Twisting all the way: From classical mechanics to quantum fields

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We discuss the effects that a noncommutative geometry induced by a Drinfeld twist has on physical theories. We systematically deform all products and symmetries of the theory. We discuss noncommutative classical mechanics, in particular its deformed Poisson bracket and hence time evolution and symmetries. The twisting is then extended to classical fields, and then to the main interest of this work: quantum fields. This leads to a geometric formulation of quantization on noncommutative space-time, i.e., we establish a noncommutative correspondence principle from $\star$-Poisson brackets to $\star$ commutators. In particular commutation relations among creation and annihilation operators are deduced.

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I. INTRODUCTION

One of the most interesting and promising fields of research in theoretical physics is the issue of space-time structure in extremal energy regimes. There are evidences from general relativity, string theory, and black hole physics which support the hypothesis of a noncommutative structure. The simplest and probably most suggestive argument which points at a failure of the classical space-time picture at high energy scales comes from the attempt of conjugating the principles of quantum mechanics with those of general relativity (see [1] and for a review [2]). If one tries to locate an event with a spatial accuracy comparable with the Planck length, space-time uncertainty relations necessarily emerge. In total analogy with quantum mechanics, uncertainty relations are naturally implied by the presence of noncommuting coordinates,

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\Theta^{\mu\nu}, \tag{1.1}$$

where $\Theta^{\mu\nu}$ is in general coordinate dependent and its specific form qualifies the kind of noncommutativity. Therefore, below Planck length the usual description of space-time as a pseudo-Riemannian manifold locally modeled on Minkowski space is not adequate anymore, and it has been proposed that it be described by a *noncommutative geometry* [3–5]. This line of thought has been pursued since the early days of quantum mechanics [6] and more recently in [7–19] (see also the recent review [20]).

In this context two relevant issues are the formulation of general relativity and the quantization of field theories on noncommutative space-time. There are different proposals for this second issue, and different canonical commutation relations have been considered in the literature [21–29]. We here frame this issue in a geometric context and address it by further developing the twist techniques used in [16–18] in order to formulate a noncommutative gravity theory. We see how noncommutative space-time induces a noncommutative phase space geometry, equipped with a deformed Poisson bracket. This leads to canonical quantization of fields on noncommutative space.

We work in the deformation quantization context; noncommutativity is obtained by introducing a $\star$ product on the algebra of smooth functions on space-time. The most widely studied form of noncommutativity is the one for which the quantity $\Theta^{\mu\nu}$ of (1.1) is a constant. This noncommutativity is obtained through the Grönewold-Moyal-Weyl $\star$ product (for a review see [30]). The product between functions (fields) is given by

$$(f \star h)(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right)f(x)h(y)\Big|_{x=y}, \tag{1.2}$$

with the $\theta^{\mu\nu}$ matrix constant and antisymmetric. In particular the coordinates satisfy the relations

$$x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}. \tag{1.3}$$

There are two approaches to study the symmetries (e.g., Poincaré symmetry) of this noncommutative space. One can consider $\theta^{\mu\nu}$ as a covariant tensor (see for example [31,32]), then the Moyal product is fully covariant under Poincaré (indeed linear affine) transformations. Poincaré symmetry is spontaneously broken by the nonzero values $\theta^{\mu\nu}$. The other approach is to consider the matrix components $\theta^{\mu\nu}$ as fundamental physical constants, like $\hbar$ or c. Since the commutator $x^\mu \star x^\nu - x^\nu \star x^\mu$ in (1.3) is not Lorentz invariant, the usual notion of Poincaré symmetry is lost. However there is still a symmetry, due to a twisted Poincaré group [33–36], a quantum Poincaré Lie algebra and Lie group invariance that implies that fields on noncommutative space are organized according to the same particle representations as in commutative space.

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We adopt this second approach and we consider the quantum Lie algebras of vector fields on noncommutative space-time, and of vector fields on the noncommutative phase spaces associated to this space-time, the quantum Lie algebra of symplectic transformations, and that of the constants of motion of a given Hamiltonian system. These noncommutative spaces and symmetries are obtained by deforming the usual ones via a Drinfeld twist [37]. For example the Drinfeld twist that implements the Moyal-Weyl noncommutativity (1.2) is $\mathcal{F} = e^{-(i/2)\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$.

In Sec. II we introduce the twist $\mathcal{F} = e^{-(i/2)\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$ and, starting from the principle that every product, and in general every bilinear map, is consistently deformed by composing it with the appropriate realization of the twist $\mathcal{F}$, we briefly review the construction of noncommutative space-time differential geometry as in [16–18]. Vector fields have a natural $\star$ action on the noncommutative algebras of functions and tensor fields, giving rise to the concept of deformed derivations. These $\star$ derivations form a quantum Lie algebra. In this way we consider the $\star$-Lie algebra of infinitesimal diffeomorphisms.

In Sec. III we study Hamiltonian mechanics on noncommutative space. The differential geometry of phase space is naturally induced from that of space-time (see Sec. II). The twist gives a noncommutative algebra of observables and here too we have the $\star$-Lie algebra of vector fields. A $\star$-Poisson bracket is introduced so that the $\star$ algebra of observables becomes a $\star$-Lie algebra. It can be seen as the $\star$-Lie subalgebra of Hamiltonian vector fields (canonical transformations). Time evolution is discussed. In particular, constants of motion of translation invariant Hamiltonians generate symmetry transformations, they close a $\star$-Lie symmetry algebra. Moreover in Sec. III B we formulate the general consistency condition between twists and $\star$-Poisson brackets (later applied in Sec. IV). In subsection we study the deformed symmetries of the harmonic oscillator, as well as a deformed harmonic oscillator that conserves usual angular momentum.

In Sec. IV we generalize the twist setting to the case of an infinite number of degrees of freedom. We lift the action of the twist from functions on space-time to functionals and study their $\star$ product (in particular a well-defined definition of $\Phi(x) \star \Phi(y)$ and $a(k) \star a(k')$ is given). We study the algebra of observables (functionals on phase space), and field theory in the Hamiltonian formalism. Our inspiring principle is that, having a precise notion of $\star$ derivation and of $\star$-Lie algebra, as in the point mechanics case, we are able to define a $\star$-Poisson bracket for functionals which is unambiguous and which gives the $\star$ algebra of observables a $\star$-Lie algebra structure. In particular we obtain the $\star$-Poisson bracket between canonically conjugated fields.

In Sec. V we similarly deform the algebra of quantum observables by lifting the action of the twist to operator valued functionals on space-time. We thus obtain a deformed $\hbar$ noncommutativity for operator valued functionals, which is in general nontrivial. Starting from the usual canonical quantization map for field theories on commutative space-time, $\Phi \xrightarrow{\hbar} \hat{\Phi}$, we uniquely obtain a quantization scheme for field theories on noncommutative space-time and show that it satisfies a correspondence principle between $\star$-Poisson brackets and $\star$ commutators. Finally in order to compare our results with the existing literature [21–29] we specialize them to the algebra of creation and annihilation operators of noncommutative quantum field theory.

Throughout this paper we consider just space noncommutativity, this restriction is in order to have a simple presentation of the Hamiltonian formalism.

II. TWIST

In this section we introduce the concept of twist and develop some of the noncommutative geometry associated to it. For the sake of simplicity we start and concentrate on the twist which gives rise to the Moyal $\star$ product (1.2), so that we deform the algebra of smooth functions $C^\infty(\mathbb{R}^d)$ on space (or space-time) $\mathbb{R}^d$. However the results presented hold for a general smooth manifold and a general twist $\mathcal{F}$ [17]. Only formulas with explicit tensor indices $\mu, \nu \ldots$ in the frame ∂_μ hold exclusively for the Moyal twist. Comments on the case of a general twist are inserted in the appropriate places throughout the paper.

The Moyal $\star$ product (1.2) between functions can be obtained from the usual pointwise product $(fg)(x) = f(x)g(x)$ via the action of a twist operator $\mathcal{F}$

$$f \star g := \mu \circ \mathcal{F}^{-1}(f \otimes g), \tag{2.1}$$

where μ is the usual pointwise product between functions, $\mu(f \otimes g) = fg$, and the twist operator and its inverse are

$$\mathcal{F} = e^{-(i/2)\theta^{\mu\nu}(\partial/\partial x^\mu)\otimes(\partial/\partial x^\nu)}, \quad \mathcal{F}^{-1} = e^{(i/2)\theta^{\mu\nu}(\partial/\partial x^\mu)\otimes(\partial/\partial x^\nu)}; \tag{2.2}$$

here $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial x^\nu}$ are globally defined vector fields on $\mathbb{R}^d$ (infinitesimal translations). Given the Lie algebra Ξ of vector fields with the usual Lie bracket

$$[u, v] := (u^\mu \partial_\mu v^\nu)\partial_\nu - (v^\nu \partial_\nu u^\mu)\partial_\mu, \tag{2.3}$$

and its universal enveloping algebra $U\Xi$, the twist $\mathcal{F}$ is an element of $U\Xi \otimes U\Xi$. The elements of $U\Xi$ are sums of products of vector fields, with the identification $uv - vu = [u, v]$.

We shall frequently write (sum over α understood)

$$\mathcal{F} = \mathsf{f}^\alpha \otimes \mathsf{f}_\alpha, \qquad \mathcal{F}^{-1} = \bar{\mathsf{f}}^\alpha \otimes \bar{\mathsf{f}}_\alpha, \tag{2.4}$$

so that

$$f \star g := \bar{\mathsf{f}}^\alpha(f)\bar{\mathsf{f}}_\alpha(g). \tag{2.5}$$

Explicitly we have

$$\mathcal{F}^{-1} = e^{(i/2)\theta^{\mu\nu}(\partial/\partial x^\mu)\otimes(\partial/\partial x^\nu)}$$
$$= \sum \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}\otimes\partial_{\nu_1}\ldots\partial_{\nu_n}$$
$$= \bar{\mathsf{f}}^\alpha \otimes \bar{\mathsf{f}}_\alpha, \tag{2.6}$$

so that α is a multi-index. We also introduce the universal $\mathcal{R}$ matrix

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}, \tag{2.7}$$

where by definition $\mathcal{F}_{21} = \mathsf{f}_\alpha \otimes \mathsf{f}^\alpha$. In the sequel we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha, \qquad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha. \tag{2.8}$$

In the present case we simply have $\mathcal{R} = \mathcal{F}^{-2}$, but for more general twists this is no more the case. The $\mathcal{R}$ matrix measures the noncommutativity of the $\star$ product. Indeed it is easy to see that

$$h \star g = \bar{R}^\alpha(g) \star \bar{R}_\alpha(h). \tag{2.9}$$

The permutation group in noncommutative space is naturally represented by $\mathcal{R}$. Formula (2.9) says that the $\star$ product is $\mathcal{R}$ commutative in the sense that if we permute (exchange) two functions using the $\mathcal{R}$-matrix action then the result does not change.

