiative tail of Lee *et al.* (115). We set

$$\varphi_+^B(\omega) = N\varphi_+^{\text{BT}}(\omega) + \varphi_+^{\text{RAD}}(\omega) \tag{135}$$

where $\varphi_+^{\text{BT}}(\omega)$ is the same function (31) or (34) as in (116) and we now regularize the radiative tail by making the replacement $\frac{1}{\omega} \to \frac{\omega}{\omega^2 + \mu^2}$, i.e. we take

$$\varphi_+^{\text{RAD}}(\omega) = \frac{C_F \alpha_s}{\pi} \frac{\omega}{\omega^2 + \mu^2} \left\{ \left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right) \right] \right.$$
$$\left. + \frac{4\bar{\Lambda}_{\text{SF}}(\mu)}{3} \frac{\omega}{\omega^2 + \mu^2} \left[2 - \log\left(\frac{\omega}{\mu}\right) - 6 \right] \right\}. \tag{136}$$

The radiative tails (117) and (136) differ at low ω. Now we do not have to determine the gluing point ω_t. To impose the OPE conditions we proceed as follows. We have two conditions to fulfill

$$\begin{aligned} M_0^{\text{model}}(\Lambda_{UV}) &= M_0^{\text{OPE}}(\Lambda_{UV}) \\ M_1^{\text{model}}(\Lambda_{UV}) &= M_1^{\text{OPE}}(\Lambda_{UV}). \end{aligned} \tag{137}$$

These conditions give, respectively

$$N = 1 + \frac{C_F \alpha_s}{4\pi} \left(\left[-2\log^2\left(\frac{\Lambda_{UV}}{\mu}\right) + 2\log\left(\frac{\Lambda_{UV}}{\mu}\right) - \frac{\pi^2}{12} \right] + \frac{16\bar{\Lambda}_{\text{SF}}}{3\Lambda_{UV}} \left[\log\left(\frac{\Lambda_{UV}}{\mu}\right) - 1 \right] \right.$$
$$\left. - \int_0^{\Lambda_{UV}} \frac{4\omega}{\omega^2 + \mu^2} \left\{ \left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right) \right] + \frac{4\bar{\Lambda}_{\text{SF}}}{3} \frac{\omega}{\omega^2 + \mu^2} \left[2 - \log\left(\frac{\omega}{\mu}\right) \right] \right\} \right) \tag{138}$$

$$M_+^{(1)\text{BT}} = -\int_0^{\Lambda_{UV}} \frac{C_F \alpha_s}{4\pi} \frac{4\omega}{\omega^2 + \mu^2} \left\{ \left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right) \right] + \frac{4\bar{\Lambda}_{\text{SF}}}{3} \frac{\omega}{\omega^2 + \mu^2} \left[2 - \log\left(\frac{\omega}{\mu}\right) \right] + \Lambda_{UV} \frac{C_F \alpha_s}{4\pi} \left[-4\log\left(\frac{\Lambda_{UV}}{\mu}\right) + 6 \right] \right.$$
$$+ \frac{4\bar{\Lambda}_{\text{SF}}}{3} \left(1 + \frac{C_F \alpha_s}{4\pi} \left\{ -6\log\left(\frac{\Lambda_{UV}}{\mu}\right) - \frac{16\bar{\Lambda}_{\text{SF}}}{3\Lambda_{UV}} \left[\log\left(\frac{\Lambda_{UV}}{\mu}\right) - 1 \right] \right\} \right)$$
$$+ \frac{C_F \alpha_s}{4\pi} \int_0^{\Lambda_{UV}} \frac{4\omega}{\omega^2 + \mu^2} \left\{ \left[\frac{1}{2} - \log\left(\frac{\omega}{\mu}\right) \right] + \frac{4\bar{\Lambda}_{\text{SF}}}{3} \frac{\omega}{\omega^2 + \mu^2} \left[2 - \log\left(\frac{\omega}{\mu}\right) \right] \right\} \right) - \frac{C_F \alpha_s}{4\pi} 6\mu \qquad (139)$$

where $M_+^{(1)\text{BT}} = 0.964$ GeV from (108).

We have to solve these two equations for N and $\bar{\Lambda}_{\text{SF}}$ in terms of Λ_{UV} and find in which region of Λ_{UV} these quantities become approximately constant. We find that this is the case for the reasonable range $\Lambda_{UV} > 3$ GeV, giving

$$N = 0.856 \qquad \bar{\Lambda}_{\text{SF}}(\mu) = 0.563 \text{ GeV}$$
$$\bar{\Lambda} = 0.338 \text{ GeV} \qquad (140)$$

i.e. a situation very close to the first model for $\bar{\Lambda}_{\text{SF}}(\mu)$.

