Winding tachyons on a Banados-Teitelboim-Zanelli spacetime

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Analyzing closed string tachyon condensation will improve our understanding of spacetime in string theory. We study the string spectrum on a Banados-Teitelboim-Zanelli black hole spacetime supported by Neveu-Schwarz-Neveu-Schwarz flux, which provides a calculable example where we would expect to find a quasilocalized tachyon. We find that there is a winding tachyon when the black hole horizon is smaller than the string scale, as expected. However, due to effects of the Neveu-Schwarz-Neveu-Schwarz B field, this tachyon is not localized in the region where the spatial circle is string scale. We also discuss the relation to the Milne orbifold in the limit near the singularity.

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I. INTRODUCTION

The study of tachyons in string theory provides an interesting window into aspects of nonperturbative dynamics. The dynamics of open string tachyon condensation is relatively well understood, but the corresponding story for closed string tachyons is far from complete. We have a nice picture for localized closed string tachyons at orbifold singularities such as $\mathbb{C}/\mathbb{Z}_N$ [1] (cf., [2] for a review), where by virtue of the tachyon dynamics being confined to a small region in spacetime, one has control over the condensation process. Recently, there has been interest in studying quasilocalized closed string tachyons [3], which have been argued to arise in several interesting contexts. The basic idea is that for a string on a circle of size smaller than the string length ℓ_s, with antiperiodic boundary conditions for fermions, there are tachyonic winding modes. If the size of this circle varies over some base space, one heuristically expects a tachyon which is confined to the region where the size of the circle $\leq \ell_s$. Such configurations arise when we consider strings propagating on a Riemann surface in corners of moduli space where handles degenerate [3], in simple time-dependent spaces [4], or in charged black string geometries [5,6]. The condensation of such tachyons is argued to provide insight into issues such as spacetime fragmentation/topology change, black hole evaporation, and spacelike singularity resolution [4,7,8]. (In the last context, the tachyon condensate provides a realization of the final state proposal of [9].)

Most of the discussion of quasilocalized tachyons so far has been based on this kind of approximate analysis, as the examples considered were too complicated for the string spectrum to be calculated explicitly. In this paper, we consider in detail the string spectrum on a Banados-Teitelboim-Zanelli (BTZ) black hole ($\times\, \mathbf{S}^3 \times \mathbf{T}^4$) [10,11]. The arguments used previously imply that the BTZ black hole has a winding tachyon when the horizon size $\sqrt{k} r_+ \leq \ell_s$ [5], and that this tachyon will be confined to the region near the horizon, where the spatial circle is smaller than the string scale. Indeed, this geometry arises as the near-horizon limit of the black string examples considered in [5,6].

In BTZ, we can calculate the perturbative string spectrum exactly, and test this heuristic analysis. The BTZ black hole is an orbifold of AdS_3 by an identification under a boost. We consider the $AdS_3 \times \mathbf{S}^3 \times \mathbf{T}^4$ geometry supported by Neveu-Schwarz-Neveu-Schwarz (NS-NS) flux, corresponding to the F1-NS5 system in Type II string theory compactified on[1] $\mathbf{T}^4$. The world-sheet theory is a conformal field theory (CFT) with a $\widehat{SL(2,\mathbf{R})}_k \times \widehat{SU(2)}_k$ supercurrent algebra, with the level k being set by the NS-NS flux, or alternatively by the number of effective strings in six dimensions. The bosonic string on the BTZ orbifold has been previously studied in [12–16]. We exploit and extend these results to determine when there is a winding string tachyon in the BTZ geometry.

We find that there is indeed a twisted sector tachyon in the spectrum, which for the superstring appears precisely when $\sqrt{k} r_+ \leq \sqrt{2}\ell_s$. In the superstring, the tachyon in odd twisted sectors will survive the Gliozzi-Scherk-Olive (GSO) projection if the spin structure on spacetime imposes antiperiodic boundary conditions on fermions around the spatial circle [17]. This is in accord with the expectations from the qualitative argument.

The major surprise of our analysis is that the tachyon wave functions are not localized! We find that the tachyon has nontrivial support all the way out to the anti–de Sitter (AdS) boundary, with a wave function very similar to that for a bulk tachyon. The NS-NS flux plays a key role in this delocalization. It is directly related to the existence of "long string" states in this geometry, which can grow arbitrarily large due to the cancellation of the string tension

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[1] We can alternately consider compactification on K3. The internal space will play no role in our analysis, and we will concentrate on $\mathbf{T}^4$ for simplicity.

by the coupling to the background B field [18]. This delocalization will make it more difficult to understand the condensation of these tachyons. However, one might hope that the AdS asymptotics might result in the tachyon condensation only appreciably changing the geometry in some compact region.

We also study the Milne limit, where we zoom in on the region near the singularity. This limit is analogous to the flat-space limit of the elliptic orbifolds of [19]. We find that with an appropriate scaling, physical states survive in both twisted and untwisted sectors in the limit. We argue that from the T-dual point of view, these twisted sectors seem to be localized near the singularity, in agreement with the expectations of [4]. We leave a detailed understanding of the relation of the twisted sectors we find here to previous work on the Milne orbifold [20–22] for future investigation.

In the next section, we briefly outline the relevant aspects of string theory on AdS_3 and the BTZ black hole. We then discuss the computation of the twisted sector tachyon for the bosonic string in Sec. III, and for the superstring in Sec. IV. We conclude with some remarks on open issues in Sec. V. Our conventions for $SL(2, \mathbf{R})$ are contained in Appendix A. We review the flat-space limit of the elliptic orbifold in Appendix B. We briefly discuss aspects of the thermal AdS partition function in Appendix C.

II. PRELIMINARIES

To set the stage for discussing string theory on the BTZ background, we collect some useful information regarding the Wess-Zumino-Witten (WZW) model with target space AdS_3 and the $\widehat{SL(2, \mathbf{R})}$ current algebra. Further details regarding our conventions can be found in Appendix A.

A. AdS_3

Bosonic string theory on AdS_3 with NS-NS flux is described by an $SL(2, \mathbf{R})$ WZW model (see, e.g., [23] for a nice discussion). The action for the WZW model is the conventional one:

$$S_{\rm WZW} = \frac{k}{8\pi\alpha'}\int d^2\sigma\, {\rm Tr}(g^{-1}\partial_a g g^{-1}\partial^a g) + \frac{ik}{12\pi}\int {\rm Tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg). \qquad (2.1)$$

The level k of the WZW model is not quantized, since $H^3(SL(2, \mathbf{R}), \mathbf{R}) = 0$. Later, when we discuss the superstring, we will quantize k, since the level of the $SL(2, \mathbf{R})$ current algebra will be tied to that of an $SU(2)$ current algebra (for strings on $AdS_3 \times \mathbf{S}^3$). For purposes of discussing the AdS_3 geometry, the $SL(2, \mathbf{R})$ group manifold is conveniently parametrized in terms of global coordinates (t, ρ, ϕ) as[2]

$$g = \begin{pmatrix} \cos\tau\cosh\rho + \sin\theta\sinh\rho & \sin\tau\cosh\rho + \cos\theta\sinh\rho \\ -\sin\tau\cosh\rho + \cos\theta\sinh\rho & \cos\tau\cosh\rho - \sin\theta\sinh\rho \end{pmatrix}, \qquad (2.2)$$

which leads to the metric

$$ds^2 = \alpha' k(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2) \qquad (2.3)$$

and NS-NS twoform

$$B = \alpha' k \sinh^2\rho d\tau \wedge d\theta. \qquad (2.4)$$

Henceforth, we will set $\alpha' = 1$, so we work in units of the string length. The AdS length scale is then $\ell = \sqrt{k}$.

The WZW model (2.1) is invariant under the action

$$g(z, \bar{z}) \to \omega(z) g(z, \bar{z}) \bar{\omega}(\bar{z})^{-1}, \qquad (2.5)$$

which leads to a set of conserved world-sheet currents[3]

$$J^a = k\,{\rm Tr}(\tau^a \partial g g^{-1}). \qquad (2.6)$$

This choice of currents ensures that in the flat-space limit $k \to \infty$, J^a reduce to the translational currents. The conformal Ward identity implies the operator product expansions (OPEs)

$$J^a(z)J^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c J^c(w)}{(z-w)}, \qquad (2.7)$$

with a similar expression for the right-movers.[4] The OPE can be translated into commutation relations by using the mode expansions

$$J^a(z) = \sum_{n=-\infty}^{\infty} J_n^a z^{-n-1}, \qquad (2.8)$$

leading to

$$[J_n^3, J_m^3] = -\frac{k}{2}n\delta_{n+m,0}, \qquad [J_n^3, J_m^\pm] = \pm J_{n+m}^\pm, \qquad [J_n^+, J_m^-] = -2J_{n+m}^3 + kn\delta_{n+m,0}. \qquad (2.9)$$

Here we have used $J^\pm = J^1 \pm iJ^2$. This choice corresponds to the elliptic basis of $SL(2, \mathbf{R})$ used for AdS_3 or

[2]This choice corresponds to the Euler angle parametrization of $SU(1, 1)$. The isomorphism between $SL(2, \mathbf{R})$ and $SU(1, 1)$ given by $g \in SL(2, \mathbf{R}) \Rightarrow h = t^{-1} g t \in SU(1, 1)$ where $t = \mathbb{1} + i\sigma_1$.

[3]We are using the τ^a generators for $SL(2, \mathbf{R})$; see Appendix A for our conventions.

[4]Our conventions for the $\widehat{SL(2, \mathbf{R})}$ are analogous to those used in [19]. As discussed there we need to redefine the right-moving currents to ensure that the standard conventions for raising and lowering operators is respected. We assume henceforth that the appropriate redefinition has been applied to the right-movers.

spacelike quotients thereof [19], and is useful if we want to diagonalize $J^3(z)$.

The world-sheet Virasoro generators are

$$L_0 = \frac{1}{k-2}\Big[(J_0^1)^2 + (J_0^2)^2 - (J_0^3)^2 + 2\sum_{m=1}^{\infty}(J_m^1 J_m^1 + J_m^2 J_m^2 - J_m^3 J_m^3)\Big], \tag{2.10}$$

$$L_{n\neq 0} = \frac{2}{k-2}\sum_{m=1}^{\infty}(J_{n-m}^1 J_m^1 + J_{n-m}^2 J_m^2 - J_{n-m}^3 J_m^3),$$

with commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{2.11}$$

and

$$[L_n, J_m^a] = -mJ_{n+m}^a. \tag{2.12}$$

The central charge c is given in terms of the level k as

$$c = \frac{3k}{k-2}. \tag{2.13}$$

Note that the contribution to L_0 from the zero modes of the currents is proportional to the quadratic Casimir c_2 of $SL(2, \mathbf{R})$.

The spectrum of strings on global AdS_3 contains the untwisted, or short string, states in the representations of the current algebra $\hat{\mathcal{C}}_j^\alpha \times \hat{\mathcal{C}}_j^\alpha$, $j = \frac{1}{2} + is$ and $\hat{\mathcal{D}}_j^\pm \times \hat{\mathcal{D}}_j^\pm$ for $\frac{1}{2} < j < \frac{k-1}{2}$. These current algebra representations are highest weight representations of the current algebra built from the corresponding $SL(2, \mathbf{R})$ representations by acting with current algebra lowering operators. The $\mathcal{C}_j^\alpha$ are continuous representations of $SL(2, \mathbf{R})$, while $\mathcal{D}_j^\pm$ are, respectively, highest and lowest weight discrete series representations. The continuous representations correspond to the bosonic string tachyon; this follows from the fact that the quadratic Casimir is $-j(j-1)$. The spectrum on global AdS_3 will also contain twisted sector states obtained by acting on these short string states with spectral flow, as described in [23]. In [24], it was shown that this spectral flow could be reexpressed in terms of twisting with respect to a twist operator which imposes the periodicity in global coordinates. In our case, we will have instead twisted sectors corresponding to the BTZ orbifold.