Note 1: The class of $\star$ products that can be obtained from a twist $\mathcal{F}$ is quite rich (for example we can obtain star products that give the commutation relations $x \star y = qy \star x$ with $q \in \mathbb{C}$ in two or more dimensions). Moreover we can consider twists and $\star$ products on arbitrary manifolds not just on $\mathbb{R}^d$. For example, given a set of mutually commuting vector fields $\{X_a\}$ $(a = 1, 2, \ldots n)$ on a d-dimensional manifold M, we can consider the twist

$$\mathcal{F} = e^{-(i/2)\theta^{ab}X_a\otimes X_b}. \tag{2.10}$$

Another example is $\mathcal{F} = e^{(1/2)H\otimes\ln(1+\lambda E)}$ where the vector fields H and E satisfy $[H, E] = 2E$. In these cases too, the $\star$ product defined via (2.1) is associative and properly normalized. In general an element $\mathcal{F}$ of $U\Xi \otimes U\Xi$ is a twist if it is invertible, satisfies a cocycle condition, and is properly normalized [37] (see [17,18] for a short introduction; see also the book [38]). The cocycle and the normalization conditions imply associativity of the $\star$ product and the normalization $h \star 1 = 1 \star h = h$.

A. Vector fields and tensor fields

We now use the twist to deform the space-time commutative geometry into a noncommutative one. The guiding principle is the one used to deform the product of functions into the $\star$ product of functions. Every time we have a bilinear map

$$\mu: X \times Y \to Z, \tag{2.11}$$

where X, Y, Z are vector spaces, and where there is an action of $\mathcal{F}^{-1}$ on X and Y we can combine this map with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ

$$\mu_\star := \mu \circ \mathcal{F}^{-1}, \tag{2.12}$$

$$\mu_\star: X \times Y \to Z \qquad (\mathsf{x}, \mathsf{y}) \mapsto \mu_\star(\mathsf{x}, \mathsf{y}) = \mu(\bar{\mathsf{f}}^\alpha(\mathsf{x}), \bar{\mathsf{f}}_\alpha(\mathsf{y})).$$

The $\star$ product on the space of functions is recovered setting $X = Y = \mathcal{A} = \mathrm{Fun}(M)$. We now study the case of vector fields, 1-forms, and tensor fields.

Vector fields $\Xi_\star$. We deform the product $\mu: \mathcal{A} \otimes \Xi \to \Xi$ between the space $\mathcal{A} = \mathrm{Fun}(M)$ of functions on space-time M and vector fields. A generic vector field is $v = v^\nu\partial_\nu$. Partial derivatives act on vector fields via the Lie derivative action

$$\partial_\mu(v) = [\partial_\mu, v] = \partial_\mu(v^\nu)\partial_\nu. \tag{2.13}$$

According to (2.12) the product $\mu: \mathcal{A} \otimes \Xi \to \Xi$ is deformed into the product

$$h \star v = \bar{\mathsf{f}}^\alpha(h)\bar{\mathsf{f}}_\alpha(v). \tag{2.14}$$

Since $\mathcal{F}^{-1} = e^{(i/2)\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$, iterated use of (2.13) (e.g., $\partial_\rho\partial_\mu(v) = \partial_\rho(\partial_\mu(v)) = [\partial_\rho, [\partial_\mu, v]]$), gives

$$h \star v = \bar{\mathsf{f}}^\alpha(h)\bar{\mathsf{f}}_\alpha(v) = \bar{\mathsf{f}}^\alpha(h)\bar{\mathsf{f}}_\alpha(v^\nu)\partial_\nu = (h \star v^\nu)\partial_\nu. \tag{2.15}$$

In particular we have

$$v^\mu \star \partial_\mu = v^\mu\partial_\mu. \tag{2.16}$$

From (2.15) it is easy to see that $h \star (g \star v) = (h \star g) \star v$, i.e., that the $\star$ multiplication between functions and vector fields is consistent with the $\star$ product of functions. We denote the space of vector fields with this $\star$ multiplication by $\Xi_\star$. As vector spaces $\Xi = \Xi_\star$, but Ξ is an $\mathcal{A}$ module while $\Xi_\star$ is an $\mathcal{A}_\star$ module.

1-forms $\Omega_\star$. Analogously, we deform the product $\mu: \mathcal{A} \otimes \Omega \to \Omega$ between the space $\mathcal{A} = \mathrm{Fun}(M)$ of functions on space-time M and 1-forms. A generic 1-form is $\rho = \rho_\nu dx^\nu$. As for vector fields we have

$$h \star \rho = \bar{\mathsf{f}}^\alpha(h)\bar{\mathsf{f}}_\alpha(\rho). \tag{2.17}$$

The action of $\bar{\mathsf{f}}_\alpha$ on forms is given by iterating the Lie derivative action of the vector field ∂_μ on forms. Explicitly, if $\rho = \rho_\nu dx^\nu$ we have

$$\partial_\mu(\rho) = \partial_\mu(\rho_\nu)dx^\nu \tag{2.18}$$

and

$$\rho = \rho_\nu dx^\nu = \rho_\nu \star dx^\nu. \tag{2.19}$$

Forms can be multiplied by functions from the left or from the right (they are a $\mathcal{A}$ bimodule). If we deform the multiplication from the right we obtain the new product

$$\rho \star h = \bar{\mathsf{f}}^\alpha(\rho)\bar{\mathsf{f}}_\alpha(h), \tag{2.20}$$

and we move h to the left with the help of the $\mathcal{R}$ matrix,

$$\rho \star h = \bar{R}^{\alpha}(h) \star \bar{R}_{\alpha}(\rho). \tag{2.21}$$

Tensor fields $\mathcal{T}_\star$. Tensor fields form an algebra with the tensor product $\otimes$ (over the algebra of functions). We define $\mathcal{T}_\star$ to be the noncommutative algebra of tensor fields. As vector spaces $\mathcal{T} = \mathcal{T}_\star$; the noncommutative and associative tensor product is obtained by applying (2.12)

$$\tau \otimes_\star \tau' := \bar{\mathrm{f}}^{\alpha}(\tau) \otimes \bar{\mathrm{f}}_{\alpha}(\tau'). \tag{2.22}$$

Here again the action of the twist on tensors is via the Lie derivative; on vectors we have seen that it is obtained by iterating (2.13), on 1-forms it is similarly obtained by iterating $\partial_\mu(h \star dg) = \partial_\mu(h) \star dg + h \star d\partial_\mu(g)$. Use of the Leibniz rule gives the action of the Lie derivative on a generic tensor.

If we consider the local coordinate expression of two tensor fields, for example of the type

$$\begin{aligned} \tau &= \tau^{\mu_1,\ldots\mu_m}\partial_{\mu_1} \otimes_\star \ldots \otimes_\star \partial_{\mu_m} \\ \tau' &= \tau'^{\nu_1,\ldots\nu_n}\partial_{\nu_1} \otimes_\star \ldots \otimes_\star \partial_{\nu_n} \end{aligned} \tag{2.23}$$

then their $\star$-tensor product is

$$\begin{aligned} \tau \otimes_\star \tau' = {}& \tau^{\mu_1,\ldots\mu_m} \star \tau'^{\nu_1,\ldots\nu_n}\partial_{\mu_1} \otimes_\star \ldots \otimes_\star \partial_{\mu_m} \otimes_\star \partial_{\nu_1} \\ & \otimes_\star \ldots \otimes_\star \partial_{\nu_n}. \end{aligned} \tag{2.24}$$

Notice that since the action of the twist $\mathcal{F}$ on the partial derivatives ∂_μ is the trivial one, we have

$$\partial_{\mu_1} \otimes_\star \ldots \partial_{\mu_n} = \partial_{\mu_1} \otimes \ldots \partial_{\mu_n}. \tag{2.25}$$

There is a natural action of the permutation group on undeformed arbitrary tensor fields

$$\tau \otimes \tau' \xrightarrow{\sigma} \tau' \otimes \tau. \tag{2.26}$$

In the deformed case it is the $\mathcal{R}$ matrix that provides a representation of the permutation group on $\star$-tensor fields

$$\tau \otimes_\star \tau' \xrightarrow{\sigma_{\mathcal{R}}} \bar{R}^{\alpha}(\tau') \otimes_\star \bar{R}_{\alpha}(\tau). \tag{2.27}$$

It is easy to check that, consistently with $\sigma_{\mathcal{R}}$ being a representation of the permutation group, we have $(\sigma_{\mathcal{R}})^2 = id$.

Consider now an antisymmetric 2-vector

$$\begin{aligned} \Lambda &= \frac{1}{2}\Lambda^{ij}(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) \\ &= \frac{1}{2}\Lambda^{ij} \star (\partial_i \otimes_\star \partial_j - \partial_j \otimes_\star \partial_i). \end{aligned} \tag{2.28}$$

Since the action of the $\mathcal{R}$ matrix on the partial derivatives ∂_μ is the trivial one, we have that Λ is both an antisymmetric 2-vector and a $\star$-antisymmetric one.

B. $\star$-Lie algebra of vector fields

The $\star$-Lie derivative on the algebra of functions $\mathcal{A}_\star$ is obtained following the general prescription (2.12). We combine the usual Lie derivative on functions $\mathcal{L}_u h = u(h)$ with the twist $\mathcal{F}$

$$\mathcal{L}^\star_u(h) := \bar{\mathrm{f}}^{\alpha}(u)(\bar{\mathrm{f}}_{\alpha}(h)). \tag{2.29}$$

By recalling that every vector field can be written as $u = u^\mu \star \partial_\mu = u^\mu \partial_\mu$ we have

$$\begin{aligned} \mathcal{L}^\star_u(h) &= \bar{\mathrm{f}}^{\alpha}(u^\mu\partial_\mu)(\bar{\mathrm{f}}_{\alpha}(h)) = \bar{\mathrm{f}}^{\alpha}(u^\mu)\partial_\mu(\bar{\mathrm{f}}_{\alpha}(h)) \\ &= u^\mu \star \partial_\mu(h), \end{aligned} \tag{2.30}$$

where in the second equality we have considered the explicit expression (2.6) of $\bar{\mathrm{f}}^{\alpha}$ in terms of partial derivatives, and we have iteratively used the property $[\partial_\nu, u^\mu\partial_\mu] = \partial_\nu(u^\mu)\partial_\mu$. In the last equality we have used that the partial derivatives contained in $\bar{\mathrm{f}}_\alpha$ commute with the partial derivative ∂_μ.

The differential operator $\mathcal{L}^\star_u$ satisfies the deformed Leibniz rule

$$\mathcal{L}^\star_u(h \star g) = \mathcal{L}^\star_u(h) \star g + \bar{R}^{\alpha}(h) \star \mathcal{L}^\star_{\bar{R}_\alpha(u)}(g). \tag{2.31}$$

This deformed Leibniz rule is intuitive: in the second addend we have exchanged the order of u and h, and this is achieved by the action of the $\mathcal{R}$ matrix, that, as observed, provides a representation of the permutation group.

The Leibniz rule is consistent (and actually follows) from the coproduct rule

$$u \mapsto \Delta_\star u = u \otimes 1 + \bar{R}^{\alpha} \otimes \bar{R}_{\alpha}(u). \tag{2.32}$$

[This formula holds also for the twist (2.10). However in the most generic twist case the term $\bar{R}^\alpha$ has to be replaced with $\mathrm{f}^\beta(\bar{R}^\alpha)\mathrm{f}_\beta$ [17]].

