PHYSICAL REVIEW D **77,** 026002 (2008)

Toric resolutions of heterotic orbifolds

Stefan Groot Nibbelink,[1,2,*] Tae-Won Ha,[1,†] and Michele Trapletti[1,‡]

[1]*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16 und 19, D-69120 Heidelberg, Germany*

[2]*Shanghai Institute for Advanced Study, University of Science and Technology of China, 99 Xiupu Rd, Pudong, Shanghai 201315, People's Republic of China*

(Received 24 July 2007; published 9 January 2008)

We investigate resolutions of heterotic orbifolds using toric geometry. Our starting point is provided by the recently constructed heterotic models on explicit blowup of $\mathbb{C}^n/\mathbb{Z}_n$ singularities. We show that the values of the relevant integrals, computed there, can be obtained as integrals of divisors (complex codimension one hypersurfaces) interpreted as (1, 1)-forms in toric geometry. Motivated by this we give a self-contained introduction to toric geometry for nonexperts, focusing on those issues relevant for the construction of heterotic models on toric orbifold resolutions. We illustrate the methods by building heterotic models on the resolutions of $\mathbb{C}^2/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_4$, and $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$. We are able to obtain a direct identification between them and the known orbifold models. In the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ case we observe that, in spite of the existence of two inequivalent resolutions, fully consistent blowup models of heterotic orbifolds can only be constructed on one of them.

DOI: 10.1103/PhysRevD.77.026002 PACS numbers: 11.25.Mj, 04.50.−h, 04.65.+e

I. INTRODUCTION

One of the main aims of string phenomenology is to build a string model reproducing, at low energies, the standard model of particle physics, or a supersymmetric extension of it. This issue has been faced from different perspectives, in particular, we remind the reader of models built using free-fermion models [1–3], intersecting D-branes in type II string theory [4–7], Gepner models [8,9], and compactifications of the heterotic string. In the latter case, in order to obtain four dimensional models with at most $N = 1$ supersymmetry, i.e. in order to have a chiral spectrum, one needs to compactify on a Calabi-Yau space [10] (see also [11–16] for recent progresses in this direction), or on a singular limit of it: an orbifold. Orbifolds are particularly convenient, since they allow fully calculable string compactifications, in terms of combinations of free conformal field theories [17,18]. Given this calculability, it is possible to produce a vast but controllable landscape of models, and scan among them for realistic ones. Indeed, this approach has been proven to be successful, and models extremely close to the minimally supersymmetric standard model (MSSM) have been built [19–26].

Orbifolds are special points in the full moduli space of the heterotic string on Calabi-Yau manifolds. In order to have a better control on the theory away from these special orbifold points, it is crucial to have a better understanding of model building on the corresponding smooth compactification spaces. As the theory is completely calculable at the orbifold point, one may also hope, that one can learn about its properties in the moduli space in the vicinity of this singularity. The underlying theme of this paper is precisely to study the interplay between the heterotic string theory at the orbifold points of the moduli space and on generic Calabi-Yau spaces.

A concrete way to probe the moduli space surrounding orbifold points is to consider blowups of orbifold singularities in an effective supergravity coupled to super Yang-Mills description. The idea is to first study the resolution of isolated singularities and after that obtain a description of a compact Calabi-Yau by gluing various orbifold resolutions together. The construction of explicit blowups is unfortunately not easy. The most known example is the Eguchi-Hanson resolution [27] of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity. Generalizations to $\mathbb{C}^n/\mathbb{Z}_n$ were discussed in the mathematical literature [28], see also [29,30]. The construction and the application of explicit blowups of these singularities to heterotic model building has been investigated in [31,32]. In particular, it was shown that all $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/\mathbb{Z}_3$ heterotic orbifold models could be recovered by considering U(1) bundle gauge backgrounds on the blowup [32]. This construction was used to explicitly verify that in blowup multiple anomalous U(1)'s are possible [33,34], even though heterotic orbifold models always have at most a single anomalous U(1). The way out of this apparent paradox is that a twisted state, with a nonvanishing vacuum expectation value (VEV), can be reinterpreted as a model dependent axion, that can cancel nonuniversal anomalies [35]. This, in particular, helped to resolve confusion [36–38] concerning the heterotic/type I duality on $\mathbb{Z}_3$ orbifolds.

Explicit blowups of $\mathbb{C}^n/\mathbb{Z}_n$ singularities were possible because both these orbifolds and their blowups have a large isometry group. However, for four dimensional string model building, these blowups can only be used to model $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/\mathbb{Z}_3$ singularities, while MSSM-like model building seems to require more complicated orbifolds, like $T^6/\mathbb{Z}_{6-II}$ or $T^6/\mathbb{Z}_{12-I}$. (See e.g. [19–26].) The singularities of these orbifolds are more complicated and might not

*grootnib@thphys.uni-heidelberg.de
†tha@tphys.uni-heidelberg.de
‡M.Trapletti@thphys.uni-heidelberg.de

allow for a simple explicit blowup construction. On the other hand, the topological properties of such resolutions can be conveniently described by toric geometry, see e.g. [39]. In this paper we explain how using toric geometry one can construct heterotic models on resolutions on arbitrarily complicated orbifold singularities.

For a general mathematical introduction to the subject of toric geometry we refer the reader to e.g. [40–44]. Applications of toric geometry to orbifold resolutions have also recently been discussed in [45,46]. The presentation of the toric geometry in this paper gives an exposition of simple toric techniques, which can be used to understand the topological properties relevant for model building. For this program we explain the construction of toric varieties that represent the resolution of orbifolds. The divisors, complex codimension one hypersurfaces, encode the topology of the resolution. We explain that one can interpret divisors as (1, 1)-forms, and integrate them over the resolution. This allows us to use divisors as field strengths, i.e. first Chern classes, of U(1) complex line bundle gauge backgrounds. These backgrounds can then be used to construct consistent heterotic models on the resolution. To cross-check this procedure we first reproduce all results obtained using the explicit blowup of $\mathbb{C}^n/\mathbb{Z}_n$. After that we extend the analysis to more complicated orbifolds, for which to our knowledge no explicit blowup has been written down.

To present our results the paper has been structured as follows: In Sec. II we first review the explicit blowup of the $\mathbb{C}^n/\mathbb{Z}_n$ orbifold. After that we introduce toric geometrical techniques to reobtain the integrals computed on the explicit blowup as integrals of certain divisors over the corresponding toric variety. In Sec. III we first give a general account of the analysis of orbifold singularities using toric geometry, and explain how this can be applied to heterotic model building on orbifold resolutions. We illustrate the various methods by two examples: The resolution of $\mathbb{C}^2/\mathbb{Z}_3$, the simplest example of blowup with two exceptional divisors, is described in subsection III B. The next subsection is devoted to the resolution of $\mathbb{C}^3/\mathbb{Z}_4$. For both these resolutions we explain how we can construct consistent models on them, and derive the conditions that ensure they have a direct orbifold interpretation as well. For the $\mathbb{C}^3/\mathbb{Z}_4$ resolution we construct models that satisfy possible Bianchi identities, and we confirm that they give rise to models free of non-Abelian anomalies in four dimension, which all can be matched to heterotic orbifolds. Section IV investigates orbifolds that do not possess a single unique resolution. We propose minimal requirements of defining integrals avoiding inconsistencies with the linear equivalence relations. The issues that arise when the resolution is not unique, are exemplified by discussing the two inequivalent resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ in subsection IV B. In the final subsection IV C we compute heterotic models on one of the resolutions, and argue that no fully consistent model can be built on the other. In Sec. V we summarize our conclusions.

II. TORIC DESCRIPTION OF EXPLICIT BLOWUPS OF ORBIFOLD SINGULARITIES

A. Blowup of $\mathbb{C}^n/\mathbb{Z}_n$ orbifold

In [32] we have given a detailed description of how to explicitly obtain a blowup of the $\mathbb{C}^n/\mathbb{Z}_n$ orbifold with possible U(1) bundles. Here we will only recall those results which will be relevant for our subsequent discussion; for details the reader may consult [31,32]. The $\mathbb{C}^n/\mathbb{Z}_n$ orbifold is defined by the $\mathbb{Z}_n$ action

$$\Theta(\tilde{Z}) = \theta \tilde{Z}, \qquad \theta = e^{2\pi i \phi}, \qquad \phi = \frac{1}{n}\,\mathrm{diag}(1, \ldots, 1), \quad (1)$$

on the orbifold coordinates $\tilde{Z}$. This defines a space with a singularity, having deficit angle of $2\pi(1 - 1/n)$. The geometry of the nonsingular blowup is described by the Kähler potential $\mathcal{K}$ given by

$$\mathcal{K}(X) = \int_1^X \frac{dX'}{X'} M(X'), \qquad M(X) = \frac{1}{n}(r + X)^{1/n}, \quad (2)$$

where $X = (1 + \bar{z}z)^n |x|^2$ is an SU(n) invariant, and the z and x are the coordinates of the space. In detail, the z form a set of inhomogeneous complex coordinates of $\mathbb{CP}^{n-1}$, and x the coordinate parametrizing the complex line over $\mathbb{CP}^{n-1}$. Finally, r is the resolution parameter, defined such that in the limit $r \to 0$ one retrieves the orbifold geometry.

From the Kähler potential all geometrical quantities can be derived in the standard way, in particular, the curvature 2-form reads

$$\mathcal{R} = \frac{r}{r + X}\begin{pmatrix} e\bar{e} - \bar{e}e + \frac{1}{n}\frac{\bar{e}\epsilon}{r+X} & \frac{\bar{\epsilon}e}{\sqrt{r+X}} \\ \frac{\bar{e}\epsilon}{\sqrt{r+X}} & n\bar{e}e - \frac{n-1}{n}\frac{\bar{\epsilon}\epsilon}{r+X} \end{pmatrix}. \quad (3)$$

Here e and ϵ are the holomorphic vielbein 1-forms of $\mathbb{CP}^{n-1}$ and its complex line bundle. It can be shown that $\mathcal{R}$ is traceless, which is consistent with the Calabi-Yau property of having vanishing first Chern class. In other words the Kähler potential (2) defines the Ricci-flat metric on the blowup showing explicitly that it is a noncompact Calabi-Yau [32]. In addition, this geometry admits a U(1) gauge background, that satisfies the Hermitian Yang-Mills equations on this Ricci-flat noncompact Calabi-Yau blowup, with field strength 2-form:

$$i\mathcal{F} = \left(\frac{r}{r + X}\right)^{1-(1/n)}\left(\bar{e}e - \frac{n-1}{n^2}\frac{1}{r + X}\bar{\epsilon}\epsilon\right). \quad (4)$$

Because both the geometry and its U(1) gauge background are given explicitly, integrals of them can be computed straightforwardly. In particular, we obtain

$$\int_{\mathbb{CP}^2} \frac{\mathrm{tr}\,\mathcal{R}^2}{(2\pi i)^2} = -n\int_{\mathbb{CP}^1 \ltimes \mathbb{C}} \frac{\mathrm{tr}\,\mathcal{R}^2}{(2\pi i)^2} = n(n + 1), \quad (5)$$

and

$$\int_{\mathbb{CP}^p}\left(\frac{i\mathcal{F}}{2\pi i}\right)^p = -n\int_{\mathbb{CP}^{p-1}\ltimes\mathbb{C}}\left(\frac{i\mathcal{F}}{2\pi i}\right)^p = 1. \tag{6}$$

The integrals over $\mathbb{CP}^p$ are taken at $X = 0$ integrating over p of the $n - 1$ inhomogeneous coordinates of $\mathbb{CP}^{n-1}$, with the others set to a fixed value, say, 0. The integral over $\mathbb{CP}^{p-1}\ltimes\mathbb{C}$ corresponds to the integral over all values of $x \in \mathbb{C}$ and over $p - 1$ inhomogeneous coordinates.

These and other integrals were relevant to determine the heterotic blowup models that satisfy the integrated version of the Bianchi identity

$$\mathrm{d}H = \mathrm{tr}\,\mathcal{R}^2 - \mathrm{tr}\,(i\mathcal{F}_V)^2, \tag{7}$$

where $i\mathcal{F}_V = i\mathcal{F}V^I H_I$ defines the embedding of the U(1) gauge background in the SO(32) or $E_8 \times E_8$ gauge group. Integrating the Bianchi identity over the full blowup of $\mathbb{C}^2/\mathbb{Z}_2$ and requiring that it vanishes, leads to the consistency condition $V^2 = 6$. In the three dimensional case the integral in the Bianchi identity over either $\mathbb{CP}^2$ or $\mathbb{CP}^2\ltimes\mathbb{C}$ lead to the same consistency condition $V^2 = 12$ for the blowup of $\mathbb{C}^3/\mathbb{Z}_3$. Both conditions in two and three complex internal dimensions are compatible with the corresponding modular invariance conditions, $(2v)^2 = 2 \bmod 4$ and $(3v)^2 = 0 \bmod 6$, of the heterotic string, respectively.

Moreover, in [32] we confirmed that the integral or half-integral solutions of this equation, gives rise to all blowups of all of the known modular invariant $T^4/\mathbb{Z}_2$ and $T^6/\mathbb{Z}_3$ heterotic orbifold models (except the $\mathbb{Z}_3$ models with unbroken SO(32) and $E_8 \times E_8$ gauge groups). We identified the gauge background $\mathcal{F}_V$ with the orbifold action on the gauge degrees of freedom $\mathfrak{A}(\theta\tilde{Z}) = U\mathfrak{A}(\tilde{Z})U^{-1}$, with $U = \exp(2\pi i v^I H_I)$ characterized by v^I. For this we computed the integral over the contour γ of the phase of x at $x \to \infty$ at fixed values of the $\mathbb{CP}^{n-1}$ coordinates z:

$$v^I H_I \equiv \int_\gamma \mathcal{A}_V = -\frac{1}{n}V^I H_I. \tag{8}$$

The equivalence sign "≡" indicates that the identification of the orbifold gauge shift vector v, and the blowup parameter V that characterizes the U(1) bundle embedding in the gauge group, is up to lattice vectors in the Spin(32) lattice.