B. BTZ

We will study the nonrotating BTZ black hole,[5] which is an orbifold of AdS_3 by a hyperbolic generator of $SL(2, \mathbf{R})$ [11]. To describe this orbifold, we use a different parametrization of the group. Describing the AdS space in BTZ coordinates amounts to writing the $SL(2, \mathbf{R})$ group element in Euler angles [12]:

$$g = e^{-2i\varphi'\tau^3}e^{-2i\rho'\tau^1}e^{-2i\psi'\tau^3} = \begin{pmatrix} e^{\varphi'} & 0 \\ 0 & e^{-\varphi'}\end{pmatrix}\begin{pmatrix} r & \sqrt{r^2-1} \\ \sqrt{r^2-1} & r\end{pmatrix}\begin{pmatrix} e^{\psi'} & 0 \\ 0 & e^{-\psi'}\end{pmatrix}, \tag{2.14}$$

where $r = \cosh\rho'$. In these coordinates, the target space metric of the WZW model (2.1) is

$$ds^2 = k\Big[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\phi^2\Big], \tag{2.15}$$

where $\phi = (\varphi' + \psi')$, $t = (\varphi' - \psi')$. The background NS-NS twoform can be written in a suitable gauge as

$$B = k(r^2-1)d\phi \wedge dt. \tag{2.16}$$

The orbifold action which generates a nonrotating BTZ black hole is then simply $\phi \sim \phi + 2\pi r_+$. Note that r_+ is dimensionless and $M_{BH} = r_+^2$. Unlike (2.2), the coordinates in (2.14) do not cover the full spacetime; they are valid outside the event horizon $r = 1$, where the proper size of the ϕ circle is $2\pi\sqrt{k}r_+$.

This choice of basis for the generators can now be translated into the current algebra. The BTZ coordinates correspond to choosing a hyperbolic basis for the current algebra, in which the generator J^2 is diagonalized, as the generators of spacetime time translation and rotation are [14]

$$Q_t = J_0^2 - \bar{J}_0^2, \qquad Q_\phi = J_0^2 + \bar{J}_0^2. \tag{2.17}$$

Since these involve J_0^2, we are interested in real eigenvalues of J_0^2. The commutation relations for the current algebra in the hyperbolic basis read

$$[J_n^2, J_m^2] = \frac{k}{2}n\delta_{n+m,0}, \qquad [J_n^2, J_m^\pm] = \pm iJ_{n+m}^\pm, \qquad [J_n^+, J_m^-] = 2iJ_{n+m}^2 + kn\delta_{n+m,0}, \tag{2.18}$$

where we have used $J^\pm = J^1 \pm J^3$. Note that $J_m^\pm$ have J_0^2 charge $\pm i$. The issues associated with this are discussed in detail in[6] [12,14]. The corresponding OPEs are (cf. (2.7))

$$J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2iJ^2}{(z-w)},$$

$$J^2(z)J^2(w) \sim \frac{k/2}{(z-w)^2}, \qquad J^2(z)J^\pm(w) \sim \pm\frac{iJ^\pm}{(z-w)}. \tag{2.19}$$

It will also be useful for later discussion to record the explicit form of the currents in the BTZ coordinates. In the parametrization (2.14) we find that the currents (2.6)

[5] This is a simpler example since the action of the orbifold is left-right symmetric. The generalization to the rotating case involves an asymmetric orbifold.

[6] See [25] for an excellent discussion of the representations in the hyperbolic basis.

take the form

$$J^1 = ik(\cosh 2\varphi' \partial\rho' - 2\sinh 2\varphi' \cosh\rho' \sinh\rho' \partial\psi'), \tag{2.20}$$

$$J^3 = ik(\sinh 2\varphi' \partial\rho' - 2\cosh 2\varphi' \cosh\rho' \sinh\rho' \partial\psi'), \tag{2.21}$$

$$J^2 = ik(\partial\varphi' + (\cosh^2\rho' + \sinh^2\rho')\partial\psi'), \tag{2.22}$$

where we write $r = \cosh\rho'$. Similarly, the antiholomorphic currents are written as

$$\bar{J}^1 = ik(\cosh 2\psi' \bar{\partial}\rho' - 2\sinh 2\psi' \cosh\rho' \sinh\rho' \bar{\partial}\varphi'), \tag{2.23}$$

$$\bar{J}^3 = ik(-\sinh 2\psi' \bar{\partial}\rho' + 2\cosh 2\psi' \cosh\rho' \sinh\rho' \bar{\partial}\varphi'), \tag{2.24}$$

$$\bar{J}^2 = ik(\bar{\partial}\psi' + (\cosh^2\rho' + \sinh^2\rho')\bar{\partial}\varphi'). \tag{2.25}$$

Bosonic strings in the BTZ background were originally studied in [12,13] and more recently in [14]. The latter analysis reproduced the spectrum by applying the spectral flow operation introduced in [23] to generate the twisted sectors. Our aim is to more explicitly identify the tachyon in these twisted sectors. We will also extend the analysis of the orbifold to the superstring.

III. THE BOSONIC STRING

As we have seen above, the BTZ black hole is obtained by a quotient of $\widehat{SL(2,\mathbf{R})}$ by a hyperbolic element. In the BTZ coordinates (2.14), the quotient is simply the identification $\phi \sim \phi + 2\pi r_+$. We want to understand the twisted sectors associated with this orbifold, and see under what circumstances we will find a tachyon in the twisted sectors.

A. Twisted sectors of the BTZ orbifold

The periodic identification along ∂_ϕ which generates the BTZ orbifold restricts the states to have quantized values of Q_ϕ. By (2.17), this restricts the $J_0^2 + \bar{J}_0^2$ eigenvalue:

$$r_+(J_0^2 + \bar{J}_0^2) \in \mathbb{Z}, \tag{3.1}$$

where J_0^2 refers to the eigenvalue of the corresponding operator on the states. In addition to this restriction on the untwisted sectors, the orbifold action will introduce appropriate twisted sectors. Following [24], we find it convenient to determine the twisted sectors by imposing the constraint (3.1) on an enlarged set of vertex operators. We implement this by first introducing an appropriate twist operator t_n, and then projecting onto the states which are mutually local with respect to this twist operator. The twisted sector vertex operators are then obtained by taking the set of operators including the twist operator which are mutually local and closed under OPE.

To construct twisted sectors, it is convenient to work with a parafermionic representation of the current algebra (analogous to the construction of [19] in the elliptic case).[7] To begin with we bosonize the J^2 current in terms of a free field X;

$$J^2 = -i\sqrt{\frac{k}{2}}\partial X, \tag{3.2}$$

where $X(z)X(w) \sim -\ln(z-w)$, and introduce parafermions to represent the remaining $\widehat{SL(2,\mathbf{R})}_k/\widehat{U(1)}$ algebra by

$$J^\pm = \xi^\pm e^{\pm\sqrt{(2/k)}X}, \tag{3.3}$$

with

$$\xi^+\xi^- \sim \frac{k}{(z-w)^{2+(2/k)}}, \qquad \xi^\pm\xi^\pm \sim (z-w)^{2/k}. \tag{3.4}$$

For chiral primary operators of the current algebra, there is a parafermionic representation

$$\Phi_{j\lambda}(w) = \Psi_{j\lambda}(w) e^{-i\sqrt{(2/k)}\lambda X}, \tag{3.5}$$

where λ is the J^2 eigenvalue, which determines the spacetime energy. Note that in the hyperbolic basis λ and j are unrelated. The primary operators have conformal dimension

$$h(\Phi_{j\lambda}) = -\frac{j(j-1)}{k-2} \tag{3.6}$$

where $c_2 = -j(j-1)$ is the Casimir of the global $SL(2,\mathbf{R})$ symmetry generated by the zero modes of the currents. For the continuous representations $c_2 \geq \frac{1}{4}$; it is bounded from above, $c_2 \leq \frac{1}{4}$, for the discrete representations. Nontachyonic modes are required to have $c_2 \leq \frac{1}{4}$ which corresponds to the Breitenlohner-Freedman (BF) bound in AdS_3. From (3.5) and (3.6) it follows that

$$h(\Psi_{j\lambda}) = -\frac{j(j-1)}{(k-2)} - \frac{\lambda^2}{k}. \tag{3.7}$$

In this parafermionic representation, the restriction (3.1) can be imposed by introducing twist operators

$$t_n = e^{ir_+\sqrt{(k/2)}n(X-\bar{X})} \quad \text{for } n \in \mathbb{Z}, \tag{3.8}$$

[7]This choice of representation is inspired by the analysis of [19], where the orbifolds $AdS_3/\mathbb{Z}_N$ involving identifications of AdS_3 (and extensions to include the orbifold also acting on the internal CFT) under the spatial rotation isometry ∂_θ were studied. In fact the parafermion OPEs written in (3.4) are the same as in the parafermionic representation of the elliptic form of $\widehat{SL(2,\mathbf{R})}_k$. In that case the J^3 current is bosonized in terms of a free field; see Appendix B for some details.

and requiring that physical vertex operators are mutually local with respect to these twist operators.

Given the twist operator it is easy to write down the vertex operators for primary states in the nth twisted sector. They are just given by the composite operator arising from the product of the untwisted sector primary with the twist, i.e.,

$$\Phi^n_{j\lambda\bar{\lambda}} = \Psi_{j\lambda}\bar{\Psi}_{j\bar{\lambda}} e^{-i\sqrt{(2/k)}[(\lambda+(k/2)nr_+)X+(\bar{\lambda}-(k/2)nr_+)\bar{X}]}, \tag{3.9}$$

where $\Psi_{j\lambda}$, $\Psi_{j\bar{\lambda}}$ are the chiral parafermions from the untwisted sector primaries. These operators have dimensions

$$\begin{aligned} h(\Phi^n_{j\lambda\bar{\lambda}}) &= -\frac{j(j-1)}{(k-2)} - \frac{\lambda^2}{k} + \frac{(\lambda + kr_+n/2)^2}{k} \\ &= -\frac{j(j-1)}{(k-2)} + \lambda r_+ n + \frac{kn^2r_+^2}{4}, \\ \bar{h}(\Phi^n_{j\lambda\bar{\lambda}}) &= -\frac{j(j-1)}{(k-2)} - \frac{\bar{\lambda}^2}{k} + \frac{(\bar{\lambda} - kr_+n/2)^2}{k} \\ &= -\frac{j(j-1)}{(k-2)} - \bar{\lambda} r_+ n + \frac{kn^2r_+^2}{4}. \end{aligned} \tag{3.10}$$

In [14], these twisted sectors were discussed using the language of spectral flow developed in [23]. For global AdS, the spectral flow is equivalent to the introduction of an appropriate twist operator, as discussed in [24]. However, for the BTZ orbifold, we think the twist operator language is more appropriate, as the twisting does not correspond to an automorphism of the full current algebra. The symmetries associated with $J^\pm$ are broken by the orbifold ($J^\pm$ are not mutually local with respect to t_n), so these operators will have different moding in the twisted sectors. This twisting is still related to a spectral flow: if we focus on the algebra of the surviving symmetries, which is the $\widehat{U(1)}$ algebra generated by J^2 and the Virasoro algebra, the spectral flow

$$\tilde{J}^2_n = J^2_n + \frac{k}{2} w\delta_{n,0}, \qquad \tilde{L}_n = L_n + wJ^2_n + \frac{k}{4} w^2\delta_{n,0} \tag{3.11}$$

for arbitrary w is an automorphism of this algebra. Taking $w = nr_+$, $\bar{w} = -nr_+$ for integer n recovers the charges of the twisted sector states described above. However, this restricted algebra is no longer spectrum generating.

The full vertex operators are formed by taking descendants of the primary operators (3.9) and combining them with some vertex operator from the internal CFT. The physical state conditions $(L_0 - 1)\,|\,\text{phys}\rangle = (\bar{L}_0 - 1)\,|\,\text{phys}\rangle = 0$ will then be

$$-\frac{j(j-1)}{(k-2)} - \frac{\lambda^2}{k} + \frac{(\lambda + kr_+n/2)^2}{k} + h_{\text{int}} + N = 1, \tag{3.12}$$

$$-\frac{j(j-1)}{(k-2)} - \frac{\bar{\lambda}^2}{k} + \frac{(\bar{\lambda} - kr_+n/2)^2}{k} + \bar{h}_{\text{int}} + \bar{N} = 1, \tag{3.13}$$

where h_{int}, $\bar{h}_{\text{int}}$ are the dimensions of the operator from the internal CFT, and N, $\bar{N}$ are oscillator numbers for the current algebra. We assume that the internal CFT is unitary, so $h_{\text{int}}, \bar{h}_{\text{int}} \geq 0$.

