Estimating the final spin of a binary black hole coalescence

Alessandra Buonanno,[1] Lawrence E. Kidder,[2] and Luis Lehner[3]

[1]*Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*
[2]*Center for Radiophysics and Space Research, Cornell University, Ithaca, New York 14853, USA*
[3]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA*

We present a straightforward approach for estimating the final black hole spin of a binary black hole coalescence with arbitrary initial masses and spins. Making some simple assumptions, we estimate the final angular momentum to be the sum of the individual spins plus the orbital angular momentum of a test particle orbiting at the last stable orbit around a Kerr black hole with a spin parameter of the final black hole. The formula we obtain is able to reproduce with reasonable accuracy the results from available numerical simulations, but, more importantly, it can be used to investigate what configurations might give rise to interesting dynamics. In particular, we discuss scenarios which might give rise to a flip in the direction of the total angular momentum of the system. By studying the dependence of the final spin upon the mass ratio and initial spins, we find that our simple approach suggests that it is not possible to spin-up a black hole to extremal values through merger scenarios irrespective of the mass ratio of the objects involved.

I. INTRODUCTION

Our understanding of the dynamics of a binary black hole system has advanced at an impressive pace in the past several years as numerical simulations have provided the missing piece for describing the complete dynamics of the system, starting from the late inspiral, and continuing through the merger and ringdown of the final black hole. While these simulations have been exploring the parameter space of possible configurations of initial masses and spins of the black holes, the process is necessarily slow due to the time consuming nature of the simulations. This has led to a number of recent works aimed at producing recipes that allow for predicting, to a certain extent, what can be expected as the final state of the merged black hole from a given initial configuration. This is important, not only in order to decide which situations might give rise to the most interesting dynamical behavior, but also for the possibility of exploiting this information within a simulation. Two examples of this are (1) in the extraction of waveforms, where choosing an adequate background can reduce the errors in determining observable effects, and (2) in helping to provide an *analytic* description of inspiral-merger ringdown to be used in the construction of templates.

To date efforts to produce such estimates have progressed on several fronts. The effective-one-body (EOB) approach [1,2], which maps the two-body problem to the dynamics of a test particle in a suitably defined spacetime, provided estimates for the final black hole spin [3–5] that turn out to agree with current numerical simulations to within ∼10%. More recently, by combining the EOB approach with test-mass limit predictions for the energy released during the merger and ringdown phases, Ref. [6]

has refined these predictions for the nonspinning case to obtain final spins that agree to within ∼2%. On another front, several fitting formulas for the final black hole spin have recently appeared in the literature. These formulas are based on the results of available numerical simulations along with a judicious exploitation of the natural symmetries in the problem and/or expectations from the test-mass limit (see e.g. Refs. [7–12]).

In this work we follow a different route based on first principles, although implicitly using numerical results along with intuitive arguments that, during the merger and ringdown phases, the mass and angular momentum of the system are roughly conserved. Beyond this assumption, however, and in contrast to the approaches mentioned above, we make no explicit use of results from post-Newtonian approximations or numerical simulations. Our straightforward approach is based on simple estimates for the quasiadiabatic inspiral, coupled with standard results for the angular momentum of a particle in a Kerr black hole spacetime corresponding to the *final* black hole. We obtain a closed form expression for the final spin of the black hole for arbitrary mass ratios and spins. We find that predictions from this simple-minded expression give results that agree reasonably well with the numerical simulations. This is yet further evidence of the rather simple behavior underlying the dynamics of binary black hole spacetimes.

In spirit, our work is related to that of Refs. [13,14] which employ similar arguments to estimate the final spin of the black hole by accounting for orbital and individual spins contributions. Our approach, however, contains an important difference, which allows us to make reasonable predictions even for comparable mass systems.

Our approach and assumptions are described in Sec. II, and we present a simple expression that estimates the spin of the final black hole. In Sec. III we illustrate our results for several interesting scenarios, including spinning, precessing black hole binary systems, and compare with numerical simulations. We conclude in Sec. IV with final comments.

II. THE APPROACH

We consider two widely separated black holes that can be well approximated by two Kerr black holes with masses and angular momentum parameters (m_i, a_i). For simplicity we will first restrict our discussion to scenarios where the orbit stays within a plane (which we will refer to as the equatorial plane). In this case the orbital angular momentum and individual spins are orthogonal to the equatorial plane. Our goal is to obtain the angular momentum of the final black hole in terms of the initial configuration of the system by a phenomenological approach rather than by evolving the system numerically. This will aid in finding particularly interesting cases that can be explored with numerical simulations.

Achieving this goal certainly requires some compromises as the system undergoes a long dynamical process that comprises several stages: inspiral, merger, and ringdown. While an accurate description of the whole process requires following the system completely, a back of the envelope estimate can be obtained by exploiting the fact that: (i) during the inspiral phase, the system evolves quasiadiabatically, and (ii) during the merger and ringdown phases the total mass and angular momentum of the system change by only a small amount. These observations are based on intuitive arguments but are strongly backed by post-Newtonian and perturbative calculations for the inspiral and ringdown, and by numerical simulations for the merger.