In addition we could use these integrals to compute the complete chiral spectrum of the blowups using index theorems. We found that the spectra were identical to the orbifold spectra in the blow down limit up to singlets and vectorlike states. The fact that we were able to obtain the blowups of all heterotic orbifold models and the chiral part of the spectra gives us confidence that, even though we are (partly) integrating over noncompact cycles, the integrals can nevertheless be trusted and used in a naive way in index theorems. In particular, we do not have to use extensions of index theorems for spaces with boundaries, when computing on the blowup of noncompact $\mathbb{C}^n/\mathbb{Z}_n$ orbifolds and comparing this with the properties of compact $T^{2n}/\mathbb{Z}_n$ orbifolds. The reason that this procedure works is that we in the end compare with the spectrum of a compact orbifold. This requires that we glue various resolutions together. In this process the boundary contributions cancel.

B. Resolution of $\mathbb{C}^n/\mathbb{Z}_n$ using toric geometry

The purpose of this subsection is to understand the topology of the resolution of $\mathbb{C}^n/\mathbb{Z}_n$ using toric geometry. In particular, we show how the integrals (5) and (6) can be obtained using this machinery. Our description explains the basic methods to obtain the results relevant for (heterotic) string model building.

As explained below Eq. (1) the orbifold $\mathbb{C}^n/\mathbb{Z}_n$ has a deficit angle. To obtain a nonsingular resolution $\mathrm{Res}(\mathbb{C}^n/\mathbb{Z}_n)$, we define a set of local coordinates

$$Z_1 = z_1 x^{1/n}, \ldots, Z_n = z_n x^{1/n}, \tag{9}$$

from the homogeneous coordinates $z_1, \ldots, z_n, x \in \mathbb{C}$. The orbifold action (1) is then extended by the transformation $x \to e^{-2\pi i}x$. As it stands we describe the n local coordinates using $n + 1$ homogeneous coordinates; we therefore need to define a $\mathbb{C}^* = \mathbb{C} - 0$ "toric" action on the homogeneous coordinates, that leave the local coordinates inert. This requirement fixes the $\mathbb{C}^*$ action uniquely to

$$\mathbb{C}^*\colon (z_1, \ldots, z_n, x) \sim (\lambda^{-1}z_1, \ldots, \lambda^{-1}z_n, \lambda^n x), \tag{10}$$

$\lambda \in \mathbb{C}^*$. The resolution of $\mathbb{C}^n/\mathbb{Z}_n$ is defined by the toric variety

$$\mathrm{Res}\,(\mathbb{C}^n/\mathbb{Z}_n) = (\mathbb{C}^{n+1} - F)/\mathbb{C}^*, \tag{11}$$

where the exclusion set F has been subtracted to ensure that the resolution is not singular. In particular, the $\mathbb{C}^*$ action should act nontrivially, hence at least the origin, $\{z_1 = \ldots = z_n = x = 0\}$, has to be excluded. Indeed, the number of coordinates set to zero in a toric variety, p, determines a subspace of complex dimension $n - p$. In particular, one expects, that the origin has "-1" dimensions, and hence are totally irrelevant. But since the $\mathbb{C}^*$ leaves it inert, it is zero dimensional, i.e., a collection of points, which do matter in general.

The resolution $\mathrm{Res}(\mathbb{C}^n/\mathbb{Z}_n)$ is topologically nontrivial, i.e. one needs more than one coordinate patch to describe it entirely. A set of coordinate patches U_i is obtained straightforwardly by taking one of the homogeneous coordinates not to be vanishing

$$U_0 = \{x \neq 0\}, \qquad U_i = \{z_i \neq 0\}, \tag{12}$$

for $i = 1, \ldots n$, defined of course in $\mathbb{C}^{n+1} - F$ only. In each of the coordinate patches we can use the rescaling (10) to set its defining nonvanishing coordinates to unity. For U_i this can be done uniquely by setting $\lambda = z_i$. But for U_0 we find a $\mathbb{Z}_n$ ambiguity because $\lambda = e^{2\pi i p/n}x^{-1/n}$. Hence on the remaining coordinates $z_1, \ldots, z_n$ the $\mathbb{C}^*$ reduces to a $\mathbb{Z}_n$ action. This is in fact the original orbifold action, and we have a singularity unless we exclude

$$F = \{z_1 = \ldots = z_n = 0\}. \tag{13}$$

To define proper patches, we need to subdivide the punctured U_0, but we will not dwell on this here.

The explicit blowup of the $\mathbb{C}^n/\mathbb{Z}_n$ orbifold, described in the previous subsection, used the coordinate patch U_n, with $z_n = 1$. In this patch the SU(n) invariant variable X is obtained from the inhomogeneous coordinates (9):

$$X^{1/n} = \bar{Z}Z = (1 + \bar{z}z)|x^{1/n}|^2. \tag{14}$$

Only here $z = (z_1, \ldots z_{n-1})$ denote a set of inhomogeneous coordinates on $\mathbb{CP}^{n-1}$. The reason is that even though the coordinate patch U_n is not sufficient to describe the whole resolution, still the integrals give the correct numbers, is that the parts of Res($\mathbb{C}^n/\mathbb{Z}_n$) not in U_n correspond to lower dimensional subspaces, irrelevant for these integrals.

We define a set of $n + 1$ hypersurfaces of complex dimension $n - 1$, which are called divisors. (For a general introduction to algebraic geometry including divisors, see e.g. [47,48].) There are two types of divisors, D_i, $i = 1, \ldots, n$, and E, defined by

$$D_i = \{z_i = 0\}, \qquad E = \{x = 0\}. \tag{15}$$

The final one, E, is called an exceptional divisor, because it defines a subspace of the resolution not present in the orbifold. Taking into account the remaining rescaling (10), we see that $E = \mathbb{CP}^{n-1}$ defined in terms of homogeneous coordinates. This means that the singularity of the orbifold $\mathbb{C}^n/\mathbb{Z}_n$ has been "blown up" to a $\mathbb{CP}^{n-1}$. In a similar fashion, it follows that $D_i = \mathbb{CP}^{n-2} \ltimes \mathbb{C}$ is defined as a complex line bundle over $\mathbb{CP}^{n-2}$. The resolution Res($\mathbb{C}^n/\mathbb{Z}_n$) itself can be thought of as a complex line bundle over $\mathbb{CP}^{n-1}$. The exceptional divisor E is obviously compact, while the other divisors are not compact.

To each of the divisors we can associate a complex line bundle. Any complex line bundle is defined by its holomorphic scalar transition functions. To determine these transition functions for the various divisors we write the defining equation of the divisor in patch U_i. This gives for the ordinary divisor D_i:

$$U_{j\neq i}\colon \frac{z_i}{z_j} = 0, \qquad U_i\colon 1 = 0, \qquad U_0\colon x^{1/n} z_i = 0, \tag{16}$$

and for the exceptional divisor E:

$$U_j\colon z_j^n x = 0, \qquad U_0\colon 1 = 0. \tag{17}$$

In the coordinate patches, where we encounter the inconsistent equation "$1 = 0$," the corresponding divisor simply does not live. From this we read off the transition functions for the associated line bundle of divisors D_i and E:

$$\begin{aligned} g_{kj}(D_i) &= \frac{z_k}{z_j}, \qquad g_{j0}(D_i) = x^{1/n} z_j \quad \text{and} \\ g_{kj}(E) &= \frac{z_j^n}{z_k^n}, \qquad g_{j0}(E) = \frac{1}{z_j^n x}. \end{aligned} \tag{18}$$

The subscripts indicate between which two coordinate patches the transition functions interpolate. It follows that the transition functions of the line bundles, associated to the divisors, D_i and E, are all related to each other:

$$g(D_1)^{-n} = \ldots = g(D_n)^{-n} = g(E). \tag{19}$$

Since the equality holds for all transition functions, we have dropped the subscripts that indicate the coordinate patches.

To understand the consequences of the fact that all transition functions of the divisors are related, we make the following brief excursion to properties of vector bundles. A vector bundle $\mathcal{V}$ can be topologically characterized by its total Chern class

$$c(\mathcal{V}) = \det\left(1 + \frac{F(\mathcal{V})}{2\pi i}\right), \tag{20}$$

where $F(\mathcal{V})$ is the curvature of the bundle. The total Chern class can be expanded in terms of its first, second, etc., Chern classes $c_1(\mathcal{V})$, $c_2(\mathcal{V})$, etc. A complex line bundle is completely determined by its first Chern class $c_1(\mathcal{V}) = F(\mathcal{V})/2\pi i$, which can be taken to be a harmonic (1, 1)-form. Because it is closed, locally its curvature can be written as $F(\mathcal{V}) = \mathrm{d}A_i(\mathcal{V})$ in terms of a connection $A_i(\mathcal{V})$ in coordinate patch U_i. Between two coordinate patches U_i and U_j the connections

$$A_j(\mathcal{V}) = A_i(\mathcal{V}) + g_{ji}(\mathcal{V})^{-1}\mathrm{d}g_{ji}(\mathcal{V}) \tag{21}$$

are related via the transition functions $g_{ji}(\mathcal{V})$.

With this in mind, we can describe the Chern classes of the line bundles associated to the divisors of the resolution Res($\mathbb{C}^n/\mathbb{Z}_n$). To each of the divisors D_i and E of the resolution we can associate a line bundle with first Chern class, $c_1(D_i)$ and $c_1(E)$, respectively. It is a convenient toric geometrical convention, to let the context determine whether the symbol for the divisor refers to the defining hypersurface, or the first Chern class of its associated line bundle. Therefore, one may write $D_i = c_1(D_i)$ and $E = c_1(E)$. The relations between the transition functions (19) imply that the divisors satisfy the following linear equivalence relations

$$D_i \sim D_j, \qquad nD_i + E \sim 0, \tag{22}$$

where the linear equivalences, $\sim$, can be replaced by equalities, provided that the symbols for the divisors refer to the first Chern classes of the line bundles, when we ignore addition of exact forms. Upon using Poincaré's duality the divisors refer to hypersurfaces, the linear equivalences mean that these surfaces can be deformed to differ by boundary surfaces. The derivation of the linear equivalence relations by first determining the relation between the transition functions (19) is proper but somewhat lengthy. It can be bypassed by requiring that the local coordinates (9) are invariant under the transformations $z_i \to e^{D_i} z_i$ and $x \to e^E x$. The reason that this works is that one can perform transformations on the homogeneous coordinates, that leave the local coordinates (9) invariant.

The (1, 1)-forms, D_i and E, can be integrated over holomorphic 1-cycles, i.e. complex curves. Similarly (2, 2)-forms, like $D_i D_j$, $D_i E$, and E^2, can be integrated over holomorphic 2-cycles, and so on. It is therefore useful to have a classification of the holomorphic p-cycles within the resolution $\mathrm{Res}(\mathbb{C}^n/\mathbb{Z}_n)$, using the divisors D_i and E interpreted as hypersurfaces. From their definition it follows immediately that D_i and E define holomorphic $(n-1)$-cycles. We can define the integral of any $(n-1, n-1)$-form, say, $D_2^{n-2}E$ over, for example, D_1, and denote it by $\int_{D_1} D_2^{n-2}E$. Moreover, the intersection of two divisors, like

$$D_i \cdot D_{j\neq i} = \{z_i = z_j = 0\} \quad \text{and} \quad D_i \cdot E = \{z_i = x = 0\}, \tag{23}$$

define $(n-2)$-dimensional holomorphic hypersurfaces. The integral over such intersection of $(n-2, n-2)$-forms can similarly be defined. This can of course be extended to the intersection of an arbitrary number of different divisors. Because E is compact, intersections that involve E, will also be compact; contrary to intersections of only noncompact divisors D_i can be noncompact. This gives us a way to identify the integration ranges used in (5) and (6):

$$\mathbb{CP}^p = ED_1 \ldots D_{n-1-p}, \quad \mathbb{CP}^{p-1} \ltimes \mathbb{C} = D_1 \ldots D_{n-p}, \tag{24}$$

with intersections of divisors.

The intersections of n different divisors are of special interest, because they define zero dimensional surfaces, i.e. sets of points. The number of such points is called the intersection number of these divisors. The intersection number of $n-1$ different D_i's and a single E can be computed directly: For example consider $D_2 \cdot \ldots \cdot D_n \cdot E$. Setting $z_2 = \ldots = z_n = x = 0$ in (10), realizing that $z_1 \neq 0$, we can choose $\lambda = z_1$ uniquely. This means that all the intersection numbers

$$E \cdot \prod_{j\neq i} D_j = \int ED_2 \ldots D_n = 1. \tag{25}$$

The middle equation shows that we can also view these intersection numbers as integrals over the whole toric variety of the n divisor interpreted as (1, 1)-forms.

This naturally leads to the following generalization the "product" or "intersection" of any n divisors can be defined as the integral over the corresponding (1, 1)-forms. The linear equivalences to relate the integral to an integral of all different divisors one of which being E. In particular, we find self-intersection number

$$E^n = (-n)^{n-1} \int D_2 \ldots D_n E = (-n)^{n-1}. \tag{26}$$

In the same way all other (self-)intersections involving at least one E may be computed. As can be seen from these simple computations the symbol $\cdot$ to indicate intersection of divisors is also essentially obsolete, and in the following we let the context decide whether, say ED_1, refers to a (2, 2)-form or a complex $(n-2)$-cycle. Employing the linear equivalence relations we can even compute integrals over n noncompact divisors, for example

$$D_1 \ldots D_n = -\frac{1}{n} \int ED_2 \ldots D_n = -\frac{1}{n}. \tag{27}$$

This brings us to a few important issues: First of all, one cannot interpret this result naively as saying that the noncompact divisors D_1 to D_n intersect $-\frac{1}{n}$ times. In fact, the exclusion set F, defined in (13), implies that this intersection does not exist in the resolution $\mathrm{Res}(\mathbb{C}^n/\mathbb{Z}_n)$. Hence, one should *only* interpret $D_1 \ldots D_n$ as the integral of the corresponding (1, 1)-forms over the whole resolution.