Finally, we should consider the relation of λ, $\bar{\lambda}$ to spacetime energy more carefully. It is clear that $J^2_0 + \bar{J}^2_0$ corresponds to momentum around the compact circle, but there are two possible contributions to $J^2_0 - \bar{J}^2_0$, coming from spacetime energy or winding around the compact circle. That is, there is an ambiguity in the definition of Q_t in the twisted sectors, analogous to the ambiguity in the definition of Q_ϕ discussed in [14]. If we apply the naive formula (2.17), the twisted sector operators have energy

$$E = \lambda - \bar{\lambda} + kr_+n, \tag{3.14}$$

since the eigenvalue of J^2_0 is $\lambda + kr_+n/2$ and the eigenvalue of $\bar{J}^2_0$ is $\bar{\lambda} - kr_+n/2$, for a twisted sector vertex operator (3.9). However, thinking of our orbifold as analogous to an ordinary translation orbifold to generate a compact circle, this twist contribution to the J^2_0, $\bar{J}^2_0$ eigenvalue is more naturally interpreted as the usual winding contribution to p^L_ϕ, p^R_ϕ. Therefore we do not think it is appropriate to interpret it as a contribution to the spacetime energy of the mode. We therefore propose to identify instead

$$Q_t = J^2_0 - \bar{J}^2_0 - kr_+n \tag{3.15}$$

as the generator of spacetime time translation, so that the spacetime energy of the mode (3.9) is simply $\lambda - \bar{\lambda}$. As explained in [14], this shift corresponds to adding the divergence of an antisymmetric tensor to the Noether current; this does not change the conservation law, but shifts the value of the charge in topologically nontrivial sectors.

This issue becomes clearer when we study the flat-space limit. In Sec. III C), we will see that (3.15) gives the usual notion of spacetime energy in the translational orbifold. It should be noted that the appropriate choice is actually gauge dependent. We will return to this issue in Sec. III E) where (2.17) is a more appropriate choice of generators in the chosen gauge.

B. Tachyons in BTZ

Having determined the spectrum of twisted sector operators in the BTZ orbifold, we want to determine which of them corresponds to a tachyon in the spacetime. We first need to consider carefully the question of how a tachyon is defined. A mode is tachyonic if it has sufficiently negative spacetime mass-squared. We want to apply this condition by thinking of our orbifold as analogous to a translational orbifold, and looking for modes which have appropriately negative mass-squared[8] in the directions orthogonal to the orbifold.

We are twisting with respect to J_0^2, so we view the Casimir

$$J_0^1 J_0^1 - J_0^3 J_0^3 = \frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) \tag{3.16}$$

for the other two components of the current as representing the directions orthogonal to the orbifold. Note that although $J_0^{\pm}$ individually do not commute with J_0^2, this Casimir will, so we can work with a basis of vertex operators which are eigenvectors for this Casimir. In the parafermionic representation, the eigenvalue of this Casimir is a multiple of the dimension of the parafermionic part of the vertex operator (3.7), so what we want to do is to view the parafermionic part of the operator as representing the contribution from the orthogonal dimensions. This is not strictly true in a naive sense, since the bosonic field X introduced to bosonize J^2 is not simply a target space coordinate on the circle. Nonetheless, we think this is a natural interpretation. We would then decompose (3.10) into the dimension of the parafermionic operator, (3.7), and a contribution

$$\frac{(\lambda + kr_+ n/2)^2}{k} \tag{3.17}$$

associated with the compact circle.

For general operators, there is a problem, as this latter term depends on the spacetime energy Q_t as well as the momentum Q_ϕ on the compact circle. This dependence on Q_t is a complicating factor, so we will focus for now on identifying tachyon operators with $Q_t = 0$, that is, $\lambda = \bar{\lambda}$. If there is a field with mass-squared violating the BF bound, it will have a mode with zero energy, so this analysis should still be sufficiently general to find all spacetime tachyons, at least in the region outside the horizon. In this case, $\lambda = Q_\phi/2$, and we can interpret (3.17) as p_L^2, the usual contribution of the momentum and winding on a compact circle to the conformal dimension. Thus in this case, an appropriate criterion to identify a tachyon is that the Casimir of the representation in the space orthogonal to the orbifold direction should be $\geq \frac{1}{4}$. That is, we claim that the appropriate criterion for a twisted or untwisted sector mode with $\lambda = \bar{\lambda}$ to be tachyonic is that the parafermionic part of the operator has positive dimension greater than $\frac{1}{4(k-2)}$.

We see that unlike in the case of the elliptic orbifolds analyzed in [19], we can only get tachyons from operators in the continuous representations, even when we are considering the twisted sectors. For (3.7) to be greater than $\frac{1}{4(k-2)}$, we need the full quadratic Casimir $-j(j-1)$ to violate the BF bound. The discrete representations of $SL(2,\mathbf{R})$ at best saturate the bound. The essential difference between the elliptic and hyperbolic cases is the sign of the second term in (3.7).

We want to construct physical states which are tachyonic. The dimensions of operators in the internal CFT will be positive, so to be able to satisfy the physical state condition, we need to require in addition that the total dimensions of the $SL(2,\mathbf{R})$ vertex operator (3.9) are h, $\bar{h} \leq 1$.[9] With our restriction to $\lambda = \bar{\lambda}$, this condition is most easily satisfied for zero momentum, $\lambda = \bar{\lambda} = 0$, when

$$h = \bar{h} = -\frac{j(j-1)}{(k-2)} + \frac{kn^2 r_+^2}{4} = \frac{\frac{1}{4} + s^2}{k-2} + \frac{kn^2 r_+^2}{4}, \tag{3.18}$$

where we have used the j value for a principal continuous representation, $j = \frac{1}{2} + is$. The condition $h \leq 1$ thus translates (for large k) to $\sqrt{k} r_+ < 2$. Thus, we conclude that there will be tachyons in the twisted sectors if and only if $\sqrt{k} r_+ < 2$. The vertex operator corresponding to the most tachyonic mode is Φ^n_{j00} with $j = \frac{1}{2} + is$. Note that in the contrary case $\sqrt{k} r_+ > 2$, we see no tachyon in the spectrum for $\lambda = \bar{\lambda}$.

The bound $\sqrt{k} r_+ < 2$ is in good agreement with what we expect based on the heuristic argument comparing this space to a Scherk-Schwarz compactification. In the next subsection, we will study the near-horizon limit, and recover the usual Scherk-Schwarz analysis [17] as a limit of the present discussion.

C. Flat-space limit of BTZ

There are two interesting flat-space limits which we can consider by sending the AdS curvature to zero. First, we can zoom in on the near-horizon region keeping the part of the spacetime outside the horizon, and second we zoom in

[8] As we are dealing with an asymptotically AdS geometry, the appropriate condition for a tachyon is that the mass-squared violates the Breitenlohner-Freedman bound, which for AdS_3 is $m^2 \leq -\frac{1}{4}$.

[9] In the more familiar case of orbifolding in the internal CFT, a tachyon is also identified with a relevant operator, but the argument is different: there, the dimension of operators in the CFT which includes the time direction could be negative, but we require it to be positive to have a tachyon, and therefore need $h \leq 1$ for the internal CFT. Here, h is the dimension of an operator in the BTZ CFT, which includes the time direction, so we need $h \leq 1$ to be able to satisfy the physical state condition for any choice of operator in the internal CFT. Note however that not any relevant operator in this BTZ CFT corresponds to a tachyon: only those which satisfy the additional condition that (3.16) is sufficiently positive do.

on the singularity. For the moment we will concentrate on the first case and return to the second later. In this limit, the generator we are orbifolding along goes over to a translation generator in flat space, and our orbifold reduces to the usual Scherk-Schwarz compactification.

In the first limit, we need to take $k \to \infty$ holding the horizon radius in AdS units $R = \sqrt{k} r_+$ fixed. Let us define coordinates

$$x^2 = \sqrt{k}\phi, \qquad \rho = \sqrt{k}\sqrt{r^2 - 1} = \sqrt{k}\sinh\rho', \tag{3.19}$$

in which the metric becomes:

$$ds^2 = -\rho^2 dt^2 + d\rho^2 + (dx^2)^2 + \mathcal{O}\left(\frac{1}{k}\right). \tag{3.20}$$

Note that x^2 is a periodic coordinate, $x^2 \sim x^2 + 2\pi R$. The metric (3.20) is just two-dimensional Rindler times a circle. Further defining coordinates $x^1 = \rho\cosh t$, $x^3 = \rho\sinh t$, the metric becomes

$$ds^2 = -(dx^3)^2 + (dx^1)^2 + (dx^2)^2. \tag{3.21}$$

The currents are to leading order simply $J^a = i\sqrt{k}\partial x^a$, $\bar{J}^a = i\sqrt{k}\bar{\partial}x^a$ which are translational currents in the flat metric.

However, to understand the time translation and momentum generators in the near-horizon region, we need to be more careful, and keep track of subleading terms in J^2, $\bar{J}^2$. Recall that the rotation generator $Q_\phi = J_0^2 + \bar{J}_0^2$; hence p_2 will have a finite value in the near-horizon limit if $\lambda + \bar{\lambda} \sim \sqrt{k}$. On the other hand, the energy is $E = \lambda - \bar{\lambda}$, so it is finite if $\lambda - \bar{\lambda} \sim 1$. We therefore need to consider the terms in J^2 which are $\mathcal{O}(1)$ to see the t-translation generator. Retaining terms to subleading order, we find

$$J^2 = i\sqrt{k}\partial x^2 - i\rho^2\partial t, \tag{3.22}$$

$$\bar{J}^2 = i\sqrt{k}\bar{\partial}x^2 + i\rho^2\bar{\partial}t. \tag{3.23}$$

Thus in this flat-space limit,

$$J^2 - \bar{J}^2 = i\sqrt{k}(\partial - \bar{\partial})x^2 - i\rho^2(\partial + \bar{\partial})t, \tag{3.24}$$

and we can see quite clearly that there are two contributions, one $\mathcal{O}(\sqrt{k})$ associated with winding, and one $\mathcal{O}(1)$ associated with time translation. This shows why we need to take a winding part out of $J_0^2 - \bar{J}_0^2$ to obtain Q_t in (3.15).

It might seem surprising that these currents (3.22) and (3.23) are conserved holomorphic and antiholomorphic currents; in flat space, the Lorentz invariance only implies

$$\bar{\partial}(\rho^2\partial t) + \partial(\rho^2\bar{\partial}t) = 0, \tag{3.25}$$

not separate conservation of the left- and right-moving parts. In fact, it is the total J^2 which is conserved, not each term separately. To see why the currents (3.22) and (3.23) are conserved, we need to work with the equations of motion to subleading order, including a term coming from the B field. In the near-horizon limit, it is convenient to work with the B field in the gauge (2.16). In the near-horizon limit we then have a B-field

$$B = \frac{1}{\sqrt{k}}\rho^2 dx^2 \wedge dt. \tag{3.26}$$

This makes a subleading contribution to the x^2 equation of motion

$$\partial\bar{\partial}x^2 + \frac{1}{2\sqrt{k}}(\partial(\rho^2\bar{\partial}t) - \bar{\partial}(\rho^2\partial t)) = 0. \tag{3.27}$$

Together with the conservation law following from Lorentz invariance (3.25), this indeed implies the conservation of J^2, $\bar{J}^2$ to the indicated order.

Now, it is clear that in this flat-space limit, a tachyon is a mode which has a negative mass-squared in the subspace spanned by x^3, x^1. That is, if we consider a vertex operator of zero momentum in the x^2 direction, with winding n, and write the conformal dimension as

$$h = \bar{h} = C + \frac{n^2R^2}{4}, \tag{3.28}$$

then the operator is a tachyon if C is positive,[10] as this is the Casimir in the x^3, x^1 directions. In AdS$_3$, if we start with an untwisted sector operator with $\lambda = \bar{\lambda} = 0$, and apply n units of twist, the conformal dimension of the resulting twisted sector state is

$$h = \bar{h} = -\frac{j(j-1)}{(k-2)} + \frac{kn^2r_+^2}{4}. \tag{3.29}$$

Comparing (3.28) and (3.29), we see that the state corresponds to a tachyon in the twisted sector if and only if it comes from a tachyon—a continuous representation—in the untwisted sector, precisely as we argued in the previous section. Thus, we see that in this near-horizon limit, the space is approximately flat, with one direction periodically identified, and the twisted sector tachyons identified in the previous section go over precisely to the usual Scherk-Schwarz winding tachyons in the flat space. This shows how the approximate Scherk-Schwarz analysis can be recovered from our exact analysis.