We obtain a simple expression, based on first principles, that can be employed to predict what the angular momentum parameter of the final black hole will be. This expression is obtained naturally when, inspired by the observations above, the following assumptions are made:

(i) To first order, the mass of the system is conserved. Thus, the final black hole will have total mass $M = m_1 + m_2$. During the whole process the total radiated energy remains small ($M_{\text{radiated}} \lesssim 10\% M_{\text{initial}}$); thus this assumption is justified to the level of approximation we seek.

(ii) The magnitude of the individual spins of the black holes will remain constant. Since both spin-spin and spin-orbit couplings are small, and radiation falling into the black holes affects the spins by a small amount, this is a safe assumption. Therefore the contribution to the final total angular momentum due to the individual black holes spins will be determined by the initial spins.

(iii) The system radiates much of its angular momentum in the long inspiral stage until it reaches the innermost stable circular orbit (ISCO), at which point the dynamics quickly leads to the merger of the black holes. During the merger the radiation of energy and angular momentum with respect to the mass and angular momentum of the system is small. Thus, to estimate the contribution of the orbital angular momentum to the final angular momentum of the black hole, we adopt the orbital angular momentum of a test-particle orbiting at the ISCO of a Kerr black hole with a spin parameter of the *final* black hole. While adopting this value makes strict sense in the extreme mass-ratio case, we will see that it leads to predictions that agree quite well with the results of numerical simulations. That the point-particle approximation is able to capture key aspects of the two-body dynamics has also been observed in Refs. [3,6,9,15] when comparing with results in Ref. [16], in Refs. [17,18] when comparing the close-limit approximation to numerical simulations, and more recently in studies of unstable circular orbits in black hole mergers [19].

Bringing all these assumptions together, we may express the angular momentum parameter of the final black hole a_f as

$$\frac{a_f}{M} = \frac{L_{\text{orb}}}{M^2}(\mu, r_{\text{ISCO}}, a_f) + \frac{m_1 a_1}{M^2} + \frac{m_2 a_2}{M^2}, \qquad (1)$$

where L_{orb} is the orbital angular momentum of a particle (of mass μ) at the ISCO of a Kerr black hole (with spin parameter a_f).

Note that our assumptions differ in several ways from those of Ref. [13]: (1) we keep the mass of the system constant, while Ref. [13] adds a contribution to the final mass from the energy at the ISCO; (2) we keep the contributions from the spins of both bodies, while Ref. [13] neglects the spin of the smaller black hole; and (3) we use the orbital angular momentum of the ISCO for a Kerr black hole with the *final* spin, while Ref. [13] uses the initial spin of the larger black hole. For extreme mass-ratio cases, both approaches give essentially the same answer (e.g. for $m_1 > 50m_2$ the difference is below 1%); however, our approach can be applied to general mass ratios and accounts for the orbital and both individual spin angular momenta when obtaining the final black hole spin. For instance, in the case of black holes with equal individual spin parameters $a_i/m_i = 0, 1/2$ or 0.99, our predicted value differs from that estimated with the results of Ref. [13] by 31%, 17%, and 1%, respectively, for $m_1 = 2m_2$ and by 17%, 11%, and 1% for $m_1 = 10m_2$. As we will see below, a comparison with available numerical results indicates the estimate

presented here is able to capture the resulting final spin quite well.

We can reexpress our formula for a_f in a more convenient form as

$$\frac{a_f}{M} = \frac{L_{\mathrm{orb}}}{M^2}(\nu, r_{\mathrm{ISCO}}, a_f) + \frac{\chi_1}{4}(1 + \sqrt{1 - 4\nu})^2$$
$$+ \frac{\chi_2}{4}(1 - \sqrt{1 - 4\nu})^2, \qquad (2)$$

where $\chi_i = a_i/m_i$ ($\in [-1, 1]$) and $\nu = (m_1 m_2)/M^2$ ($\in [0, 1/4]$), and without loss of generality it is assumed that $m_1 \gtrsim m_2$. This equation provides a way to obtain a_f given m_i and χ_i. Since the expression for L_{orb} at the ISCO is known in closed form for equatorial orbits, we concentrate first on such cases. This will allow us to investigate the viability of Eq. (2) by comparing it to results obtained with numerical simulations.

Adopting the expression for equatorial orbits in Ref. [20], one deduces

$$\frac{L_{\mathrm{orb}}}{M^2} = \pm \frac{\nu(r^2 \mp 2a_f M^{1/2} r^{1/2} + a_f^2)}{M^{1/2} r^{3/4}(r^{3/2} - 3Mr^{1/2} \pm 2a_f M^{1/2})^{1/2}}, \quad (3)$$

where the upper signs correspond to prograde orbits and the lower signs to retrograde orbits. This function is to be evaluated at $r = r_{\mathrm{ISCO}}$ with

$$r_{\mathrm{ISCO}} = M\{3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\}, \quad (4)$$

where

$$Z_1 \equiv 1 + \left(1 - \frac{a_f^2}{M^2}\right)^{1/3}\left[\left(1 + \frac{a_f}{M}\right)^{1/3} + \left(1 - \frac{a_f}{M}\right)^{1/3}\right],$$

$$Z_2 \equiv \left(3\frac{a_f^2}{M^2} + Z_1^2\right)^{1/2}.$$

The use of the prograde or retrograde case depends on whether the final spin is aligned or antialigned with the initial orbital angular momentum. Indeed one particularly interesting application of the above expression is to understand the direction of the final spin as a function of initial spins and the mass ratio.