In the commutative case the commutator of two vector fields is again a vector field, we have the Lie algebra of vector fields. In this $\star$-deformed case we have a similar situation. We first calculate

$$\begin{aligned} \mathcal{L}^\star_u\mathcal{L}^\star_v(h) &= \mathcal{L}^\star_u(\mathcal{L}^\star_v(h)) \\ &= u^\mu \star \partial_\mu(v^\nu) \star \partial_\nu(h) + u^\mu \star v^\nu \star \partial_\nu\partial_\mu(h). \end{aligned}$$

Then instead of considering the composition $\mathcal{L}^\star_v\mathcal{L}^\star_u$ we consider $\mathcal{L}^\star_{\bar{R}^\alpha(v)}\mathcal{L}^\star_{\bar{R}_\alpha(u)}$. Indeed the usual commutator is constructed permuting (transposing) the two vector fields, and we have just remarked that the action of the permutation group in the noncommutative case is obtained using the $\mathcal{R}$ matrix. We have

$$\begin{aligned} \mathcal{L}^\star_{\bar{R}^\alpha(v)}\mathcal{L}^\star_{\bar{R}_\alpha(u)}(h) = {}& \bar{R}^{\alpha}(v^\nu) \star \bar{R}_{\alpha}(\partial_\nu u^\mu) \star \partial_\mu h \\ & + \bar{R}^{\alpha}(v^\nu) \star \bar{R}_{\alpha}(u^\mu) \star \partial_\nu\partial_\mu h. \end{aligned}$$

In conclusion

$$\mathcal{L}^\star_u \mathcal{L}^\star_v - \mathcal{L}^\star_{\bar{R}^\alpha(v)} \mathcal{L}^\star_{\bar{R}_\alpha(u)} = \mathcal{L}^\star_{[u,v]_\star}, \tag{2.33}$$

where we have defined the new vector field

$$[u, v]_\star := (u^\mu \star \partial_\mu v^\nu)\partial_\nu - (\partial_\nu u^\mu \star v^\nu)\partial_\mu. \tag{2.34}$$

A more telling definition of the $\star$ bracket is

$$[u, v]_\star := [\bar{\mathrm{f}}^\alpha(u), \bar{\mathrm{f}}_\alpha(v)], \tag{2.35}$$

again as in (2.12) the deformed bracket is obtained from the undeformed one via composition with the twist

$$[,]_\star = [,] \circ \mathcal{F}^{-1}. \tag{2.36}$$

Therefore, in the presence of twisted noncommutativity, we replace the usual Lie algebra of vector fields, Ξ, with $\Xi_\star$, the algebra of vector fields equipped with the $\star$ bracket (2.35) or equivalently (2.36).

It is not difficult to see that the bracket $[,]_\star$: $\Xi_\star \times \Xi_\star \to \Xi_\star$ is a bilinear map and verifies the $\star$ antisymmetry and the $\star$-Jacoby identity

$$[u, v]_\star = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star, \tag{2.37}$$

$$[u, [v, z]_\star]_\star = [[u, v]_\star, z]_\star + [\bar{R}^\alpha(v), [\bar{R}_\alpha(u), z]_\star]_\star. \tag{2.38}$$

For example we have

$$\begin{aligned}[u, v]_\star &= [\bar{\mathrm{f}}^\beta(u), \bar{\mathrm{f}}_\beta(v)] = -[\bar{\mathrm{f}}_\beta(v), \bar{\mathrm{f}}^\beta(u)] \\ &= [\bar{\mathrm{f}}^\delta \mathrm{f}^\gamma \bar{\mathrm{f}}_\beta(v), \bar{\mathrm{f}}_\delta \mathrm{f}_\gamma \bar{\mathrm{f}}^\beta(u)] = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star,\end{aligned}$$

where in the third passage we inserted $1 \otimes 1$ in the form $\mathcal{F}^{-1}\mathcal{F}$.

We have constructed the deformed Lie algebra of vector fields $\Xi_\star$. As vector spaces $\Xi = \Xi_\star$, but $\Xi_\star$ is a $\star$-Lie algebra. We stress that a $\star$-Lie algebra is not a generic name for a deformation of a Lie algebra. Rather it is a quantum Lie algebra of a quantum (symmetry) group [39], (see [40] for a short introduction and further references). In this respect the deformed Leibniz rule (2.31), that states that only vector fields (or the identity) can act on the second argument g in $h \star g$ (no higher order differential operators are allowed on g) is of fundamental importance (for example it is a key ingredient for the definition of a covariant derivative along a generic vector field).

Usually in the literature concerning twisted symmetries the Hopf algebra $U\Xi^{\mathcal{F}}$ is considered. This has the same algebra structure as $U\Xi$ so that the Lie bracket is the undeformed one. Also the action of $U\Xi^{\mathcal{F}}$ on functions and tensors is the undeformed one (so that no $\star$-Lie derivative $\mathcal{L}^\star$ is introduced). It is the coproduct $\Delta^{\mathcal{F}}$ of $U\Xi^{\mathcal{F}}$ that is deformed: for all $\xi \in U\Xi$,

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}.$$

The $\star$-Lie algebra $\Xi_\star$ we have constructed gives rise to the universal enveloping algebra $U\Xi_\star$ of sums of products of vector fields, with the identification $u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u) = [u, v]_\star$ and coproduct (2.32) [17,18]. The Hopf (or symmetry) algebras $U\Xi^{\mathcal{F}}$ and $U\Xi_\star$ are isomorphic. Therefore to some extent it is a matter of taste which algebra one should use. We prefer $U\Xi_\star$ because $U\Xi_\star$ naturally arises from the general prescription (2.12): the product $u \star v$ in $U\Xi_\star$ is just $u \star v = \bar{\mathrm{f}}^\alpha(u)\bar{\mathrm{f}}_\alpha(v)$, and because it is in $U\Xi_\star$ (not in $U\Xi^{\mathcal{F}}$) that vector fields have the geometric meaning of *infinitesimal* generators, for example, the coproduct $\Delta_\star(t)$ is a minimal deformation of the usual coproduct $\Delta(t) = t \otimes 1 + 1 \otimes t$. Also, from (2.30), we have the $\mathcal{A}_\star$ linearity property $\mathcal{L}^\star_{f\star u} h = f \star \mathcal{L}^\star_u h$.

III. CLASSICAL MECHANICS

In this section we apply the program we outlined to classical mechanics, thus building a $\star$-*classical mechanics*. A main motivation is the construction of a deformed Poisson bracket and the study of its geometry. The Poisson bracket will be generalized to field theory in the next section.

In subsection III A we briefly review the geometry of usual phase space, then we lift the action of the twist $\mathcal{F}$ from space-time to phase space. The structures introduced in Sec. II immediately give the differential geometry on noncommutative phase space. The deformation of the standard Poisson bracket on $\mathbb{R}^{2n}$ and the $\star$-Lie algebra of Hamiltonian vector fields are then studied. The general case of an arbitrary Poisson bracket deformed by an arbitrary twist $\mathcal{F}$ is considered in subsection III B, there we see that a compatibility requirement between the twist $\mathcal{F}$ and the Poisson bracket emerges.

In subsection III C we study Hamiltonian dynamics. The constants of motion of translation invariant Hamiltonians generate symmetry transformations and close a $\star$-Lie subalgebra under the $\star$-Poisson bracket. We also study the harmonic oscillator as an example of noncommutative Hamiltonian dynamics that is not translation invariant.

A. $\star$-Poisson bracket

In the Hamiltonian approach the dynamics of a classical finite-dimensional mechanical system is defined through a Poisson (usually symplectic) structure on phase space and the choice of a Hamiltonian function. The Poisson structure is a bilinear map

$$\{,\}: \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \tag{3.1}$$

where $\mathcal{A}$ is the algebra of smooth functions on phase space. It satisfies

$$\{f, g\} = -\{g, f\} \qquad \text{antisymmetry} \tag{3.2}$$

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \qquad \text{Jacobi identity} \tag{3.3}$$

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad \text{Leibniz rule.} \tag{3.4}$$

The first two properties show that the Poisson bracket $\{,\}$ is a Lie bracket. The last property shows that the map $\{f,\}$: $\mathcal{A} \to \mathcal{A}$ is a derivation of the algebra $\mathcal{A}$, it therefore defines a vector field

$$X_f := \{f, \}, \tag{3.5}$$

so that $\{f, g\} = X_f(g) = \langle X_f, dg\rangle$. X_f is the Hamiltonian vector field associated to the "Hamiltonian" f. We will also use the notation $\{f,\} = \mathcal{L}_{X_f}$ where $\mathcal{L}_{X_f}$ is the Lie derivative. The antisymmetry property shows that the vector field X_f actually depends on f only through its differential df, and we thus arrive at the Poisson bivector field Λ that maps 1-forms into vector fields according to

$$\langle \Lambda, df\rangle = X_f. \tag{3.6}$$

We therefore have

$$\langle \Lambda, df \otimes dg\rangle = X_f(g) = \{f, g\}. \tag{3.7}$$

Notice that we use the pairing $\langle u \otimes v, df \otimes dg\rangle = \langle v, df\rangle\langle u, dg\rangle$ (u and v vector fields) that is obtained by first contracting the innermost elements. We use this onionlike structure pairing because it naturally generalizes to the noncommutative case.

To be definite let us consider the canonical bracket on the phase space $T^*\mathbb{R}^n$ with the usual coordinates $x^1, \ldots x^n$, $p_1, \ldots p_n$,

$$\{f, g\} := \frac{\partial f}{\partial x^\ell}\frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell}\frac{\partial g}{\partial x^\ell}, \tag{3.8}$$

sum over repeated indices (which takes the values $1, \ldots n$) is assumed.

Because of the onionlike structure of the pairing and since $\langle \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_i}, df\rangle = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i}$, we have that the Poisson bivector field is

$$\Lambda = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i} = \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_i}, \tag{3.9}$$

while

$$X_f = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i}. \tag{3.10}$$

The symplectic form associated to the nondegenerate Poisson tensor Λ satisfies $\{f, h\} = \langle X_f \otimes X_h, \omega\rangle$ and explicitly reads

$$\omega = dp_i \wedge dx^i. \tag{3.11}$$

A Hamiltonian H is a function on phase space. Motion of a point in phase space describes the time evolution of the dynamical system. Infinitesimally it is given by the vector field X_H, and on the algebra $\mathcal{A}$ of observables (not explicitly dependent on time), we have Hamilton's equation

$$\dot{f} = -\{H, f\} = -X_H(f). \tag{3.12}$$

We denote with σ_t the integral flow of $-X_H$. If the system at time $t_0 = 0$ is described by the point P_0 in phase space, at a later time t has evolved to the point $P_t = \sigma_t(P_0)$. Correspondingly the time evolution of any observable is

$$\sigma_t^*(f) = f \circ \sigma_t, \tag{3.13}$$

where σ_t^* is the pullback of the integral flow. In particular the coordinates of the point P_t are $x^i(t) = x^i(\sigma_t(P_0))$ and $p_i(t) = p_i(\sigma_t(P_0))$. Hamilton's equation can be equivalently rewritten as an equation for the pull-back flow σ_t^*,

$$\frac{d}{dt}\sigma_t^* = -\sigma_t^* \circ X_H. \tag{3.14}$$

Now we twist commutative space-time into noncommutative space-time (actually we consider just noncommutative space coordinates, no time noncommutativity). Correspondingly the configuration space and the phase space of a mechanical system will be noncommutative. For example if space is $\mathbb{R}^3$ and we consider an unconstrained mechanical system of r points then the configuration space will be $\mathbb{R}^{3r}$. Noncommutativity on $\mathbb{R}^{3r}$ is induced from noncommutativity on $\mathbb{R}^3$. Recall that $\mathbb{R}^{3r}$ should be considered as r copies of $\mathbb{R}^3$, therefore a transformation on $\mathbb{R}^3$ induces a simultaneous transformation on all the r copies of $\mathbb{R}^{3r}$. Infinitesimally, if the transformation on $\mathbb{R}^3$ (with coordinates x^k, $k = 1, 2, 3$) is given by the vector field $\frac{\partial}{\partial x^i}$, then the corresponding infinitesimal transformation on $\mathbb{R}^{3r}$ is given by the vector field

$$\frac{\partial}{\partial x_1^i} + \frac{\partial}{\partial x_2^i} \ldots + \frac{\partial}{\partial x_r^i} \tag{3.15}$$