Although the radiative tails are different, the final functions are almost identical, as shown in Fig. 4. The reason is that the positive part of the radiative tail in the latter model is compensated by the lower value of N (140), imposed by the OPE constraints.

In this second model we have, for the quantities of interest,

$$\lambda_B^{-1} = 1.432 \text{ GeV}^{-1} \qquad \sigma_B = 1.219. \qquad (141)$$

These values are very close to the ones of the first model (133) and (134).

We conclude that our results, due to the OPE constraints, are invariant relatively to the way of adding the radiative tail to the nonperturbative part.

X. PROPOSALS FOR LCDA IN OTHER APPROACHES

The alternative theoretical method for the calculation of the LCDA is essentially the one of QCD sum rules. Here we will distinguish between the work that considers the LCDA at leading order and the one incorporating the radiative tail. Our aim is not a critical one, but only to show the great variety of ansätze that one can find in the literature for the functions $\varphi_\pm^B(\omega)$, and the corresponding varied results for the parameters $\bar{\Lambda}$ and λ_B^{-1}.

A. LCDA at leading order

A word of caution is in order here. In this part, that involves the LCDA at leading order, without radiative corrections, $\bar{\Lambda}$ is considered to be related to the center-of-gravity of the function $\varphi_+^B(\omega)$, i.e. to its first moment through

$$\langle \omega \rangle_+ = \frac{4\bar{\Lambda}}{3}. \qquad (142)$$

Actually, in the calculation of the present paper, this first moment is assimilated to $\frac{4\bar{\Lambda}_{\text{SF}}}{3}$, since the parameters are obtained from the fit of the roman shape function to $B \rightarrow X_s \gamma$, that provides $\bar{\Lambda}_{\text{SF}}$. This will make easier the comparison with the other approaches in the absence of radiative corrections.

Before going to specific theoretical schemes, it is worth to quote the bound found by Korchemsky, Pirjol, and Yan [5], that is independent of the precise form of $\varphi_+^B(\omega)$, and follows from assuming positivity $\varphi_+^B(\omega) > 0$

$$\lambda_B^{-1} \geq \frac{3}{4\bar{\Lambda}}. \qquad (143)$$

Of course, the positivity condition $\varphi_+^B(\omega) > 0$ is violated by the radiative tail, as we have seen in the preceding section.

1. QCD Sum Rules

Grozin and Neubert did obtain from the QCDSR result the simple form for the functions $\varphi_\pm^B(\omega)$ [2],

$$\varphi_+^B(\omega) = \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0} \qquad \varphi_-^B(\omega) = \frac{1}{\omega_0^2} e^{-\omega/\omega_0} \qquad (144)$$

that satisfy relation (39) and give, for the positive moments (see also the analysis of Ref. [31]),

$$\langle \omega \rangle_+ = 2\langle \omega \rangle_- = 2\omega_0 = \frac{4\bar{\Lambda}}{3} \qquad (145)$$

and for the parameter λ_B^{-1}

$$\lambda_B^{-1} = \frac{1}{\omega_0} = \frac{3}{2\bar{\Lambda}}. \qquad (146)$$

From the value $\bar{\Lambda} \cong 0.55$ GeV used in this paper one gets

$$\lambda_B^{-1} \cong 2.72 \text{ GeV}^{-1}. \qquad (147)$$

On the other hand, Braun et al. [7] have obtained from QCDSR the following simple form for the normalized long distance shape of the LCDA

$$\varphi_+(\omega) = \frac{3}{4\varepsilon_c^3} \omega(2\varepsilon_c - \omega)\theta(2\varepsilon_c - \omega) \qquad (148)$$

where ε_c is the continuum threshold, and obtain

$$\langle\omega\rangle_+ = 2\langle\omega\rangle_- = \frac{4\bar{\Lambda}}{3} \qquad \lambda_B^{-1} = \frac{3}{2\varepsilon_c} = \frac{9}{8\bar{\Lambda}} \qquad (149)$$

that gives, for $\varepsilon_c = (0.9 \pm 0.1)$ GeV [7]

$$\bar{\Lambda} \cong (0.67 \pm 0.07) \text{ GeV}$$
$$\lambda_B^{-1} \cong (1.68 \pm 0.18) \text{ GeV}^{-1}. \qquad (150)$$

Also, within the QCDSR approach, Ball and Kou [8] find the above relation and the numerical value, adopting the value $\bar{\Lambda} = 0.68$ GeV [34,35]

$$\lambda_B^{-1} = \frac{9}{8\bar{\Lambda}} \cong 1.67 \text{ GeV}^{-1}. \qquad (151)$$

Both Refs. [7,8] find the same expression (148). However, we must emphasize that in the work of Braun *et al.* [7], formula (148) represents just the simplest contribution to the QCD sum rules. The full result of [7] is much more complicated, and the formula (110) represents a simple parametrization of the full expression. In Ref. [7] the long distance part and the radiative tail follow together from the calculation, and the latter is not added by hand.