III. REPRESENTATIVE CASES

We now concentrate on several representative cases that explore different interesting scenarios, and make contact with available numerical results.

A. Equal spin case

A simple case that can be compared with existing simulations is for equal spins (i.e. $\chi_1 = \chi_2 = \chi$). In this case Eq. (2) reduces to

$$\frac{a_f}{M} = \frac{L_{\mathrm{orb}}}{M^2} + (1 - 2\nu)\chi. \qquad (5)$$

This equation allows us to determine the value of a_f as a function of ν and χ and answer specific questions. Figure 1 illustrates the behavior of the final spin parameter as a function of mass ratio when the individual spins of the initial black holes are maximal. The largest spin for the final black hole is achieved for the extreme mass-ratio case. This coincides with the intuitive picture that a particle falling into an extreme black hole will have a negligible effect on the final spin, while a head-on collision of equal-mass extreme Kerr black holes will give rise to a final black hole with $a_f/M = 1/2$.

This behavior, however, varies when considering initial spins less than maximal. For initially nonspinning black holes, intuitively the final black hole will also be essentially nonspinning for the extreme mass-ratio case while it would have a nontrivial final spin for the equal-mass case. Figure 2 illustrates the spin of the final black hole as a function of the mass ratio for different values of χ for spins that are aligned with the orbital angular momentum. We see that there is a critical value for the initial spins, $\chi = 0.948$, above which the maximum final spin will increase as the mass ratio $q = m_1/m_2 \geq 1$ increases. Below the critical value, the final spin will increase as the equal-mass limit is approached. Finally, at the critical value any merger will leave the final spin essentially unchanged irrespective of the mass ratio.

The case of black holes which are initially nonspinning can be compared directly with a number of simulations [21–25]. For equal masses, the value predicted by Eq. (5) is
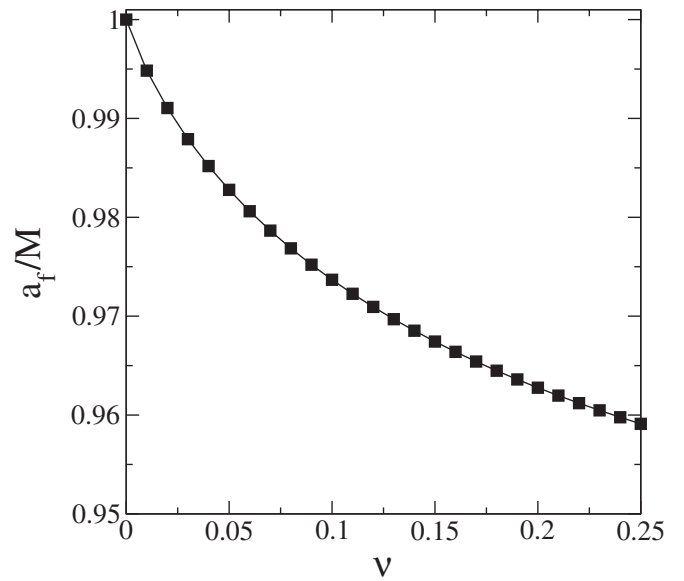