(with $x_1^k, x_2^k, \ldots x_r^k$ coordinates of $\mathbb{R}^{3r}$). We therefore have the following lift of the action of the twist $\mathcal{F}$ from $C^\infty(\mathbb{R}^3)$ to $C^\infty(\mathbb{R}^{3r}) \otimes C^\infty(\mathbb{R}^{3r})$,

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha = e^{-(\mathrm{i}/2)\theta^{ij}(\partial/\partial x_1^i + \ldots \partial/\partial x_r^i)\otimes(\partial/\partial x_1^j + \ldots \partial/\partial x_r^j)}, \tag{3.16}$$

and correspondingly the following $\star$ product on configuration space, for all $a, b \in C^\infty(\mathbb{R}^{3r})$,

$$a \star b(x_1, \ldots x_r) = \exp\Big(\frac{\mathrm{i}}{2}\theta^{ij}\Big(\frac{\partial}{\partial x_1^i} + \ldots \frac{\partial}{\partial x_r^i}\Big)\Big(\frac{\partial}{\partial y_1^j} + \ldots \frac{\partial}{\partial y_r^j}\Big)\Big) \times a(x_1, \ldots x_r)b(y_1, \ldots y_r)|_{x=y}. \tag{3.17}$$

On the subalgebra $C^\infty(\mathbb{R}^3) \otimes \ldots \otimes C^\infty(\mathbb{R}^3)$ (r times) of $C^\infty(\mathbb{R}^{3r})$ the $\star$ product (3.17) coincides with the one defined in [29].

We further lift the twist $\mathcal{F}$ to the tangent bundle $T\mathbb{R}^{3r}$ and to the phase space $T^*\mathbb{R}^{3r}$. A point of the manifold $T\mathbb{R}^{3r} \simeq \mathbb{R}^{6r}$ has coordinates (x^A, v^A), $(A = 1, \ldots, 3r)$ where v^A are the components of the vector $v = v^A\frac{\partial}{\partial x^A}$ tangent to the point of coordinates x^A. Under the translation generated by $(\frac{\partial}{\partial x_1^i} + \ldots \frac{\partial}{\partial x_r^i})$ we have that (x^A, v^A) is translated into (x'^A, v^A), where x'^A are the new coordinates of the translated point, while the coefficients v^A do not change because we are considering a constant translation. Therefore the action of $(\frac{\partial}{\partial x_1^i} + \ldots \frac{\partial}{\partial x_r^i})$, and of the twist $\mathcal{F}$, on the tangent bundle $T\mathbb{R}^{3r}$ is the usual one on the base space and the trivial one on the fibers. Similarly for the phase space $T^*\mathbb{R}^{3r}$. Let x^A, p_A be phase space coordinates, the explicit expression of $\mathcal{F}$ on $C^\infty(T^*\mathbb{R}^{3r}) \otimes C^\infty(T^*\mathbb{R}^{3r})$ is again (3.16). In particular $f \star h = fh$ if f or h is only a function of the momenta p_A.

Note 2: This result holds just because of the particular twist we have considered. In general the lift of a vector field $u = u^A \frac{\partial}{\partial x^A}$ from $\mathbb{R}^{3r}$ to $T\mathbb{R}^{3r}$ is given by $u_* = u^A \frac{\partial}{\partial x^A} + v^B \frac{\partial u^A}{\partial x^B} \frac{\partial}{\partial v^A}$ (here x^A, v^A are the coordinates of $T\mathbb{R}^{3r}$). Notice the linearity of u_* in the fiber coordinates v^A, indeed the lift u_* can be obtained from its flow $T\sigma_t^u$, that is linear on the fibers because it is a tangent flow, precisely the differential of the flow σ_t^u associated to the vector field u. Similarly the lift of u to the phase space $T^*\mathbb{R}^{3r}$ (with coordinates x^A, p_A), is given by the vector field

$$u^* = u^B \frac{\partial}{\partial x^B} - p_B \frac{\partial u^B}{\partial x^C} \frac{\partial}{\partial p_C}. \tag{3.18}$$

We have seen how noncommutativity of space-time induces noncommutativity of phase space. Let us consider a system with n degrees of freedom with phase space $M = \mathbb{R}^{2n}$, and $\mathcal{A}_\star = C^\infty(M)_\star$ the noncommutative algebra of functions on M with twist

$$\mathcal{F} = e^{-(i/2)\theta^{\ell s}(\partial/\partial x^\ell)\otimes(\partial/\partial x^s)} \qquad \ell, s = 1, \ldots n. \tag{3.19}$$

It can be easily checked that the Poisson bracket does not define a derivation of the algebra $\mathcal{A}_\star = C^\infty(M)_\star$,

$$\{f, g \star h\} \neq \{f, g\} \star h + g \star \{f, h\}, \tag{3.20}$$

or, in different words,

$$\mathcal{L}_{X_f}(g \star h) \neq (\mathcal{L}_{X_f} g) \star h + g \star (\mathcal{L}_{X_f} h). \tag{3.21}$$

On the other hand, according to (2.12), we are led to deform the Poisson structure into a noncommutative Poisson structure $\{,\}_\star$. We define the $\star$-Poisson bracket

$$\{f, g\}_\star := \{\bar{f}^\alpha(f), \bar{f}_\alpha(g)\}. \tag{3.22}$$

A simple calculation, that exploits the fact that the Poisson structure is invariant under the partial derivatives appearing in the twist, shows that this twisted Poisson bracket can be expressed as

$$\{f, g\}_\star = \frac{\partial f}{\partial x^\ell} \star \frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell} \star \frac{\partial g}{\partial x^\ell}. \tag{3.23}$$

This bracket is linear in both arguments, it is $\mathcal{R}$ antisymmetric and it satisfies the $\star$-Leibniz rule and $\star$-Jacobi identity

$$\{f, g\}_\star = -\{\bar{R}^\alpha(g), \bar{R}_\alpha(f)\}_\star \tag{3.24}$$

$$\{f, g \star h\}_\star = \{f, g\}_\star \star h + \bar{R}^\alpha(g) \star \{\bar{R}_\alpha(f), h\}_\star \tag{3.25}$$

$$\{f, \{g, h\}_\star\}_\star = \{\{f, g\}_\star, h\}_\star + \{\bar{R}^\alpha(g), \{\bar{R}_\alpha(f), h\}_\star\}_\star. \tag{3.26}$$

We conclude from (3.25) that $\{f, \}$ is a $\star$ derivation. We can write

$$\{f, \}_\star = \mathcal{L}_v^\star \tag{3.27}$$

for some vector field v. From (3.23) and the definition of $\star$-Lie derivative, we deduce that the vector field v is the undeformed Hamiltonian vector field $v = X_f = \{f, \}$, therefore we obtain

$$\{f, \}_\star = \mathcal{L}_{X_f}^\star = \mathcal{L}_{\{f,\}}^\star. \tag{3.28}$$

The Leibniz rule (3.25) can be rewritten as

$$\mathcal{L}_{X_f}^\star(g \star h) = \mathcal{L}_{X_f}^\star(g) \star h + \bar{R}^\alpha(g) \star \mathcal{L}_{X_{\bar{R}_\alpha(f)}}^\star(h) \tag{3.29}$$

and is consistent (and actually follows) from the coproduct rule

$$X_f \mapsto \Delta_\star X_f = X_f \otimes 1 + \bar{R}^\alpha \otimes X_{\bar{R}_{\alpha(f)}}. \tag{3.30}$$

Property (3.26), the $\star$-Jacobi identity, can be rewritten as

$$\mathcal{L}_{X_f}^\star \mathcal{L}_{X_g}^\star - \mathcal{L}_{\bar{R}^\alpha(X_g)}^\star \mathcal{L}_{\bar{R}_\alpha(X_f)}^\star = \mathcal{L}_{X_{\{f,g\}_\star}}^\star. \tag{3.31}$$

Recalling (2.33) we equivalently have

$$[X_f, X_g]_\star = X_{\{f,g\}_\star}. \tag{3.32}$$

Because of this property and of the Leibniz rule (3.29) [or better the coproduct rule (3.30)] Hamiltonian vector fields are a $\star$-Lie subalgebra of the $\star$-Lie algebra of vector fields.

B. General twist and Poisson bracket

These results, obtained in the case of the θ-constant twist (3.16) or (3.19) on $M = \mathbb{R}^{2n}$, can be generalized to a twist $\mathcal{F}$ on an arbitrary Poisson manifold M (phase space). We comment on this general case because it is in this context that the compatibility relation between twist and Poisson structure most clearly emerges. The twist deforms the algebra of functions on M into the $\star$ algebra $\mathcal{A}_\star = C_\star^\infty(M)$, where $f \star g = \bar{f}^\alpha(f)\bar{f}_\alpha(g)$. According to the general principles we have set in Sec. II, first we define the $\star$ pairing between vector fields and 1-forms

$$\langle u, \vartheta\rangle_\star := \langle\bar{f}^\alpha(u), \bar{f}_\alpha(\vartheta)\rangle. \tag{3.33}$$

It can be proven that this pairing has the $\mathcal{A}_\star$-linearity properties

$$\langle f \star u, \vartheta \star h\rangle_\star = f \star \langle u, \vartheta\rangle_\star \star h, \tag{3.34}$$

(where $\vartheta \star h := \bar{f}^\alpha(\vartheta)\bar{f}_\alpha(h)$) and

$$\langle u, f\star\vartheta\rangle_\star = \bar{R}^\alpha(f)\star\langle\bar{R}_\alpha(u), \vartheta\rangle_\star. \tag{3.35}$$

We extend the pairing to covariant tensors, τ, and contravariant ones, ρ, via the definition

$$\langle\tau, \rho\rangle_\star := \langle\bar{\mathrm{f}}^\alpha(\tau), \bar{\mathrm{f}}_\alpha(\rho)\rangle. \tag{3.36}$$

It can be shown that this definition, and the onionlike structure of the undeformed pairing [cf. after (3.7)], imply the property

$$\langle u\otimes_\star v, \vartheta\otimes_\star\eta\rangle_\star := \langle u, \langle v, \vartheta\rangle_\star\star\eta\rangle_\star \tag{3.37}$$

(where η is a 1-form). This equation gives an equivalent definition of the pairing between covariant and contravariant 2-tensors. From (3.37) it follows that the $\mathcal{A}_\star$-linearity properties are preserved:

$$\begin{aligned}\langle f\star u\otimes_\star v, \vartheta\otimes_\star\rho\star h\rangle_\star &= f\star\langle u\otimes_\star v, \vartheta\otimes_\star\rho\rangle_\star\star h\\ \langle u\otimes_\star v, f\star\vartheta\otimes_\star\rho\rangle_\star &= \bar{R}^\alpha(f)\\ &\quad\star\langle\bar{R}_\alpha(u\otimes_\star v), \vartheta\otimes\star\rho\rangle_\star.\end{aligned} \tag{3.38}$$

Finally, following (2.12), we define the $\star$-Poisson bracket as

$$\{f, g\}_\star := \langle\Lambda, df\otimes_\star dg\rangle_\star. \tag{3.39}$$

Using the fact that Λ is $\star$ antisymmetric the $\star$-antisymmetry property (3.24) can be proven. However from the definition (3.39) it follows that

$$\begin{aligned}\{f, g\star h\}_\star &= \{f, g\}_\star\star h\\ &\quad+\bar{R}^\alpha\bar{R}^\beta(g)\star\langle\bar{R}_\alpha(\Lambda), d\bar{R}_\beta(f)\otimes_\star dh\rangle_\star.\end{aligned} \tag{3.40}$$

This equality becomes the deformed Leibniz rule (3.25) if

$$\bar{R}^\alpha\otimes\bar{R}_\alpha(\Lambda) = 1\otimes\Lambda \tag{3.41}$$

(recall that 1 and $\bar{R}^\alpha$ are elements in $U\Xi$). This is a compatibility relation between the Poisson structure and the twist.

