Composite strings in (2 + 1)-dimensional anisotropic weakly coupled Yang-Mills theory

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The small-scale structure of a string connecting a pair of static sources is explored for the weakly coupled anisotropic SU(2) Yang-Mills theory in (2 + 1) dimensions. A crucial ingredient in the formulation of the string Hamiltonian is the phenomenon of color smearing of the string constituents. The quark-antiquark potential is determined. We close with some discussion of the standard, fully Lorentz-invariant Yang-Mills theory.

I. INTRODUCTION

Recently, the author has established the existence of confinement and a mass gap in a version of (2 + 1)-dimensional SU(N) Yang-Mills theory, in which the coupling constants are anisotropic and small. The understanding of the interquark potential and the mass gap is elementary [1,2] though finding precise values for the string tension [3], and the mass spectrum [4] requires detailed information of an integrable (1 + 1)-dimensional quantum field theory. This integrable field theory is the SU(N) × SU(N) principal-chiral nonlinear sigma model. For $N = 2$, exact knowledge of certain matrix elements makes it possible to perturb away from integrability.

Though the gauge theory we consider is not spatially rotation invariant, it has features one expects of real (3 + 1)-dimensional QCD; it is asymptotically free and confines quarks at weak coupling.

One can formally remove the regulator in strong-coupling expansions of (2 + 1)-dimensional gauge theories; the vacuum state in this expansion yields a string tension and a mass gap which have formal continuum limits. This is possible because of purely dimensional considerations in this number of dimensions. Such strong-coupling analyses can be done in a Hamiltonian lattice formalism [5], or with an ingenious choice of degrees of freedom and point-splitting regularization [6]. There are even formal improvements of the vacuum state using the point-splitting cutoff [7] or the lattice cutoff [8], which do not confine adjoint sources. It is important to know whether these results can be justifiably extrapolated to the limit of no regularization (more discussion of this issue can be found in the introduction of Ref. [3]). In contrast, the approach we have taken is a weak-coupling method. It is, thus far, the only method yielding quark

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confinement with no strong-coupling assumptions in more than two dimensions, without dynamical matter. There is a hint of another weak-coupling approach in (2 + 1) dimensions [9,10] based on general properties of gauge-orbit space.

Simple intuitive formulas for the potential between a static quark and antiquark were found quite early for our anisotropic theory [1]. String tensions for higher representations can also be worked out, and adjoint sources are not confined [2]. The string tensions for the cases of *horizontally* and *vertically* separated quarks, *i.e.* separated in the x^1- and x^2-directions, respectively, have corrections, however. For gauge group SU(2), the leading correction to the horizontal string tension was found in Ref. [3]. In this paper, the vertical potential is shown to be the ground-state energy of a certain Hamiltonian in one spatial dimension. This Hamiltonian describes the dynamics of a string with both coordinate and color degrees of freedom. The correction to the potential of a vertically separated quark-antiquark pair is thereby determined.

The connection between the gauge theory and integrable systems using the Kogut-Susskind lattice formalism was explained in Refs. [1,3]. A quicker derivation was given in Refs. [4,11]. Here we simply present the axial-gauge Hamiltonian formalism and refer the reader to these papers for its derivation.

The 2-coordinate is discrete, so that x^2 takes the values $x^2 = a, 2a, 3a \ldots, L^2$, where a is a lattice spacing. All fields are considered functions of $x = (x^0, x^1, x^2)$. The boundary condition is periodic in x^2, which means that any function $f(x^0, x^1, x^2)$ satisfies $f(x^0, x^1, x^2 + L^2) = f(x^0, x^1, x^2)$. The boundary condition in the x^1-direction is open, so that space is a cylinder. In this paper, we assume the thermodynamic limit is already taken, so we will not worry too much about the boundaries. The gauge fields are SU(N)-Lie-algebra valued. We chose generators of this Lie algebra t_b, which satisfy Tr$t_b t_c = \delta_{bc}$ and define structure coefficients f^d_{bc} by $[t_b, t_c] = if^d_{bc} t_d$. We have set the gauge