FIG. 1. The final spin a_f/M vs $\nu = m_1 m_2/M^2$ for initial black holes with extreme spin parameters $\chi = 1$. Clearly, as the mass ratio approaches $\nu = 1/4$ (equal-mass case), the final expected spin parameter decreases.
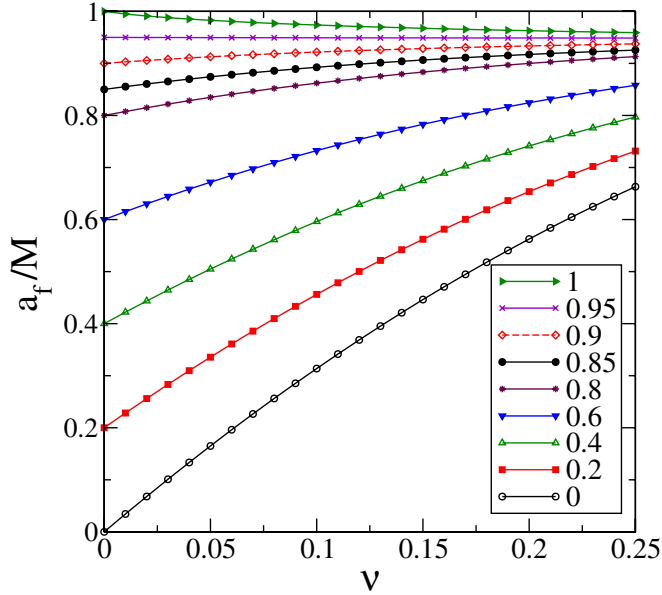
FIG. 2 (color online). Final spin a_f/M vs ν for initial black holes for several spin parameters $\chi = 0$, 0.2, 0.4, 0.6, 0.8, 0.85, 0.9, 0.95, 1. For spins $\gtrsim 0.948$ the maximum final spin is achieved as the extreme mass-ratio case is approached while for initial spins $\lesssim 0.948$ the equal-mass case would give rise to the maximum final spin.
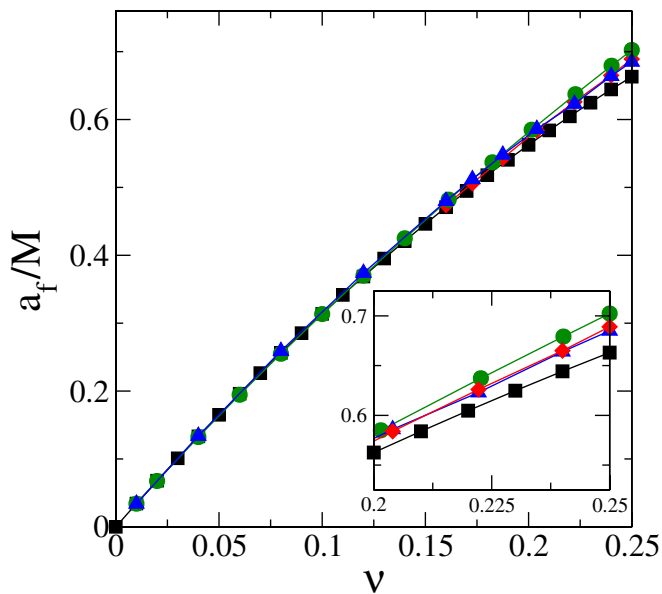
$a_f/M = 0.66$, which is quite close to the value $a_f/M = 0.68$ obtained by simulations of equal-mass, nonspinning black holes. Figure 3 illustrates our predicted values for the final spin as the mass ratio is varied from the extreme mass ratio to the equal-mass case, along with the results from Refs. [6,8,9]. Excellent agreement is found with results from numerical simulations. Notice that the final black hole with the highest spin occurs in the equal-mass case as expected since the orbital angular momentum is maximized in that case.

Another case that can be compared to simulations is for equal masses where the initial spins are either aligned or antialigned with respect to the orbital angular momentum. While simulations for close to maximally spinning black holes are a challenge, robust results exist for $\chi_i \in [-0.7, 0.7]$. Figure 4 shows the final value for the spin parameter predicted by Eq. (5) as χ is varied, along with the values available from existing simulations. Clearly, a quite reasonable agreement is found for $\chi_i \in [-0.7, 0.7]$. Furthermore, we obtain for the extreme cases $a_f(\chi = -1) = 0.2909M$ and $a_f(\chi = 1) = 0.9591M$, which are, respectively, 14% and 0.01% away from the values reported by the fit formulas employed in Ref. [11]. Figure 4 also shows results from the EOB model used in
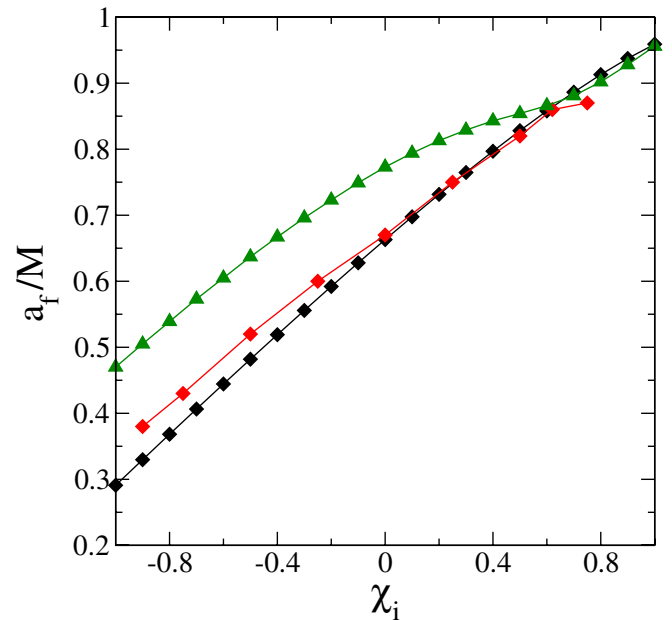

FIG. 3 (color online). Final spin a_f/M vs ν for initial black holes with initially nonspinning black holes ($\chi = 0$) as predicted by our simple model (filled squares), and as obtained numerically in Ref. [8] (red diamonds), by the EOB model combined with nonspinning test-particle predictions in Ref. [6] (green circle) and by numerical relativity combined with nonspinning test-particle expectations in Ref. [9] (blue triangle). The largest final black hole spin is obtained in the equal-mass case.

FIG. 4 (color online). Final spin a_f/M vs χ for equal-mass black holes as predicted by our expression (filled squares), numerically obtained (filled diamonds) in Ref. [11] and from the EOB model (filled triangle) used in Ref. [5], but without taking into account the angular momentum released during merger-ringdown (thus, the latter curve overestimates the results and should be considered as an upper limit). Irrespective of the alignment or antialignment of the individual spins, the final black hole spin is aligned with the initial orbital angular momentum.