2. QCD factorization models

Within the QCD factorization approach of Beneke *et al.* [3] for the calculation of charmless nonleptonic B decays ($\bar{B} \to \bar{K}\pi$, $\bar{B} \to \pi\pi$, $\cdots$) the following range is adopted:

$$\lambda_B^{-1} = (3.5 \pm 1.5) \text{ GeV}^{-1}. \qquad (152)$$

This is a guess essentially based on the determination (146) of [2], using $\bar{\Lambda} \cong 0.4$ GeV. The same range is adopted in [6] for the description of the decay $B^- \to \gamma\ell\bar{\nu}_\ell$.

3. pQCD factorization models

A number of models have been proposed for the function $\Phi_+^B(\xi)$ defined in (3) within the framework of the perturbative QCD factorization (pQCD factorization) approach, needed in the description of nonleptonic charmless B decays. Integrating over $\mathbf{p}_\perp$ or equivalently taking impact parameter $\mathbf{b} = 0$, the following models were proposed [35–37] for the function $\Phi_+^B(\xi)$ defined by (3),

$$\Phi_+^B(\xi) = NF(\xi)\exp\left[-\frac{1}{2}\left(\frac{m_B\xi}{\omega_B}\right)^2\right] \qquad (153)$$

with a range of values for ω_B in the interval

$$0.25 \leq \omega_B \leq 0.65 \qquad (154)$$

and different models for the function $F(\xi)$,

$$F(\xi) = \xi^2(1-\xi)^2, \qquad \xi(1-\xi) \quad \text{or} \quad \sqrt{\xi(1-\xi)} \qquad (155)$$

with N determined by the normalization condition (4).

To compare with the present work we have to make the change of variables (5) and take the limit $m_b \to \infty$ at ω fixed. Using relation (7) one finds the following models for $\varphi_+^B(\omega)$,

$$\varphi_+^B(\omega) = NG(\omega)\exp\left[-\frac{1}{2}\left(\frac{\omega}{\omega_B}\right)^2\right] \qquad (156)$$

with

$$G(\omega) = \omega^2, \qquad \omega \quad \text{or} \quad \sqrt{\omega} \qquad (157)$$

and N determined by the normalization condition (8).

Just a few comments are in order to compare with the present work. First, in the BT scheme with an harmonic oscillator potential, $\varphi_+^B(\omega)$ is not proportional to a Gaussian, but given by the function (106). Second, the models (153)–(157) give generically low values for the first positive moment and very large values for the first inverse moment

$$0.38 \text{ GeV} \leq \langle\omega\rangle_+ = \frac{4\bar{\Lambda}}{3} \leq 0.48 \text{ GeV}$$
$$2.66 \text{ GeV}^{-1} \leq \langle\omega^{-1}\rangle_+ = \lambda_B^{-1} \leq 4.18. \qquad (158)$$

4. Results in the BT quark model

The Bakamjian-Thomas quark model of the present paper with harmonic oscillator wave function and parameters taken from the fit to the $B \to X_s\gamma$ spectrum using the roman shape function yields

$$\langle\omega\rangle_+ = 2\langle\omega\rangle_- = \frac{4\bar{\Lambda}_{\text{SF}}}{3} = 0.964 \text{ GeV} \qquad (159)$$

and

TABLE I. Results for the parameters $\bar{\Lambda}$ and λ_B^{-1} in the different theoretical approaches for $\varphi_+^B(\omega)$ in the absence of the radiative tail.