D. (Non)localization of tachyon

One of our main aims is to say something about the localization of this winding tachyon. It is difficult to analyze this precisely, as we need to understand the spacetime dependence of the twisted sector vertex operators. We have seen in the previous section that the tachyons all come from operators in the continuous representations of $SL(2, \mathbf{R})$. In [12], the radial profile of the vertex operator

[10]Of course, in taking the flat-space limit we are no longer sensitive to the finite k piece coming from the BF bound. The criterion espoused in Sec. III B), $h(\Psi_{j\lambda}) \geq \frac{1}{4(k-2)}$, simply reduces to the positivity of the Casimir in the two dimensions.

wave function for untwisted sectors was analyzed in terms of hypergeometric functions. From this analysis, we can see that as expected, the untwisted sector tachyon of the bosonic string is not localized in the radial direction.

It is not completely straightforward to extend this analysis to the twisted sectors, as the twisted sector vertex operators Φ^n_{j00} differ from the untwisted vertex operator by a phase factor $e^{-i(\sqrt{k}/2)r_+ n(X-\bar{X})}$, and the field X is not simply related to the target space coordinates. However, using the definition of X (3.2) and the currents in BTZ (2.22) and (2.25), we can see that $\partial X \propto (r^2 \partial\phi - (r^2 - 1)\partial t)$, so we would expect that there is no exponential damping with the radial direction r coming from the twist field. So the radial profile of the wave function is roughly the same as the untwisted vertex operator. As a result, it appears that the twisted sector tachyons are also not localized!

This conclusion can be further supported and understood by considering the analysis in the T-dual description of the CFT. The winding mode then becomes an ordinary momentum mode, and the analysis in the T-dual geometry can be performed at a supergravity level. Note however that in the full geometry the ϕ circle has a size determined by the radial coordinate r, and therefore the T-dual has a varying dilaton that becomes strongly coupled deep inside the bulk. This would invalidate working with tree level string theory. Nonetheless, this T-dual analysis provides some indication of the behavior of the vertex operator wave functions, and gives some more intuitive understanding of the failure of the mode to be localized. See [26] for a related discussion in the context of the two-dimensional black hole.

The T-dual of the BTZ black hole was worked out in [27]. The geometry is

$$ds^2 = -\frac{k(r^2-1)}{r^2}dt^2 + \frac{2}{r^2}(r^2-1)dtd\theta + \frac{d\theta^2}{r^2 k} + \frac{kdr^2}{(r^2-1)}, \qquad (3.30)$$

the dilaton is

$$e^{-2\phi} = kr_+^2 r^2, \qquad (3.31)$$

and the B field vanishes in this T-dual description. The coordinate θ parametrizes the T-dual circle, and has periodic identifications $\theta \sim \theta + 2\pi/r_+$. The determinant of the metric is $g = -1/r^2$, and the inverse metric is

$$g^{-1} = \begin{pmatrix} -\frac{1}{k(r^2-1)} & 1 & 0 \\ 1 & k & 0 \\ 0 & 0 & \frac{(r^2-1)}{k} \end{pmatrix}. \qquad (3.32)$$

We want to consider a mode with one unit of momentum on θ, which is T-dual to the first winding mode. As a warm-up, we can consider the geodesics. The geodesic equation reduces to

$$\dot{r}^2 - E^2 = \frac{(r^2-r_+^2)}{k}(-m^2 - kL^2 + 2kEL), \qquad (3.33)$$

where E, L are the conserved quantities associated to ∂_t, ∂_θ, and m is the particle's rest mass. We can see that the effect of the angular momentum is to effectively shift the mass-squared by a finite amount; in particular, the effect is independent of radius. The r dependence comes solely from red-shifting of the radial momentum. Considering the wave equation for a scalar field T of mass m, if we set $T = f(r)e^{i\omega t}e^{iL\theta}$, we have

$$r\partial_r\left(\frac{(r^2-1)}{kr}\partial_r f\right) + \left(\frac{\omega^2}{k(r^2-1)} - 2\omega L - kL^2\right)f = m^2 f, \qquad (3.34)$$

and again the angular momentum acts just as a shift on the effective mass. In both cases, the essential point is that the inverse metric component $g^{\theta\theta} = k$, so the contribution of this momentum is independent of radius. Since $L = nr_+$ for integer n, this is precisely reproducing the contribution from the winding modes in the original description. If we consider a mode with $\omega = 0$, the effective mass $\tilde{m}^2 = m^2 + kL^2$ corresponds to the mass of the mode in a Kaluza-Klein reduced two-dimensional theory. Hence, the tachyonic modes are those for which $\tilde{m}^2 < \tilde{m}^2_{\rm BF}$, and they behave in exactly the same way for $L = 0$ and $L \neq 0$: the winding tachyons have the same radial wave function as a nonwinding tachyon with the same value of $\tilde{m}^2$. Hence, our winding tachyons are not localized in the near-horizon region.

This T-dual analysis makes it clear that the failure of the tachyon to be localized is due to the coupling to the B field in the original spacetime. If we considered a BTZ geometry with no B field (for example, the S-dual D1-D5 geometry), the T-dual metric is

$$ds^2 = -k(r^2-1)dt^2 + \frac{kdr^2}{(r^2-1)} + \frac{d\theta^2}{kr^2}, \qquad (3.35)$$

and it is clear that momentum modes will be localized: for example, the geodesic equation is

$$\dot{r}^2 - E^2 = \frac{(r^2-1)}{k}(-m^2 - L^2 kr^2). \qquad (3.36)$$

Here we expect that the winding modes of the fundamental string in the BTZ geometry are localized within an AdS scale of the horizon.

The B field makes it possible for winding modes to propagate to large r because there is a cancellation between the positive energy from the tension of the string and a negative contribution to the energy from the coupling between the string world sheet and the background B field. This is the same effect that is responsible for the existence of long strings in the AdS_3 world-sheet theory. If we have any winding mode which is delocalized on the AdS scale, it has no potential barrier from moving out all the way to the boundary.

This failure of the tachyon to be localized is a striking result. A negative consequence is that it will likely be difficult to control the deformation of the spacetime caused by tachyon condensation. However, we expect the endpoint of tachyon condensation to be just the global AdS_3 geometry, which would indicate that the tachyon condensation process only modifies the geometry significantly in the interior of the spacetime. If this is correct, it may still be possible to analyze the tachyon condensation.

E. Milne limit

The other flat-space limit of interest is near the singularity. Getting a better understanding of the tachyon in this time-dependent region is important to understand its effect on singularity resolution. In this region, the geometry looks locally like a Milne orbifold of flat space; the generator we are orbifolding along will go over to a boost generator, rather than a translation generator. In [4], it was argued that there would be a tachyon localized in the region near the singularity, where the circle is becoming small. However, this seems to contradict the study of the Milne orbifold in [20,21], where it was found that there are no physical states in twisted sectors. On the other hand, it has been argued that there will be physical states in a different quantization of the string [22]. We have physical twisted sector states in the full BTZ geometry; it is clearly interesting to ask what happens to them in this limit.

This limit is analogous to the flat-space limit of the elliptic orbifold in [19]. To make this analogy clear, we give a brief discussion of that case in Appendix B. The scalings required to get a regular solution in this limit are different from in the previous case. We must take $k \to \infty$ with r_+ fixed to get a finite-size identification. The appropriate coordinates in the limit are $x^2 = \sqrt{k}(t - i\pi/2)$, $\tau = \sqrt{k}r = \sqrt{k}\cosh\rho'$, so we need to take $\sqrt{k}t$ and $\sqrt{k}r$ fixed. Then the metric becomes

$$ds^2 = -d\tau^2 + \tau^2 d\phi^2 + (dx^2)^2 + \mathcal{O}(1/k), \tag{3.37}$$

where ϕ is still a periodic coordinate, $\phi \sim \phi + 2\pi r_+$. If we define coordinates $x^3 = \tau\cosh\phi$, $x^1 = \tau\sinh\phi$, the metric becomes

$$ds^2 = -(dx^3)^2 + (dx^1)^2 + (dx^2)^2, \tag{3.38}$$

and the currents are to leading order simply $J^a = i\sqrt{k}\partial x^a$, $\bar{J}^a = i\sqrt{k}\bar{\partial} x^a$. Thus, the orbifold is reducing to the usual Milne orbifold in this limit.

If we took the B field in the gauge (2.16) and scaled it in this way, the constant term would blow up. Therefore, we must first make a gauge transformation to rewrite the B field as

$$B = kr^2 d\phi \wedge dt, \tag{3.39}$$

which becomes

$$B = \frac{1}{\sqrt{k}}\tau^2 d\phi \wedge dx^2. \tag{3.40}$$

This vanishes in the limit, but will contribute subleading terms to the equation of motion, as in the previous flat-space analysis. We again need to keep track of the subleading terms in J^2, $\bar{J}^2$, as we need to consider the terms which are $\mathcal{O}(1)$ to see the ϕ-translation generator. To subleading order,

$$J^2 = i\sqrt{k}\partial x^2 + i\tau^2\partial\phi, \tag{3.41}$$

$$\bar{J}^2 = i\sqrt{k}\bar{\partial} x^2 - i\tau^2\bar{\partial}\phi. \tag{3.42}$$

Again, the Lorentz invariance only implies

$$\bar{\partial}(\tau^2\partial\phi) + \partial(\tau^2\bar{\partial}\phi) = 0, \tag{3.43}$$

and we need a subleading term in the equations of motion coming from the B field. The x^2 equation of motion, including this subleading term, is

$$\partial\bar{\partial}x^2 - \frac{1}{2\sqrt{k}}(\partial(\tau^2\bar{\partial}\phi) - \bar{\partial}(\tau^2\partial\phi)) = 0. \tag{3.44}$$

Together with the above equation, this indeed implies the conservation of J^2, $\bar{J}^2$ to the indicated order.

The important point, however, is that the gauge transformation of the B field will affect the relation between $J_0^2 - \bar{J}_0^2$ and the spacetime energy.[11] In this gauge, we should define the spacetime energy by (2.17) rather than (3.15). This is clearer from the T-dual perspective. The B field gives rise to an electric field under dimensional reduction; in the T-dual (3.30), this is the Kaluza-Klein electric field coming from the metric, and the above gauge transformation is implemented by a coordinate transformation

$$\theta' = \theta - kt, \qquad t' = t. \tag{3.45}$$

A mode of the scalar field T with energy ω and momentum L with respect to the original coordinates will have

$$L' = L, \qquad \omega' = \omega + kL \tag{3.46}$$

with respect to these coordinates. Recalling that $L = nr_+$, this is precisely the difference between (3.15) and (2.17), so ω' corresponds to the energy (3.14).

Since we hold ϕ and $\sqrt{k}t$ fixed as we take $k \to \infty$, we should take $Q_\phi = J_0^2 + \bar{J}_0^2 \sim 1$ and $Q_t = J_0^2 - \bar{J}_0^2 \sim \sqrt{k}$. The J_0^2 ($\bar{J}_0^2$) eigenvalue for the twisted sectors is $\lambda + kr_+n/2$ ($\bar{\lambda} - kr_+n/2$), so this implies that

[11] We thank Eva Silverstein for discussions which clarified this point.

$$\lambda \to \frac{1}{2}(p_\phi + \sqrt{k}p_2 - kr_+ n), \qquad \bar{\lambda} \to \frac{1}{2}(p_\phi - \sqrt{k}p_2 + kr_+ n) \tag{3.47}$$

as $k \to \infty$.

The vertex operators (3.9) will then have regular limits as $k \to \infty$. Because the J^2, $\bar{J}^2$ parts are translation in x^2 (to leading order) in this limit, the boson parts go over to just a momentum mode vertex operator in the x^2 direction. That is, from (3.41) and (3.42), we see that to leading order, $X \approx \sqrt{2}x_L^2(z)$, $\bar{X} \approx \sqrt{2}x_R^2(\bar{z})$, and (3.9) becomes

$$\Phi^n_{j\lambda\bar{\lambda}} \approx \Psi_{j\lambda}\Psi_{j\bar{\lambda}}e^{-ip_2x^2}. \tag{3.48}$$

The parafermion parts represent the dependence on the x^1, x^3 directions. For the untwisted sector operators, (3.47) implies λ, $\bar{\lambda} \sim \sqrt{k}$, and the parafermions will have finite dimensions in the limit if $j \sim \sqrt{k}$ as well. This reproduces the ordinary untwisted sector vertex operators in the limit. Note that $h(\Psi_{j\lambda}) - \bar{h}(\Psi_{j\bar{\lambda}}) = -(\lambda^2 - \bar{\lambda}^2)/k \to 0$ in the limit.