Ref. [5], where the final black hole spin is computed at the end of the EOB plunge, disregarding spin-spin corrections and without taking into account the angular momentum released during merger ringdown;[1] thus, the values should be considered as an upper limit.[2]

A final interesting example that we report here is where the individual spins are antialigned with the *initial* orbital angular momentum and the spin of the final black hole is zero. This occurs in our model when the orbital angular momentum remaining at the ISCO is exactly counteracted by the individual black hole spins. This scenario determines the border between cases in which the direction of the final angular momentum (or final spin) is determined by the initial orbital angular momentum, or by the direction of the initial black hole spins. In the latter cases the direction of the total angular momentum will "flip," and inertial frames will see the direction they are dragged reverse as the system goes through the merger. This should be reflected in the waveforms, which would likely display an interesting behavior [26]. Figure 5 shows the mass ratios and initial value of the individual antialigned spins required to yield a final nonspinning black hole. For a realizable situation of $a_i = 0.8m_i$, the mass ratio adopted should be 4:1, while for maximally spinning holes, we would predict a final spin of zero for the mass ratio $q \simeq 3.15$. Note that this value is smaller than that predicted by Ref. [13] of $q \simeq 4.23$. By our arguments, any system with mass ratio $q > 6.78$ will undergo a flip of the total angular momentum if the individual spins are equal and antialigned with the initial orbital angular momentum and their spin parameters obey $a_i/m_i \gtrsim 1/2$. Notice, however, that the orbital separation at which this flip takes place also depends upon the mass ratio. This location can be estimated via simple Newtonian arguments so as to obtain a rough idea on where the flip may take place. A particle of reduced mass $\mu = \nu M$ in a circular orbit about a central object of mass M has an associated orbital angular momentum given by

$$L_{\mathrm{approx}} = \mu \sqrt{rM}. \qquad (6)$$

Thus, the distance at which L_{approx} will be canceled by the contribution of the individual spins is determined by

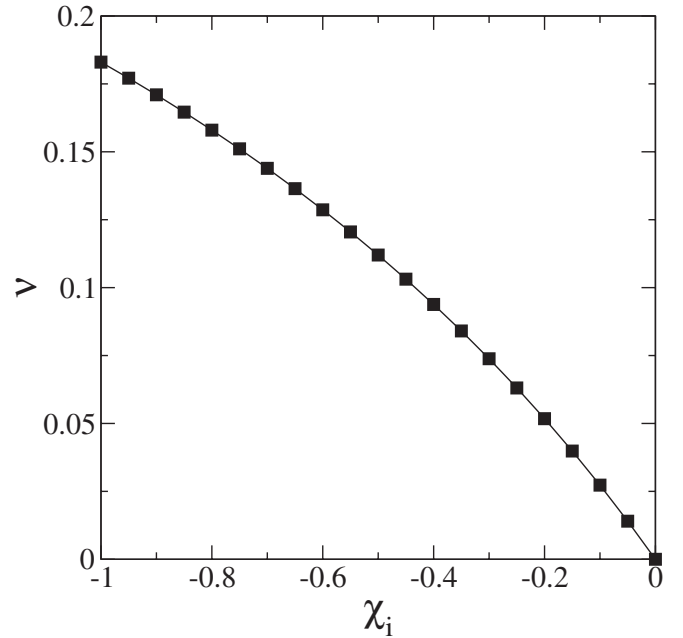$$\frac{L_{\mathrm{approx}}}{M^2} = (1 - 2\nu)|\chi|. \qquad (7)$$

FIG. 5. Relation between the mass ratio ν and the individual spins of the black holes to achieve a final nonspinning black hole. Only mass ratios $\nu \lesssim 0.183$ ($q \gtrsim 3.15$) will allow for the final spin to be antialigned with the initial orbital angular momentum direction if the individual black holes spin fast enough.

Thus, the following estimate,

$$\frac{r}{M} = \frac{(1 - 2\nu)^2}{\nu^2}|\chi|^2, \qquad (8)$$

can be used to determine the approximate distance at which the flip will occur. Notice in the limit $\nu \to 0$ ($q \to \infty$) that $r \to \infty$ so the flip has essentially taken place prior to any astrophysically interesting initial configuration. In such cases the final spin direction is determined by the spin of the large black hole. We stress that this is a simple Newtonian-based estimate for the distance at which the flip might occur; sharper values can be obtained by employing Post-Newtonian expressions.