Theoretical frame	$\bar{\Lambda}$	λ_B^{-1}
Positivity $\varphi_+^B(\omega) \geq 0$ [5]		$\geq \frac{3}{4\bar{\Lambda}}$
QCDSR [2]	0.55 GeV	$\frac{3}{2\bar{\Lambda}} = 2.72$ GeV^{-1}
QCD Factorization [3,6]	~0.45 GeV	$\frac{3}{2\bar{\Lambda}} = (3.5 \pm 1.5)$ GeV^{-1}
QCDSR [7,8]	(0.67 ± 0.07) GeV	$\frac{9}{8\bar{\Lambda}} = (1.7 \pm 0.2)$ GeV^{-1}
pQCD [35–37]	(0.32 ± 0.04) GeV	(3.42 ± 0.76) GeV^{-1}
BT model with roman shape function parameters	0.49 GeV	1.52 GeV^{-1}
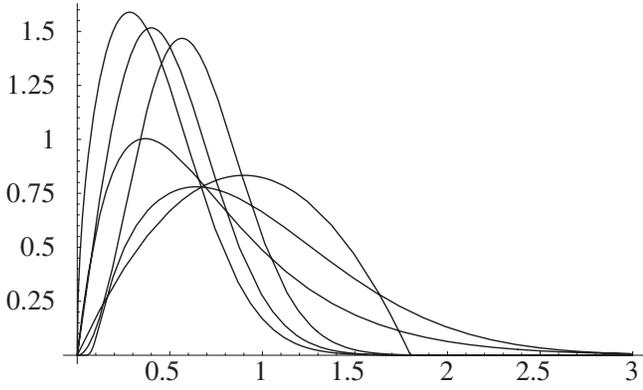
FIG. 5. Different models for the function $\varphi_+^B(\omega)$ in leading order. From higher to lower curves: the heavy quark limit of the three models of pQCD [35–37] (156) and (157), with $\omega_B = 0.4$; follows the models of QCDSR [2] (144) with $\bar{\Lambda} = 0.49$ GeV and [7,8] (148); the wider curve is the model of the present paper (31) with the harmonic oscillator wave function (85) and the parameters (98), that give $\bar{\Lambda} = 0.49$ GeV.

$$\lambda_B^{-1} = \langle \omega^{-1} \rangle_+ = 1.521 \text{ GeV}^{-1} \tag{160}$$

that satisfies the bound (142). The comparison between λ_B^{-1} and $\langle \omega \rangle_+^{-1}$ is different than in the Lee *et al.* model (144), since we find approximately

$$\langle \omega^{-1} \rangle_+ \cong \frac{3}{4} \langle \omega \rangle_+^{-1}. \tag{161}$$

λ_B^{-1} is smaller than in the Lee-Neubert model due to the effect of the dynamical light quark mass m_2, that has a large value from the fit with the roman shape function, and depresses $\langle \omega^{-1} \rangle_+$ due to the behavior (42).

From the value for $\bar{\Lambda}_{\text{SF}} = 0.723$ GeV from (159), not inconsistent with (120) we obtain, for $\mu = 1.5$ GeV, from (118) and (119):

$$\bar{\Lambda} = 0.49 \text{ GeV}. \tag{162}$$

We summarize the values obtained for the parameters $\bar{\Lambda}$ and λ_B^{-1}, guessed or used in other approaches, that we compare with the results of the present paper (Fig. 1), in Table I and in Fig. 5.

A remark is in order here in the comparison in Fig. 5 between $\varphi_+^B(\omega)$ from, e.g. the QCDSR result (144) and our model. While in the former $\langle \omega \rangle_+ = \frac{4\bar{\Lambda}}{3}$ with $\bar{\Lambda} = 0.55$ GeV, we have in our model (159), $\langle \omega \rangle_+ = \frac{4\bar{\Lambda}_{\text{SF}}}{3}$ with $\bar{\Lambda}_{\text{SF}} = 0.723$ GeV, since it is $\bar{\Lambda}_{\text{SF}}$ that is determined by the Shape Function. This explains why our $\varphi_+^B(\omega)$ is more spread than (144).

B. LCDA with a radiative tail

Let us now discuss the impact of adding a radiative tail to the LCDA in the various approaches. Braun *et al.* [7] have proposed the parametrization (110) of $\varphi_+^B(\omega, \mu)$ for $\mu = 1$ including the radiative tail. Lee and Neubert [9]
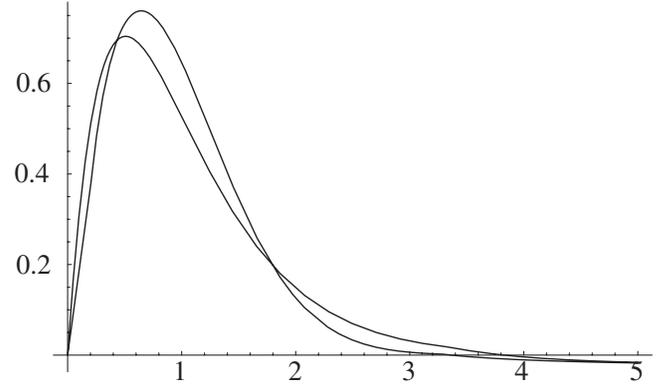
FIG. 6. The function $\varphi_+^B(\omega)$ including the radiative tail for $\mu = 1.5$ GeV. From higher to lower curves: model of the present paper; Lee-Neubert model [9].

take as a model the long distance piece expression (144) adding the tail (117) with $\bar{\Lambda}_{\text{SF}}(\mu)$ replaced by $\bar{\Lambda}_{\text{DA}}(\mu, \mu)$,

$$\varphi_+^B(\omega) = N \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0} + \theta(\omega - \omega_t)\varphi_+^{\text{RAD}}(\omega). \tag{163}$$