But even when one interprets $D_1 \ldots D_n$ as an integral only, one may still wonder what fixes its values, because being noncompact it seems not to be topological. To pursue this question, we explain how to recover the results for integrals (6) using toric geometry. To obtain the latter integrals we need to identify the gauge background $i\mathcal{F}$ with a divisor interpreted as a first Chern class (1, 1)-form. The linear equivalences (22) imply that there is in fact only one independent (1, 1)-form, hence it is determined up to an overall normalization. To fix the overall normalization we look for the (1, 1)-form which integral is unity on compact curves, like $ED_2 \ldots D_n$, which according to (24) corresponds to $\mathbb{CP}^n$. In this way we obtain the identification

$$\frac{\mathcal{F}}{2\pi} = D_i = -\frac{1}{n}E, \qquad \int_{ED_2\ldots D_n} \frac{\mathcal{F}}{2\pi} = 1. \tag{28}$$

Using the identification of the cycles (24) and the linear equivalences (22) we find the toric formulation

$$\int_{ED_1\ldots D_{n-1-p}} \left(\frac{i\mathcal{F}}{2\pi i}\right)^p = -n \int_{D_1\ldots D_{n-p}} \left(\frac{i\mathcal{F}}{2\pi i}\right)^p = 1, \tag{29}$$

in agreement with the integrals (6). The reason for this agreement is that these integrals define topological invariants to which one has access using toric geometry. The explicit expression for the gauge background (4) satisfying the Hermitian Yang-Mills equations on the Ricci-flat background (2), defines a special representative of the corresponding characteristic class. This shows that it is the boundary conditions on $ED_3 \ldots D_n$ or at the boundary of $D_2 \ldots D_n$ at infinity, which fixes the values of these integrals. By patching various resolutions together, one can turn the noncompact divisors and curves into compact ones, and then the standard intersection theory works, see [45].

Similarly, to obtain a representation of the integrals (5) involving the curvature $\mathcal{R}$, we can employ the splitting principle [49], which says that the total Chern class $c(\mathcal{R})$ of the tangent bundle is given as the product of $1 + D$ over all compact and noncompact divisors D. For the resolution of $\mathbb{C}^n/\mathbb{Z}_n$ this amounts to [40]

$$c(\mathcal{R}) = (1 + E)\prod_{i=1}^{n}(1 + D_i). \tag{30}$$

The first, second, etc., Chern classes of the tangent bundle can be determined by expanding this to the appropriate order. As the resolution represents a (noncompact) Calabi-Yau space, the first Chern class should vanish. This can be confirmed easily:

$$c_1(\mathcal{R}) = E + \sum_{i=1}^{n} D_i = 0, \tag{31}$$

by virtue of the linear equivalence relations (22). By expanding the general formula for the total Chern class (20) to second order gives

$$-\frac{1}{2}\,\mathrm{tr}\Big(\frac{\mathcal{R}}{2\pi i}\Big)^2 = c_2(\mathcal{R}) = E\sum_i D_i + \sum_{i<j} D_i D_j = \frac{n+1}{2} E D_1, \tag{32}$$

using that the first Chern class vanishes. From this it is straightforward to confirm the integrals (5) of $\mathrm{tr}\,\mathcal{R}^2$ as well.

Next, we want to relate the toric geometry to heterotic orbifolds. In particular we explain how from it the blowup models characterized by the vector V of only integers or half-integers, the corresponding orbifold models defined by the gauge shift v can be recovered. The relation between V and v was made in (8) by computing the contour integral over the gauge connection $\mathcal{A}_V$ far away from the singularity. Using Stoke's theorem this can be translated to an integral of $\mathcal{F}_V$ over a curve like $D_2 \ldots D_n$:

$$v^I H_I \equiv \int_{D_2 \ldots D_n} \mathcal{F}_V = -\frac{1}{n} V^I H_I. \tag{33}$$

Hence the fractional nature of the orbifold gauge shift vector v is obtained by integrating over a noncompact curve. The integrated version Bianchi Identity is easily computed. For $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ we find

$$V^2 = -2\int \mathrm{tr}(\mathcal{F}_V)^2 = -2\int \mathrm{tr}\,\mathcal{R}^2 = 6, \tag{34}$$

when integrated over the whole resolution. For $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_3)$ we obtain

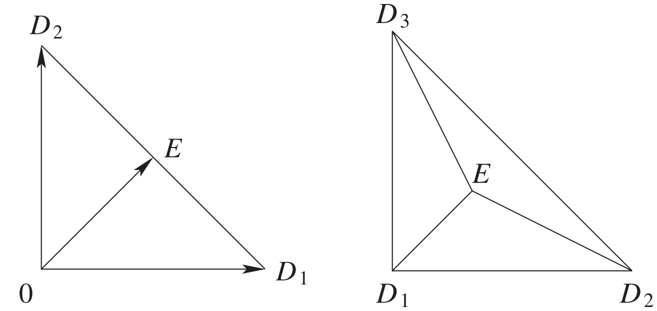

FIG. 1. The left picture displays the toric diagram of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$. The right picture displays a projected view of the toric diagram of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_3)$. Because the latter is a projection, there are no arrows from the origin pointing to the divisors as in the former toric diagram.

$$V^2 = \int_E \mathrm{tr}(\mathcal{F}_V)^2 = -3\int_{D_i} \mathrm{tr}(\mathcal{F}_V)^2 = -3\int_{D_i} \mathrm{tr}\,\mathcal{R}^2 = \int_E \mathrm{tr}\,\mathcal{R}^2 = 12, \tag{35}$$

using (32). Hence, we have retrieved the conditions mentioned in the previous subsection. Moreover the toric approach shows that the integrals over the compact and noncompact 2-cycles E and D_i lead to the same condition, and is a simple consequence of the fact that these divisors are linearly equivalent (22).

There is a convenient way to represent the properties of toric varieties including the properties of the divisors: the toric diagram. To build the toric diagram of $\mathrm{Res}(\mathbb{C}^n/\mathbb{Z}_n)$ first give n vectors $v_1, \ldots, v_n$ that represent the n ordinary divisors $D_1, \ldots, D_n$. For example we can take the basis $v_1 = (1, 0, \ldots, 0)$, to $v_n = (0, \ldots, 0, 1)$. The exceptional divisor E is represented by the vector

$$w = \sum_i \phi_i v_i, \tag{36}$$

which in this basis takes the form $w = (1, \ldots, 1)/n$. This basis $v_1, \ldots, v_n$ and w precisely dictate how to construct the local coordinates (9). The toric diagram of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ is given in the left picture of Fig. 1. The toric diagram of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_3)$ is three dimensional; to obtain a simple representation of it we can take a two dimensional projection of the three dimensional toric diagram. We choose the basis $v_1 = (0, 0, 1)$, $v_2 = (1, 0, 1)$, and $v_3 = (0, 1, 1)$, so that the exceptional divisor E is then represented by $w = (\frac{1}{3}, \frac{1}{3}, 1)$. Because the last entry in both v_i and w are identical, we only need to use the first two entries, which defines a projection. The resulting projected toric diagram is given in the right picture in Fig. 1. The exceptional divisor E lies in the interior of the toric diagram. A theorem in toric geometry guarantees that such a divisor is compact. We see this theorem confirmed in this example. Toric geometry also tells us that the basic cones, the smallest possible cones inside a (projected) toric diagram, correspond to the intersection of divisors with unit intersection number. This is consistent with (25), for example, $D_1E = 1$ and $D_1D_2E = 1$, in the resolution, $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ and $\text{Res}(\mathbb{C}^3/\mathbb{Z}_3)$, respectively. Together with the linear equivalences (22) we can determine the intersections of a compact curve with the divisors. We construct the table:

Divisor	D_1	...	D_n	E
$ED_2 \ldots D_n$	1	...	1	$-n$

Notice that the values in this table precisely correspond to minus the powers of the rescaling parameter λ in (10), hence we read off the $\mathbb{C}^*$ scaling charges from the toric diagram, by computing the intersection numbers of a compact curve with the divisors.

To summarize, we have shown that all the results for the integrals obtained using the explicit blowup of the $\mathbb{C}^n/\mathbb{Z}_n$ orbifold singularity can be obtained using toric geometrical techniques, without ever having to compute any integral explicitly. This procedure shows that the integrals all have a topological origin, which is compatible with the fact that these integrals are used in the integrated Bianchi identities to select consistent blowup models. All this information can be obtained uniquely from the toric diagram, which was directly determined form the orbifold action.

III. ORBIFOLD RESOLUTIONS WITH MULTIPLE EXCEPTIONAL DIVISORS

A. Generalities of orbifold resolutions

In the previous section we have seen how we can obtain all topological relevant information of the resolution of $\mathbb{C}^n/\mathbb{Z}_n$ orbifolds using toric geometrical techniques. (For related discussions see e.g. [43,45,46].) In this section we would like to show that this machinery can be used to treat resolutions of more complicated orbifolds as well. This requires us to be able to analyze resolutions with more than one exceptional divisor.

We begin to formalize the toric geometrical method to construct the resolution of an orbifold singularity by defining the toric diagram. Consider noncompact orbifolds $\mathbb{C}^n/G$, where G is a finite group, Abelian for simplicity, and $n = 2, 3$. The action of an element $\theta \in G$ on the orbifold coordinates $\tilde{Z}_1, \ldots \tilde{Z}_n$ can be written as

$$\theta: (\tilde{Z}_1, \ldots, \tilde{Z}_n) \rightarrow (e^{2\pi i \phi_1(\theta)}\tilde{Z}_1, \ldots, e^{2\pi i \phi_n(\theta)}\tilde{Z}_n), \quad (37)$$

such that all $0 \leq \phi_i(\theta) < 1$. The elements θ and θ^{-1} lead to the same orbifold action up to complex conjugation. They have to be identified, when θ acts nontrivially on three complex dimensions, but not when it only acts on two complex coordinates. (A $\mathbb{Z}_2$ group element θ, for which all $\phi_i(\theta) = 0, 1/2$, is self conjugate.) We define the corresponding representative $[\theta]$ to be the element that satisfies $\sum_i \phi_i(\theta) = 1$. To each representative $[\theta]$ we can associate an exceptional divisor E_θ. The total number of exceptional divisors is denoted as N. For even and odd ordered orbifolds we encounter $N(\mathbb{Z}_{2k}) = N(\mathbb{Z}_{2k+1}) = k$ exceptional divisors. If we let $v_1, \ldots v_n$ define a basis for the toric diagram of the orbifold, then the vector

$$w_\theta = \sum_i \phi_i(\theta) v_i \quad (38)$$

identifies the exceptional divisor E_θ in the toric diagram of the resolution for each representative $[\theta]$. This definition of exceptional divisors of the resolution is in one-to-one correspondence to the twisted sectors in orbifold string theory: Also there each representative $[\theta]$ corresponds to a distinct, e.g. first, second, etc., twisted sector. In particular, as is well known the $\mathbb{C}^n/\mathbb{Z}_n$ orbifolds, with $n = 2, 3$, have only a single-twisted sector, in agreement with the previous section where we only had a single exceptional divisor. The set of vectors v_i and w_θ define the points in the toric diagram corresponding to divisors of the resolution.

Next, we describe how to associate to the toric diagram a toric variety which represents the resolution of $\mathbb{C}^n/G$. Each of the vectors v_i and w_θ correspond to a homogeneous coordinate z_i and x_θ of the resolution $\text{Res}(\mathbb{C}^n/G)$, respectively. As in the previous section, the divisors are defined by setting the corresponding coordinate to zero:

$$D_i = \{z_i = 0\}, \qquad E_\theta = \{x_\theta = 0\}. \quad (39)$$

The ordinary divisors D_i are never compact, while the exceptional divisors are compact only when they lie in the interior of the toric diagram. We introduce a set of local coordinates

$$Z_j = \prod_i z_i^{(v_i)_j} \prod_\theta x_\theta^{(w_\theta)_j}, \quad (40)$$

where $(v_i)_j$ denotes the jth component of the vector v_i. We can read off the n linear equivalence relations of the divisors from them:

$$\sum_i (v_i)_j D_i + \sum_\theta (w_\theta)_j E_\theta \sim 0. \quad (41)$$

At the same time the $(\mathbb{C}^*)^N$ group of scaling of homogeneous coordinates z_i and x_θ is defined, such that it leaves the local coordinates (40) invariant. This means that if one substitutes the scaling charges as values of the divisors in the linear equivalence relations (41) one obtains zero. The action $(\mathbb{C}^*)^N$ of scaling is in general not well defined on $\mathbb{C}^{n+N}$. The resolution of the $\mathbb{C}^n/G$ orbifold is defined as

$$\text{Res}(\mathbb{C}^n/G) = (\mathbb{C}^{n+N} - F)/(\mathbb{C}^*)^N, \quad (42)$$

where exclusion set F is defined, as in the previous section, such that in none of the coordinate patches singularities arise. This coincides with the definition of the exclusion set given in [42].