B. Unequal spin case

We now illustrate the case with equal masses, but unequal spins. Setting $\chi_1 = \chi = \alpha\chi_2$ and $\nu = 1/4$ in Eq. (2) yields

$$\frac{a_f}{M} = \frac{L_{\mathrm{orb}}}{M^2} + \frac{\chi}{4}(1 + \alpha). \qquad (9)$$

Figure 6 illustrates the value of the final spin parameter for equal-mass black holes and $\chi = 0.584$ and varying $\alpha \in [-1, 1]$, while Fig. 7 illustrates the case with $\chi = -0.584$. In both cases we compare directly against the results of the simulations in Ref. [10]. Our results differ at most by 8% with the reported results. We have also compared with
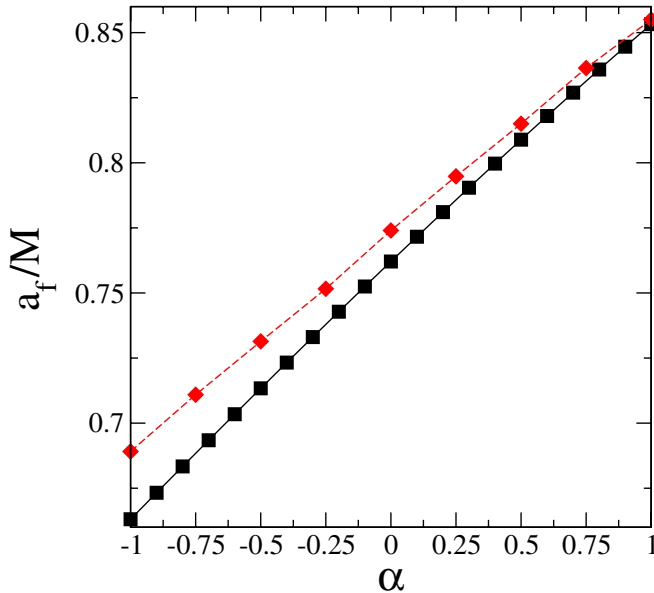
FIG. 6 (color online). Final spin a_f/M vs α for equal-mass black holes with $\chi_1 = 0.584$ and $\chi_2 \in [-0.584, 0.584]$ as predicted by our simple model (filled squares) and as obtained numerically in Ref. [10] (red diamonds).

simulations with mass ratio 3:2 and either $a_1/m_1 = -0.194$ and $a_2/m_2 = 0.201$, or $a_1/m_1 = 0.193$ and $a_2/m_2 = -0.201$ [27]. In these cases our simple model predicts a final spin of $0.617M$ and $0.671M$, respectively, while Ref. [28] obtains numerically a final spin of $0.640M$ and $0.704M$. Finally, for equal masses and $a_1/m_1 = -0.198$ and $a_2/m_2 = 0.198$, Refs. [27,28] obtain a final spin of $0.697M$, whereas we predict $0.663M$.
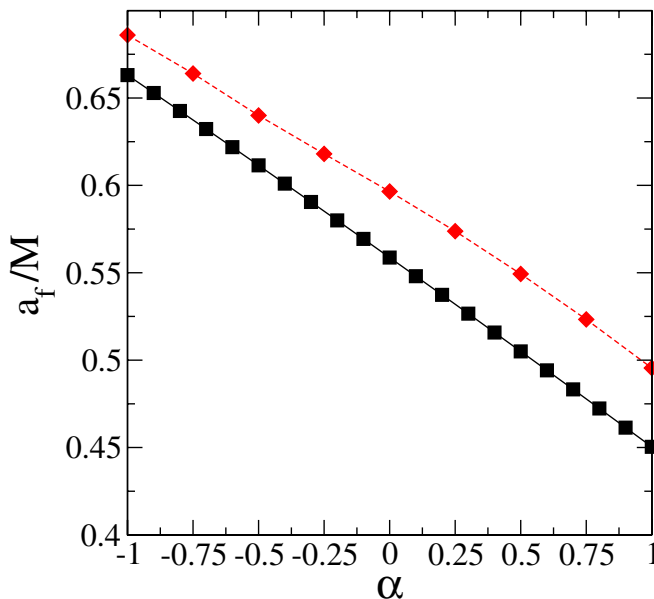

FIG. 7 (color online). Final spin a_f/M vs α for equal-mass black holes with $\chi_1 = -0.584$ and $\chi_2 \in [-0.584, 0.584]$ as predicted by our simple model (filled squares) and as obtained numerically in Ref. [10] (red diamonds).

C. Generic spin configurations

Until now our analysis has been restricted to cases where the orbital plane does not change in time. However, we expect that the same arguments which lead us to Eq. (1) are applicable to more generic scenarios with precessing orbits and arbitrary directions for the individual spin. A key difference in generic cases is that the orbit at the ISCO will, in general, be inclined with respect to the final total angular momentum. For these cases the expression for the orbital contribution to the total angular momentum would require either the numerical integration of generic geodesics in a Kerr spacetime or the use of the radial potential for quasiadiabatic spherical orbits [29]. Alternatively, one can make use of the fit formulas presented in Ref. [13] to express $\vec{L}_{\rm orb}$ in analytic form.

The simplest possible extension of our method to more generic spin configurations can be formulated when the following assumptions (in addition to the assumptions we made in Sec. II) are adopted:

(i) The following quantities are known:

$$\{m_1, m_2, \vec{S}_1, \vec{S}_2, \hat{L}_{\rm orb}\}$$

at some point of the inspiral (prior to the ISCO), where $\hat{L}_{\rm orb}$ is a unit vector parallel to the orbital angular momentum.