The consistency of (163) with the first moments with the OPE (112) and (113) imposes constraints on N and ω_0 that only depend on ω_t and $\bar{\Lambda}_{\text{DA}} = \bar{\Lambda}_{\text{DA}}(\mu, \mu)$ [9]. For $\mu = 1$, using the relations between $\bar{\Lambda}_{\text{DA}}(\mu, \mu)$, $\bar{\Lambda}_{\text{SF}}(\mu_*, \mu_*)$ and $\bar{\Lambda}$ one finds from $\bar{\Lambda}_{\text{SF}}(\mu_*, \mu_*) = 0.65$ GeV (120), $\bar{\Lambda}_{\text{DA}}(\mu, \mu) = 0.52$ GeV ($\mu = 1$ GeV), $\omega_t = 2.33$ GeV, $N = 0.963$, $\omega_0 = 0.438$ and the results of Table II. On the other hand, Lee *et al.* have shown that the model of Braun *et al.* for $\varphi_+^B(\omega)$, for $\mu = 1$ GeV is quite close to their own.

We show in Fig. 6 and Table II the results of our models for adding the radiative tail of sections IX A and IX B, compared to the results of [7,9].

Some comments are in order here on the row of Ref. [7] in Table II. The range for $\bar{\Lambda}$ is the choice given in [7], below formula (15). The value of λ_B^{-1} (1.5 GeV) is obtained from λ_B^{-1} (1.0 GeV) (formula (39) of [7]) using $\varphi_+(\omega, \mu = 1$ GeV) [formulas (39), (43), and (44)] and the scale dependence for $\lambda_B^{-1}(\mu)$ given by (41).

TABLE II. Results for the parameters $\bar{\Lambda}$ and λ_B^{-1} for $\mu = 1.5$ GeV in the QCDSR approach and in our models (1) and (2) of Secs. IX A and IX B including the radiative tail.

Method	$\bar{\Lambda}$ (GeV)	λ_B^{-1} (GeV^{-1})
Braun *et al.* [7]	0.4–0.5	1.98 ± 0.52
Lee-Neubert [9]	0.34	1.86 ± 0.17
BT model (1)	0.35	1.43
BT model (2)	0.34	1.43

XI. PHENOMENA SENSITIVE TO THE LCDA

Let us now review the observables that are related to the LCDA. In this respect there is essentially the work based on QCD factorization and the one based on QCD sum rules. We will briefly review:

(1) The decay $B^- \to \gamma \ell \bar{\nu}_\ell$.
(2) Hard scattering contribution to nonleptonic decay amplitudes like $A(\bar{B}_d \to \pi^+ \pi^-)$ in the framework of QCD factorization.
(3) Asymptotic behavior of the Isgur-Wise function $\xi(w)$ and the subleading form factor $\xi_3(w)$.
(4) Heavy-to-light form factors like $B \to \pi$ at $q^2 = 0$.

A. The decay $B^- \to \gamma \ell \bar{\nu}_\ell$

This decay is described by two form factors $F_V(E_\gamma)$ and $F_A(E_\gamma)$ that, at tree level, are related, in the heavy quark limit, to the LCDA $\varphi_+^B(\omega)$ by

$$F_V(E_\gamma) = F_A(E_\gamma) = \frac{f_B m_B Q_u}{2E_\gamma} \langle \omega^{-1} \rangle_+ = \frac{f_B m_B Q_u}{2E_\gamma} \frac{1}{\lambda_B}$$
(164)

This process is directly related to the parameter λ_B^{-1} and would be the most direct way of measuring it. Conversely, having a good theoretical estimate of λ_B^{-1}, the process $B^- \to \gamma \ell \bar{\nu}_\ell$ allows us to measure $|V_{ub}|$.

A considerable effort has been devoted to the study of this decay going beyond the tree result (164). Korchemsky *et al.* [5] have computed the form factors for photon energies larger than $\Lambda_{\rm QCD}$ combining QCD methods for exclusive processes with HQET. They have written the leading twist form factors as the convolution of the B meson light-cone amplitude $\varphi_+^B(\omega)$ with a hard-scattering term, and computed also Sudakov contributions. In a later paper, Braun *et al.* [7] have considered the radiative tail for the LCDA $\varphi_+^B(\omega)$, reviewed in Sec. IX and X. Descotes-Genon and Sachrajda [6] have studied the decay $B^- \to \gamma \ell \bar{\nu}_\ell$ in the framework of QCD factorization, demonstrating that indeed at the one-loop order the amplitude can be written as a convolution of a perturbatively calculable hard-scattering amplitude with $\varphi_+^B(\omega)$. For the parameter λ_B, they use the guess of [3] $\lambda_B = (350 \pm 150)$ MeV.