Led by this observation we require, as compatibility condition, that the action of the twist $\mathcal{F}$ on the Poisson tensor Λ be the trivial one,

$$\bar{\mathrm{f}}^\alpha\otimes\bar{\mathrm{f}}_\alpha(\Lambda) = 1\otimes\Lambda, \tag{3.42}$$

$$\bar{\mathrm{f}}^\alpha(\Lambda)\otimes\bar{\mathrm{f}}_\alpha = \Lambda\otimes 1. \tag{3.43}$$

Any two of the last three equations imply the third one. If we consider a twist of the form $\mathcal{F} = e^{-(\mathrm{i}/2)\theta^{ab}X_a\otimes X_b}$, where the X_a's are arbitrary commuting vector fields (and θ^{ab} is antisymmetric), then these three equations are equivalent. They are satisfied if (and when θ^{ab} is nondegenerate only if) the vector fields X_a leave invariant the Poisson structure (in particular this happens if they are Hamiltonian vector fields). The semiclassical limit of Eqs. (3.42) and (3.43) implies that the Poisson structure P associated with the twist $\mathcal{F}$ is compatible with the Poisson structure Λ on the manifold M. Explicitly $[P, \Lambda] = 0$, where $[,]$ is the Schouten-Nijenhuis bracket.

Condition (3.42) implies that

$$\{f, g\}_\star = \{\bar{\mathrm{f}}^\alpha(f), \bar{\mathrm{f}}_\alpha(g)\}, \tag{3.44}$$

and that Hamiltonian vector fields are undeformed,

$$X_f^\star := \langle\Lambda, df\rangle_\star = \langle\Lambda, df\rangle = X_f. \tag{3.45}$$

It can be proven that conditions (3.42) and (3.43) imply the following compatibility between the twist and Hamiltonian vector fields

$$\bar{\mathrm{f}}^\alpha\otimes\bar{\mathrm{f}}_\alpha(X_h) = \bar{\mathrm{f}}^\alpha\otimes X_{\bar{\mathrm{f}}_\alpha(h)}, \tag{3.46}$$

$$\bar{\mathrm{f}}^\alpha(X_h)\otimes\bar{\mathrm{f}}_\alpha = X_{\bar{\mathrm{f}}^\alpha(h)}\otimes\bar{\mathrm{f}}_\alpha. \tag{3.47}$$

The $\star$-Jacoby identity that is equivalent to property (3.32), easily follows from these equations because of linearity

$$\begin{aligned}[X_f, X_g]_\star &= [\bar{\mathrm{f}}^\alpha(X_f), \bar{\mathrm{f}}_\alpha(X_g)] = [X_{\bar{\mathrm{f}}^\alpha(f)}, X_{\bar{\mathrm{f}}_\alpha(g)}]\\ &= X_{\{\bar{\mathrm{f}}^\alpha(f), \bar{\mathrm{f}}_\alpha(g)\}} = X_{\{f,g\}_\star}.\end{aligned} \tag{3.48}$$

Because of this property and of the Leibniz rule (3.29) [or better the coproduct rule (3.30)] we have that also for a general twist with a compatible Poisson bracket Hamiltonian vector fields are a $\star$-Lie subalgebra of the $\star$-Lie algebra of vector fields.

C. Time evolution and constants of motion

The study of the noncommutative phase space geometry is here applied to briefly discuss time evolution and symmetries in deformed mechanics. We consider point particles on space with usual Moyal-Weyl noncommutativity given by the θ-constant twist $\mathcal{F} = e^{-(\mathrm{i}/2)\theta^{ij}\partial_i\otimes\partial_j}$.

A natural definition of time evolution is

$$\dot{f} = -\mathcal{L}^\star_{X_H}f = -\{H, f\}_\star. \tag{3.49}$$

As noticed in (3.45), we see that the time evolution generator $X_H = \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i}$ is the same as the undeformed one; it is its action $\mathcal{L}^\star$ on functions that are deformed. Indeed in general $\{H, f\}_\star\neq\{H, f\}$ and therefore time evolution is different from the undeformed one. Equation (3.49) should be considered as an equation for the deformed pull-back flow $(\sigma_t^*)_\star$ [cf. (3.14)],

$$\frac{d}{dt}(\sigma_t^*)_\star = -(\sigma_t^*)_\star\circ\mathcal{L}^\star_{X_H}. \tag{3.50}$$

Equation (3.49), [or (3.50)] can be formally integrated

$$\begin{aligned}(\sigma_t^*)_\star f &= \exp(-t\mathcal{L}^\star_{X_H})f\\ &= f - t\mathcal{L}^\star_{X_H}f + \frac{1}{2}t^2\mathcal{L}^\star_{X_H}(\mathcal{L}^\star_{X_H}f) + \ldots.\end{aligned} \tag{3.51}$$

A more explicit expression of this formula is obtained if we denote by ξ^a the phase space coordinates x^i, p_j, and if we correspondingly expand the Hamiltonian vector field as $X_H = X_H^a \partial_a$, where $\partial_a = \frac{\partial}{\partial \xi^a}$. Then we have

$$(\sigma_t^*)_\star f = \exp(-t\mathcal{L}^\star_{X_H})f$$
$$= f - tX_H^a \star \partial_a f + \frac{1}{2}t^2 X_H^a \star \partial_a(X_H^a \star \partial_a f) + \ldots. \quad (3.52)$$

Another expression for $(\sigma_t^*)_\star$ is $(\sigma_t^*)_\star = \mathcal{L}^\star_{e_\star^{tX_H}}$ where the $\star$ exponential $e_\star^{tX_H}$ is obtained with the $\star$ product in $U\Xi_\star$, and $\mathcal{L}^\star$ represents $e_\star^{tX_H}$ as a differential operator on functions.

It is easy to verify the one parameter group property $(\sigma_t^*)_\star \circ (\sigma_s^*)_\star = (\sigma_{t+s}^*)_\star$. On the other hand the deformed Leibniz rule for $\mathcal{L}^\star_{X_H}$ implies $(\sigma_t^*)_\star(f \star g) \neq (\sigma_t^*)_\star f \star (\sigma_t^*)_\star g$, as well as

$$(\sigma_t^*)_\star f(x, p) \neq f(x(t), p(t)), \quad (3.53)$$

where $x^i(t) = (\sigma_t^*)_\star x^i$, $p_j(t) = (\sigma_t^*)_\star p_j$.

A constant of motion is a function Q on phase space that satisfies

$$\{H, Q\}_\star = 0. \quad (3.54)$$

If

$$\{Q, H\}_\star = 0 \quad (3.55)$$

we say that the Hamiltonian is invariant under the vector field X_Q (because $\{Q, H\}_\star = \mathcal{L}^\star_{X_Q} H$). Since the $\star$-Poisson bracket is not antisymmetric (3.54) and (3.55) are independent equations.

Notice that for translation invariant Hamiltonians the time evolution equation as well as the notion of constant of motion are undeformed. Then (3.54) and (3.55) coincide. Using the $\star$-Jacoby identity we have that the $\star$ bracket $\{Q, Q'\}_\star$ of two constants of motion is again a constant of motion. We conclude that the subspace of Hamiltonian vector fields X_Q that $\star$ commute with X_H form a $\star$-Lie subalgebra of the $\star$-Lie algebra of Hamiltonian vector fields: The $\star$-symmetry algebra of constants of motion.

Examples of translation invariant Hamiltonians include all point particle Hamiltonians whose potential depends only on the relative distance of the point particles involved. We also see that this formalism is quite well suited for field theory Hamiltonians that have potentials like $\int d^3x \bar{\phi}(x) \star \phi(x) \star \bar{\phi}(x) \star \phi(x)$ and are translation invariant.

1. Example: The harmonic oscillator

In this subsection we see our deformed point mechanics at work on a simple example that does not admit translation invariance.

We consider the harmonic oscillator in two noncommutative space dimensions. We study its equation of motion, the constants of motion and the invariances of the Hamiltonian. Angular momentum is not conserved, but a deformed version is. Vice versa, a deformation of this oscillator conserves usual angular momentum.

The results here presented are not used in the later sections on field theory.

Let

$$H = \frac{1}{2}(x^i \star x^j \delta_{ij} + p_i \star p_j \delta^{ij}) = \frac{1}{2}(x^i x^j \delta_{ij} + p_i p_j \delta^{ij}) \quad (3.56)$$

$$L = \varepsilon_i^j x^i \star p_j = \varepsilon_i^j x^i p_j \quad (3.57)$$

be the Hamiltonian and the angular momentum of the 2-dimensional harmonic oscillator.

Since

$$\{h, f\}_\star = \{h, f\} \quad (3.58)$$

if h and f are sums of functions that depend only on the coordinates x^i or the momenta p_j, we have the undeformed equations $\{H, H\}_\star = \{H, H\} = 0$ and

$$\dot{x}^i = -\{H, x^i\}_\star = -\{H, x^i\},$$
$$\dot{p}_j = -\{H, p_j\}_\star = -\{H, p_j\}. \quad (3.59)$$

On the other hand neither the angular momentum is a constant of motion

$$\dot{L} = -\{H, L\}_\star = -\mathcal{L}^\star_{X_H} L = -\frac{\mathrm{i}}{2}\varepsilon_{ij}\theta^{ij} = -\mathrm{i}\theta \quad (3.60)$$

(we have defined $\theta^{ij} = \theta\varepsilon^{ij}$), nor the Hamiltonian is rotation invariant, indeed we have $\mathcal{L}^*_{X_L} H = \{L, H\}_\star = -\mathrm{i}\theta$. From (3.52) the time evolution of the angular momentum is $(\sigma_t^*)_\star L = L - \mathrm{i}\theta t$.

We recall that the classical harmonic oscillator is a maximally superintegrable system, that is, it has $3(= 2d - 1)$ constants of motion which are functionally independent. For example we can consider

$$H, L, K = (x^1)^2 - (x^2)^2 + (p_1)^2 - (p_2)^2,$$
$$T = x^1 x^2 + p_1 p_2. \quad (3.61)$$

Only three of the above constants of motion are functionally independent. The third and fourth constants have the interesting property of being preserved in our twist-deformed setting. Indeed from (3.58) it immediately follows

$$\{H, K\}_\star = \{K, H\}_\star = 0, \qquad \{H, T\}_\star = \{T, H\}_\star = 0. \quad (3.62)$$

Therefore the $\star$-harmonic oscillator remains a superintegrable system, but loses rotational invariance.