(ii) Both the magnitude of the total spin $\vec{S}_{\rm tot} = \vec{S}_1 + \vec{S}_2$, and the angle $\theta_{\rm LS}$ between the total spin and the direction of the orbital angular momentum $\hat{L}_{\rm orb}$ will remain constant up to the ISCO.

We notice that in general $\theta_{\rm LS}$ and $|\vec{S}_{\rm tot}|$ can change during the evolution so this assumption might not hold to a tolerable level. However, we know from the post-Newtonian spin-precession equations [30,31] that this assumption is valid in two cases, notably (i) equal-mass double-spin binary systems (when spin-spin terms are neglected) and (ii) unequal-mass single-spin binary systems [26].

From the initial conditions, we compute

$$S_{\rm tot} \equiv |\vec{S}_{\rm tot}|, \tag{10}$$

$$\cos\theta_{\rm LS} = \frac{\hat{L}_{\rm orb} \cdot \vec{S}_{\rm tot}}{|\vec{S}_{\rm tot}|}. \tag{11}$$

In Fig. 8, we show the total spin and orbital angular momentum at the ISCO, where $\vec{J}_f = \vec{L}_{\rm orb} + \vec{S}_{\rm tot}$. As before, we can then obtain the final spin of the black hole as $\vec{a}_f = \vec{J}_f/M$. More explicitly, if we decompose the vectors along the directions parallel and orthogonal to the final spin, we obtain

$$L_{\rm orb}(\iota, a_f)\cos\iota + S_{\rm tot}\cos(\theta_{\rm LS} - \iota) = Ma_f, \tag{12}$$

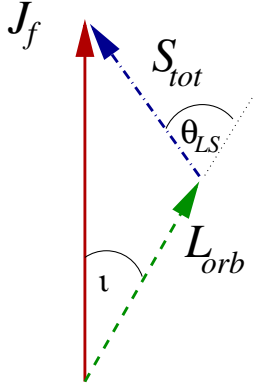$$L_{\rm orb}(\iota, a_f)\sin\iota - S_{\rm tot}\sin(\theta_{\rm LS} - \iota) = 0, \tag{13}$$

FIG. 8 (color online).   At ISCO, the total angular momentum will be the sum of the orbital angular momentum and the spin vectors. The angle θ_{LS} is assumed to be given and remain fixed during the inspiral prior to ISCO. The inclination angle ι is solved for along with the magnitude of the total angular momentum.

where $L_{\rm orb} = |\vec{L}_{\rm orb}|$. Equations (12) and (13) can be solved to derive the magnitude of the final spin a_f and the inclination angle ι at the ISCO (see also Ref. [14]). For simplicity, we can compute the orbital angular momentum of the inclined orbit using the fit formula of Ref. [13]:

$$L_{\rm orb}(\iota, a_f) = \tfrac{1}{2}(1 + \cos\iota)L_{\rm orb}^{\rm pro}(\nu, r_{\rm ISCO}^{\rm pro}, a_f)$$
$$+ \tfrac{1}{2}(1 - \cos\iota)|L_{\rm orb}^{\rm ret}(\nu, r_{\rm ISCO}^{\rm ret}, a_f)|, \quad (14)$$

where $L_{\rm orb}^{\rm pro}$ and $L_{\rm orb}^{\rm ret}$ are given by Eq. (3) for prograde and retrograde orbits, respectively, and $r_{\rm ISCO}^{\rm pro}$ and $r_{\rm ISCO}^{\rm ret}$ are the corresponding ISCOs.

With the above procedure, we can compare the obtained estimates with available numerical results. In Table I, we list the estimates from our approach together with the results obtained from numerical simulations in Refs. [7,32–34] where several equal-mass double-spin and a few unequal-mass single-spin precessing configurations have been studied. Note that in cases C_1 and C_3, the total spin is zero (i.e. the individual spins are equal, but opposite) and the numerical results give quite similar values for a_f/M. The total spin is also zero for the "B-series" presented in Ref. [34] where the final spins for all configurations[3] are the same $a_f/M = 0.66$. This behavior is predicted by our approach since when the total spin is zero, it will predict the same result (independent of the magnitude of the individual spins), which is given as a function of mass ratio by the bottom curve in Fig. 2. Note that the cases C_6, C_7, and C_8 all have the same $S_{\rm tot}$ and θ_{LS}, thus our approach predicts the same a_f/M and ι. Moreover, in cases C_1 and C_3, the masses are equal and

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[3]The final spins in Ref. [34] are quoted in terms of a mass parameter M that is not the mass of the final black hole M_f. We estimate $M_f = M(1 - E_{\rm rad})$, where $E_{\rm rad}$ is the reported energy radiated.

TABLE I.   In the second and third columns, we list results from numerical simulations of precessing, spinning black holes, for the magnitude of the final spin and the angle between the final spin and the direction of the orbital angular momentum (if available). We use data from Ref. [32] (we indicate with C_i, $i = 1, \ldots 8$ the spin configurations from left to right in Table I of Ref. [32]), from Refs. [7,33] (see Table 1 in each case), and representative cases from Ref. [34] (see Table 2). In the last two columns we list the estimates obtained with our approach. The cases of Ref. [32] are equal mass with $\chi_i = 0.8$. The cases of Ref. [7] are also equal mass with $\chi_i \approx 0.50$. The case of Ref. [33] is for $q \approx 2$, $\chi_1 = 0.885$, and $\chi_2 = 0$. The cases of Ref. [34] are also equal mass with $\chi_i = 0.6$.

