# All static spherically symmetric anisotropic solutions of Einstein's equations

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An algorithm recently presented by Lake to obtain all static spherically symmetric perfect fluid solutions is extended to the case of locally anisotropic fluids (principal stresses unequal). As expected, the new formalism requires the knowledge of two functions (instead of one) to generate all possible solutions. To illustrate the method some known cases are recovered.

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## I. INTRODUCTION

As is well known, static spherically symmetric perfect fluid distributions in general relativity are described by a system of three independent Einstein equations for four variables (two metric functions, the energy density, and the isotropic pressure). Thus, additional information in the form of an equation of state or a heuristic assumption involving metric and/or physical variables has to be provided in order to integrate the system. This situation suggests the possibility of obtaining any possible solution, giving a single generating function. A formalism to obtain solutions in this way has been recently presented by Lake [1] (see also [2]).

The purpose of this work is to extend the abovementioned formalism to the case of locally anisotropic fluids.

The motivation for doing so is provided by the fact that the assumption of local anisotropy of pressure, which seems to be very reasonable for describing the matter distribution under a variety of circumstances, has been proved to be very useful in the study of relativistic compact objects (see [3–13] and references therein).

In the next section we shall present the general equations and the formalism to obtain the solutions, then we shall apply the method to analyze some specific cases.

## II. THE EINSTEIN EQUATIONS FOR STATIC LOCALLY ANISOTROPIC FLUIDS

In curvature coordinates the line element reads

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (1)$$

which has to satisfy the Einstein equations. For a locally anisotropic fluid they are

$$8\pi\rho = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \quad (2)$$

$$8\pi P_r = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \quad (3)$$

$$8\pi P_\perp = \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \quad (4)$$

where primes denote derivative with respect to $r$, and $\rho$, $P_r$, and $P_\perp$ are the proper energy density, radial pressure, and tangential pressure, respectively.

### A. The algorithm

From (3) and (4) it follows:

$$8\pi(P_r - P_\perp) = e^{-\lambda}\left(-\frac{\nu''}{2} - \left(\frac{\nu'}{2}\right)^2 + \frac{\nu'}{2r} + \frac{1}{r^2}\right) + e^{-\lambda}\frac{\lambda'}{2}\left(\frac{\nu'}{2} + \frac{1}{r}\right) - \frac{1}{r^2}. \quad (5)$$

Then, introducing the variables

$$e^{\nu(r)} = e^{\int(2z(r)-2/r)dr} \quad (6)$$

and

$$e^{-\lambda} = y(r) \quad (7)$$

and feeding back into (5) we get

$$y' + y\left[\frac{2z'}{z} + 2z - \frac{6}{r} + \frac{4}{r^2 z}\right] = -\frac{2}{z}\left(\frac{1}{r^2} + \Pi(r)\right), \quad (8)$$

with $\Pi(r) = 8\pi(P_r - P_\perp)$.

Integrating (8) we obtain for $\lambda$:

$$e^{\lambda(r)} = \frac{z^2(r)e^{\int([4/r^2 z(r)]+2z(r))dr}}{r^6\left(-2\int\frac{z(r)(1+\Pi(r)r^2)e^{\int([4/r^2 z(r)]+2z(r))dr}}{r^8}dr + C\right)}. \quad (9)$$

where $C$ is a constant of integration. Then, using (6) and (9) in (1) we get

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$$ds^2 = -e^{\int(2z(r)-2/r)dr}dt^2 + \frac{z^2(r)e^{\int([4/r^2z(r)]+2z(r))dr}}{r^6(-2\int\frac{z(r)(1+\Pi(r)r^2)e^{\int([4/r^2z(r)]+2z(r))dr}}{r^8}dr + C)}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2. \tag{10}$$

Thus any solution describing a static anisotropic fluid distribution is fully determined by means of the two generating functions $\Pi$ and $z$.

It will be convenient to express the physical variables in terms of metric and generating functions, in order to impose conditions leading to physically meaningful solutions. Thus we have

$$4\pi P_r = \frac{z(r-2m) + m/r - 1}{r^2} \tag{11}$$

$$4\pi\rho = \frac{m'}{r^2} \tag{12}$$

and

$$4\pi P_\perp = \left(1 - \frac{2m}{r}\right)\left(z' + z^2 - \frac{z}{r} + \frac{1}{r^2}\right) + z\left(\frac{m}{r^2} - \frac{m'}{r}\right), \tag{13}$$

where the mass function $m(r)$ is defined as usual by

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}. \tag{14}$$

Physically meaningful solutions must be regular at the origin, and should satisfy the conditions $\rho > 0$, $\rho > P_r, P_\perp$. If stability is required then $\rho$ and $P_r$ must be monotonically decreasing functions of $r$.