For the twisted sector operators, one might be concerned because the twist operator (3.8) is becoming ill-defined in this limit. This does not prevent us from constructing regular twisted sector states in the limit. We can regard the twist operator as just a mathematical device to obtain the physical twisted sector states. However, this does have an interesting consequence: the twisted sector states of the orbifold geometry do not arise by twisting the untwisted sector states surviving the projection. This is because we need different values for j for each sector to get regular parafermion operators in the limit.[12] For the parafermion parts of the twisted sector operators to remain regular in the Milne limit, we need to take

$$j \to \frac{1}{2} + \frac{i}{2}((k-1)r_+ n - \sqrt{k}p_2 + \alpha) \tag{3.49}$$

for some constant α,[13] so that

$$h(\Psi_{j\lambda}) \to \frac{1}{2}r_+ n(\alpha + p_\phi), \tag{3.50}$$

$$\bar{h}(\Psi_{j\bar{\lambda}}) \to \frac{1}{2}r_+ n(\alpha - p_\phi). \tag{3.51}$$

With this scaling, the parafermions should have a regular limit as $k \to \infty$. These are distinct from the parafermions arising in the untwisted sector operators. In particular, we see that

$$h(\Psi_{j\lambda}) - \bar{h}(\Psi_{j\bar{\lambda}}) \to r_+ n p_\phi. \tag{3.52}$$

This looks like what we would expect for operators carrying n units of winding and p_ϕ units of momentum on a spatial circle, and indicates that the Milne limit of the BTZ twisted sectors can be interpreted as describing twisted sectors on the Milne orbifold. This identification is further supported by the fact that the currents $J^\pm$ which reduce to $i\sqrt{k}\partial x^\pm = i\sqrt{k}\partial(x^1 \pm x^3)$ have the correct monodromies to (3.54) be twisted sectors of the Milne orbifold. Unlike the flat-space limit of the elliptic orbifolds reviewed in Appendix B, we can choose j so as to get a regular limit for all the twisted sectors. Thus, the spectrum in the Milne limit includes both the usual untwisted sectors and physical twisted sector states constructed by the above scaling.

Since we have physical twisted sector states, it would be interesting to know which of them are tachyonic. Our previous analysis will not be helpful here, as we restricted our consideration to states with $\lambda = \bar{\lambda}$, whereas the twisted sector modes which have a regular limit have $\bar{\lambda} - \lambda \sim kr_+ n$. Clearly here identifying the tachyons will involve disentangling the contribution to the conformal dimension from winding around the ϕ circle. In this limit as the winding is hidden in the parafermion parts of the operator, we do not see how to isolate the winding contribution. Perhaps some other representation of the vertex operators will be more helpful here.

For similar reasons, we have difficulty in understanding how localized these twisted sector modes are. We can attempt to address this question again from the T-dual point of view. Taking the wave Eq. (3.34) and inserting the change of basis (3.46), we have

$$r\partial_r\left(\frac{(r^2-1)}{kr}\partial_r f\right) + \frac{1}{r^2-1}\left(\frac{\omega'^2}{k} - 2r^2\omega' L' + kr^2 L'^2\right)f = m^2 f. \tag{3.53}$$

Thus, we can see that for modes with $\omega' \sim \sqrt{k}$ and $L' \sim 1$, near $r = 0$ there is a positive contribution to the effective mass-squared which goes like $kr^2L'^2$. This should effectively restrict these modes to the region where $\sqrt{k}r \sim 1$, near the singularity, as expected by [4].

An important goal for the future is to understand the relation to the analysis of [20–22]. In [21], it was argued that a modular-invariant partition function for the Milne orbifold could be expressed in terms of a spectrum which only includes untwisted sector states. In [22], it was argued that the same partition function could be given a different interpretation, which involved scattering states in twisted sectors. Our results are closer to those of the latter analysis, but this is surprising to us, as the approach we have adapted on BTZ is a standard quotient construction, and does not appear to involve any analogue of the nonstandard quanti-

[12] This is similar to the situation arising in the flat-space limit of the elliptic orbifold $AdS_3/\mathbb{Z}_N$, as reviewed in Appendix B.

[13] The factor of $(k-1)$ multiplying r_+n is introduced for convenience, to cancel a subleading term coming from expanding the $(k-2)$ denominator in $h(\Psi_{j\lambda}) = -j(j-1)/(k-2) - \lambda^2/k^2$. This would be simply k in the superstring case.

zation advocated in [22]. Note that we are assuming that parafermionic operators with the dimensions (3.50) and (3.51) exist; if no such regular operators could be constructed, we would be back with [21]. From the BTZ point of view, we would not expect there to be any problem with the construction of these parafermion operators, but it should be checked explicitly. These issues clearly deserve further investigation.

F. Remarks about the spacetime algebra

It is well known that asymptotically AdS_3 spacetimes have an enlarged asymptotic symmetry group, which forms two copies of a Virasoro algebra [28]. As a first step towards relating our perturbative world-sheet study of strings on BTZ to the description in terms of a dual CFT living on the boundary of the spacetime, it would be useful to see how this enlarged asymptotic symmetry group emerges from the world-sheet point of view. For global AdS_3, this was addressed in [29], where it was shown that the spacetime $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ isometries obtained from the world-sheet currents could be extended to construct the spacetime Virasoro generators $\mathcal{L}_n$ by exploiting a special field γ,[14] which has zero conformal dimension and the right charge to fill out the isometry algebra into a complete Virasoro algebra. This construction is easy to generalize to elliptic orbifolds of AdS_3 as discussed in [19]; for $AdS_3/\mathbb{Z}_N$ one just keeps the Virasoro generators $\mathcal{L}_n$ which are multiples of N. These give again a complete Virasoro algebra. The BTZ spacetime is asymptotically AdS, so it should be possible to extend the construction to this case as well. This case is a little more subtle, since we do not have a global $\widehat{SL(2,\mathbf{R})}_k$ to provide clues; the orbifold action leaves only a $\widehat{U(1)}$ algebra. Also, the algebra will not arise as a restriction of the Virasoro algebra of the covering space in this case, as none of those generators commute with the orbifold action. As a result, all that we can do is to suggest the form that the Virasoro generators should take.

We assume that the construction will proceed in much the same way as in the AdS_3 case [29], identifying a physical vertex operator that has dimension zero and J^2 charge 1, to play the role of the field γ. The monodromies of the currents in the nth twisted sector are

$$J^2(e^{2\pi i}z) = J^2(z), \qquad J^{\pm}(e^{2\pi i}z) = e^{\mp 2\pi r_+ n}J^{\pm}(z), \tag{3.54}$$

which could be realized by giving the free boson X (3.2) monodromy $X(e^{2\pi i}z) = X(z) - 2\pi r_+ n\sqrt{\frac{k}{2}}$. This would imply that the monodromies of the untwisted sector vertex operators are

$$\Phi_{j\lambda}(e^{2\pi i}z) = e^{2\pi i r_+ n\lambda}\Phi_{j\lambda}(z). \tag{3.55}$$

The spacetime $\mathcal{L}_0$ generator is $\mathcal{L}_0 = -r_+\oint dz J^2(z)$, where we have introduced a normalization factor r_+, which is required to make the charges work out correctly, but perhaps also seems natural from the spacetime point of view. With this normalization, the spacetime symmetry generators $\mathcal{L}_0 - \bar{\mathcal{L}}_0$ will generate angular momentum with respect to the 2π periodic coordinate ϕ/r_+.[15] We want the $\mathcal{L}_n$ to have $\mathcal{L}_0$ charge n, and we need to integrate a well-defined (trivial monodromy) world-sheet operator of conformal dimension one. We see that $\Phi_{0(m/r_+)}$ has the right properties to be identified as $(\gamma_{\rm BTZ})^m$, so an appropriate ansatz is

$$\mathcal{L}_n = -r_+\oint dz[f_1(n)J^2\Phi_{0(n/r_+)} + f_2(n)J^-\Phi_{0(n/r_+ + i)} + f_3(n)J^+\Phi_{0(n/r_+ - i)}], \qquad n\in\mathbb{Z}. \tag{3.56}$$

Note the factors of $\pm i$ in the vertex operators, which are required to cancel the J^2 charges of $J^{\pm}$. The functions $f_i(n)$ are to be fixed by the requirement that the algebra of the $\mathcal{L}_n$s closes correctly into a Virasoro algebra.

Morally, this is how the spacetime Virasoro algebra should arise from the world-sheet point of view. To check this in detail, we would need to know the OPEs of the primary operators $\Phi_{0\lambda}$ to evaluate the commutators and work out the appropriate choices for the coefficients, which we leave as an interesting exercise for the future. We postpone further discussion of the relation of our world-sheet analysis to the dual CFT point of view to the discussion in Sec. V.

IV. THE SUPERSTRING

We would now like to extend our discussion to the superstring. This will eliminate the bulk tachyon of the bosonic theory; as noted before, our winding tachyons are not well localized, so it is still difficult to obtain control of the decay of our spacetime, even after eliminating the bulk tachyon. However, we consider it useful to verify that there is a GSO projection which eliminates the tachyon in the untwisted sector but retains it in twisted sectors. For BTZ, there are two possible choices of spin structure, and we will see that the tachyon in odd twisted sectors survives the GSO projection if we take an antiperiodic spin structure on spacetime. Also, the world-sheet theory has some interesting technical features. We will consider Type II string theory on $\mathrm{BTZ}\times\mathbf{S}^3\times\mathbf{T}^4$ for simplicity.

A. The superstring WZW model

The superstring on AdS_3 is described by a $\widehat{SL(2,\mathbf{R})}_k$ super-WZW model [29]. We begin by reviewing some

[14]The field γ is the weight zero part of the $\beta-\gamma$ system involved in writing a Wakimoto representation of $SL(2,\mathbf{R})$.

[15]The normalization was fixed in the AdS case by considering the global $SL(2,\mathbf{R})$.

aspects of this model in the hyperbolic basis, which is adapted to the orbifold we want to consider. The world-sheet WZW model with $\widehat{SL(2,\mathbf{R})}_k$ current algebra has generators[16] J^a at level k. This can be decomposed into a bosonic $\widehat{SL(2,\mathbf{R})}_{k+2}$ at level $\tilde{k} = k + 2$, whose generators we denote as j^a, and a set of free fermions. Our conventions for the supercurrent algebra are

$$J^a = j^a - \frac{i}{k}\epsilon^a{}_{bc}\psi^b\psi^c, \tag{4.1}$$

with

$$\psi^a(z)\psi^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)}, \qquad j^a(z)\psi^b(w) \sim 0, \tag{4.2}$$

and as before

$$j^a(z)j^b(w) \sim \frac{\tilde{k}}{2}\frac{\eta^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c j^c}{(z-w)}, \tag{4.3}$$

so

$$J^a(z)J^b(w) \sim \frac{k}{2}\frac{\eta^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c J^c}{(z-w)}. \tag{4.4}$$

The world-sheet $\mathcal{N} = 1$ supercurrent in these conventions is then given as

$$G(z) = \frac{2}{k}\left(g_{ab}\psi^a j^b - \frac{i}{3k}\epsilon_{abc}\psi^a\psi^b\psi^c\right). \tag{4.5}$$

Our conventions for the $\widehat{SL(2,\mathbf{R})}$ are $\epsilon^{123} = 1$ and $\eta_{ab} = \mathrm{diag}(1, 1, -1)$.

The internal CFT which has target space $\mathbf{S}^3 \times \mathbf{T}^4$ will have more of a role in the superstring than previously, as we need to work out the appropriate spin fields. The $\mathbf{S}^3$ part is a world-sheet $\mathcal{N} = 1$ $SU(2)_k$ WZW model while the $\mathbf{T}^4$ is a free $\mathcal{N} = 1$ super-conformal field theory. The $\widehat{SU(2)}_k$ algebra is generated by (again K^a are the total currents and the k^a represent the bosonic contribution)

$$K^a = k^a - \frac{i}{k}\epsilon^a{}_{bc}\chi^b\chi^c, \tag{4.6}$$

with

$$\begin{aligned} \chi^a(z)\chi^b(w) &\sim \frac{k}{2}\frac{g^{ab}}{(z-w)}, \qquad k^a(z)\chi^b(w) \sim 0, \\ k^a(z)k^b(w) &\sim \frac{\tilde{k}}{2}\frac{g^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c k^c}{(z-w)}, \end{aligned} \tag{4.7}$$

where now $\tilde{k} = k - 2$, so

$$K^a(z)K^b(w) \sim \frac{k}{2}\frac{g^{ab}}{(z-w)^2} + i\frac{\epsilon^{ab}{}_c K^c}{(z-w)}, \tag{4.8}$$

and the world-sheet supercurrent is

$$G(z) = \frac{2}{k}\left(g_{ab}\chi^a k^b - \frac{i}{3k}\epsilon_{abc}\chi^a\chi^b\chi^c\right). \tag{4.9}$$

Of course, the major difference is that the metric is positive definite: $g_{ab} = \mathrm{diag}(1, 1, 1)$.