The scheme of the present paper predicts a value $\lambda_B^{-1} \cong 1.43$ GeV^{-1} that is in the lower range given in the different schemes of the literature, as reviewed in Sec. X. This feature is due to the rather large value of the dynamical mass of the constituent light quark.

It is important to underline that the decay $B^- \to \gamma \ell \bar{\nu}_\ell$ is the only process that could allow, in principle, to directly measure λ_B^{-1}, modulo radiative and $1/m_Q$ corrections.

Recently, in a search for the decay $B^+ \to \gamma \ell^+ \gamma_\ell$, with $\ell = e$ or μ *BABAR* has found the following upper bounds [38], depending on the way of analyzing the data:

$$\lambda_B^{-1} < 1.49 \text{ GeV}^{-1}$$
(165)

or

$$\lambda_B^{-1} < 1.69 \text{ GeV}^{-1}.$$
(166)

These bounds are fulfilled by our value (133) $\lambda_B^{-1} = 1.43$ GeV^{-1} while they seem at odds with the predictions of other schemes (Tables I and II). The predictions of Refs. [7,9] are within 1σ or 2σ in agreement with the bounds (165) or (166).

B. Hard-scattering in nonleptonic two-body B decays

The correction to factorization to the decays with an emitted light meson $\bar{B} \to D\pi, \pi\pi, \dots$, that comes from a gluon attached to the spectator quark, called the hard-scattering amplitude, depends directly on $\Phi_+^B(\xi)$ or $\varphi_+^B(\omega)$ [3]. In the case of the decays to two light mesons, it scales in terms of dimensionful quantities like $G_F \alpha_s m_b \Lambda_{\rm QCD}^{5/2}$, having the same behavior as the leading term $G_F m_b \Lambda_{\rm QCD}^{5/2}$. In this case of decays of B to two light mesons, this results in a contribution to the effective QCD factors like a_1 [3]

$$a_{1,\parallel} = \frac{C_2}{N_c} \frac{C_F \pi \alpha_s}{N_c} H_{K\pi}$$
(167)

with

$$H_{K\pi} = \frac{1}{\lambda_B} \frac{f_B f_\pi}{m_B F_0^{B \to \pi}(0)} [\langle (1-x)^{-1} \rangle_K \langle (1-y)^{-1} \rangle_\pi + r_\chi^\pi \langle x^{-1} \rangle_K X_H^\pi]$$
(168)

where r_χ^π is the well-known chiral enhancement factor [3]. This contribution to nonleptonic decays is proportional to λ_B^{-1}. BBNS [3] propose the number $\lambda_B = (350 \pm 150)$ MeV, based on the relation to $\bar{\Lambda}$ obtained in [2] and the bound [5]. The range adopted by BBNS $\lambda_B^{-1} = (3.5 \pm 1.5)$ GeV^{-1} is larger than the prediction of the model of the present paper, $\lambda_B^{-1} \cong 1.43$ or $\lambda_B^{-1} \cong 1.52$ with or without the radiative tail. However, in the phenomenological analysis of two-body nonleptonic decays, as we can see in expression (168), λ_B^{-1} is affected by a subleading although important term, chirally enhanced and proportional to the unknown logarithmically divergent factor $\int_0^1 \frac{dy}{1-y} \Phi_p^\pi(y)$, where $\Phi_p^\pi(y) \cong 1$ is the twist-3 pion light-cone amplitude. The second term in (168) is parametrized phenomenologically by $X_H^\pi = (1 + \rho_H e^{i\varphi_H}) \log(m_B/\mu)$ ($\rho_H \lesssim 1$), and fitted to the data. The modulus of the ratio between the two terms in (168) is roughly $|r_\chi^\pi X_H^\pi| = \frac{2m_\pi^2}{m_b(m_u+m_d)} |1 + \rho_H e^{i\varphi_H}| \log(m_B/\mu)$, of $O(1)$. This term results in a large uncertainty on $H_{K\pi}$ (Fig. 5 of [3]), that for the real part is $0.5 \lesssim \text{Re}H_{K\pi} \lesssim 2.5$, with a somewhat smaller uncertainty for the imaginary part. Therefore, strictly speaking, due to this unknown term, that plays a non-negligible role in the description of the data, two-body nonleptonic decays do not allow us to make a model-independent extraction of the parameter λ_B^{-1}. We can conclude that data on nonleptonic

decays into two light mesons, although dependent on λ_B^{-1}, are not a model-independent determination of this quantity.