To avoid singular behavior of physical variables on the boundary of the source ($\Sigma$), solutions should also satisfy the Darmois conditions on the boundary. Implying $(P_r)_\Sigma = 0$ and

$$e^{\nu_\Sigma} = e^{-\lambda_\Sigma} = 1 - \frac{2M}{r_\Sigma} \tag{15}$$

with $m_\Sigma = M$, and $r_\Sigma$ denotes the radius of the fluid distribution.

### B. The locally isotropic case

If we impose the isotropic condition on pressure

$$\Pi = 8\pi(P_r - P_\perp) = 0 \tag{16}$$

in (10) we obtain

$$ds^2 = -e^{\int(2z(r)-2/r)dr}dt^2 + \frac{z^2(r)e^{\int([4/r^2z(r)]+2z(r))dr}}{r^6(-2\int\frac{z(r)e^{\int([4/r^2z(r)]+2z(r))dr}}{r^8}dr + C)}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \tag{17}$$

which is the same result obtained in [1], with $z(r) = \Phi(r)' + \frac{1}{r}$.

## III. SOME EXAMPLES

We shall next apply the algorithm to reproduce some known situations.

### A. Conformally flat anisotropic fluids

Instead of giving two generating functions, we may provide one generating function and an additional *ansatz*. Thus, for example, in the spherically symmetric case we know that there is only one independent component of the Weyl tensor. Therefore the conformally flat condition reduces to a single equation which reads

$$\frac{\nu''}{2} + \left(\frac{\nu'}{2}\right)^2 - \frac{\nu'\lambda'}{4} - \frac{\nu' - \lambda'}{2r} + \frac{1 - e^\lambda}{r^2} = 0. \tag{18}$$

Equation (18) has been integrated in [14], giving:

$$e^{\nu/2} = cr\cosh\left(\int\frac{e^{\lambda/2}}{r}dr\right), \tag{19}$$

which, in terms of $z$ becomes

$$z = \frac{2}{r} + \frac{e^{\lambda/2}}{r}\tanh\left(\int\frac{e^{\lambda/2}}{r}dr\right). \tag{20}$$

On the other hand, from (4) and (18) it follows:

$$\Pi = r\left(\frac{1 - e^{-\lambda}}{r^2}\right)'. \tag{21}$$

Thus the system is completely determined (in this case) provided a single generating function $z$ is known.

### B. Bowers–Liang solution

This solution corresponds to an anisotropic fluid with a homogeneous energy density distribution $\rho = \rho_0 =$ const. [15], and is given by

$$e^\nu = \left[\frac{3(1 - 2M/r_\Sigma)^{h/2} - (1 - 2m/r)^{h/2}}{2}\right]^{2/h} \tag{22}$$

$$m(r) = \frac{4\pi}{3}r^3\rho_0; \qquad M = \frac{4\pi}{3}r_\Sigma^3\rho_0. \tag{23}$$

The two generating functions for this metric are

$$z = \frac{\frac{2m}{r^2}(1-\frac{2m}{r})^{(h/2)-1}}{3(1-\frac{2M}{r_\Sigma})^{h/2}-(1-\frac{2m}{r})^{h/2}}+\frac{1}{r} \tag{24}$$

and

$$\Pi = -6C\frac{(z-\frac{1}{r})^2(1-\frac{2M}{r_\Sigma})^{h/2}}{(1-\frac{2m}{r})^{(h/2)-1}} \tag{25}$$

with $h = 1 - 2C = \text{const}$. The case $h = 1$ reproduces the well-known Schwarzschild interior solution, whereas the case $h = 0$ describes the Florides solution [16].

### C. Anisotropic solutions with a nonlocal equation of state

An interesting family of solutions may be found from the assumption that the energy density and the radial pressure are related by a nonlocal equation of state of the form [17]

$$P_r(r) = \rho(r) - \frac{2}{r^3}\int_0^r \tilde{r}^2\rho(\tilde{r})d\tilde{r} + \frac{C}{2\pi r^3} \tag{26}$$

or, using (12)

$$P_r(r) = \frac{m'}{4\pi r^2} - \frac{m}{2\pi r^3} + \frac{C}{2\pi r^3}. \tag{27}$$

From (11) and (27) it follows that these solutions are defined by the generating function $z$ of the form:

$$z = \frac{rm' - 3m + 2C + r}{r(r-2m)}. \tag{28}$$

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