Bosonization of the free fermions: To write down spin fields it is useful to bosonize the fermions. Since we wish to work in the hyperbolic basis we want to diagonalize J^2 for the $\widehat{SL(2,\mathbf{R})}$ current algebra. From the definition of the total current (4.1), we see that the fermionic current involved in J^2 is made up of $\psi^1\psi^3$ so we want to bosonize this combination into a single free boson. The natural extension of the story for the elliptic basis of [29] would be to consider the following bosonization rules:

$$\begin{aligned} \partial H_1 &= J^2 - j^2 = -\frac{2i}{k}\psi^1\psi^3, \\ i\partial H_2 &= K^2 - k^2 = +\frac{2i}{k}\chi^1\chi^3, \\ i\partial H_3 &= -\frac{2i}{k}\psi^2\chi^2, \\ i\partial H_4 &= -i\lambda^1\lambda^2, \\ i\partial H_5 &= -i\lambda^3\lambda^4, \end{aligned} \tag{4.10}$$

where the λ^i are the free fermions for the $\mathbf{T}^4$ part of the story, and the bosons $H_i(z)$ are all canonically normalized

$$H_i(z)H_j(w) = -\delta_{ij}\log(z-w). \tag{4.11}$$

For future reference we also give the expression for the fermions directly in terms of the bosonic fields,

$$\begin{aligned} \psi^3 &= -i\frac{\sqrt{k}}{2}(e^{iH_1} + e^{-iH_1}), \\ \psi^1 &= i\frac{\sqrt{k}}{2}(e^{iH_1} - e^{-iH_1}), \end{aligned} \tag{4.12}$$

which imply that

$$\psi^{\pm} \equiv \psi^1 \pm \psi^3 = \mp i\sqrt{k}e^{\mp iH_1}. \tag{4.13}$$

The superparafermions: As in the discussion of the bosonic string we find it useful to work with a superparafermion representation of the $\widehat{SL(2,\mathbf{R})}_k$ and the $\widehat{SU(2)}_k$ current algebras. Concentrating on the $\widehat{SL(2,\mathbf{R})}_k$ current algebra we introduce a bosonic representation for the currents,

$$J^2(z) = -i\sqrt{\frac{k}{2}}\partial X. \tag{4.14}$$

As before, we also have a bosonic representation for the bosonic current,

[16]In this section the total current including the fermionic contribution will be denoted as J^a; since we will no longer talk about the bosonic theory this notation should hopefully cause no confusion.

$$j^2(z) = -i\sqrt{\frac{\tilde{k}}{2}}\partial X, \tag{4.15}$$

and the bosons $\mathcal{X}$ and X are both canonically normalized, so $\mathcal{X}(z)\mathcal{X}(w) = -\log(z-w)$. Clearly, by virtue of (4.1) we have

$$iH_1 = \sqrt{\frac{k}{2}}\mathcal{X} - \sqrt{\frac{\tilde{k}}{2}}X. \tag{4.16}$$

It is useful to introduce another canonically normalized boson, $\mathcal{H}_1(z)$, which is orthogonal to $\mathcal{X}$, so that we can write

$$H_1 = i\sqrt{\frac{2}{k}}\mathcal{X} + \sqrt{\frac{\tilde{k}}{k}}\mathcal{H}_1, \qquad X = \sqrt{\frac{\tilde{k}}{k}}\mathcal{X} - i\sqrt{\frac{2}{k}}\mathcal{H}_1. \tag{4.17}$$

Note that in the flat-space limit $k \to \infty$, $\mathcal{X} = X$ and $\mathcal{H}_1 = H_1$.

The remainder of the currents $j^\pm$ are written by introducing parafermions

$$j^\pm = j^1 \pm j^3 = \xi^\pm e^{\pm\sqrt{(2/\tilde{k})}X} = \xi^\pm e^{\pm\sqrt{2/k}(\mathcal{X} - i\sqrt{(2/\tilde{k})}\mathcal{H}_1)}, \tag{4.18}$$

using the fact that $j^\pm$ carry imaginary j^2 charge $\pm i$ respectively. The fermions which are bosonized as in (4.12) can be written in terms of the bosons $\mathcal{X}$, $\mathcal{H}_1$ as

$$\psi^\pm = \psi^1 \pm \psi^3 = \mp i\sqrt{k}e^{\mp iH_1} = \mp i\sqrt{k}e^{\mp(i\sqrt{(\tilde{k}/k)}\mathcal{H}_1 - \sqrt{(2/k)}\mathcal{X})}. \tag{4.19}$$

Note that this implies that the fermions carry imaginary J^2 charge. In the hyperbolic basic one linear combination of the J^2 charge measures the spacetime energy. Fermions in this basis therefore have imaginary spacetime energy! This is a consequence of the transformation properties of the spacetime fermions and vector fields under the hyperbolic generator of $SL(2, \mathbf{R})$.

Finally, we can write down the supercurrent in terms of the parafermion representation used above,

$$\sqrt{k}G(z) = i\xi^+ e^{i\sqrt{(k/\tilde{k})}\mathcal{H}_1} - i\xi^- e^{-i\sqrt{(k/\tilde{k})}\mathcal{H}_1} - \sqrt{2}i\psi^2\partial\mathcal{X}. \tag{4.20}$$

As in the elliptic basis, the boson $\mathcal{X}$ associated with the total current only appears differentiated in the expression for the supercurrent. This implies that the supercurrent will be mutually local with respect to the twist operator we will introduce to implement the orbifold.

The spin fields: The simplest set of spin fields we can write down are

$$S_\alpha = e^{(i/2)\epsilon_I H_I}, \tag{4.21}$$

where the H_I are the canonically normalized bosons introduced in (4.10). Note that the spin fields only involve H_1, and not $\mathcal{H}_1$. To determine the OPE with the world-sheet supercurrent $G(z)$ given in (4.20) is straightforward. The most singular terms in the OPE come from the three-fermion piece—this has to cancel to ensure that the $G(z)S_\alpha(w)$ OPE has as its leading singularity a square root branch cut. This calculation works along the same lines as in [29], with the $(z-w)^{3/2}$ singularity being cancelled by an interplay between the contributions from the $\widehat{SL(2, \mathbf{R})}_k$ part and the $\widehat{SU(2)}_k$ part. This leads to the condition derived by [29],

$$\prod_{I=1}^{3} \epsilon_I = 1. \tag{4.22}$$

Furthermore the $S_\alpha(z)S_\beta(w)$ OPE is local provided

$$\prod_{I=1}^{5} \epsilon_I = 1. \tag{4.23}$$

To do this calculation it is useful to write down a parafermionic representation for the $\widehat{SU(2)}_k$ theory as well; up to some signs and factors of i one defines $(\mathcal{Y}, Y, \mathcal{H}_2, H_2)$ which are analogous to the set of bosons $(\mathcal{X}, X, \mathcal{H}_1, H_1)$ described above. For details see [19]. The conditions (4.22) and (4.23) define the set of spin fields on $AdS_3 \times \mathbf{S}^3 \times \mathbf{T}^4$. Note that while our focus is on a particular choice of internal CFT, the considerations here can be easily generalized to a more general internal space as in [30].

Vertex operators: The NS ground states are

$$\mathcal{T}_{jj'} = e^{-\varphi}\Phi^{SL(2)}_{j\lambda}\Phi^{SU(2)}_{j'm}e^{iq\cdot Y}, \tag{4.24}$$

where $\Phi^{SL(2)}_{j\lambda}$ is a primary operator of the $\widehat{SL(2, \mathbf{R})}$ current algebra considered in the bosonic string discussion (3.5), $\Phi^{SU(2)}_{j'm}$ is a primary of the $\widehat{SU(2)}$ algebra associated with the $\mathbf{S}^3$, $e^{iq\cdot Y}$ represents the winding and momentum on the T^4, and φ is the bosonized super-reparametrisation ghost. These operators have dimension

$$h = \frac{-j(j-1)}{k} + \frac{j'(j'+1)}{k} + \frac{q\cdot q}{2}. \tag{4.25}$$

The first excited states in the NS sector are constructed by adding a world-sheet fermion to this operator, so for example

$$\mathcal{V}^i_{jj'} = e^{-\varphi}\lambda^i\Phi^{SL(2)}_{j\lambda}\Phi^{SU(2)}_{j'm}e^{iq\cdot Y}. \tag{4.26}$$

The R ground states are constructed by adding a spin field,

$$\mathcal{Y}_R = e^{-\varphi/2}S\Phi^{SL(2)}_{j\lambda}\Phi^{SU(2)}_{j'm}e^{iq\cdot Y}. \tag{4.27}$$

A detailed discussion of which states survive in the Becchi-Rouet-Stora-Tyutin cohomology, and of the charges associated with the R states, can be found in [24]. Since we are interested in the tachyons, we will be content with observing that the only physical vertex operators involving continuous representations of $SL(2, \mathbf{R})$ are NS ground states.

B. The BTZ orbifold

We can now consider the orbifold of the $\widehat{SL(2, \mathbf{R})}_k$ super-WZW model to construct the BTZ spacetime in the superstring. We begin by presenting a twist field that implements the BTZ orbifold projection and then go on to discuss the GSO projection involved in the superstring, showing that for antiperiodic spin structure, the projection retains the tachyon in odd twisted sectors.

In the superstring, it is natural to construct the twist field for the BTZ orbifold from the total $\widehat{SL(2, \mathbf{R})}_k$ current $\mathcal{X}$, as in the elliptic case [19]. We therefore consider

$$t_n = e^{ir_+ n\sqrt{k/2}(\mathcal{X}-\bar{\mathcal{X}})}. \tag{4.28}$$

As mentioned previously, the supercurrent $G(z)$ (4.20) is mutually local with respect to this twist operator, so the orbifold will preserve the world-sheet supersymmetry.[17] This performs the same twist as before on the bosonic currents. The additional part of the twist involving the field H_1 which bosonizes the spinor fields can be understood as implementing the correct transformation properties for the spacetime spinor and vector indices (which are carried by the world-sheet fermions) under the $SL(2, \mathbf{R})$ generator that we are orbifolding by.

The spin fields will not be mutually local with the twist operator (4.28): the $t_n(z)S_\alpha(w)$ OPE has a logarithmic branch cut. Thus, spacetime supersymmetry is completely broken by the orbifolding, as we would expect. We can construct spacetime fermions by combining $S_\alpha(w)$ with a bosonic vertex operator with an imaginary value for $\lambda + \bar{\lambda}$, producing a compensating branch cut to give a Ramond-Neveu-Schwarz (R-NS) vertex operator which is mutually local with respect to t_n. These correspond to the modes of the spacetime fermions which are invariant under the combined action of the translation and an $\widehat{SL(2, \mathbf{R})}$ rotation of the spinor indices.

We can now implement the orbifold and obtain the twisted sector states. Naively, we should proceed as in the bosonic case, imposing mutual locality with respect to the twist operator (4.28) and including all the twisted sector states required to achieve closure of the OPE. However, there is a small subtlety: in the superstring, the states in untwisted sectors are not mutually local until we impose the GSO projection. They can have square root branch cuts in the OPE. Therefore, at this stage we may need to allow square root branch cuts in the OPE with t_n. Having constructed a general set of twisted sectors in this way, we will seek a GSO projection which gives a mutually local spectrum.