Deformations $L_\star$ of the angular momentum L can however be constants of motion. For example we have the two functionally independent deformations

$$L'_\star = L - \mathrm{i}\theta \arctan\left(\frac{x^1}{p_1}\right), \qquad L''_\star = L - \mathrm{i}\theta \arctan\left(\frac{x^2}{p_2}\right) \tag{3.63}$$

that satisfy $\{H, L'_\star\}_\star = 0$, $\{H, L''_\star\}_\star = 0$. In order to prove this statement it is instructive to consider an arbitrary θ deformation of L,

$$L_\star = \sum_{n=0}^{\infty} \theta^n L_n, \tag{3.64}$$

where $L_0 = L$ and all coefficients L_n are θ-independent functions on phase space. We determine these coefficients by requiring $L_\star$ to be a constant of motion,

$$\{H, L_\star\}_\star = \sum_{n=0}^{\infty} \theta^n \{H, L_n\}_\star = 0. \tag{3.65}$$

Since for any function f on phase space we have

$$\{H, f\}_\star = \{H, f\} - \frac{\mathrm{i}}{2}\theta \varepsilon^i_j \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_j} f, \tag{3.66}$$

$L_\star$ is a constant of motion if

$$\sum_{n=0}^{\infty} \theta^n \{H, L_n\} - \frac{\mathrm{i}}{2}\theta^{n+1}\varepsilon^i_j \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_j} L_n = 0. \tag{3.67}$$

All the coefficients in this θ expansion have to vanish and we then obtain the recursive relation

$$\{H, L_{n+1}\} = \frac{\mathrm{i}}{2}\varepsilon^i_j \frac{\partial}{\partial x^i}\frac{\partial}{\partial p_j} L_n \tag{3.68}$$

with the initial condition $L_0 = L$. At first order in θ we have $\{H, L_1\} = \mathrm{i}$, that is

$$\left(x^j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial x^j}\right) L_1 = \mathrm{i}. \tag{3.69}$$

Since the left-hand side preserves the degree of any homogeneous polynomial in the coordinates x^i and p_j, no analytic function on phase space can solve this equation. If we relax the analyticity condition we find two independent solutions

$$L'_1 = -\mathrm{i}\arctan\left(\frac{x^1}{p_1}\right), \tag{3.70}$$

$$L''_1 = -\mathrm{i}\arctan\left(\frac{x^2}{p_2}\right). \tag{3.71}$$

In order to solve (3.68) we can choose all higher order coefficients L_n with $n \geq 2$ to be zero. We thus obtain the two solutions (3.63). Notice that, unlike H and T, the constants of motion (3.63) do not $\star$ commute with themselves.

As an instance of our general comment on the independence of (3.54) and (3.55), that is to say, on the independence of the notions of constant of motion and invariance, we observe that the two constants of motion (3.63) do not generate symmetries of the Hamiltonian. It can be easily verified that solutions of (3.55) are given instead by the complex conjugates of (3.63).

We find also interesting to study deformations $H_\star$ of the harmonic oscillator Hamiltonian that admit the undeformed angular momentum L as constant of motion. The aim, like in [41], is to consider new dynamical systems that may be highly nontrivial if thought in commutative space [the equation of motion (3.49) or (3.59) can just be seen as a partial differential equation on commutative space-time], but that analyzed in the noncommutative Hamiltonian mechanics framework shows the same constants of motion, and possibly richness of symmetries and integrability, as the undeformed ones.

We therefore consider the power series

$$H_\star = \sum_{n=0}^{\infty} \theta^n H_n \tag{3.72}$$

with $H_0 = H$, and determine the coefficients H_n (that are functions on phase space) by requiring $\{H_\star, L\}_\star = \sum_{n=0}^{\infty} \theta^n \{H_n, L\}_\star = 0$. Since

$$\{f, L\}_\star = \{f, L\} - \frac{\mathrm{i}}{2}\theta(\partial_1^2 + \partial_2^2) f, \tag{3.73}$$

by setting $f = H_\star$ we obtain the recursion relation

$$\{H_{n+1}, L\} = \frac{\mathrm{i}}{2}(\partial_1^2 + \partial_2^2) H_n, \tag{3.74}$$

and in particular

$$\{H_1, L\} = \frac{\mathrm{i}}{2}(\partial_1^2 + \partial_2^2) H_0 = \mathrm{i}. \tag{3.75}$$

This yields a partial differential equation similar to (3.69)

$$\varepsilon^k_j \left(x^j \frac{\partial}{\partial x^k} - p_k \frac{\partial}{\partial p_j}\right) H_1 = \mathrm{i}. \tag{3.76}$$

As in the previous calculation since the operator on the left-hand side preserves the degree of a homogeneous polynomial in x^i and p_j, no analytic solution is possible. Comparison with (3.69) however gives the solutions,

$$H'_1 = -\mathrm{i}\arctan\left(\frac{p_1}{p_2}\right) \tag{3.77}$$

$$H''_1 = -\mathrm{i}\arctan\left(\frac{x^1}{x^2}\right). \tag{3.78}$$

Again it can be checked that all the subsequent equations in (3.74) are satisfied with the choice $H_i = 0$, $i \geq 2$, therefore we have two possible deformations of the Hamiltonian which admit the angular momentum as a constant of motion

$$H'_\star = H - \mathrm{i}\theta \arctan\left(\frac{p_1}{p_2}\right) \qquad H''_\star = H - \mathrm{i}\theta \arctan\left(\frac{x^1}{x^2}\right). \tag{3.79}$$

Notice however that $\{L, H'_\star\}_\star \neq 0$ and $\{L, H''_\star\}_\star \neq 0$, that is, (3.79) are not invariant under rotations. Rotational invariance is fulfilled if we consider the complex conjugates of (3.79).

It is interesting to note that, unlike the deformations of the angular momentum (3.63), both the deformations (3.79) $\star$ commute with themselves. The first Hamiltonian $H'_\star$ is nonlocal, while the second one is local. They are both real if we consider the parameter θ to be purely imaginary. We will not deepen their analysis here because it goes beyond the scope of the present article.

IV. CLASSICAL FIELD THEORY

We generalize the twist setting to the case of an infinite number of degrees of freedom. In this case the position and momenta generalize to the fields $\Phi(x)$ and $\Pi(x)$ with $x \in \mathbb{R}^d$ ($\mathbb{R}^{d+1}$ being space-time). The algebra A is an algebra of functionals, it is the algebra of functions on N where in turn N is the function space

$$N = \mathrm{Maps}(\mathbb{R}^d \to \mathbb{R}^2). \tag{4.1}$$

Here we are considering a scalar field theory, in a more general case $\mathbb{R}^2$ (with its coordinates Φ and Π) is substituted by the proper target space. The generalization to $\mathbb{R}^{2s}$ (with s scalar fields) is immediate. Particle mechanics with phase space $\mathbb{R}^{2d}$ is recovered by considering that $\mathbb{R}^d$ in (4.1) collapses to d points.

We define the Poisson bracket between the functionals $F, G \in \mathsf{A}$ to be

$$\{F, G\} = \int d^d x \frac{\delta F}{\delta \Phi}\frac{\delta G}{\delta \Pi} - \frac{\delta F}{\delta \Pi}\frac{\delta G}{\delta \Phi} \tag{4.2}$$

The fields $\Phi(x)$ and $\Pi(x)$ for fixed x can be considered themselves a family of functionals parametrized by $x \in \mathbb{R}^n$, for fixed x, $\Phi(x)$ is the functional that associates to Φ and Π the value $\Phi(x)$; similarly with $\Pi(x)$). Their brackets are[1]

$$\begin{gathered}\{\Phi(x), \Phi(y)\} = 0, \qquad \{\Pi(x), \Pi(y)\} = 0,\\ \{\Phi(x), \Pi(y)\} = \delta(x-y).\end{gathered} \tag{4.3}$$

Now let space $\mathbb{R}^d$ become the noncommutative Moyal space. The algebra of functions on $\mathbb{R}^d$ and the algebra (4.1) become noncommutative with noncommutativity given by the twist (2.2), $\mathcal{F} = e^{-(\mathrm{i}/2)\theta^{ij}(\partial/\partial x^i)\otimes(\partial/\partial x^j)}$.

The twist lifts to the algebra A of functionals [42] so that this latter too becomes noncommutative. This is achieved by lifting to A the action of infinitesimal translations. Explicitly $\frac{\partial}{\partial x^i}$ is lifted to ∂_i^* acting on A as

$$\partial_i^* G := -\int d^d x \partial_i \Phi(x) \frac{\delta G}{\delta \Phi(x)} + \partial_i \Pi(x) \frac{\delta G}{\delta \Pi(x)}. \tag{4.4}$$

Therefore on functionals the twist is represented as

$$\mathcal{F} = e^{-(\mathrm{i}/2)\theta^{ij}\int d^d x(\partial_i\Phi(\delta/\delta\Phi(x)) + \partial_i\Pi(\delta/\delta\Pi(x)))\otimes\int d^d y(\partial_j\Phi(\delta/\delta\Phi(y)) + \partial_j\Pi(\delta/\delta\Pi(y)))}. \tag{4.5}$$

The associated $\star$ product is

$$F \star G = \bar{\mathsf{f}}^\alpha(F)\bar{\mathsf{f}}_\alpha(G). \tag{4.6}$$

We can regard $\Phi(x)$ as the functional $\Phi(x) = \int d^d z \delta(x - z)\Phi(z)$ that associates to the function Φ its value in x. In particular we can consider the $\star$ product between functionals $\Phi(x) \star \Phi(y)$. If $x = y$ then $\Phi(x) \star \Phi(y) = (\Phi \star \Phi)(x)$ where this latter $\star$ product is the usual one with the *function* Φ.

Note 3. The twist $\mathcal{F} = e^{(\mathrm{i}/2)\theta^{ij}\partial_i\otimes\partial_j}$ gives rise to the $\star$-Lie algebra of infinitesimal diffeomorphisms of subsection II B; similarly the twist (4.5) yields the $\star$-Lie algebra of infinitesimal functional variations. The former $\star$-Lie algebra is generated by the $\star$-Lie derivatives along vector fields $\mathcal{L}_u^\star$, the latter $\star$-Lie algebra is generated by the $\star$-functional variations $\delta_\varepsilon^\star$. We briefly discuss this $\star$-Lie algebra in the Appendix.