component $A_1(x)$ to be zero and replace $A_2(x)$ by a field $U(x)$ lying in SU(N), via

$$U(x) = \exp i \int_{x^2}^{x^2+a} dy^2 A_2(x^0, x^1, y^2).$$

The left-handed and right-handed currents are

$$j_\mu^L(x)_b = i \operatorname{Tr} t_b \partial_\mu U(x) U(x)^\dagger,$$

$$j_\mu^R(x)_b = i \operatorname{Tr} t_b U(x)^\dagger \partial_\mu U(x),$$

respectively, where $\mu = 0, 1$. The Hamiltonian is $H_0 + H_1$, where

$$H_0 = \sum_{x^2} \int dx^1 \frac{1}{2g_0^2} \{[j_0^L(x)_b]^2 + [j_1^L(x)_b]^2\}, \quad (1.1)$$

and

$$H_1 = -\sum_{x^2} \int dx^1 \int dy^1 \frac{(g_0')^2}{4g_0^4 a^2} |x^1 - y^1| [j_0^L(x^1, x^2)_b$$
$$- j_0^R(x^1, x^2 - a)_b - \bar{q}_b \delta(x^1 - u^1)\delta_{x^2 u^2}$$
$$+ q_b \delta(x^1 - v^1)\delta_{x^2 v^2}][j_0^L(y^1, x^2)_b - j_0^R(y^1, x^2 - a)_b$$
$$- \bar{q}_b \delta(y^1 - u^1)\delta_{x^2 u^2} + q_b \delta(y^1 - v^1)\delta_{x^2 v^2}], \quad (1.2)$$

where we have inserted two color charges—a quark with charge q at site v and an antiquark with charge $\bar{q}$ at site u. These charge operators satisfy $[q_b, q_c] = i f_{bc}^d q_d$ and $[\bar{q}_b, \bar{q}_c] = i f_{bc}^d \bar{q}_d$. A constraint remains after the axial-gauge fixing, namely, that for each x^2

$$\int dx^1 [j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b - \bar{q}_b \delta(y^1 - u^1)\delta_{x^2 u^2}$$
$$+ q_b \delta(y^1 - v^1)\delta_{x^2 v^2}]\Psi = 0, \quad (1.3)$$

where Ψ is any physical state. For more details on the derivation of the term in the Hamiltonian (1.2) and the constraint (1.3), see Refs. [1,3]. The Hamiltonian H_0 given in (1.1) is a sum of principal-chiral sigma-model Hamiltonians.

The anisotropic regime of $(2 + 1)$-dimensional Yang-Mills theory is

$$(g_0')^2 \ll \frac{1}{g_0} e^{-4\pi/(g_0^2 N)}. \quad (1.4)$$

The point where the regulator can be removed in the theory is the same as that of the standard isotropic theory $g_0' = g_0 = 0$. The left-hand side and ride-hand side are proportional to the two energy scales in the theory (the latter comes from the two-loop beta function of the sigma model). For more discussion of these matters, see Refs. [2–4].

The excitations of H_0, which we call Fadeev-Zamolodchikov (FZ) particles, behave like solitons, though they are not quantized versions of classical solutions. Some of these FZ particles are elementary and others are bound states of the elementary FZ particles. An ele-

mentary FZ particle has an adjoint charge and mass m_1. An elementary one-FZ-particle state is a superposition of color-dipole states, with a quark (antiquark) charge at x^1, x^2, and an antiquark (quark) charge at x^1, $x^2 + a$. The interaction H_1 produces a linear potential between color charges with the same value of x^2. Residual gauge invariance (1.3) requires that at each value of x^2, the total color charge is zero. If there are no quarks with coordinate x^2, the total right-handed charge of FZ particles in the sigma model at $x^2 - a$ is equal to the total left-handed charge of FZ particles in the sigma model at x^2.