To obtain the integrals of the various divisors over the resolution, loosely speaking the intersection numbers, assume that the definition of the toric diagram has to be completed by giving a triangulation. In this section we assume that the triangulation is unique. In Sec. IV we return to the complication when more than one triangulation is possible. The triangulation defines the basic cones, i.e. the smallest possible cones, inside the toric diagram. The intersection of the divisors, or the corresponding integral, that form the corners of the basic cones, are defined to have unity intersection number. In other words, the triangulation defines the compact curves of the resolution as the interior lines in the toric diagram. The intersection number with the divisor of the basic cone of which such a compact curve forms the edge is equal to one. In addition, the intersection of divisors that are linearly dependent vanishes. In the projected toric diagram in three complex dimensions this corresponds to the case when three or more divisors are aligned. The set of basic cones, together with the linear equivalence relations, determine all other integrals of the divisors uniquely. In total there are

$$\#_2(D, E) = \frac{(N+2)(N+3)}{2}, \qquad \#_3(D, E) = \frac{(N+5)(N+4)(N+3)}{6} \tag{43}$$

such integrals in two and three complex dimensions, respectively. When there are a large number of exceptional divisors, this means that the total number of integrals grows rapidly. Indeed, in three complex dimensions we have $\#_3(D, E) = 20, 35, 56, 84$, for $N = 1, 2, 3, 4$ exceptional divisors. (The resolution of the $\mathbb{Z}_{6-II}$ singularity provides an example of the case with $N = 4$.) Fortunately, we do not need to give all these integrals explicitly, because we can use the linear equivalences to express integrals involving ordinary divisors in terms of those involving exceptional divisors only. The number of integrals of exceptional divisors in two and three complex dimensions, grows like

$$\#_2(E) = \frac{N(N+1)}{2}, \qquad \#_3(E) = \frac{(N+2)(N+1)N}{6}, \tag{44}$$

with the number of exceptional divisors N. In particular, in three complex dimensions we find the more manageable numbers $\#_3(E) = 1, 4, 10, 20$ for $N = 1, 2, 3, 4$. This completes the purely geometrical description of resolutions of $\mathbb{C}^n/G$ singularities.

For applications in model building of heterotic orbifold blowups we need to specify the gauge background. The simplest gauge backgrounds, apart from the standard embedding, are U(1) line bundle backgrounds $\mathcal{F}_V$. As we have seen above, complex line bundles play a prominent role in the toric geometrical description of orbifold resolution. Taking the linear equivalence relations (41) into account, a basis for U(1) gauge backgrounds is given by the exceptional divisors that correspond to (1, 1)-forms. Given this, the Yang-Mills equations of motion reduce to

$$J \wedge J \wedge \mathcal{F}_V = 0, \tag{45}$$

J being a Kähler form on the resolved manifold. Such a requirement imposes a set of constraints on the Kähler moduli (see e.g. [33]); we assume that these constraints can be satisfied in our specific cases without further restrictions on the gauge bundles.

In general, exceptional divisors do not represent the minimal line bundles of the resolution. A basis of the smallest line bundles is obtained by requiring that each of the elements integrated on all compact curves, that form a basis for all compact curves, either gives zero or one. In the two dimensional case all exceptional divisors are compact. In three complex dimensions all curves, represented by lines between two adjacent divisors, that go through the interior of the toric diagram, are compact. Taking into account the linear equivalences again, one can define such a basis of N minimal compact curves C_θ of the resolution. After that it is a straightforward exercise in linear algebra to find those linear combinations ω_θ of exceptional divisors, that are orthonormal to the basis of compact curves

$$\int_{C_\theta} \omega_{\theta'} = \delta_{\theta,\theta'}. \tag{46}$$

This basis of N compact curves can be used to compute the weights of the N scalings defining the $(\mathbb{C}^*)^N$. To find the relevant charges, one may compute the intersections between these compact curves and all divisors.

After this basis has been determined, the general U(1) gauge bundle embedded in the SO(32) or $E_8 \times E_8'$ gauge group, can be represented as

$$\frac{\mathcal{F}_V}{2\pi} = \sum_{[\theta]} V_\theta^I \omega_\theta H_I. \tag{47}$$

For each representative $[\theta]$ the vector V_θ either contains only integers or only half-integers. This ensures, that we have well-defined eigenvalues on the roots of the adjoint of SO(32) super Yang-Mills theory. (When we want to discuss compactification of $E_8 \times E_8$ SYM or either heterotic string, we need the entries of V_θ to sum to an even number.) In analogy to (33) we can make identifications of the vectors V_θ and the orbifold gauge shift vectors v_i for each of the Abelian factors inside the orbifold group G. The integral of $\mathcal{F}_V$ over each noncompact divisor D_i gives rises to such a relation. This procedure does not work when on a face of the toric diagram, one or more exceptional divisors are located. In such a case, the face defines the resolution of a suborbifold $\mathbb{C}^2/G'$, $G' \subset G$. To make the identification of the orbifold and line bundle shifts, one has to perform the matching on this subvariety. To restrict the divisors to this resolution of the suborbifold, one needs to put some exceptional divisors to zero. This means ignoring the corresponding extra homogeneous coordinate and its associated $\mathbb{C}^*$ scaling. In this way all properties, including e.g. the total Chern class, can be reduced to the subresolution.

Only those gauge configurations which in addition satisfy the integrated Bianchi identity

$$\int_{C_2} \mathrm{tr}\, \mathcal{R}^2 = \int_{C_2} \mathrm{tr}\, \mathcal{F}_V^2, \tag{48}$$

for all compact 2-cycles C_2, define consistent background on the resolution. In this work we will often require that the integrated Bianchi identity also vanishes for noncompact 2-cycles. The latter requirement is not necessary, but we will see in examples that with this condition we are able to recover many of the modular invariant heterotic orbifold models. In particular, for $\mathrm{Res}(\mathbb{C}^2/G')$, the resolution is itself the only 2-cycle, which obviously is noncompact. For the three dimensional case, the (non)compact holomorphic 2-cycles correspond to the (non)compact divisors.

As a final cross-check on the validity of the application of toric methods to obtain resolutions of heterotic orbifold, we compute the four dimensional spectra. We only compute the spectra of those models that satisfy all possible consistency conditions. (For the other models, there is H flux flowing out of the singularity, this means that the resolution has locally torsion. Therefore the standard index theorems for computing the spectra do not apply.) The four dimensional spectrum on the resolution with the U(1) gauge background can be computed using the multiplicity operator

$$N_V = \int \left\{ \frac{1}{3!}\left(\frac{\mathcal{F}_V}{2\pi}\right)^3 + \frac{1}{12} c_2(\mathcal{R}) \frac{\mathcal{F}_V}{2\pi} \right\}. \tag{49}$$

This operator can then be applied to the branching of the adjoint representation due to the gauge background to determine the multiplicity factors. As we are considering resolutions of noncompact orbifolds, the multiplicities often take fractional values.

After this general digression of the use of toric geometrical techniques to obtain resolutions of heterotic orbifold models, we give in the following two subsections interesting examples of orbifold resolutions, $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_3)$ and $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$, which both have two exceptional divisors.

B. Resolution of $\mathbb{C}^2/\mathbb{Z}_3$

To illustrate the resolutions with more than one exceptional divisor in two dimensions, we consider the resolution of $\mathbb{C}^2/\mathbb{Z}_3$, as an example. The orbifold action reads

$$\theta\colon (\tilde{Z}_1, \tilde{Z}_2) \to (e^{2\pi i \phi_1}\tilde{Z}_1, e^{2\pi i \phi_2}\tilde{Z}_2), \qquad \phi = \tfrac{1}{3}(1,2). \tag{50}$$

Taking the vectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$ to represent the ordinary divisors D_1 and D_2 in the toric diagram, we find that $w_1 = \frac{1}{3}(1,2)$ and $w_2 = \frac{1}{3}(2,1)$ indicate the two exceptional divisors E_1 and E_2, respectively. The resulting toric diagram of the resolution is given in Fig. 2. From the local coordinates (40)

$$Z_1 = z_1 x_1^{1/3} x_2^{2/3}, \qquad Z_2 = z_2 x_1^{2/3} x_2^{1/3}, \tag{51}$$

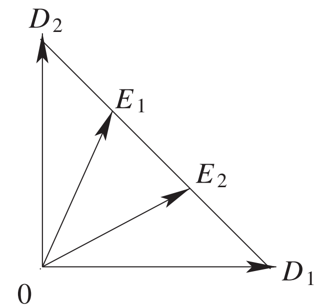

FIG. 2. The toric diagram of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_3)$ is displayed. Both exceptional divisors E_1 and E_2 are compact.

we read off the linear equivalence relations

$$3D_1 + E_1 + 2E_2 \sim 0, \qquad 3D_2 + 2E_1 + E_2 \sim 0, \tag{52}$$

and the $(\mathbb{C}^*)^2$ scalings

$$(z_1, z_2, x_1, x_2) \sim (\lambda_1^{-1} z_1, \lambda_2^{-1} z_2, \lambda_2^2 \lambda_1^{-1} x_1, \lambda_1^2 \lambda_2^{-1} x_2). \tag{53}$$

The exclusion set reads

$$F = \{z_1 = x_1 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_1 = z_2 = 0\}, \tag{54}$$

as can be seen from the toric diagram displayed in Fig. 2.

From this toric diagram one can read off the basic cones:

$$D_1 E_2 = E_1 E_2 = D_2 E_1 = 1. \tag{55}$$

Because the toric variety is two complex dimensional the divisors are the same as the curves of the resolution, all intersection of curves with divisors can be compactly displayed in a single table, see Table I. From the intersection table we infer that D_2 and D_1 define (1, 1)-forms that are orthonormal to the compact curves E_1 and E_2, respectively. Hence we can expand a U(1) gauge background as

$$\frac{\mathcal{F}_V}{2\pi} = (V_1^I D_1 + V_2^I D_2) H_I, \tag{56}$$

where, V_1 and V_2, are either integer or half-integer vectors. Using methods explained above, we can make identifications between the orbifold gauge shift v, and the vectors V_1 and V_2, by computing the integrals over $\mathcal{F}_V$ over non-

TABLE I. The upper half of the table gives intersection numbers of the compact curves E_1 and E_2, with all divisors of the resolution $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_3)$. The bottom half of the table gives the values of the integrals over the product of the (1, 1)-forms corresponding to the divisors, which are not necessarily integral.

Divisor	D_1	D_2	E_1	E_2
E_1	0	1	−2	1
E_2	1	0	1	−2
D_1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	1
D_2	$-\frac{1}{3}$	$-\frac{2}{3}$	1	0

compact curves D_1 and D_2, respectively:

$$v^I H_I \equiv \int_{D_1} \frac{\mathcal{F}_V}{2\pi} = -\frac{1}{3}(2V_1^I + V_2^I)H_I, \\ -v^I H_I \equiv \int_{D_2} \frac{\mathcal{F}_V}{2\pi} = -\frac{1}{3}(V_1^I + 2V_2^I)H_I. \quad (57)$$

It follows that $V_1 \equiv -V_2 \equiv 3v$, in order that the line bundle background can be interpreted in the blow down limit.

To determine the consequences of the Bianchi identity, we compute the integral of the second Chern class over the resolution

$$-\frac{1}{2}\int \frac{\mathrm{tr}\,\mathcal{R}^2}{(2\pi i)^2} = \int c_2(\mathcal{R}) = \frac{8}{3}. \quad (58)$$

Requiring that the integrated Bianchi identity vanishes leads to the consistency condition

$$V_1^2 + V_2^2 + V_1 \cdot V_2 = 8. \quad (59)$$

This is the analog of the modular invariance consistency condition of the heterotic string

$$(3v)^2 = 2 \text{ mod } 6. \quad (60)$$

In Table II we give the inequivalent modular invariant orbifold gauge shifts v, and indicate the vectors V_1 and V_2, of the corresponding blowup model(s). The first four orbifold models in this table can be realized in blowup with the choice: $V_2 = -V_1$. For the orbifold standard embedding $3v = (1^2, 0^{14})$ can also be realized in an alternative way, in which the vectors are not simply equal and opposite, but nevertheless satisfy the condition that they can be identified with the orbifold gauge shift.

The final orbifold model in Table II cannot be realized by any combination of resolution vectors V_1 and V_2, satisfying all conditions. For this reason we have separated it from the rest of the table. We give two proposals of vectors that could realize the blowup of the orbifold model: The first realization has vectors V_1 and V_2, that can be directly identified with the orbifold one, but do not have a vanishing Bianchi identity. The second realization has vectors V_1 and V_2, that lead to the vanishing of the Bianchi identity, but cannot be linked directly to an orbifold shift. For this model and all the others where we can compute the spectrum, they coincide with the ones that were identified in [50].

TABLE II. This table compares the $\mathbb{C}^2/\mathbb{Z}_3$ orbifold gauge shift vector v with the blowup vectors V_1 and V_2, that topologically characterize gauge background of the resolution $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_3)$. The blowup vectors under the double line do not satisfy all possible conditions simultaneously. The upper proposal gives a nonvanishing Bianchi, while the vectors of the bottom one cannot be identified with the orbifold shift.