C. Asymptotic behavior of the functions $\xi(w)$ and $\xi_3(w)$

Grozin and Neubert [2] have given the behavior of the elastic Isgur-Wise function $\xi(\cosh\theta)$ in the large recoil limit $w \gg 1$ ($\theta \gg 0$),

$$\xi(\cosh\theta) = 16\pi\alpha_s \frac{C_F}{N_c} f^2 \langle\omega^{-2}\rangle_+ \langle\omega^{-1}\rangle_- e^{-2\theta} \quad (169)$$

where $2f = f_M\sqrt{m_Q}$. In our notation, this writes

$$\xi(w) \cong \pi \frac{C_F\alpha_s}{N_c} (f_B\sqrt{m_B})^2 \langle\omega^{-2}\rangle_+ \langle\omega^{-1}\rangle_- \frac{1}{w^2} \quad (170)$$

where

$$\langle\omega^{-2}\rangle_+ = \int_0^\infty \frac{d\omega}{\omega^2} \varphi_+^B(\omega) \qquad \langle\omega^{-1}\rangle_- = \int_0^\infty \frac{d\omega}{\omega} \varphi_-^B(\omega) \quad (171)$$

From the description of the behavior of the functions $\varphi_\pm^B(\omega)$ in the different limits in Secs. VI and X, one realizes that these two integrals diverge for massless light quarks. However, the nonvanishing light quark mass of the BT model, implying the behavior (42), provides a natural infrared cutoff.

Using expressions (31) or (34) for nonvanishing light quark mass and the wave function (85) with the parameters (98), one finds the finite results

$$\langle\omega^{-2}\rangle_+ = 3.826 \text{ GeV}^{-2} \qquad \langle\omega^{-1}\rangle_- = 2.322 \text{ GeV}^{-1}. \quad (172)$$

This gives, for $\mu = 1$ ($C_F\alpha_s \cong 0.624$), an order of magnitude $\xi(w) \sim \frac{1}{w^2}$, smaller but of the same order of magnitude as with a function $\xi(w) \cong (\frac{2}{w+1})^2$ and a slope of the order $\rho^2 \cong 1$.

A comment is in order here. Grozin and Neubert [2] argue that the logarithmic singularities of $\langle\omega^{-2}\rangle_+$ and $\langle\omega^{-1}\rangle_-$ would be cutoff by the transverse momenta and virtualities of the light quarks in the mesons. In our scheme, the transverse momenta do not play such a role, since the divergence remains taking those into account, as shown by the explicit expressions for $\varphi_\pm^B(\omega)$ in the massless limit that can be read from (31) or (34). It is the nonvanishing dynamical light quark mass, that one can consider as a "virtuality" of the massless quark due to the $\langle\bar{q}q\rangle$ condensate, that ensures the finiteness of the moments $\langle\omega^{-2}\rangle_+$ and $\langle\omega^{-1}\rangle_-$.

The subleading function $\xi_3(w)$ [39], coming from $1/m_Q$ perturbations to the current, behaves [2] in the large recoil limit $\theta \gg 0$ like

$$\xi_3(\cosh\theta) = 4\pi\alpha_s \frac{C_F}{N_c} f^2 \langle\omega^{-1}\rangle_+^2 e^{-\theta} \quad (174)$$

or, in our notation,

$$\xi_3(w) \cong \frac{\pi}{2} \frac{C_F\alpha_s}{N_c} (f_B\sqrt{m_B})^2 \langle\omega^{-1}\rangle_+^2 \frac{1}{w} \quad (175)$$

and from the value (133) or (141) for $\langle\omega^{-1}\rangle_+ = \lambda_B^{-1}$,

$$\langle\omega^{-1}\rangle_+ \cong 1.521 \text{ GeV}^{-1} \quad (176)$$

we find

$$\xi_3(w) \cong 0.174\pi \frac{C_F\alpha_s}{N_c} \frac{1}{w} \text{ GeV} \quad (177)$$

that gives

$$\xi_3(w) \sim \frac{0.1}{w}. \quad (178)$$

D. $B \to \pi$ form factor at $q^2 = 0$

From Eq. (23) of [1] one has up to terms in $1/(1-u)$,

$$F_{+,0}^{B\to\pi}(0) \cong \frac{\pi\alpha_s C_F}{N_c} \frac{f_\pi f_B}{m_b^2} \int_0^1 d\xi du \Phi_-^B(\xi) \Phi_\pi(u)$$
$$\times \frac{1}{\xi(1-u)^2} \quad (179)$$

i.e.,

$$F_{+,0}^{B\to\pi}(0) \cong \frac{\pi\alpha_s C_F}{N_c} \frac{f_\pi f_B}{m_b} \int_0^\infty \frac{d\omega}{\omega} \varphi_-(\omega)$$
$$\times \int_0^1 \frac{du}{(1-u)^2} \Phi_\pi(u). \quad (180)$$

However, this expression is not usable due to the infrared divergence of the integral over u, since $\Phi_\pi(u) \cong 6u(1-u)$.