The particles of the principal-chiral sigma model carry a quantum number r, with the values $r = 1, \dots, N - 1$ [12]. Each particle of label r has an antiparticle of the same mass, with label $N - r$. The masses are given by

$$m_r = m_1 \frac{\sin\frac{r\pi}{N}}{\sin\frac{\pi}{N}},$$

$$m_1 = K\Lambda (g_0^2 N)^{-1/2} e^{-4\pi/g_0^2 N} + \text{nonuniversal corrections},$$
$$(1.5)$$

where K is a nonuniversal constant and Λ is the ultraviolet cutoff of the sigma model.

Lorentz invariance in each x^0, x^1 plane is manifest. For this reason, the linear potential is not the only effect of H_1. The interaction creates and destroys pairs of elementary FZ particles. This effect is quite small, provided that g_0' is small enough. Specifically, this means that the string tension in the x^1-direction coming from H_1 is small compared to the square of the mass of the fundamental FZ particle; this is just the condition (1.4). The effect is important, however, in that it is responsible for the correction to the horizontal string discussed in the next paragraph in Eq. (1.7).

Simple arguments readily show that at leading order in g_0', the vertical and horizontal string tensions are given by

$$\sigma_V = \frac{m_1}{a}, \qquad \sigma_H = \frac{(g_0')^2}{a^2} C_N, \quad (1.6)$$

respectively, where C_N is the smallest eigenvalue of the Casimir of SU(N). These naive results for the string tension have further corrections in g_0', which were determined for the horizontal string tension for SU(2) [3]:

$$\sigma_H = \frac{3}{2} \frac{(g_0')^2}{a^2} \left[1 + \frac{4(g_0')^2}{3\pi^2 m_1^2 a^2} \right.$$
$$\left. \times \exp{-2\int_0^\infty \frac{d\xi}{\xi} e^{-\xi} \tanh^2 \frac{\xi}{2}} \right]^{-1}$$
$$= \frac{3}{2} \frac{(g_0')^2}{a^2} \left[1 + \frac{4(g_0')^2}{3\pi^2 m_1^2 a^2} 0.7296 \right]^{-1}. \quad (1.7)$$

The leading term agrees with (1.6). This calculation was done using the exact form factor for sigma-model currents obtained by Karowski and Weisz [13]. In this paper, we shall use the form factor to study corrections of order $(g_0')^2$
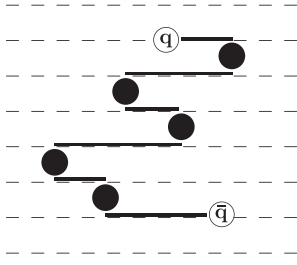
FIG. 1. A low-lying quark-antiquark-pair state. The horizontal coordinate is x^1 and the vertical coordinate is x^2. The quark lies at a larger value of x^2 than the antiquark. Between the pair is a collection of FZ particles. All the particles are bound together by horizontal strings.

to the vertical string tension. A review of integrability and form-factor methods is in the appendix of Ref. [3].

A picture of a gauge-invariant state for the gauge group SU(2) with a single quark and a single antiquark at different values of x^2 is given in Fig. 1. For $N > 2$, there are more complicated ways in which strings can join particles. The lightest states have the smallest number of particles, by virtue of $\sigma_{\rm H} \ll \sigma_{\rm V}$. Thus, there is a single FZ particle in each layer between the quark and the antiquark. There is a piece in H_1 which can create and destroy FZ particles, but this can safely be neglected in a nonrelativistic approximation. We shall treat the quarks as static, nondynamical sources in this paper.

In the next section we show how the color of FZ particles is smeared by radiative corrections, with the aid of the exact matrix elements of the current operator. We use this to derive the Hamiltonian of a string in Sec. III. The ground-state energy of this string, and thus the potential between static color sources is found in Sec. IV. In Sec. V, we argue that the functional form of this potential extends to the standard Lorentz-invariant SU(2) Yang-Mills theory. We present our conclusions in Sec. VI.