Orbifold shift $3v$	Blowup vector V_1	Blowup vector V_2
$(1^2, 0^{14})$	$(2^2, 0^{14})$	$-(2^2, 0^{14})$
	$(2, 1, 0^{14})$	$(1, -1, 0^{14})$
$(2, 1^4, 0^{11})$	$(2, 1^4, 0^{11})$	$-(2, 1^4, 0^{11})$
$(1^8, 0^8)$	$(1^8, 0^8)$	$-(1^8, 0^8)$
$(1^{14}, 0^2)$	$\frac{1}{2}(1^{14}, 3^2)$	$-\frac{1}{2}(1^{14}, 3^2)$
$(2, 1^{10}, 0^5)$	$(2, 1^{10}, 0^5)$	$-(2, 1^{10}, 0^5)$
	$\frac{1}{2}(-3, 1^{10}, 1^5)$	$(1, 0^{10}, -1^5)$

C. Resolution of $\mathbb{C}^3/\mathbb{Z}_4$

The second example of a resolution with two exceptional divisors is obtained from the three dimensional orbifold $\mathbb{C}^3/\mathbb{Z}_4$:

$$\theta: (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) \to (e^{2\pi i\phi_1}\tilde{Z}_1, e^{2\pi i\phi_2}\tilde{Z}_2, e^{2\pi i\phi_3}\tilde{Z}_3), \\ \phi = \tfrac{1}{4}(1, 1, 2). \quad (61)$$

The elements θ and θ^3 are each other's complex conjugates, hence there are two exceptional divisors E_1 and E_2. The vectors

$$w_1 = \tfrac{1}{4}v_1 + \tfrac{1}{4}v_2 + \tfrac{1}{2}v_3, \qquad w_2 = \tfrac{1}{2}v_1 + \tfrac{1}{2}v_2, \quad (62)$$

take the form $\frac{1}{4}(1, 1, 2)$ and $\frac{1}{2}(1, 1, 0)$, in the basis, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, respectively. This leads to the local coordinates

$$Z_1 = z_1 x_1^{1/4} x_2^{1/2}, \quad Z_2 = z_2 x_1^{1/4} x_2^{1/2}, \quad Z_3 = z_3 x_1^{1/2}, \quad (63)$$

which imply the linear equivalence relations

$$4D_1 + E_1 + 2E_2 \sim 0, \qquad 4D_2 + E_1 + 2E_2 \sim 0, \\ 2D_3 + E_1 \sim 0. \quad (64)$$

The $(\mathbb{C}^*)^2$ scalings

$$(z_1, z_2, z_3, x_1, x_2) \\ \sim (\lambda_1^{-1}z_1, \lambda_1^{-1}z_2, \lambda_3^{-1}z_3, \lambda_3^2 x_1, \lambda_1^2\lambda_3^{-1}x_2), \quad (65)$$

require that the exclusion set is given by

$$F = \{z_1 = z_2 = 0\} \cup \{z_3 = x_2 = 0\}, \quad (66)$$

in order to avoid singularities in any of the coordinate patches. The projected toric diagram was composed using the basis, $v_1 = (0, 0, 1)$, $v_2 = (1, 0, 1)$, $v_=(0, 1, 1)$, in which $w_1 = (\frac{1}{4}, \frac{1}{2}, 1)$ and $w_2 = (\frac{1}{2}, 0, 1)$.

The projected toric diagram 3 implies that the basic cones

$$D_1E_1E_2 = D_2E_1E_2 = D_1D_3E_1 = D_2D_3E_1 = 1, \quad (67)$$

all have unit intersection number, and that the integrals

$$D_1D_2E_2 = D_3E_1E_2 = 0 \quad (68)$$

vanish. The total number of integrals of divisors on this resolution is 35, but as discussed above it suffices to only give the 4 integrals of the exceptional divisors

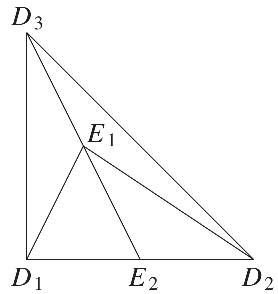

FIG. 3. This figure gives the projected toric diagram of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$. Only the exceptional divisor E_1 is compact, all other divisors are noncompact.

$$E_1^2E_2 = 0, \quad E_2^2E_1 = -2, \quad E_1^3 = 8, \quad E_2^3 = 2, \quad (69)$$

as all other integrals can be determined from them using the linear equivalences (64).

The exceptional divisor E_1 lies in the interior of the projected toric diagram, and hence is compact. This can be easily confirmed explicitly. The divisor E_1 is embedded as

$$E_1 = (\lambda_1^{-1}z_1, \lambda_1^{-1}z_2, \lambda_3^{-1}z_3, 0, \lambda_1^2\lambda_3^{-1}x_2), \quad (70)$$

inside the toric variety $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$. By fixing the scaling such that $|\lambda_1|^2 = |z_1|^2 + |z_2|^2$ and $|\lambda_2|^2 = |z_3|^2 + |\lambda_1^2x_2|^2$, it is obvious that E_1 is bounded and hence compact. Moreover, notice the coordinates z_1 and z_2 have a scaling factor λ_1^{-1} and the coordinates z_3 and x_2 have a scaling factor λ_3^{-1}. Ignoring the factor λ_1^2, that also scales x_2, E_1 would be a direct product of two $\mathbb{CP}^1$'s. However, precisely this additional scaling of x_2 with λ_1^2 means that E_1 is not simply a direct product of two $\mathbb{CP}^1$'s, but rather a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$. Such a surface is called the Hirzebruch surface $\mathbb{F}_2$ in the mathematical literature.

The exceptional divisor E_2 is noncompact in three complex dimension. It equals a direct product $\mathbb{CP}^1 \times \mathbb{C}$, which signals that we should view the situation from a two dimensional complex perspective instead. The edge of the toric diagram, in Fig. 3, spanned by D_1 and D_2, is itself precisely the toric diagram of the resolution $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$, as depicted on the left of Fig. 1. Therefore, the integrals computed in subsection II B, for $n = 2$, can be directly applied to the divisors D_1, E_2, and D_2. Hence, in particular, we have $D_1E_2 = D_2E_2 = 1$.

Next, we want to find a basis of orthonormal (1, 1)-forms, that can be used to expand the U(1) gauge background around. To determine this basis, we note that there exist four compact curves: D_1E_1, D_2E_1, D_3E_1, and E_1E_2. Using the linear equivalences (64) we infer that if we have constructed an orthonormal basis of (1, 1)-forms on the curves D_1E_1 and E_1E_2, they are integer on all these compact curves. Such a basis of (1, 1)-forms is spanned by D_1 and D_3, see the same Table III. This means that we can expand the gauge background as

TABLE III. The first part of the table gives all possible intersection numbers of the compact curves with all divisors of the resolution $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$. As the curve D_3E_2 is excluded, the final row of this table can only be interpreted as giving (fractional) values of the integrals of the corresponding forms.

	D_1	D_2	D_3	E_1	E_2
D_1E_1	0	0	1	-2	1
D_2E_1	0	0	1	-2	1
D_3E_1	1	1	2	-4	0
E_1E_2	1	1	0	0	-2
D_3E_2	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	1

$$\frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2}E_1H_1 - \frac{1}{4}(E_1 + 2E_2)H_2, \quad (71)$$

where $H_1 = V_1^IH_I$ and $H_2 = V_2^IH_I$, respectively. We have used the linear equivalences (64) to express D_1 and D_3 in terms of the exceptional divisors only.

In order that this gauge background (71) defines a consistent compactification, we have to require that the Bianchi identity vanishes when integrated over the compact divisor E_1. To determine the resulting condition we evaluate the second Chern class

$$c_2(\mathcal{R}) = D_1^2 - 2D_1D_3 - 2D_3^2 + 2D_1E_2 - D_3E_2, \quad (72)$$

which leads to the necessary consistency condition

$$V_1^2 + V_1 \cdot V_2 = 4. \quad (73)$$

This condition ensures that the gauge background, defined by V_1 and V_2, is consistent.

In addition to this necessary condition, we may also require that the integrated Bianchi vanishes on E_2, and on the subvariety $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$. As noted above, the edge of the toric diagram, Fig. 3, spanned by D_1 and D_2, defines the toric diagram of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$. This tell us that we should do the computation on two complex dimensional toric variety, with the divisors D_1, D_2, and the exceptional one E_2. All properties of this subvariety are inherent from $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$ by setting $E_1 = 0$, i.e. simply ignoring the homogeneous coordinate x_1 and its associated scaling λ_3. Indeed, the scaling (65) reduces to

$$(z_1, z_2, z_3, x_2) \sim (\lambda_1^{-1}z_1, \lambda_1^{-1}z_2, z_3, \lambda_1^2x_2), \quad (74)$$

which defines the space $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$. It is also not difficult to check that the total Chern class of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_4)$ with vanishing E_1 reduces to that of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$. Similarly, taking $E_1 = 0$ in (71) gives us the gauge background on this subresolution. This gives rise to the additional conditions

$$V_1 \cdot V_2 = -2 \quad \text{and} \quad V_2^2 = 6, \quad (75)$$

respectively.

Finally, we can make a partial matching with the orbifold gauge shift. From the six dimensional perspective we can use the identification of the orbifold and blowup shifts on the subresolution of $\mathbb{C}^2/\mathbb{Z}_2$. Integrating the bundle background over D_1 within $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$ gives

$$2v^I H_I \equiv \int_{D_1} \frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2} V_2^I H_I. \tag{76}$$

We can identify this integral with the $\mathbb{Z}_2$ gauge orbifold shift $2v$. The identification from the four dimensional perspective is more complicated, and will not be discussed here.

We can give a complete classification of all consistent models on the resolution of $\mathbb{C}^3/\mathbb{Z}_4$, using *all* the conditions described above. Table IV gives the gauge shift vectors of the possible heterotic orbifold models, and the vectors V_1 and V_2, that define the U(1) bundle background on the resolution. Only for the orbifold model numbered 4 in Table IV we have not found a blowup model. This orbifold model has no matter in the first twisted sector. Since the blowup modes are precisely the twisted states of the string, we expect that no complete resolution of this orbifold model exists.

TABLE IV. This table compares the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold gauge shift vector v, with the blowup vectors V_1 and V_2, that characterize the line bundle gauge background on the resolution. We provide a complete classification of U(1) fluxes compatible with the resolution of a $\mathbb{C}^3/\mathbb{Z}_4$ singularity, i.e. fulfilling the orbifold matching (76) and the Bianchi identities (73) and (75).

Orbifold shift $4v$	Blowup vector V_2	Blowup vector V_1	Nr.
$(0^{13}, 1^2, 2)$	$(0^{13}, 1^2, 2)$	$(0^{13}, 1^2, -2)$	1a
	$(0^{13}, 1^2, 2)$	$(0^{12}, 2, -1^2, 0)$	1b
	$(0^{13}, 1^2, 2)$	$(0^{11}, 2, 1, 0^2, -1)$	1c
$(0^{11}, 1^2, 2^3)$	$(0^{13}, 1^2, 2)$	$(0^{10}, 1^4, -1^2)$	2a
	$(0^{13}, 1^2, 2)$	$(0^{11}, 1^2, -2, 0^2)$	2b
$(0^9, 1^2, 2^5)$	$(0^{13}, 1^2, 2)$	$(0^8, 1^5, 0^2, -1)$	3a
	$(0^{13}, 1^2, 2)$	$(0^9, 1^4, -1^2, 0)$	3b
$(0^7, 1^2, 2^7)$	—	—	4
$(0^{10}, 1^6)$	$(0^{10}, 1^6)$	$(0^{10}, 1^2, -1^4)$	5a
	$(0^{10}, 1^6)$	$(0^{13}, 1, -1, -2)$	5b
$(0^{10}, 1^5, 3)$	$(0^{10}, 1^6)$	$(0^9, 2, -1^2, 0^4)$	6
$(0^8, 1^6, 2^2)$	$(0^{10}, 1^6)$	$(0^8, 1^3, -1^3, 0^2)$	7a
	$(0^{10}, 1^6)$	$(0^8, 1^2, -2, 0^5)$	7b
$(0^6, 1^6, 2^4)$	$(0^{10}, 1^6)$	$(0^6, 1^4, -1^2, 0^4)$	8
$(0^5, 1^{10}, 2)$	$(0^{10}, 1^6)$	$\frac{1}{2}(-3, 1^{10}, -1^5)$	9
$(0^3, 1^{10}, 2^3)$	$(0^{10}, 1^6)$	$\frac{1}{2}(1^{12}, -1^3, -3)$	10
$(1^{14}, 2^2)$	$(0^{13}, -2, 1^2)$	$\frac{1}{2}(1^{15}, -3)$	11
$(1^{13}, -1, 2^2)$	$(0^{13}, 1^2, 2)$	$\frac{1}{2}(1^{15}, -3)$	12a
	$(0^{13}, 1^2, 2)$	$-\frac{1}{2}(-3, 1^{15})$	12b
$\frac{1}{2}(1^3, 3^{12}, -3)$	$\frac{1}{2}(-3, 1^{15})$	$-(0^{13}, 1^2, 2)$	13a
	$\frac{1}{2}(1^{15}, -3)$	$(0^{13}, 1^2, 2)$	13b
	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(1^3, -1^{11}, 3, 1)$	13c
$\frac{1}{2}(1^7, 3^8, -3)$	$\frac{1}{2}(1^{15}, -3)$	$(-1^5, 1, 0^{10})$	14a
	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(1^6, -1^8, -3, 1)$	14b
	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(1^8, -1^7, 3)$	14c
$\frac{1}{2}(1^{11}, 3^4, -3)$	$\frac{1}{2}(1^{15}, -3)$	$(0^{10}, 1^3, -1^3)$	15
$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(1^{15}, -3)$	$(0^{13}, -2, 1^2)$	16a
	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(-1^{14}, 3, -1)$	16b

For each of the other models, we compute the spectrum using (49), and compare it with the spectrum of the corresponding orbifold model. The multiplicity operator takes the form

$$N_V = \frac{1}{6}\Big[\frac{3}{2}\Big(\frac{1}{2} - H_1^2\Big)H_2 + (1 - H_1^2)H_1\Big], \tag{77}$$

where we employed the short-hand notation $H_i = V_i^I H_I$. The resulting spectra in the SO(32) theory are given in Table V. The multiplicity factors of $1/8$ and $1/4$ can be easily understood from the heterotic orbifold point of view: In paper [51] the local anomalies at four and six dimensional fixed points of $T^6/\mathbb{Z}_4$ were computed, using general trace formulae on orbifolds [52]: The ten dimensional states contribute $1/8$ of an anomaly at a $\mathbb{Z}_4$ fixed point, the six dimensional second-twisted sector contributes $1/4$, and the four dimension single-twisted sector gives integral contributions. The matter representations can also be traced back to the orbifold model. The six and four dimensional spectra of the heterotic string on $\mathbb{C}^3/\mathbb{Z}_4$ can be found in [53,54]. The spectra in Table V are obtained from simple branching with respect to the unbroken gauge group, up to possible mismatches due to vectorlike states. Mostly only a single scalar is not part of the charged chiral spectrum on the resolution (as explained in [35] this state has become a model dependent axion part of the expansion of B_2). Some models have SU(N) gauge groups and therefore non-Abelian gauge anomalies could arise. However, from Table V it can be confirmed that all pure SU(N), $N \geq 3$, anomalies cancel. The models contain a bunch of U(1)'s that are all potentially anomalous, we expect that their anomalies are canceled via the Green-Schwarz mechanism involving universal and nonuniversal axions [34,35,55,56].