For the NS-NS states, the OPE with the twist operator (4.28) will give

$$t_n \mathcal{T}\bar{\mathcal{T}} \sim \frac{1}{(z-w)^{r_+ n(\lambda+\bar{\lambda})}}, \tag{4.29}$$

so demanding mutual locality of these operators with respect to the twist operator imposes $r_+(\lambda + \bar{\lambda}) \in \mathbb{Z}$, as in the bosonic case, quantizing the momentum around the circle. For the R-NS states, the OPE with the twist operator gives

$$t_n \mathcal{Y}_R \bar{\mathcal{T}} \sim \frac{1}{(z-w)^{r_+ n[(\lambda+\bar{\lambda})-(i/2)]}}. \tag{4.30}$$

As explained earlier, the factor of i here comes from the transformation properties of spacetime spinors under the hyperbolic generator which we orbifold along. Requiring mutual locality with respect to t_n will then impose $r_+[(\lambda + \bar{\lambda}) - \frac{i}{2}] \in \mathbb{Z}$, which corresponds to choosing modes of the spacetime spinor which are invariant under the orbifold action. However, in the BTZ spacetime, there are two possible choices of spin structure; since the ∂_ϕ circle is not contractible, fermions can be either periodic or antiperiodic around this circle. Considering antiperiodic fermions corresponds to imposing $r_+[(\lambda + \bar{\lambda}) - \frac{i}{2}] - \frac{1}{2} \in \mathbb{Z}$, giving a square root branch cut in the OPE with t_n. For the Ramond-Ramond (R-R) states, the OPE with the twist operator gives

$$t_n \mathcal{Y}_R \bar{\mathcal{Y}}_R \sim \frac{1}{(z-w)^{r_+ n[(\lambda+\bar{\lambda})-i]}}, \tag{4.31}$$

and we should impose mutual locality to obtain $r_+[(\lambda + \bar{\lambda}) - i] \in \mathbb{Z}$, corresponding to choosing modes of the spacetime fields which are invariant under the orbifold action. The analysis for the excited NS-NS states is similar.

Twisted sector states are constructed by taking the composite operators arising from the product of the t_n with the invariant untwisted sector operators. For the NS-NS ground states, the twisted sector operators are

$$\mathcal{T}^n_{j,j'} = e^{-\varphi}e^{-\bar{\varphi}}\Phi^{SL(2)}_{j\lambda}\bar{\Phi}^{SL(2)}_{j\bar{\lambda}}\Phi^{SU(2)}_{j'm}\bar{\Phi}^{SU(2)}_{j'\bar{m}}e^{iq\cdot Y}e^{i\bar{q}\cdot\bar{Y}} \times e^{ir_+ n\sqrt{k/2}(\mathcal{X}-\bar{\mathcal{X}})}. \tag{4.32}$$

To calculate the dimensions of these operators, it is useful to rewrite them in terms of parafermions or superparafermions, but we will not do so explicitly here; the construction closely parallels the elliptic case discussed in [19]. The dimensions of these operators are

[17] One might also argue as in [19] that the boundary Virasoro algebra (i.e., the spacetime theory currents) is generated from the total currents J^a and not the bosonic currents j^a. For the hyperbolic $\widehat{SL(2, \mathbf{R})}_k$ unlike the elliptic case we do not have a clean expression for the spacetime algebra, but one expects the structure to be maintained.

$$h = \frac{-j(j-1)}{k} + \frac{j'(j'+1)}{k} + \frac{q\cdot q}{2} - \frac{\lambda^2}{k} + \frac{(\lambda + kr_+ n/2)^2}{k}, \tag{4.33}$$

$$\bar{h} = \frac{-j(j-1)}{k} + \frac{j'(j'+1)}{k} + \frac{q\cdot q}{2} - \frac{\bar{\lambda}^2}{k} + \frac{(\bar{\lambda} - kr_+ n/2)^2}{k}. \tag{4.34}$$

We adopt the same definition of a tachyon as in the bosonic case, so a mode with $\lambda = \bar{\lambda}$ is considered tachyonic if and only if $-j(j-1) - \lambda^2 > \frac{1}{4}$, so that the $\widehat{SL(2,\mathbf{R})}$ superparafermion part of the operator is of sufficiently positive dimension. As in the untwisted sector, only the NS-NS ground states can be both physical states and tachyonic; in the other sectors, the positive contribution to the conformal dimension from the fermions or spin fields makes it impossible to satisfy the physical state condition for $-j(j-1) - \lambda^2 > \frac{1}{4}$. For large k, we can find tachyons which satisfy the physical state condition $h = \bar{h} = \frac{1}{2}$ only if $\sqrt{k}r_+ < \sqrt{2}$.

Turning to the GSO projection, we assume that we make the standard projection in the untwisted sector, projecting out the ground states in the NS-NS sector, and defining a mutually local set of operators in the untwisted sector. In the case where the R-NS vertex operators are mutually local with respect to t_n, corresponding to the periodic spin structure on spacetime, this extends trivially to the twisted sectors, projecting out the NS-NS ground states in every sector. By contrast, when the R-NS vertex operators have a square root branch cut with respect to t_n, corresponding to the antiperiodic spin structure on spacetime, the NS-NS ground states in odd twisted sectors are mutually local with respect to the states we keep in the untwisted sector, so they will be retained under GSO projection. In summary, when we choose an antiperiodic spin structure for the fermions on spacetime, the tachyons which survive the GSO projection are (4.32) for the odd twisted sectors.

In the flat-space limit of Sec. C, this GSO projection reduces to the usual Scherk-Schwarz GSO projection on the translational orbifold, so we recover the usual flat-space analysis in this limit. In the Milne limit of Sec. E, the twist operator (4.28) becomes ill-defined, as previously noted. However, since the vertex operators have regular limits, we should be able to consider either choice of GSO projection in this limit. Thus, there should exist a GSO projection on the Milne orbifold corresponding to an antiperiodic spin structure on the orbifold, in which we keep the NS-NS ground states in odd twisted sectors.

Finally, let us remark on the asymptotic symmetry algebra for the superstring. As in the bosonic theory, the asymptotic symmetries of asymptotically AdS_3 spaces are enlarged to two copies of the Virasoro algebra. From the dual CFT point of view, we would expect this to now be embedded in a superconformal algebra. In [24], the extension of the spacetime supersymmetry to obtain the full set of asymptotic superisometries from the world-sheet point of view was sketched. In the present case, the spacetime supersymmetry is broken, but we would expect the BTZ geometry will have the same asymptotic superconformal symmetry algebra, since the spacetime is still asymptotically AdS_3, and hence has asymptotic Killing spinors. It should be possible to construct this asymptotic superisometry algebra from the world-sheet point of view following [24] and our discussion of the Virasoro algebra in Sec. III F), but we will not explore this further here. It would be interesting to understand the relation of our construction to the asymptotic superisometry algebra constructed from a supergravity point of view (along the lines of [28]) by [31].

V. DISCUSSION

We have studied the closed string tachyons on the BTZ black hole with NS-NS flux, by treating it as an orbifold of AdS_3. We used a parafermion representation of the current algebra. We found that for the superstring, there is no closed string tachyon if we choose the spacetime spin structure which imposes periodic boundary conditions for fermions on the spatial circle. For the spin structure with antiperiodic boundary conditions, we showed that there is a tachyon in odd twisted sectors if the proper size of the circle at the event horizon is small enough. We focused on operators with $\lambda = \bar{\lambda}$, which corresponds to zero spacetime energy in the usual gauge, and argued that the appropriate definition of a tachyon for such operators was that the conformal dimension of the parafermionic part of the operator should be positive. In the superstring, this condition can be satisfied if $\sqrt{k}r_+ < \sqrt{2}\ell_s$.

Surprisingly, this tachyon is not localized in the region where the spatial circle is small. The wave functions for twisted sector states have the same radial falloff as for the corresponding untwisted sector states. This is due to coupling to the background B field, which cancels the positive energy coming from stretching the string as the circle becomes large. That is, the tachyon is a long string mode, and as such, can propagate out to infinity. It would be interesting to examine the asymptotically flat black strings constructed from the F1-NS5 system, which approach this geometry in the near-horizon limit: our results suggest that the tachyonic winding strings in these backgrounds should be localized near the black string, but on a scale set by the charges, rather than in the much smaller region where the size of the circle they wrap is of order the string scale.

Since the failure of this tachyon to be localized is associated with the presence of NS-NS flux, one might hope to construct an example in which it is localized by considering instead a BTZ black hole with R-R flux. We should note first that this is considerably more difficult than the present case, as the WZW methods we have used here will not be available. It is possible that one can use the D1-D5 world-sheet CFT [32] and show that the BTZ geometry in that system indeed has a tachyon when the horizon size is less than the string scale, and that the tachyon wave function is supported within an AdS radius. It may also be possible to make some progress by studying the supergravity spectrum in the *T*-dual geometry. However, one can also observe that the BTZ black hole with R-R flux is *S*-dual to the case we have considered here, so even if the winding tachyons of the fundamental string on that background are localized near the horizon, one would expect it to have instabilities to the condensation of winding *D*-strings, *S*-dual to the fundamental strings we have considered, and by our analysis, this *D*-string instability will not be localized in the near-horizon region.

One could also consider simpler single-charge black string solutions, where the near-horizon region is not a BTZ black hole. Here we have no reason to expect that the winding tachyon will not be localized in the near-horizon region, as suggested by an approximate analysis. One might take our results as counselling caution in over-reliance on such approximate arguments, however. A more careful analysis in such cases will be quite difficult.

There are a number of directions for further investigation arising from this work. Although our tachyon is not localized, it is clearly important to try to understand its condensation. The natural endpoint for condensation of the twisted sector tachyons on BTZ is the global AdS_3 spacetime. Since this only requires a change in the geometry in the interior of the spacetime, there is some hope that we can gain some insight into the tachyon condensation process. Perhaps tachyon condensation produces $\mathcal{O}(1)$ changes in the geometry everywhere, which are negligible compared to the $\mathcal{O}(r^2)$ behavior of the background metric at large distances.

It is also important to understand in detail the relation between the Milne limit of our spectrum and the spectra calculated directly in flat space in [21,22]. In particular, it would be useful to have a more explicit description of the Milne limit of the twisted sector vertex operators. It may be that adopting a different realization of the current algebra, such as the Wakimoto representation used in [13,29], would be helpful.

It would also be interesting to extend our analysis to other orbifolds of AdS_3. First, we could extend our work to the rotating BTZ black hole, which would involve considering an asymmetric orbifold, which acts differently on left- and right-movers. In this case, it would be difficult to rigorously establish modular invariance, but one can simply extend our analysis by introducing an appropriate twist operator, and hope that the resulting spectrum is modular invariant. The main example of interest is the supersymmetric BTZ black hole, which corresponds to orbifolds of AdS_3 by a parabolic generator. If we choose the supersymmetry-breaking spin structure on this spacetime, we expect to have a winding tachyon. This case is analogous to the quotient of AdS_5 considered in [7], and near the singularity, it will approach the null orbifold of flat space, so there are interesting connections to explore here. Another interesting extension would be to consider the "swedish geons" [33], which are BTZ-like orbifolds of AdS_3 with a single exterior region. These can potentially provide examples where the tachyon condensation can lead to disconnected target space geometry with intricate topology, essentially providing a rich set of examples to probe baby universes in string theory.

The other central issue for future development is to understand the description of this tachyon and its condensation in the dual CFT on the boundary of the spacetime. We have tried to make some first steps in this direction by exploring the construction of the asymptotic isometry algebra from the world-sheet point of view. However, there are significant barriers to going further: First, we do not understand the theory on the F1-NS5 worldvolume, so there is no first principles construction of the dual CFT. Second, we do not know how to interpret the twisted sectors, which correspond to long strings wrapping the spatial circle in the boundary, from the dual CFT point of view. This is a problem even in pure AdS_3. From a technical point of view, in [29] it was shown that the spectral flowed vertex operators have unconventional transformation properties with respect to the Virasoro algebra of the dual CFT. It will be an interesting problem for the future to make progress on the interpretation of the construction of the spacetime Virasoro algebra and the behavior of physical twisted sector vertex operators from this perspective.

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Note added.—After this paper was completed, we learned that tachyons in BTZ have also been investigated from a Euclidean perspective in [35,36].