Case [reference]	a_f/M_f	ι (deg)	a_f/M	ι (deg)
C_1 [32]	0.67	$\cdots$	0.66	0
C_2 [32]	0.72	$\cdots$	0.71	23
C_3 [32]	0.68	$\cdots$	0.66	0
C_4 [32]	0.73	$\cdots$	0.71	23
C_5 [32]	0.64	$\cdots$	0.61	34
C_6 [32]	0.81	$\cdots$	0.82	14
C_7 [32]	0.80	$\cdots$	0.82	14
C_8 [32]	0.80	$\cdots$	0.82	14
SP_3 [7]	0.72	18	0.70	21
SP_4 [7]	0.81	10	0.80	13
SP_6 [33]	0.50	33	0.48	35
S-0 [34]	0.75	$\cdots$	0.78	11
S-75 [34]	0.69	$\cdots$	0.69	0.4
S-150 [34]	0.58	$\cdots$	0.57	7
S-225 [34]	0.61	$\cdots$	0.64	24
S-300 [34]	0.72	$\cdots$	0.76	22

the total spin is zero, but the individual spins are not on the orbital plane (corresponding to the large kick configurations in Refs. [33,35,36]). In these cases our approach predicts the same a_f/M and ι. Once again in these more complex physical scenarios the agreement of our simple approach is reasonably good.

We find it important to stress again that it may be that, for more general precessional cases—in which both black holes carry spin and their masses are not equal, or when the system undergoes a *transitional precession* [26]—the above approach may not provide a good approximation. We plan to investigate more generic cases in more detail in the future.

IV. CONCLUSIONS

We have presented an approach to obtain a simple expression to estimate the spin of the final black hole produced through a merger of orbiting binary black holes. In this work we have concentrated on several especially interesting cases, but others can certainly be explored as well.

Notice that our work is complementary to recent works aimed at giving fit formulas for different physical quantities based on the results of numerical simulations (see e.g.,

Refs. [7,9–12]), and to the EOB predictions [3–6]. Our expression, however, does not rely directly on the simulations, but rather on a simple approach based on first principles. On the other hand, it has an inherent amount of error due to its simple assumptions. Confronting our predictions with available results, we find that they agree rather well considering the limitations of our simplistic approach. This fact gives further evidence to the rather simple behavior describing the dynamics of orbiting black holes. The expression presented in this work can be employed to predict the outcome of the simulation for a large number of cases not yet studied and can help determine which parameter choices might give the most interesting results.

For example:

(i) For individual spins aligned with the orbital angular momentum and $a_i \gtrsim 0.948m_i$, the final black hole spin is larger as ν decreases. However, for individual spins $a_i \lesssim 0.948m_i$, as ν is increased the final spin will be greater. This *transition* number has an inherent error due to our approximations, and thus should not be considered sharp. Nevertheless, we would expect a transition to occur near this value. Hence, our simple-minded model suggests that it is impossible to spin-up a black hole through mergers to its extremal value even in the *ideal* case where both spins and the orbital angular momentum are aligned, and no dissipative effects exist to reduce the final spin. As indicated by Fig. 2, the only way to get a final maximal spin requires an already maximally spinning black hole merging in extreme mass-ratio situations. Any other alternative in the highly spinning cases ($a_i \gtrsim 0.948m_i$) will cause the final spin to decrease. If ($a_i \lesssim 0.948m_i$), the final spin will only increase up to a value $a \approx 0.95$. After this state, any further merger will essentially leave this value unchanged.[4] Consequently, binary black hole systems would not give rise to an orbital hang-up due to having $J > M^2$ after the ISCO.

(ii) The direction of the orbital angular momentum of a system with arbitrary spins (perpendicular to the orbital plane) determines the final spin for $\nu \gtrsim 0.183$. For $\nu \lesssim 0.183$, however, the final spin direction will depend on how large the individual spins

are; this can give rise to a final black hole whose spin opposes the initial orbital angular momentum direction. This scenario should give rise to an interesting phenomenology in the resulting waveforms. We stress again that this critical value is approximate given our simple assumptions, but such a critical value must exist.

As discussed in Sec. III C, our approach can be generalized to spinning, precessing binaries. As long as the total spin of the system and the angle between the total spin and the orbital angular momentum are preserved during the evolution, we have been able to compare our results to numerical simulations of precessing binary systems, obtaining good agreement. We plan to carry out a more thorough study of more generic spinning, precessing binaries in the future, and investigate further extensions of our approach.

Some other applications of this approach would be to employ the predicted values in order to adopt a reasonably close background for perturbative approaches to study the after-merger epoch or to compute physical quantities with respect to this background like gravitational radiation. Additionally, it can be used to aid in providing the quasi-normal mode frequencies (which are a function of the final black hole mass and spin) to be used in analytically matching the inspiral to the merger to the ringdown.

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[4]For a related discussion see Ref. [13].

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