IV. ORBIFOLDS WITH MULTIPLE RESOLUTIONS

A. Generalities of multiple triangulations

In the general discussion and in the examples so far we have avoided one further complication of generic resolutions of orbifold singularities in three (or more) complex dimensions: The resolution of a given $\mathbb{C}^3/G$ orbifold might be nonunique. This difficulty arises precisely when more than one triangulation of the toric diagram is possible. For clarity we first indicate which properties of orbifold resolutions described and illustrated in Sec. III still hold, and after that focus on novelties, that arise from the possibility of having multiple triangulations.

Essentially all the properties of a resolution, discussed in subsection III A, that do not depend on the triangulation of the toric diagram, can be extended to orbifolds which have nonunique resolutions. In particular, the definition of the (exceptional) divisors (39), the construction of a set of

TABLE V. This table gives the chiral part of the spectrum of the resolution models of the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold. The models, defined by the blowup vectors V_1 and V_2, are numbered according to the convention defined in Table IV.

Nr.	4D gauge group	$\frac{1}{8}\times$ "untwisted"	$\frac{1}{4}\times$ "2nd twisted"	"1st twisted"
1a	$SO(26)\times U(2)\times U(1)$	$(26, 2) + 2(1, 2)$	$(26, 1) + 2(1, 2) + (1, 1)$	$(26, 1) + 2(1, 2) + 3(1, 1)$
1b	$SO(24)\times U(2)\times U(1)^2$	$(24, 2) + 4(1, 2)$	$(24, 1) + 2(1, 2) + 3(1, 1)$	$(24, 1) + 2(1, 2) + 5(1, 1)$
1c	$SO(22)\times U(2)\times U(1)^3$	$(22, 2) + 6(1, 2)$	$(22, 1) + 2(1, 2) + 5(1, 1)$	$(22, 1) + 2(1, 2) + 5(1, 1)$
2a	$SO(20)\times U(3)\times U(1)^3$	$2(20, 1) + 2(1, 3)$ $+2(1, \bar{3}) + 4(1, 1)$	$(20, 1) + (1, 3)$ $+(1, \bar{3}) + 3(1, 1)$	$2(1, 3) + 2(1, \bar{3}) + 2(1, 1)$
2b	$SO(22)\times U(2)\times U(1)^3$	$2(22, 1) + 4(1, 2) + 4(1, 1)$	$(22, 1) + 2(1, 2) + 3(1, 1)$	$2(1, 2) + 7(1, 1)$
3a	$SO(16)\times U(2)\times U(5)\times U(1)$	$(16, 2, 1) + (1, 2, 5)$ $+(1, 2, \bar{5}) + 2(1, 2, 1)$	$(16, 1, 1) + (1, 1, 5)$ $+(1, 1, \bar{5}) + (1, 1, 1)$	$(1, 1, 10) + (1, 1, \bar{5})$
3b	$SO(18)\times U(2)\times U(4)\times U(1)$	$(18, 2, 1) + (1, 2, 4)$ $+(1, 2, \bar{4}) + 2(1, 2, 1)$	$(18, 1, 1) + (1, 1, 4) + (1, 1, \bar{4})$ $+(1, 2, 1) + (1, 1, 1)$	$(1, 1, 1) + (1, 6, 1)$
5a	$SO(20)\times U(4)\times U(2)$	$(20, 4, 1) + (20, 1, 2)$	$(1, \bar{4}, 2) + (1, 6, 1) + (1, 1, 1)$	$(1, \bar{4}, 2) + (1, 6, 1) + 3(1, 1, 1)$
5b	$SO(20)\times U(3)\times U(1)^3$	$3(20, 1) + (20, 3)$	$3(1, \bar{3}) + (1, 3) + 3(1, 1, 1)$	$2(1, \bar{3}) + 5(1, 1)$
6	$SO(18)\times U(4)\times U(2)\times U(1)$	$(18, 4, 1) + (18, 1, 2)$ $+2(1, 4, 1) + 2(1, 1, 2)$	$(1, \bar{4}, 2) + (1, 6, 1) + (1, 1, 1)$	$2(1, \bar{4}, 1) + (18, 1, 1)$ $+2(1, 1, 2) + (1, 1, 1)$
7a	$SO(16)\times U(3)\times U(2)^2\times U(1)$	$(16, 1, 1, 1) + (16, 1, 1, 2)$ $+(16, \bar{3}, 1, 1) + 2(1, 3, 2, 1)$ $+2(1, 1, 2, 2) + 2(1, 1, 2, 1)$	$(1, 3, 1, 1) + (1, \bar{3}, 1, 1)$ $+(1, 1, 1, 2) + (1, \bar{3}, 1, 2)(1, 1, 1, 1)$	$2(1, 3, 1, 1) + (1, 1, 1, 1)$
7b	$SO(16)\times U(2)\times U(5)\times U(1)$	$(16, 1, 5) + (16, 1, 1)$ $+2(1, 2, \bar{5}) + 2(1, 2, 1)$	$(1, 1, 10) + (1, 1, 5)$	$2(1, 1, \bar{5}) + (1, 1, 1)$
8	$SO(12)\times U(4)\times U(2)\times U(4)$	$(12, 1, 2, 1) + (12, 4, 1, 1)$ $+(1, \bar{4}, 1, 4) + (1, \bar{4}, 1, \bar{4})$ $+(1, 1, 2, 4) + (1, 1, 2, \bar{4})$	$(1, 6, 1, 1) + (1, \bar{4}, 2, 1)$ $+(1, 1, 1, 1)$	$(1, 1, 1, 6) + (1, 1, 1, 1)$
9	$U(5)\times U(9)\times U(1)^2$	$(5, 9) + (\bar{5}, 9)$ $+(5, 1) + (\bar{5}, 1)$ $+2(1, \bar{9}) + 2(1, 1)$	$(10, 1) + (\bar{5}, 1)$	$(1, \bar{9}) + 2(1, 1)$
10	$U(3)\times U(10)\times U(2)\times U(1)$	$(3, 10, 1) + (\bar{3}, 10, 1)$ $+2(1, \overline{10}, 2) + 2(1, \overline{10}, 1)$	$2(\bar{3}, 1, 1) + (\bar{3}, 1, 2) + (1, 1, 2)$ $+(1, 1, 1)$	$(3, 1, 1) + (1, 1, 2)$
11	$U(13)\times U(1)^3$	$4(13) + 4(1)$	$2(\overline{13}) + 5(1)$	$2(1)$
12a	$U(13)\times U(2)\times U(1)$	$2(13, 2) + 2(1, 2)$	$2(13, 1) + 2(1, 2) + (1, 1)$	$(\overline{13}, 1)$
12b	$U(12)\times U(2)\times U(1)^2$	$2(12, 2) + 4(1, 2)$	$2(12, 1) + 2(1, 2) + 3(1, 1)$	$(\overline{12}, 1) + 3(1, 1)$
13a	$U(12)\times U(2)\times U(1)^2$	$(66, 1) + (12, 1)$ $+(\overline{12}, 1) + (\overline{12}, 2)$ $+2(1, 2) + 2(1, 1)$	$(12, 1) + (1, 2) + (1, 1)$	$(\overline{12}, 1) + 2(1, 2) + 3(1, 1)$
13b	$U(13)\times U(2)\times U(1)$	$(78, 1) + (\overline{13}, 2) + (\overline{13}, 1)$ $+(1, 2) + (1, 1)$	$(13, 1) + (1, 2)$	$(\overline{13}, 1) + 2(1, 2) + 2(1, 1)$
13c	$U(11)\times U(3)\times U(1)^2$	$(\overline{55}, 1) + (11, 3) + 2(\overline{11}, 1)$ $+3(1, 3) + (1, 1)$	$(\overline{11}, 1) + (1, 3) + (1, 1)$	$(11, 1) + 2(1, \bar{3})$
14a	$U(5)\times U(9)\times U(1)^2$	$(10, 1) + 2(5, 1) + (\bar{5}, \bar{9})$ $+2(1, \bar{9}) + (1, 36) + (1, 1)$	$(\bar{5}, 1) + (1, 9) + (1, 1)$	$(5, 1)$
14b	$U(6)\times U(8)\times U(1)^2$	$(\overline{15}, 1) + (\bar{6}, 1) + (6, 1)$ $+(6, \bar{8}) + (1, 8) + (1, \bar{8})$ $+(1, 28) + (1, 1)$	$(6, 1) + (1, 8) + (1, 1)$	$(\bar{6}, 1) + (1, 1)$
14c	$U(8)\times U(7)\times U(1)$	$(\overline{28}, 1) + (\bar{8}, 1) + (8, \bar{7})$ $+(1, 21) + (1, \bar{7})$	$(\bar{8}, 1) + (1, \bar{7})$	$(1, 7)$
15	$U(10)\times U(3)\times U(2)\times U(1)$	$(45, 1, 1) + (10, 1, 1)$ $+(\overline{10}, \bar{3}, 1) + (\overline{10}, 1, 2)$ $+2(1, \bar{3}, 1) + (1, 3, 2)$ $+(1, 1, 2) + (1, 1, 1)$	$(\overline{10}, 1, 1) + (1, 3, 1) + (1, 1, 2)$	$(1, \bar{3}, 1) + (1, 3, 2) + 2(1, 1, 1)$
16a	$U(13)\times U(1)^3$	$(78) + 2(13) + (\overline{13}) + 3(1)$	$(\overline{13}) + 2(1)$	$(\overline{13}) + 4(1)$
16b	$U(14)\times U(1)^2$	$(91) + (14) + (\overline{14}) + (1)$	$(\overline{14}) + (1)$	$(\overline{14}) + 3(1)$

local coordinates (40), the linear equivalences (41), and the $(\mathbb{C}^*)^N$ scaling, are uniquely defined for any triangulation. As we have seen resolutions of three dimensional orbifolds may contain two dimensional resolutions as subvarieties. These subvarieties are identified as the faces of the toric diagram. Even though the resolution of three dimensional orbifolds may not be unique, the toric diagrams corresponding to the faces is uniquely defined by the divisors on them. Hence these subvarieties are the same for each resolution.

The exclusion set F does depend on the triangulation [42]: As before, the exclusion is defined such that the resolution is by definition nonsingular. In addition, the curves that are not realized as lines within the triangulation are part of the exclusion set. The latter makes the exclusion set dependent on the triangulation of the toric diagram.

The integrals of the divisors over the resolution also crucially depend on the triangulation: As described in subsection III A the triangulation identifies that the compact curves have unit intersection number with some divisors of the resolution. Hence, if the triangulation is not unique, one can assign a different intersection of the compact curves with the divisors. The problem is that there are more basic cones possible in the toric diagram given the divisors only than can be realized in a given triangulation. This issue is illustrated by the toric diagrams of the resolution of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$: Of the ten possible basis cones, only four are realized within a triangulation, as we discuss in detail in subsection IV B. To define the integrals of the divisors, interpreted as (1, 1)-forms, over the resolution, we employ the following rules for any given triangulation:

(i) The basic cones, that do exist within the triangulation, are formed by divisors with unity intersection number.
(ii) All other non-self-intersections of divisors with strictly compact curves, i.e. curves that either contain a compact divisor or two exceptional divisors, vanish.
(iii) All other integrals of three divisors are obtained from these defining ones, using linear equivalence relations.

These rules give consistent assignments that do not clash with the linear equivalence relations. Even though these rules might in general be insufficient to determine all integrals of the exceptional divisors, they are sufficient for the resolutions considered in this paper. As in the previous sections, it may happen that the integral over some divisors is nonvanishing due to the linear equivalence relations, even though, as hypersurface the intersection of these divisors is excluded. As we will show in the examples of resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ below, using the definition of the integral of divisors given here, we are able to obtain blowup versions of all heterotic models on this orbifold. In addition, we obtain their spectra, which are all free of non-Abelian anomalies.

B. Resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$

We consider $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ as an example of an orbifold that admits more than one resolution. To clearly separate which statements are triangulation dependent, and which are not, we first describe those properties that are valid for each resolution. After that we compute the integrals of the divisors on the two inequivalent resolutions separately. Finally we study the relation between heterotic models on this orbifold and its possible resolutions.