One way out of this problem is the use of the light-cone QCD sum rules approach by Khodjamirian *et al.* [4], that provides a simple and explicit expression for the heavy-to-light form factors in terms of LCDA. Taking as an example the form factor $F_+^{B\to\pi}(0)$ one has,

$$F_+^{B\to\pi}(0) = \frac{f_B}{f_\pi m_B} \int_0^{s_0^\pi} ds \exp\left(-\frac{s}{M^2}\right) \varphi_-^B\left(\frac{s}{m_B}\right). \quad (181)$$

An approximation to this expression has been used in the first reference of [4],

$$\int_0^{s_0^\pi} ds \exp\left(-\frac{s}{M^2}\right) \varphi_-^B\left(\frac{s}{m_B}\right) \cong \varphi_-^B(0) \int_0^{s_0^\pi} ds \exp\left(-\frac{s}{M^2}\right) \quad (182)$$

that gives, from the integral relation (40), valid only for *vanishing light quark mass*,

$$F_+^{B\to\pi}(0) = \frac{1}{\lambda_B} \frac{f_B}{f_\pi m_B} M^2 \left[1 - \exp\left(-\frac{s_0^\pi}{M^2}\right)\right]. \quad (183)$$

Some remarks are in order here:

(1) If $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ are related by (40), the numerical results for $F_+^{B \to \pi}(0)$ from (181) and (183) are very close.

(2) As pointed out in [4] and we observe from (183), for a given window of the Borel parameter M^2 and a value of s_0^π,

$$0.5 \text{ GeV}^2 \le M^2 \le 1.2 \text{ GeV}^2 \qquad s_0^\pi = 0.7 \text{ GeV}^2. \tag{184}$$

$F_+^{B \to \pi}(0)$ is very sensitive to the precise value of λ_B^{-1}. The value adopted in [4] is $\lambda_B^{-1} \cong 2 \text{ GeV}^{-1}$.

(3) The differential relation (40) between $\varphi_+^B(\omega)$ and $\varphi_-^B(\omega)$ is only valid for $m_2 = 0$, and is badly violated in the BT model examined here, where the value of m_2 extracted from the $B \to X_s \gamma$ spectrum is rather large.

(4) A nonvanishing dynamical light quark mass m_2 has a dramatic impact since then $\varphi_-^B(0) = 0$. We are not allowed then to use (183) but must rely on the relation (181). Using this expression and the parameters (184) one obtains a very small value for $F_+^{B \to \pi}(0) \sim 0.015$, the reason being that $\varphi_-^B(\omega)$ vanishes at $\omega = 0$. One would need a much larger value of s_0^π to get an appreciable contribution of $\varphi_-^B(\omega)$ to the integral (181). Our conclusion is that with our prediction for $\varphi_-^B(\omega)$ (Fig. 2), relation (181) and the set of parameters (184), one cannot describe the form factor $F_+^{B \to \pi}$. This feature deserves further investigation.

XII. CONCLUSIONS

In conclusion, within the Bakamjian-Thomas relativistic quark model, that in the heavy quark limit yields covariant form factors and Isgur-Wise scaling, we have computed the B meson light cone distribution amplitudes $\varphi_\pm^B(\omega)$, that are also covariant in this scheme, and satisfy, in the limit of vanishing dynamical light quark mass, the integral relation given by QCD in the valence quark-antiquark sector. We have also computed the shape function $S(\omega)$ that enters in the description of the decay $B \to X_s \gamma$. The light cone distribution amplitudes and the shape function are related in the BT class of models and given in terms of the $Q\bar{q}$ internal wave function. Using a Gaussian wave function, we have shown that the shape function is identical to the so-called roman shape function. Using the parameters of the latter that fit the $B \to X_s \gamma$ spectrum, we have predicted the LCDA $\varphi_\pm^B(\omega)$. We have discussed the role played by the dynamical mass of the light constituent quark and included the short distance behavior of QCD for $\varphi_+^B(\omega)$. Compared to most schemes in the literature, our model predicts a rather small value for the parameter $\lambda_B^{-1} \cong 1.43 \text{ GeV}^{-1}$, due to the rather large value of the constituent light quark mass, fitted from the $B \to X_s \gamma$ spectrum. This value for λ_B^{-1} fulfills the upper bounds obtained by *BABAR* from the search of $B^+ \to \gamma \ell^+ \nu_\ell$. Moreover, the nonvanishing constituent light quark mass has the important implication $\varphi_-^B(0) = 0$. We have compared with other theoretical approaches and discussed the phenomena that are sensitive to the LCDA.

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