Let us consider the canonical Poisson tensor

$$\Lambda = \int d^d x\left(\frac{\delta}{\delta\Phi(x)} \otimes \frac{\delta}{\delta\Pi(x)} - \frac{\delta}{\delta\Pi(x)} \otimes \frac{\delta}{\delta\Phi(x)}\right) \tag{4.7}$$

and verify that it is compatible with the twist (4.5), i.e., that relations (3.41), (3.42), and (3.43) hold. We unify the phase space coordinates notation by setting

$$\Psi^a = (\Phi, \Pi). \tag{4.8}$$

Then the action of infinitesimal translations on functionals is rewritten as

$$\partial_i^* = -\int d^d y \partial_{y^i}\Psi^a(y)\frac{\delta}{\delta\Psi^a(y)}. \tag{4.9}$$

We compute

[1]In order to avoid considering distributions we should work with smeared fields $\Phi(f) = \int d^d x f(x)\Phi(x)$ and $\Pi(g) = \int d^d x g(x)\Pi(x)$. The smeared version of the Poisson bracket is then $\{\Phi(f), \Pi(g)\} = \int d^d x f(x)g(x)$.

$$\begin{aligned}\partial_i^*\Big(\int d^dx\frac{\delta}{\delta\Psi^b(x)}\otimes\frac{\delta}{\delta\Psi^c(x)}\Big) &= \int d^dx\partial_i^*\Big(\frac{\delta}{\delta\Psi^b(x)}\Big)\otimes\frac{\delta}{\delta\Psi^c(x)}+\frac{\delta}{\delta\Psi^b(x)}\otimes\partial_i^*\Big(\frac{\delta}{\delta\Psi^c(x)}\Big)\\ &= \int d^dx\Big[\partial_i^*,\frac{\delta}{\delta\Psi^b(x)}\Big]\otimes\frac{\delta}{\delta\Psi^c(x)}+\frac{\delta}{\delta\Psi^b(x)}\otimes\Big[\partial_i^*,\frac{\delta}{\delta\Psi^c(x)}\Big]\\ &= \int d^dxd^dy\partial_{y^i}\delta(x-y)\frac{\delta}{\delta\Psi^b(y)}\otimes\frac{\delta}{\delta\Psi^c(x)}+\frac{\delta}{\delta\Psi^b(x)}\otimes\partial_{y^i}\delta(x-y)\frac{\delta}{\delta\Psi^c(y)}=0,\end{aligned}$$

where in the last equality we have exchanged the dummy x and y variables of the second addend and used that $\partial_{y^i}\delta(x-y)=-\partial_{x^i}\delta(x-y)$. The vanishing of this expression implies the compatibility relations (3.41), (3.42), and (3.43).

The compatibility between the Poisson tensor and the twist assures that we have a well-defined notion of deformed Poisson bracket, $\{,\}_\star$: $\mathsf{A}\otimes\mathsf{A}\to\mathsf{A}$,

$$\{F,G\}_\star := \{\bar{\mathsf{f}}^\alpha(F),\bar{\mathsf{f}}_\alpha(G)\}. \tag{4.11}$$

This bracket satisfies

$$\{F,G\}_\star = -\{\bar{R}^\alpha(G),\bar{R}_\alpha(F)\}_\star \tag{4.12}$$

$$\{F,G\star H\}_\star = \{F,G\}_\star\star H+\bar{R}^\alpha(G)\star\{\bar{R}_\alpha(F),H\}_\star \tag{4.13}$$

$$\{F,\{G,H\}_\star\}_\star = \{\{F,G\}_\star,H\}_\star+\{\bar{R}^\alpha(G),\{\bar{R}_\alpha(F),H\}_\star\}_\star. \tag{4.14}$$

In particular the $\star$ brackets among the fields are undeformed

$$\{\Phi(x),\Pi(y)\}_\star = \{\Phi(x),\Pi(y)\}=\delta(x-y), \tag{4.15}$$

$$\{\Phi(x),\Phi(y)\}_\star = \{\Phi(x),\Phi(y)\}=0, \tag{4.16}$$

$$\{\Pi(x),\Pi(y)\}_\star = \{\Pi(x),\Pi(y)\}=0. \tag{4.17}$$

We prove the first relation

$$\begin{aligned}\{\Phi(x),\Pi(y)\}_\star &= \{\bar{\mathsf{f}}^\alpha(\Phi(x)),\bar{\mathsf{f}}_\alpha(\Pi(y))\}\\ &= \{\Phi(x),\Pi(y)\}-\frac{\mathrm{i}}{2}\theta^{ij}\Big\{\int d^dz\partial_i\Phi(z)\delta(x-z),\int d^dw\partial_j\Pi(w)\delta(y-w)\Big\}+O(\theta^2)\\ &= \{\Phi(x),\Pi(y)\}-\frac{\mathrm{i}}{2}\theta^{ij}\partial_{y^j}\partial_{x^i}\delta(x-y)+O(\theta^2)=\{\Phi(x),\Pi(y)\};\end{aligned} \tag{4.18}$$

the second term in the third line vanishes because of symmetry, as well as higher terms in θ^{ij}.

We conclude that for Moyal-Weyl deformations also in the field theoretical case the $\star$-Poisson bracket just among coordinates is unchanged. It is however important to stress that this is not the case in general. For nontrivial functionals of the fields we have

$$\{F,G\}_\star \neq \{F,G\}. \tag{4.19}$$

We now expand Φ and Π in Fourier modes

$$\begin{aligned}\Phi(x) &= \int\frac{d^dk}{(2\pi)^d\sqrt{2E_k}}(a(k)e^{\mathrm{i}kx}+a^*(k)e^{-\mathrm{i}kx})\\ \Pi(x) &= \int\frac{d^dk}{(2\pi)^d}(-\mathrm{i}\hbar)\sqrt{\frac{E_k}{2}}(a(k)e^{\mathrm{i}kx}-a^*(k)e^{-\mathrm{i}kx}),\end{aligned} \tag{4.20}$$

where $E_k=\sqrt{m^2+\vec{p}^2}=\sqrt{m^2+\hbar^2\vec{k}^2}$ and $kx=\vec{k}\cdot\vec{x}=\sum_{i=1}^d k^ix^i$. We use the usual undeformed Fourier decomposition because indeed are the usual exponentials that, once we also add the time dependence part, solve the free field equation of motion on noncommutative space $(\hbar^2\partial^\mu\partial_\mu+m^2)\Phi=0$. This equation is the same as the one on commutative space because the $\star$ product enters only the interaction terms.

The expressions of the fields Φ and Π in terms of the Fourier coefficients a and of their complex conjugate a^* can be inverted to give

$$\begin{aligned}a(k) &= \int d^dx\Big(\sqrt{\frac{E_k}{2}}\Phi(x)+\frac{\mathrm{i}}{\hbar}\sqrt{\frac{1}{2E_k}}\Pi(x)\Big)e^{-\mathrm{i}kx}\\ a^*(k) &= \int d^dx\Big(\sqrt{\frac{E_k}{2}}\Phi(x)-\frac{\mathrm{i}}{\hbar}\sqrt{\frac{1}{2E_k}}\Pi(x)\Big)e^{\mathrm{i}kx}.\end{aligned} \tag{4.21}$$

From these formulas we see that for each value of k, $a(k)$, and $a^*(k)$ are functionals of Φ and Π. We therefore can consider the $\star$ product between these functionals as defined in (4.6). In order to explicitly calculate the $\star$ product we observe that the action (4.4) of the infinitesimal translations $\frac{\partial}{\partial x^i}$ on the functionals a and a^* (that for ease of notation we here just denote by ∂_i) is

$$\partial_i a(k) = -\mathrm{i}k^i a(k) = \mathrm{i}k_i a(k),$$
$$\partial_i a^*(k) = \mathrm{i}k^i a(k) = -\mathrm{i}k_i a^*(k). \tag{4.22}$$

We find it instructive to write the $\star$ product in few simple cases

$$a(k) \star a(k') = e^{-(\mathrm{i}/2)\theta^{ij}k_i k'_j} a(k)a(k'),$$
$$a^*(k) \star a^*(k') = e^{-(\mathrm{i}/2)\theta^{ij}k_i k'_j} a^*(k)a^*(k'),$$
$$a^*(k) \star a(k') = e^{(\mathrm{i}/2)\theta^{ij}k_i k'_j} a^*(k)a(k'),$$
$$a(k) \star a^*(k') = e^{(\mathrm{i}/2)\theta^{ij}k_i k'_j} a(k)a^*(k'),$$

and more in general

$$a(k^{(1)}) \star a(k^{(2)}) \star \ldots a(k^{(m)}) = e^{-(\mathrm{i}/2)\theta^{ij}\sum_{r<s} k_i^{(r)}k_j^{(s)}} a(k^{(1)})a(k^{(2)})\ldots a(k^{(m)}),$$

where $r, s = 1, 2 \ldots m$. A similar formula holds for mixed a and a^* products.

We finally easily calculate the Poisson bracket among the Fourier modes using the definition (4.11) and the functional expressions of $a(k)$, $a^*(k)$ in terms of Φ and Π (4.21), or equivalently from (4.11) and (4.22). We obtain

$$\{a(k), a^*(k')\}_\star = e^{(\mathrm{i}/2)\theta^{ij}k_i k'_j}\{a(k), a^*(k')\} = -\frac{\mathrm{i}}{\hbar}(2\pi)^d\delta(k-k'), \tag{4.23}$$

where we used the undeformed relation $\{a(k), a^*(k')\} = -\frac{\mathrm{i}}{\hbar}(2\pi)^d\delta(k-k')$. The phase drops out in (4.23) because the δ contributes only for $k = k'$, in which case the antisymmetry of θ forces the exponent to be zero. We similarly have

$$\{a(k), a(k')\}_\star = 0, \qquad \{a^*(k), a^*(k')\}_\star = 0. \tag{4.24}$$

As for our comment related to (4.19), this is a good place to check nontriviality of the twisted Poisson bracket. Although it is equal to the untwisted one for linear combinations of the Fourier modes, it is easily verified that it yields a different result, involving nontrivial phases, as soon as we consider Poisson brackets of powers of a, a^*.

V. FIELD QUANTIZATION

We now formulate the canonical quantization of scalar fields on noncommutative space. Associated to the algebra A of functionals $G[\Phi, \Pi]$ there is the algebra $\hat{\mathsf{A}}$ of functionals $\hat{G}[\hat\Phi, \hat\Pi]$ on operator valued fields. We lift the twist to $\hat{\mathsf{A}}$ and then deform this algebra to $\hat{\mathsf{A}}_\star$ by implementing once more the twist deformation principle (2.12). We denote by $\hat\partial_i$ the lift to $\hat{\mathsf{A}}$ of $\frac{\partial}{\partial x^i}$; for all $\hat G \in \hat{\mathsf{A}}$,

$$\hat\partial_i\hat G := -\int d^dx\,\partial_i\hat\Phi(x)\frac{\delta\hat G}{\delta\hat\Phi(x)} + \partial_i\hat\Pi(x)\frac{\delta\hat G}{\delta\hat\Pi(x)}; \tag{5.1}$$

here $\partial_i\hat\Phi(x)\frac{\delta\hat G}{\delta\hat\Phi(x)}$ stands for $\int d^d\ell\,\partial_i\Phi_\ell(x)\frac{\delta\hat G}{\delta\Phi_\ell(x)}$, where like in (4.20) we have expanded the operator $\hat\Phi(x)$ as $\int d^d\ell\,\Phi_\ell(x)\hat{\mathsf{a}}(\ell)$ (and similarly for $\hat\Pi(x)$).