1. Triangulation independent properties of the resolutions

The orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ is defined by the three $\mathbb{Z}_2$ orbifold actions:

$$\begin{aligned} \theta&: (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) \to (\tilde{Z}_1, -\tilde{Z}_2, -\tilde{Z}_3), & \phi &= \tfrac{1}{2}(0,1,1), \\ \theta'&: (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) \to (-\tilde{Z}_1, \tilde{Z}_2, -\tilde{Z}_3), & \phi' &= \tfrac{1}{2}(1,0,1), \\ \theta\theta'&: (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) \to (-\tilde{Z}_1, -\tilde{Z}_2, \tilde{Z}_3), & \phi + \phi' &= \tfrac{1}{2}(1,1,0), \end{aligned} \tag{78}$$

where the latter can be viewed as the combination of the first two. This orbifold has three twisted sectors, and hence three exceptional divisors E_1, E_2, and E_3, defined by the vectors

$$w_1 = \tfrac{1}{2}v_2 + \tfrac{1}{2}v_3, \qquad w_2 = \tfrac{1}{2}v_1 + \tfrac{1}{2}v_3, \qquad w_3 = \tfrac{1}{2}v_1 + \tfrac{1}{2}v_2. \tag{79}$$

In the standard basis for v_i, they lead to the local coordinates

$$Z_1 = z_1 x_1^{1/2} x_3^{1/2}, \qquad Z_2 = z_2 x_1^{1/2} x_3^{1/2}, \qquad Z_3 = z_3 x_1^{1/2} x_2^{1/2}, \tag{80}$$

on the resolutions. This determines the linear equivalences

$$2D_1 + E_2 + E_3 \sim 2D_2 + E_1 + E_3 \sim 2D_3 + E_1 + E_2 \sim 0. \tag{81}$$

Using these linear equivalences we can represent the second Chern class as

$$c_2(\mathcal{R}) = -\tfrac{3}{4}(E_1^2 + E_2^2 + E_3^2) - \tfrac{1}{4}(E_1E_2 + E_2E_3 + E_3E_1). \tag{82}$$

The $(\mathbb{C}^*)^3$ action on the homogeneous coordinates can be parametrized as

$$(z_1, z_2, z_3, x_1, x_2 x_3) \sim (\lambda_2^{-1}\lambda_3^{-1}z_1, \lambda_1^{-1}\lambda_3^{-1}z_2, \lambda_1^{-1}\lambda_2^{-1}z_3, \lambda_1^2 x_1, \lambda_2^2 x_2, \lambda_3^2 x_3). \tag{83}$$

The integrals

$$D_1D_2E_3 = D_2D_3E_1 = D_3D_1E_2 = 0 \tag{84}$$

all vanish: they are aligned in the projected toric diagram, see Fig. 4. But precisely these edges of the projected toric diagrams define resolutions of $\mathbb{C}^2/\mathbb{Z}_2$ orbifolds, discussed in Sec. II. Hence each of these edges correspond to a six dimensional model. There are two inequivalent triangulations, which are displayed in Fig. 4, which we now in turn describe.

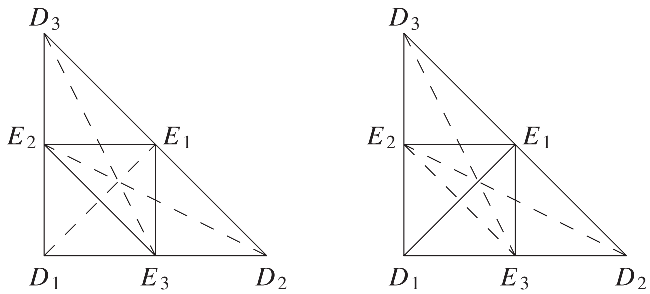

FIG. 4. The two inequivalent (projected) toric diagrams of $\mathrm{Res}(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2')$ are displayed. The left one we call the symmetric resolution, while the right one the "E_1" resolution.

2. The resolution with the "symmetric" triangulation

We investigate the topological properties of the symmetric triangulation, defined on the left side of Fig. 4. First of all, the exclusion set is defined as

$$F = \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_1 = x_1 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_3 = x_3 = 0\}. \quad (85)$$

This ensures that there are no singularities and that the dashed lines in the left projected toric diagram in Fig. 4 correspond to nonexisting curves. We read off that the basic cones are given by

$$D_1E_2E_3 = D_2E_3E_1 = D_3E_1E_2 = E_1E_2E_3 = 1, \quad (86)$$

while the other possible basic cones that are nonexistent in this triangulation vanish:

$$D_1E_1E_2 = D_1E_1E_3 = D_2E_1E_2 = 0, \quad D_2E_2E_3 = D_3E_1E_3 = D_3E_2E_3 = 0. \quad (87)$$

As we observed in Sec. III A all 56 possible integrals can be conveniently summarized by giving the 10 involving the exceptional divisors only. Because of the high amount of symmetry within the toric diagram, we can summarize all integrals over the exceptional divisors as

$$E_p^3 = -E_p^2E_{q\neq p} = E_1E_2E_3 = 1. \quad (88)$$

From these integrals we easily compute the integrals over all compact curves of all divisors. The result is tabulated in Table VI.

The curves that are not part of the triangulation do not exist in the resolution as hypersurfaces. Nevertheless, we see in Table VI below the double line, that even though curves like D_1E_1 do not exist, the integral D_1E_1X, of the dual (2, 2)-form over X (X being D_2 or D_3 or E_1) does not vanish.

TABLE VI. The upper part of the table gives the intersection numbers of the compact curves with all divisors of the symmetric resolution of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$. The lower part gives the values of the integrals of the divisors corresponding to curves that are not realized in the symmetric resolution.

	D_1	D_2	D_3	E_1	E_2	E_3
E_1E_2	0	0	1	-1	-1	1
E_1E_3	0	1	0	-1	1	-1
E_2E_3	1	0	0	1	-1	-1
D_1E_1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	0
D_2E_2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	0
D_3E_3	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1

3. The resolution with the "E_1" triangulation

Next we discuss the "E_1" triangulation. There are in fact two other possible triangulations, "E_2" and "E_3", but they are simply obtained from this one by interchanging the labels 1, 2, and 3, hence they do not constitute truly different resolutions. The exclusion set reads in this case

$$F = \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{x_1 = x_2 = 0\} \cup \{z_2 = x_2 = 0\} \cup \{z_3 = x_3 = 0\}. \quad (89)$$

All the basic cones of the"E_1" triangulation contain the exceptional divisor E_1:

$$D_1E_1E_2 = D_1E_1E_3 = D_2E_1E_3 = D_3E_1E_2 = 1. \quad (90)$$

In addition, we have the nonexisting basic cones

$$D_1E_2E_3 = E_1E_2E_3 = D_2E_1E_2 = 0, \quad D_2E_2E_3 = D_3E_1E_3 = D_3E_2E_3 = 0. \quad (91)$$

From this input data we obtain the following integrals of the exceptional divisors:

$$E_1^2E_2 = E_1^2E_3 = E_2^2E_3 = E_3^2E_2 = 0, \quad E_1E_2E_3 = E_1^3 = 0, \quad E_2^2E_1 = E_3^2E_1 = -2, \quad E_2^3 = E_3^3 = 2. \quad (92)$$

The integrals over the compact curves of the divisors can again be computed straightforwardly, using the linear equivalences. The resulting integrals are listed in Table VII. Also from this table we see that setting all integrals that involve (2, 2)-forms dual to curves, that are not part of the triangulation of the toric diagram, to zero, leads to inconsistencies. In this case only the curve E_2E_3 has only vanishing integrals, and hence is not in conflict with the linear equivalence relations. Note that also the divisor E_1 does not intersect with any of the curves listed in Table VII.

TABLE VII. The upper part of the table gives the intersection numbers of the compact curves with all divisor of the "E_1" resolution of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$. The lower part gives the values of the integrals of the divisors corresponding to curves that are not realized in the "E_1" resolution.

	D_1	D_2	D_3	E_1	E_2	E_3
E_1E_2	1	0	1	0	−2	0
E_1E_3	1	1	0	0	0	−2
D_1E_1	−1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	1
E_2E_3	0	0	0	0	0	0
D_2E_2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	0
D_3E_3	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1

3. Heterotic models from resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$

As described at the beginning of subsection IV B many topological properties are the same for all resolutions of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$. In particular, the six dimensional analysis corresponding to the edges of the projected toric diagrams, Fig. 4, are independent on the resolution chosen. Therefore, we begin with the resolution independent properties in our construction of heterotic models on these resolutions.

The gauge background on the resolution can in general be expanded as

$$\frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2}(H_1E_1 + H_2E_2 + H_3E_3), \tag{93}$$

where $H_1 = V_1^I H_I$, etc. To obtain the gauge configurations on the three edges of the projected toric diagram, we only take the exceptional divisor into account which lives on that particular edge. Using the analysis of $\mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2)$, presented in Sec. II B, we infer that V_i have either only integer or half-integer entries. In addition, we make the identification between the orbifold gauge shift vectors v_1, v_2 and $v_3 \equiv v_1 + v_2$. For example, on the edge spanned by D_2 and D_3, we have

$$\int_{E_1}\frac{\mathcal{F}_V}{2\pi} = V_1^I H_i, \qquad v_1^I H_I \equiv \int_{D_2}\frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2}V_1^I H_i. \tag{94}$$

The orbifold gauge shift vectors satisfy the modular invariance conditions

$$(2v_1)^2 = 2 \bmod 4, \qquad (2v_2)^2 = 2 \bmod 4, \qquad (2v_3)^2 = 2 \bmod 4. \tag{95}$$

Similarly, we know from the discussion in Sec. II B that the integrated Bianchi identities on the three edges do not necessarily have to vanish, but if they do, we find the conditions

$$V_1^2 = V_2^2 = V_3^2 = 6. \tag{96}$$

1. Heterotic model building on the symmetric resolution

We turn to the specific properties of the heterotic model construction on the symmetric resolution. First of all we check the quantization conditions

$$\int_{E_1E_2}\frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2}(-V_1^I - V_2^I + V_3^I)H_I, \tag{97}$$

and cyclic permutation of the labels 1, 2, and 3. The factor 1/2 might seem worrying, but it is in fact harmless because we know that in order to have an orbifold interpretation, we need that $\frac{1}{2}V_3 = \frac{1}{2}(V_1 + V_2)$ modulo a vector in the adjoint or in the spinorial representation of SO(32), and in both cases the Dirac quantization condition (97) is satisfied. The integrated Bianchi identities on the divisors E_1, E_2, and E_3 give rise to the requirements:

$$V_1^2 + 2V_2 \cdot V_3 = V_2^2 + 2V_1 \cdot V_3 = V_3^2 + 2V_1 \cdot V_2 = 8. \tag{98}$$

When combining this with the six dimensional Bianchi requirements, we conclude that

$$V_1 \cdot V_2 = V_2 \cdot V_3 = V_1 \cdot V_3 = 1. \tag{99}$$

The solution of these conditions and the corresponding orbifold models are listed in Table VIII. It is remarkable that the orbifold shift vectors $2v_i$ and the vectors V_i characterizing the gauge bundle are almost identical. Indeed, a sign flip in some entries of an orbifold shift is irrelevant, as well as the addition of vectors in the lattice of the adjoint or the spinorial representations of SO(32). The four dimensional chiral spectrum on this resolution of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ can be computed from the multiplicity operator

$$N_V = \tfrac{1}{6}(H_1 + H_2 + H_3)[\tfrac{1}{2}(H_1H_2 + H_2H_3 + H_3H_1) - \tfrac{1}{8}(H_1^2 + H_2^2 + H_3^2) - \tfrac{1}{4}] - \tfrac{3}{8}H_1H_2H_3. \tag{100}$$

The resulting spectra are rather elaborate because of multiple branchings, and not very illuminating; we refrain from giving them explicitly in the paper. However, by direct inspection of these spectra we confirmed that all the models listed in Table VIII are free of irreducible anomalies.

2. Heterotic model building on the "E_1" resolution

For the other resolution, the quantization requires that, easily:

$$\int_{E_1E_2}\frac{\mathcal{F}_V}{2\pi} = V_2^I H_I, \qquad \int_{D_1E_1}\frac{\mathcal{F}_V}{2\pi} = -\frac{1}{2}(H_2 + H_3), \qquad \int_{E_1E_3}\frac{\mathcal{F}_V}{2\pi} = V_3^I H_I. \tag{101}$$

The quantization condition can only be satisfied if $\frac{1}{2}(V_2 + V_3)$ is a vector containing either only *even* or only *odd* numbers. Moreover, in order to have an identification with

TABLE VIII. This table compares the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold gauge shift vectors v_2 and v_3, with the blowup vectors V_1, V_2, and V_3, that characterize gauge background of the symmetric resolution of this orbifold. The blowup vectors satisfy all the flux quantization conditions (97) and all the Bianchi identities (96) and (98). The identification of the orbifold and blowup shifts is performed up to lattice vectors.

Orbifold shift $2v_1$	Orbifold shift $2v_2$	Blowup vector V_1	Blowup vector V_2	Blowup vector V_3
$(1^2, 0^{14})$	$(0, 1^2, 0^{13})$	$(1^2, 0, 2, 0^{12})$	$(0, 1^2, 0, 2, 0^{11})$	$(1, 0, 1, 0, 0, 2, 0^{10})$
		$(1^2, 2, 0^{13})$	$(0, -1, 1, 2, 0^{12})$	$(-1, 0, 1, 0, 2, 0^{11})$
$(1^2, 0^{14})$	$(0, 1^6, 0^9)$	$(1^2, 0^{13}, 2)$	$(0, 1^6, 0^9)$	$(1, 0, 1^3, -1^2, 0^9)$
		$(1^2, 2, 0^{13})$	$(0, -1, 1^5, 0^9)$	$(-1, 0, 1^3, -1^2, 0^9)$
$(1^6, 0^{10})$	$(0^3, 1^6, 0^7)$	$(1^6, 0^{10})$	$(0^3, -1, 1^5, 0^7)$	$(1^2, -1, 0^3, 1^2, -1, 0^7)$
$(1^6, 0^{10})$	$(0^5, 1^6, 0^5)$	$(1^6, 0^{10})$	$(0^5, 1^6, 0^5)$	$(0^5, 1, 0^5, 1^5)$
$(1^2, 0^{14})$	$\frac{1}{2}(1^{15}, -3)$	$(-1, 1, 2, 0^{13})$	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(-1^2, 1^{12}, -3, 1)$
$(1^6, 0^{10})$	$\frac{1}{2}(-3, 1^{15})$	$(1^6, 0^{10})$	$\frac{1}{2}(-3, 1^{15})$	$\frac{1}{2}(-3, 1^5, -1^{10})$
		$(1^4, -1^2, 0^{10})$	$\frac{1}{2}(1^{15}, -3)$	$\frac{1}{2}(1^6, -1^8, 3, -1)$

the orbifold models, we need $\frac{1}{2}V_1 = \frac{1}{2}(V_2 + V_3)$ up to lattice vectors of the adjoint or spinorial representation of SO(32). This implies that V_1 contains either only odd or only even numbers. When all entries are odd $V_1^2 \geq 16$, while in the even case V_1^2 is a multiple of four. In either case the Bianchi identity $V_1^2 = 6$ cannot be satisfied. Thus, no model can be built in such a resolution of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifold singularity that fulfils all the consistency conditions listed above.