APPENDIX A: THE $SL(2, \mathbf{R})$ LIE ALGEBRA

The group $SL(2, \mathbf{R})$ is the group of 2×2 matrices with unit determinant and the elements taking real values. The

Lie algebra of $SL(2, \mathbf{R})$ is given by the commutation relations:

$$[T^a, T^b] = \epsilon^{ab}{}_c T^c, \tag{A1}$$

where $\epsilon^{123} = 1$, and the index is lowered with the metric $\eta_{ab} = \mathrm{diag}(1, 1, -1)$. Explicitly, one can choose a representation in terms of Pauli matrices:

$$T^1 = \frac{1}{2}\sigma^3, \qquad T^2 = \frac{1}{2}\sigma^1 \qquad T^3 = -\frac{i}{2}\sigma^2. \tag{A2}$$

We will however find it convenient to work with a different set of generators $\tau^a = iT^a$ in terms of which we can express (A1) in a more familiar form,

$$[\tau^a, \tau^b] = i\epsilon^{ab}{}_c \tau^c, \tag{A3}$$

similar to the $SU(2)$ commutation relations. This version is more appropriate because when $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ occurs as the isometry algebra of AdS_3, we are interested in real eigenvalues for the generators τ^a. For the elliptic basis, which is natural when thinking of the group as $SU(1, 1)$, we take $\tilde{\tau}^3 = \tau^3$ and $\tilde{\tau}^{\pm} = \tau^1 \pm i\tau^2$, so one has the commutation relations

$$[\tilde{\tau}^3, \tilde{\tau}^{\pm}] = \pm\tilde{\tau}^{\pm}, \qquad [\tilde{\tau}^+, \tilde{\tau}^-] = -2\tilde{\tau}^3. \tag{A4}$$

On the other hand, if we think of the $SL(2, \mathbf{R})$ description, it is more natural to use the hyperbolic basis, $T^{\pm} = T^1 \pm T^3$, in terms of which the commutation relations are

$$[T^2, T^{\pm}] = \pm T^{\pm}, \qquad [T^+, T^-] = 2T^2. \tag{A5}$$

Note that $T^{\pm}$ are not related to $\tilde{\tau}^{\pm}$, despite the similarity in the commutators. Again, even when working in the hyperbolic basis, we use the τ^a, not T^a. The quadratic Casimir of $SL(2, \mathbf{R})$ is

$$c_2 = -(\tau^3)^2 + (\tau^1)^2 + (\tau^2)^2. \tag{A6}$$

APPENDIX B: FLAT-SPACE LIMIT OF THE ELLIPTIC ORBIFOLD

In studying the Milne limit of our orbifold, we found it useful to compare to the flat-space limit of the elliptic orbifold studied in [19]. Since the flat-space limit is not discussed very explicitly in that reference, we give some formulae here to enable comparison with Sec. E. The elliptic orbifold is the quotient of global AdS_3 (2.3) by the $\mathbb{Z}_N$ group generated by $\theta \to \theta + 2\pi/N$. In [19], this was studied by bosonizing J^3,

$$J^3 = -\sqrt{\frac{k}{2}}\partial X, \qquad J^{\pm} = \xi^{\pm} e^{\pm\sqrt{(2/k)}X}, \tag{B1}$$

and adopting a parafermionic representation for the current algebra primaries,

$$\Phi_{jm\bar{m}} = \Psi_{jm\bar{m}} e^{\sqrt{(2/k)}(mX + \bar{m}\bar{X})}. \tag{B2}$$

The dimensions of the parafermions $\Psi_{jm\bar{m}}$ are

$$h = -\frac{j(j-1)}{(k-2)} + \frac{m^2}{k}, \qquad \bar{h} = -\frac{j(j-1)}{(k-2)} + \frac{\bar{m}^2}{k}. \tag{B3}$$

The orbifold can then be implemented by introducing a twist operator

$$t_w = e^{(q/N)\sqrt{(k/2)}(X+\bar{X})}. \tag{B4}$$

This gives twisted sector operators

$$\Phi^q_{jm\bar{m}} = \Psi_{jm\bar{m}} e^{\sqrt{(2/k)}[(m+(k/2)(q/N))X + (\bar{m}+(k/2)(q/N))\bar{X}]}. \tag{B5}$$

For $q \in \mathbb{Z}_N$, these are "fractional spectral flowed" operators associated with the orbifold. For $q \in N\mathbb{Z}$, these are the long string states in the global AdS_3 covering space (2.3).

To take the flat-space limit, we let $k \to \infty$, holding $x^3 = \sqrt{k}t$, $r = \sqrt{k}\rho$ fixed. The metric becomes

$$ds^2 = -(dx^3)^2 + dr^2 + r^2 d\theta^2, \tag{B6}$$

so the orbifold goes over to the usual flat-space orbifold in this limit. If we define $x^1 = r\cos\theta$, $x^2 = r\sin\theta$, we have $J^a = i\sqrt{k}\partial x^a$ and $\bar{J}^a = i\sqrt{k}\bar{\partial}x^a$. Since we hold θ and $\sqrt{k}t$ fixed, we should take $Q_\theta = J^3_0 - \bar{J}^3_0 \sim 1$ and $Q_t = J^3_0 + \bar{J}^3_0 \sim \sqrt{k}$. This implies that

$$\begin{aligned} m &\to \frac{1}{2}\left(p_\theta + \sqrt{k}p_3 - k\frac{q}{N}\right), \\ \bar{m} &\to \frac{1}{2}\left(-p_\theta + \sqrt{k}p_3 - k\frac{q}{N}\right). \end{aligned} \tag{B7}$$

In the flat-space limit, $X \approx -i\sqrt{2}x^3_L(z)$, $\bar{X} \approx i\sqrt{2}x^3_R(\bar{z})$, and the boson part of the operator (B5) becomes just a momentum mode in the x^3 direction,

$$\Phi^q_{jm\bar{m}} \approx \Psi_{jm\bar{m}} e^{-ip_3 x^3}. \tag{B8}$$

We also need to require that the parafermion part has a regular limit. For the untwisted sectors, this requires $j \sim \sqrt{k}$. For the twisted sectors, we need

$$j \to \frac{1}{2}\left(k\frac{q}{N} - \sqrt{k}p_3 + \alpha\right) \tag{B9}$$

for some constant α. However, recall that there is a bound on j: we only allow current algebra representations with $\frac{1}{2} < j < \frac{k-1}{2}$ [23]. This is consistent with the required scaling (B9) only for $q < N$. Hence, in the flat-space limit, the twisted sector operators for $q \in \mathbb{Z}_N$ have a regular limit, and should give us the usual twisted sectors for the flat-space orbifold, but the long string states with $q \geq N$ go off to infinite conformal dimension as we take the limit. Thus, we regain precisely the expected flat-space spectrum.

APPENDIX C: COMMENTS ON THE AdS_3 PARTITION FUNCTION

In this paper, we have approached the calculation of the twisted sector spectrum on BTZ through a vertex operator construction. It would be more satisfying to construct a modular-invariant partition function for the string theory on BTZ and extract the spectrum for this partition function. In fact, the appropriate partition function has already been constructed for the bosonic string: Since Euclidean BTZ is the same spacetime as thermal AdS_3, the thermal AdS_3 partition function calculated in [34] can be reinterpreted as a BTZ partition function. Alas, technical difficulties have prevented us from extracting the spectrum from this partition function. In this appendix, we will explain why this route is obstructed. Identifying the tachyon from the Euclidean BTZ partition function is also discussed in [35,36].

First, let us recall the relation between the two spacetimes. We have the Euclidean metric

$$ds^2 = k(\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2). \tag{C1}$$

Interpreting τ as time and θ as the spatial circle gives us the global AdS_3 metric; a further identification $\tau \sim \tau + \beta$ gets us to thermal AdS_3. On the other hand we can take θ to be the temporal direction and τ as the spatial circle. This is then the Euclidean BTZ black hole, as the temporal circle shrinks smoothly to zero (at $\rho = 0$); a simple coordinate change $\cosh\rho = r$ maps this back to usual BTZ coordinates (2.15). Formally, given a $\mathbf{T}^2$ parametrized by (τ, θ), we want a hyperbolic three-manifold which has the $\mathbf{T}^2$ as its boundary. Which circle of the torus we choose to make contractible in the bulk geometry determines the spacetime. By choosing to make a particular combination of the boundary one-cycles contractible in the bulk we can construct a full $SL(2,\mathbb{C})$ family of black holes [37]. The different geometries are related by an $SL(2,\mathbb{C})$ transformation of the boundary complex structure. Note that this is a modular transformation from the point of view of the dual boundary CFT. From the world-sheet point of view, Euclidean BTZ and thermal AdS_3 are the same spacetime, given different interpretations corresponding to different ways in which we can analytically continue to a Lorentzian spacetime.

Hence, the thermal AdS partition function calculated in [34] can also be interpreted as the BTZ partition function. The contribution to the torus partition function from the thermal AdS factor is, for a fixed world-sheet modular parameter [34,38],[18]

$$\mathcal{Z}_{\rm AdS}(\beta,\mu;\tau) = \frac{\beta\sqrt{k-2}}{2\pi\sqrt{\tau_2}} \times \sum_{n,m} \frac{e^{-k\beta^2|m-n\tau|^2/4\pi\tau_2 + 2\pi\Im(U_{n,m})^2/\tau_2}}{|\vartheta_1(\tau, U_{n,m})|^2} \tag{C2}$$

with

$$U_{n,m}(\tau) = \frac{i}{2\pi}\beta(1-i\mu)(n\bar{\tau}-m). \tag{C3}$$

To obtain the full partition function, we need to add internal and ghost contributions, and sum over the fundamental domain for the world-sheet modular parameter. The sum over n above can be traded for a sum over copies of the fundamental domain, allowing us to write the full partition function as

$$Z_{\rm AdS}(\beta,\mu) = \int_0^\infty \frac{d\tau_2}{\tau_2}\int_{-1/2}^{1/2} d\tau_1 e^{4\pi\tau_2(1-(1/(4(k-2))))} \times \sum_{h,\bar{h}} D(h,\bar{h}) q^h \bar{q}^{\bar{h}} \mathcal{Z}_{\rm AdS}(\beta,\mu;\tau), \tag{C4}$$

where $D(h,\bar{h})$ are the degeneracies in the internal CFT, and $\mathcal{Z}_{\rm AdS}(\beta,\mu;\tau)$ should be understood as now only involving a sum on m. In [34], the spectrum on AdS_3 was extracted from this partition function by expanding in terms of single string energy eigenstates. To do so, we write the free energy as

$$\begin{aligned} F(\beta,\mu) &= -\frac{1}{\beta}Z(\beta,\mu) \\ &= \frac{1}{\beta}\sum_{\text{string}\in\mathcal{H}} \log(1 - e^{\beta(E_{\rm string}+i\mu\ell_{\rm string})}) \\ &= \sum_{m=1}^{\infty} f(m\beta, m\mu), \end{aligned} \tag{C5}$$

where

$$f(\beta,\mu) = \frac{1}{\beta}\sum_{\text{string}\in\mathcal{H}} e^{-\beta(E_{\rm string}+i\mu\ell_{\rm string})} \tag{C6}$$

and $\mathcal{H}$ is the single string Hilbert space. This would allow us to read off the spectrum. The calculation is relatively straightforward, as the sum on m in (C5) can be identified with the sum on m in (C2).

To perform the same calculation in BTZ, we would want to write

$$\begin{aligned} F(\beta_{\rm BTZ},\mu_{\rm BTZ}) &= -\frac{1}{\beta_{\rm BTZ}}Z(\beta_{\rm BTZ},\mu_{\rm BTZ}) \\ &= \sum_{m'=1}^{\infty} f(m'\beta_{\rm BTZ}, m'\mu_{\rm BTZ}), \end{aligned} \tag{C7}$$

where $\beta_{\rm BTZ}$ is the inverse temperature of the black hole, and $f(\beta_{\rm BTZ},\mu_{\rm BTZ})$ is as in (C6). The problem with the calculation is that we can no longer identify the sum on m' in (C7) with the sum on m in (C2). Let us consider for simplicity the case of zero chemical potential, $\mu = 0$. Then reinterpreting the thermal AdS_3 space as the Euclidean BTZ black hole gives $\beta_{\rm BTZ} = 4\pi^2/\beta$, $\mu_{\rm BTZ} = 0$. Thus, the sum on m' in (C7) is a sum in m'/β, whereas the sum on m in (C2) is a sum on $m\beta$. To extract the BTZ

[18]In writing this expression we have corrected a typo in [34]. We would like to thank James Lucietti for discussions on this issue.

spectrum, we need to rewrite (C2) in terms of a sum on integer$/\beta$.

First, note that it is not possible to achieve the desired rewriting by a world-sheet modular transformation. The two descriptions are related by a modular transformation in the boundary theory, but not from the world-sheet point of view. In the world sheet, we are considering the same Euclidean target space; we are only changing our interpretation of it.

What we need to do is to make a Poisson resummation over (n, m) in (C2); this will replace the sum over $m\beta$ by a sum over p/β for an integer p. We have not been able to carry out this Poisson resummation because of the theta function $\vartheta_1(\tau, U_{n,m})$ appearing in the denominator of (C2). One can expand this factor in a power series, and perform the Poisson resummation term by term, but it is then not possible to resum the power series to obtain the desired information. Thus, although the partition function in principle contains all the information we want, we have had to take a vertex operator approach to identify the tachyons on the Lorentzian BTZ black hole.

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