Consequently the twist on operator valued functionals reads

$$\hat{\mathcal{F}} = e^{-(\mathrm{i}/2)\theta^{ij}\int d^dx(\partial_i\hat\Phi(\delta/\delta\hat\Phi(x))+\partial_i\hat\Pi(\delta/\delta\hat\Pi(x))\otimes\int d^dy(\partial_j\hat\Phi(\delta/\delta\hat\Phi(y))+\partial_j\hat\Pi(\delta/\delta\hat\Pi(y))}. \tag{5.2}$$

In $\hat{\mathsf{A}}_\star$ there is a natural notion of $\star$ commutator, according to the general prescription (2.12)

$$[\,,\,]_\star = [\,,\,]\circ\mathcal{F}^{-1}. \tag{5.3}$$

This $\star$-commutator is $\star$-antisymmetric, is a $\star$-derivation in $\hat{\mathsf{A}}_\star$ and satisfies the $\star$-Jacoby identity

$$[\hat F, \hat G]_\star = -[\bar R^\alpha(\hat G), \bar R_\alpha(\hat F)]_\star \tag{5.4}$$

$$[\hat F, \hat G \star \hat H]_\star = [\hat F, \hat G]_\star \star \hat H + \bar R^\alpha(\hat G) \star [\bar R_\alpha(\hat F), \hat H]_\star \tag{5.5}$$

$$[\hat F, [\hat G, \hat H]_\star]_\star = [[\hat F, \hat G]_\star, \hat H]_\star + [\bar R^\alpha(\hat G), [\bar R_\alpha(\hat F), \hat H]_\star]_\star. \tag{5.6}$$

Finally, recalling the definition of the $\mathcal{R}$ matrix it can be easily verified that

$$[\hat F, \hat G]_\star = \hat F \star \hat G - \bar R^\alpha(\hat G) \star \bar R_\alpha(\hat F), \tag{5.7}$$

which is indeed the $\star$ commutator in $\hat{\mathsf{A}}_\star$. This $\star$ commutator (5.3) has been considered in [26] (and was introduced in [43]).

We studied four algebras and brackets: $(\mathsf{A}, \{\,,\,\})$, $(\hat{\mathsf{A}}, [\,,\,])$, $(\mathsf{A}_\star, \{\,,\,\}_\star)$, $(\hat{\mathsf{A}}_\star, [\,,\,]_\star)$. Canonical quantization on noncommutative space is the map $\hbar_\star$ in the diagram

$$\begin{tikzcd} \mathsf{A} \arrow[r, "\hbar"] \arrow[d, "\mathcal{F}"'] & \widehat{\mathsf{A}} \arrow[d, "\hat{\mathcal{F}}"'] \\ \mathsf{A}_\star \arrow[r, "\hbar_\star"] & \widehat{\mathsf{A}}_\star . \end{tikzcd} \tag{5.8}$$

We define canonical quantization on nocommutative space by requiring this diagram to be commutative. Notice that the vertical maps, that with abuse of notation we have called $\mathcal{F}$ and $\hat{\mathcal{F}}$, are the identity map, indeed $\mathsf{A} = \mathsf{A}_\star$ and $\hat{\mathsf{A}} = \hat{\mathsf{A}}_\star$ as vector spaces. Therefore we have $\hbar_\star = \hbar$. The map $\hbar_\star$ satisfies a $\star$-correspondence principle because $\star$-Poisson brackets go into $\star$ commutators at leading order in $\hbar$

$$\begin{tikzcd} \{F, G\} \arrow[r, "\hbar"] \arrow[d, "\mathcal{F}"'] & -\frac{\mathrm{i}}{\hbar}[\hat{F}, \hat{G}] \arrow[d, "\hat{\mathcal{F}}"'] \\ \{\boldsymbol{F}, \boldsymbol{G}\}_\star \arrow[r, "\hbar_\star"] & -\frac{\mathrm{i}}{\hbar}[\hat{\boldsymbol{F}}, \hat{\boldsymbol{G}}]_\star . \end{tikzcd} \tag{5.9}$$

Indeed recall the definitions of the $\star$-Poisson bracket and of the $\star$ commutator and compute

$$\{F, G\}_\star = \{\bar{\mathrm{f}}^\alpha(F), \bar{\mathrm{f}}_\alpha(G)\} \xrightarrow{\hbar} -\frac{\mathrm{i}}{\hbar}[\widehat{\bar{\mathrm{f}}^\alpha(F)}, \widehat{\bar{\mathrm{f}}_\alpha(G)}] = -\frac{\mathrm{i}}{\hbar}[\bar{\mathrm{f}}^\alpha(\hat{F}), \bar{\mathrm{f}}_\alpha(\hat{G})] = -\frac{\mathrm{i}}{\hbar}[\hat{F}, \hat{G}]_\star. \tag{5.10}$$

The second equality holds because the lifts (4.4) and (5.1) of $\frac{\partial}{\partial x^i}$ satisfy

$$\widehat{\partial_i^* G} = \hat{\partial}_i \hat{G}, \tag{5.11}$$

[as is most easily seen from (4.22) and (5.14)].

From (5.3), repeating the passages of (4.18) we obtain [in accordance with (5.10)] the $\star$ commutator of the fields $\hat{\Phi}$ and $\hat{\Pi}$,

$$[\hat{\Phi}(x), \hat{\Pi}(y)]_\star = i\hbar\delta(x - y). \tag{5.12}$$

As a further confirmation that our quantization map $\hbar_\star = \hbar = \hat{}$ implements the $\star$-correspondence principle between $\star$-Poisson brackets and $\star$ commutators we notice that to all orders in $\hbar$

$$\widehat{\{\Phi, \Pi\}_\star} = \frac{\mathrm{i}}{\hbar}[\hat{\Phi}, \hat{\Pi}]_\star. \tag{5.13}$$

Concerning the creation and annihilation operators, they are functionals of the operators $\hat{\Phi}$, $\hat{\Pi}$ through the quantum analogue of the classical functional relation (4.21). Using (5.1) we have [cf. (4.22)]

$$\partial_i \hat{a}(k) = \mathrm{i}k_i \hat{a}(k), \qquad \partial_i \hat{a}^\dagger(k) = -\mathrm{i}k_i \hat{a}^\dagger(k), \tag{5.14}$$

where here for ease of notation we have just denoted the lift of the infinitesimal translations $\frac{\partial}{\partial x^i}$ by ∂_i. Their $\star$ commutator follows from (5.12) and the quantum analogue of (4.21) [or from (5.3) and (5.14), or also from (4.23) and linearity of (5.13)],

$$[\hat{a}(k), \hat{a}^\dagger(k')]_\star = (2\pi)^d \delta(k - k'). \tag{5.15}$$

In order to compare this expression with similar ones which have been found in the literature [21–24,27–29] it is useful to recall (5.7) and realize the action of the $\mathcal{R}$ matrix. Since $\mathcal{R} = \mathcal{F}^{-2}$ we obtain that (5.15) is equivalent to

$$\hat{a}(k) \star \hat{a}^\dagger(k') - e^{-\mathrm{i}\theta^{ij}k'_i k_j} \hat{a}^\dagger(k') \star \hat{a}(k) = (2\pi)^d \delta(k - k'). \tag{5.16}$$

This relation first appeared in [44]. In the noncommutative quantum field theory context it appears in [27,28], and implicitly in [26] (it is also contemplated in [29] as a second option). On the other hand [22–24,29], starting from a different definition of $\star$ commutator, $[A \star, B] := A \star B - B \star A$, obtain deformed commutation relations of the kind $a_k a^\dagger_{k'} - e^{-(\mathrm{i}/2)\theta^{ij}k_i k'_j} a^\dagger_k a_{k'} = (2\pi)^d \delta(k - k')$. These are different from (5.16), indeed if we expand also the $\star$ product in (5.16) we obtain the usual commutation relations $\hat{a}(k)\hat{a}^\dagger(k') - \hat{a}^\dagger(k')\hat{a}(k) = (2\pi)^d \delta(k - k')$.

As in the case of the $\star$-Poisson bracket, we have found that the $\star$ commutator of coordinate fields (5.12), and of creation and annihilation operators (5.15), are equal to the usual undeformed ones. Once again, we warn the reader that this is not true anymore for more complicated functionals of the coordinate fields, in general $[\hat{F}, \hat{G}]_\star \neq [\hat{F}, \hat{G}]$.

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APPENDIX: $\star$-LIE ALGEBRA OF FUNCTIONAL VARIATIONS

In Sec. II, we remarked that the twist $\mathcal{F} = e^{(-\mathrm{i}/2)\theta^{ij}\partial_i \otimes \partial_j}$ is an element of the tensor product of $U\Xi$ by itself, the universal enveloping algebra of the Lie algebra Ξ of vector fields. Similarly the lifted twist (4.5) is an element of the universal enveloping algebra $U\Upsilon$ of the Lie algebra Υ of infinitesimal functional variations (on phase space). In order to fully understand the lift (4.5) we have to clarify the way the Lie algebra of infinitesimal diffeomorphisms is a subalgebra of the Lie algebra of infinitesimal functional variations.

Undeformed infinitesimal functional variations δ_ε are defined by

$$\delta_\varepsilon G := \int d^d x \varepsilon^a(x) \frac{\delta}{\delta \Psi^a(x)} G, \tag{A1}$$

where $\Psi^a = (\Phi, \Pi)$ (more in general Ψ^a are target space coordinates), and where $\varepsilon^a(x)$ themselves can be functionals.

Consider the map between vector fields and infinitesimal functional variations (with slight abuse of notation we denote this map by the symbol δ)

$$\delta\colon \Xi \to \Upsilon \qquad u \mapsto \delta_u \tag{A2}$$

$$\delta_u G := -\int d^d x u(\Psi^a)(x) \frac{\delta}{\delta\Psi^a(x)} G. \tag{A3}$$

This map is a Lie algebra map,

$$\delta_{[u,v]} = [\delta_u, \delta_v]. \tag{A4}$$

If $u = \frac{\partial}{\partial x^i}$ then δ_u is just the lifted partial derivative ∂_i^* defined in (4.9). In order to proceed in the construction of the $\star$-Lie algebra of functional variations we define $\star$-functional variations. According to (2.12),

$$\delta^\star_\varepsilon(G) := \bar{\mathrm{f}}^\alpha(\delta_\varepsilon)(\bar{\mathrm{f}}_\alpha(G)); \tag{A5}$$

where the action of $\bar{\mathrm{f}}^\alpha$ on δ_ε is the adjoint action in $U\Upsilon$, $\delta_\sigma(\delta_\varepsilon) = [\delta_\sigma, \delta_\varepsilon]$, $(\delta_{\sigma_1}\delta_{\sigma_2})(\delta_\varepsilon) = [\delta_{\sigma_1}, [\delta_{\sigma_2}, \delta_\varepsilon]]$, and similarly for higher products of variations δ_{σ_i}.

The functional variation $\delta^\star_\varepsilon$ satisfies the Leibniz rule [cf. (2.31) and (3.30)]

$$\delta^\star_\varepsilon(F \star G) = \delta_\varepsilon(F) \star G + \bar{R}^\alpha(F) \star (\bar{R}_\alpha(\delta_\varepsilon))^*(G), \tag{A6}$$

where $\bar{R}_\alpha(\delta_\varepsilon)$ is itself a functional variation, say δ_σ, and $(\bar{R}_\alpha(\delta_\varepsilon))^\star = \delta^\star_\sigma$. The Leibniz rule is consistent (and actually follows) from the coproduct rule

$$\delta_\varepsilon \mapsto \Delta_\star(\delta_\varepsilon) = \delta_\varepsilon \otimes 1 + \bar{R}^\alpha \otimes \bar{R}_\alpha(\delta_\varepsilon). \tag{A7}$$

Finally also the formulas in this appendix hold for the most generic twist; just replace $\bar{R}^\alpha$ with $\mathrm{f}^\beta(\bar{R}^\alpha)\mathrm{f}_\beta$ in (A7).

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