V. CONCLUSIONS

We have investigated resolutions of heterotic orbifolds using toric geometry. Our initial motivation was to understand the topology behind the recently constructed heterotic models on explicit blowup of $\mathbb{C}^n/\mathbb{Z}_n$ singularities. We showed how the values of the integrals relevant to determine the consistent models and their spectra can be obtained as integrals of divisors on the corresponding toric variety. Unfortunately, only for the special $\mathbb{C}^n/\mathbb{Z}_n$ singularities explicit blowups are known; for more complicated and phenomenologically more relevant orbifolds explicit constructions remain a difficult task.

Luckily, toric geometry does not require that one has explicitly constructed the metric of the noncompact Calabi-Yau blowup of orbifold singularity: The geometrical orbifold action essentially uniquely determines the toric variety that describes the resolution of the orbifold singularity. The only caveat is that the resolution might not be topologically unique. The main advantage of having the resolution of the orbifold compared to the orbifold itself is that one is able to determine the structure inside the singularity. This is encoded by the exceptional divisors, which were needed to desingularize the toric variety. From the very definition of these exceptional divisors it is clear that they are in one-to-one correspondence to the twisted sectors of orbifold string theories. Motivated by this, we gave a self-contained introduction to toric geometry for nonexperts, emphasizing the methods relevant to obtain heterotic models on toric orbifold resolutions. As it is rather cumbersome to describe these procedures in general, we have illustrated the toric geometrical tools by constructing heterotic models on the resolutions of $\mathbb{C}^2/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_4$, and $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2'$ orbifolds. During our investigations the following issues came up:

We used the homogeneous coordinate approach to the construction of toric varieties and the corresponding exclusion set [42]. We found, however, that integrals of divisors that as hypersurfaces are excluded, can nevertheless give rise to nonvanishing values. Already for the simple resolution of $\mathbb{C}^n/\mathbb{Z}_n$ the intersection of all ordinary divisors is part of the exclusion set. However, both using linear equivalences and integrating the corresponding background field strength on the explicit blowup, we showed that such integrals are nonzero, but rather fractional. Even though intersection theory of noncompact divisors might be ill-defined,[1] the integrals of the first Chern classes of the line bundles associated to the divisors do give unambiguous results in the cases considered. The reason is that the integrands are uniquely defined up to exact terms, which means that the integrals over the noncompact resolution are defined up to boundary terms. For applications to blowups of compact orbifolds, one needs to glue various noncompact resolutions together. The boundary contributions are then canceled among themselves automatically, and the result is uniquely defined. Hence, an alternative way to deal with this complication is to consider the intersection theory of resolutions of compact orbifolds [45,46].)

After these mathematical issues we turned to the applications in heterotic model building. We aimed to find a matching between models constructed using conformal field theory on orbifolds and blowup models defined on their resolutions described by toric geometry. It is non-

[1] As D. Cox pointed out to us, the intersection of noncompact divisors is problematic because the corresponding Chow group is trivial.

trivial that such a matching exists because the supergravity description breaks down when orbifold singularities appear in blow down. However, since the comparison was made on the level of the chiral spectrum, which only relies on topological information and gauge group branchings, we have confidence that this approach can be trusted.

There are many consistency conditions which can be enforced on heterotic models on a resolution of an orbifold. There are the minimal requirements to construct a sensible model on the resolution of the orbifold: The U(1) gauge bundles have to be integral on all compact curves, both in three dimensional complex resolutions and all compact curves of the two dimensional subresolutions. In addition, the integrals of the Bianchi identity over all compact exceptional divisors (compact four dimensional real cycles) of the resolution have to vanish as well. To be able to compute the spectrum of the model on the resolution, one needs to ensure that the Bianchi identity integrated over all noncompact 4-cycles, and all subresolutions, i.e. the Bianchi identity in six dimensions, vanish. Surprisingly, satisfying all these conditions on the resolution of the orbifold seems to guarantee that in the blow down limit the model can be directly interpreted as a heterotic orbifold. A direct identification of the orbifold gauge shift vector with the U(1) gauge background can be obtained by computing integrals over noncompact curves. By Stoke's theorem we can turn it into a contour integral at infinity, which can be identified with the same integral of the orbifold model.

For each of the resolution models we have computed the spectra. To this end we used the conventional index theorem dropping possible boundary contributions. This can be justified by imagining resolutions of compact orbifolds: the boundary contributions from the local resolutions of the various fixed points precisely cancel in the gluing procedure. In any event we have confirmed that we are able to reproduce the complete spectra of the heterotic orbifold models up to vectorlike matter. All in all we have obtained a detailed dictionary of how to translate between orbifold and blowup model properties.

As explained above, not all requirements are necessary, hence one may wonder what happens if some of them are not fulfilled. In particular, we could have nonvanishing Bianchi identities, when integrated on noncompact 4-cycles. This is very natural when one thinks of obtaining blowup models of compact orbifolds: Then one only has compact 4-cycles; on each of them the integrated Bianchi needs to vanish. From a local perspective this means that there is H-flux exchanged between the resolutions of the various fixed points. Using the results of [45] one should be able to analyze such situations globally. However, one knows from orbifold field and string theory that the spectra can be determined locally at each of the fixed points (even in the presence of Wilson lines). However, the standard index theorem used in the work to compute the chiral spectrum fails because it does not take local H-fluxes into account. Using a modified index theorem that is valid in the presence of such fluxes, one may hope to be able to compute the local spectra at any of the resolution models that only satisfy the necessary vanishing Bianchi conditions.

Another natural extension of our work is to determine the blowup models of the $T^6/\mathbb{Z}_{6-II}$ orbifold. As was emphasized in [23,24] such orbifolds with Wilson lines seem to be able to give a relatively large class of MSSM-like models. It would therefore be very interesting to study these models in blowup. The $T^6/\mathbb{Z}_{6-II}$ orbifold contains various orbifold singularities that are of the types $\mathbb{C}^2/\mathbb{Z}_2$, $\mathbb{C}^2/\mathbb{Z}_3$, and $\mathbb{C}^3/\mathbb{Z}_{6-II}$. The construction of resolution models for the first two singularities have been discussed in this paper; for the first one we have constructed an explicit blowup in [32]. The final singularity type can be investigated using the methods explained here. In fact, there are five topologically inequivalent resolutions and any resolution involves four exceptional divisors. Therefore, each inequivalent resolution is characterized by 20 integrals number of the exceptional divisors. As the full analysis will therefore be rather involved, we postpone it to a future publication.

ACKNOWLEDGMENTS

We would like to thank O. Lebedev, H. P. Nilles, and S. Raby for stimulating discussions that encouraged us to pursue the investigation of general orbifold resolution. We are grateful to D. Cox for enlightening correspondence. We would also like to thank F. Plöger for reading the manuscript and giving useful suggestions.

[1] A. E. Faraggi, D. V. Nanopoulos, and K.-j. Yuan, Nucl. Phys. **B335**, 347 (1990).
[2] A. E. Faraggi, Phys. Lett. B **278**, 131 (1992).
[3] G. B. Cleaver, A. E. Faraggi, and D. V. Nanopoulos, Phys. Lett. B **455**, 135 (1999).
[4] M. Berkooz, M. R. Douglas, and R. G. Leigh, Nucl. Phys. **B480**, 265 (1996).
[5] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, J. High Energy Phys. 10 (2000) 006.
[6] G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan, and A. M. Uranga, J. Math. Phys. (N.Y.) **42**, 3103 (2001).
[7] G. Honecker and T. Ott, Phys. Rev. D **70**, 126010 (2004).
[8] T. P. T. Dijkstra, L. R. Huiszoon, and A. N. Schellekens, Nucl. Phys. **B710**, 3 (2005).

[9] T. P. T. Dijkstra, L. R. Huiszoon, and A. N. Schellekens, Phys. Lett. B **609**, 408 (2005).
[10] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985).
[11] B. Andreas, G. Curio, and A. Klemm, Int. J. Mod. Phys. A **19**, 1987 (2004).
[12] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, Phys. Lett. B **618**, 252 (2005).
[13] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, J. High Energy Phys. 06 (2005) 039.
[14] V. Braun, Y.-H. He, B. A. Ovrut, and T. Pantev, J. High Energy Phys. 05 (2006) 043.
[15] R. Blumenhagen, G. Honecker, and T. Weigand, J. High Energy Phys. 10 (2005) 086.
[16] R. Blumenhagen, S. Moster, and T. Weigand, Nucl. Phys. **B751**, 186 (2006).
[17] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985).
[18] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B274**, 285 (1986).
[19] S. Forste, H. P. Nilles, P. K. S. Vaudrevange, and A. Wingerter, Phys. Rev. D **70**, 106008 (2004).
[20] T. Kobayashi, S. Raby, and R.-J. Zhang, Nucl. Phys. **B704**, 3 (2005).
[21] T. Kobayashi, S. Raby, and R.-J. Zhang, Phys. Lett. B **593**, 262 (2004).
[22] W. Buchmuller, K. Hamaguchi, O. Lebedev, and M. Ratz, Nucl. Phys. **B712**, 139 (2005).
[23] O. Lebedev *et al.*, Phys. Lett. B **645**, 88 (2007).
[24] O. Lebedev *et al.*, Phys. Rev. Lett. **98**, 181602 (2007).
[25] J. E. Kim and B. Kyae, Nucl. Phys. **B770**, 47 (2007).
[26] J. E. Kim, J.-H. Kim, and B. Kyae, J. High Energy Phys. 06 (2007) 034.
[27] T. Eguchi and A. J. Hanson, Phys. Lett. B **74**, 249 (1978).
[28] E. Calabi, Ann. Scient. Ec. Norm. Sup. **12**, 269 (1979).
[29] J. Polchinski, *String Theory Vol. 2: Superstring Theory and Beyond*, Cambridge Monographs On Mathematical Physics (Cambridge University Press, Cambridge, England, 1998), p. 531.
[30] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs (Oxford University, New York, 2000), p. 436.
[31] O. J. Ganor and J. Sonnenschein, J. High Energy Phys. 05 (2002) 018.
[32] S. Groot Nibbelink, M. Trapletti, and M. Walter, J. High Energy Phys. 03 (2007) 035.
[33] R. Blumenhagen, G. Honecker, and T. Weigand, J. High Energy Phys. 06 (2005) 020.
[34] R. Blumenhagen, G. Honecker, and T. Weigand, J. High Energy Phys. 08 (2005) 009.
[35] S. Groot Nibbelink, H. P. Nilles, and M. Trapletti, Phys. Lett. B **652**, 124 (2007).
[36] Z. Kakushadze, Nucl. Phys. **B512**, 221 (1998).
[37] Z. Kakushadze, G. Shiu, and S. H. H. Tye, Nucl. Phys. **B533**, 25 (1998).
[38] Z. Lalak, S. Lavignac, and H. P. Nilles, Nucl. Phys. **B559**, 48 (1999).
[39] J. Erler and A. Klemm, Commun. Math. Phys. **153**, 579 (1993).
[40] W. Fulton, *Introduction to Toric Varieties* (Princeton University, Princeton, NJ, 1993).
[41] T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties* (Springer, New York, 1988).
[42] D. A. Cox, arXiv:alg-geom/9210008.
[43] K. Hori *et al.*, *Mirror Symmetry* (American Mathematical Society, Providence, 2003), p. 929.
[44] V. Bouchard, arXiv:hep-th/0702063.
[45] D. Lust, S. Reffert, E. Scheidegger, and S. Stieberger, arXiv:hep-th/0609014.
[46] S. Reffert, arXiv:0706.1310.
[47] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (John Wiley & Sons, New York, 1978).
[48] D. Huybrechts, *Complex Geometry* (Springer-Verlag, Berlin, 2004).
[49] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology* (Springer-Verlag, Berlin, 1982).
[50] G. Honecker and M. Trapletti, J. High Energy Phys. 01 (2007) 051.
[51] S. Groot Nibbelink, M. Hillenbach, T. Kobayashi, and M. G. A. Walter, Phys. Rev. D **69**, 046001 (2004).
[52] S. Groot Nibbelink, J. High Energy Phys. 07 (2003) 011.
[53] K.-S. Choi, S. Groot Nibbelink, and M. Trapletti, J. High Energy Phys. 12 (2004) 063.
[54] H. P. Nilles, S. Ramos-Sanchez, P. K. S. Vaudrevange, and A. Wingerter, J. High Energy Phys. 04 (2006) 050.
[55] J. J. Atick, L. J. Dixon, and A. Sen, Nucl. Phys. **B292**, 109 (1987).
[56] M. Dine, I. Ichinose, and N. Seiberg, Nucl. Phys. **B293**, 253 (1987).