II. COLOR SMEARING

Consider a static quark-antiquark pair for the SU(2) gauge theory, as in Fig. 1. We will assume that the x^1-coordinate of the quark and antiquark is the same and that the x^2-coordinate of the quark is v^2 and the x^2-coordinate of the antiquark is u^2, where $v^2 > u^2$. The string tension is

$$\sigma_{\rm V} = \lim_{v^2 - u^2 \to \infty} \frac{E_{\rm string}}{v^2 - u^2},$$

where $E_{\rm string}$ is the lowest possible energy of the Hamiltonian H projected on the subspace of states with exactly one FZ particle for layers with $x^2 \geq u^2$ and $x^2 < v^2$ and no FZ particles in any other layer. To leading order $\sigma_{\rm V} = m/a$, where $m = m_1$ [for SU(2) there is only one mass]. The projection of the Hamiltonian on this subspace is

$$H_{\rm proj} = \sum_{x^2 = u^2}^{v^2 - a} \sum_{k=1}^{4} \left\{ m + \int \frac{dp}{2\pi} \frac{p^2}{2m} \mathfrak{A}(p, x^2)_k^\dagger \mathfrak{A}(p, x^2)_k \right\} + H_1,$$

(2.1)

where $\mathfrak{A}(p, x^2)_k$, $\mathfrak{A}(p, x^2)_k^\dagger$ are the Fadeev-Zamolodchikov destruction and creation operators (the field operator of the FZ particles), respectively; with x^1-momentum p, the index $k = 1, \ldots, 4$ denotes the particle species [the Hamiltonian is invariant under rotations in O(4) = SU(2) × SU(2)] where H_1 is given by (1.2), as before. We are making a nonrelativistic approximation. This approximation should be valid, provided $(g_0')^2 \ll ma$ and we consider the lowest-lying states.

Particle states are produced on the vacuum by the application of FZ operators, e.g. a one particle state with momentum p and species index k is

$$|p, k\rangle = \mathfrak{A}^\dagger(p)_k |0\rangle,$$

where the index x^2 is suppressed. In a theory of relativistic particles, these states are normalized according to the rule

$$\langle p', k'|p, k\rangle = \frac{1}{\sqrt{p^2 + m^2}} \delta_{k'k} \delta(p' - p).$$

To find the spectrum of $H_{\rm proj}$, we need the matrix elements

$$\langle z_1, k_1|j_0^{\rm L,R}(y)|z_2, k_2\rangle = \langle z_1 - y, k_1|j_0^{\rm L,R}(0)|z_2 - y, k_2\rangle,$$

where, for now, we have dropped the index x^2 and where the particle states are given by $|z, k\rangle = \mathfrak{A}(z)_k|0\rangle$, $|0\rangle$ being the true vacuum of the SU(2) × SU(2) sigma model.

The matrix elements of currents may be written terms of momentum-space eigenstates by Fourier transformation:

$$\langle z_1, k_1|j_0^{\rm L,R}(y)|z_2, k_2\rangle = \int \frac{dp_1}{2\pi} \frac{1}{\sqrt{2E_1}} \int \frac{dp_2}{2\pi} \frac{1}{\sqrt{2E_1}}$$
$$\times e^{-ip_1(z_1 - y) + ip_2(z_2 - y)}$$
$$\times \langle p_1, k_1|j_0^{\rm L,R}(0)|p_2, k_2\rangle, \quad (2.2)$$

where $E_{1,2} = \sqrt{p_{1,2}^2 + m^2}$. The momentum-space matrix elements have the exact expression

$$\langle p_1, k_1|j_0^{\rm L,R}(0)|p_2, k_2\rangle = \frac{i}{\sqrt{2}} (\delta_{k_1 4} \delta_{k_2 b} - \delta_{k_2 4} \delta_{k_1 b}$$
$$\pm \epsilon_{bk_1 k_2})(p_1 + p_2)$$
$$\times F(\theta_1 - \theta_2 + i\pi), \quad (2.3)$$

where the plus or minus sign corresponds to the left-handed (L) or right-handed (R) current, respectively, the rapidities $\theta_{1,2}$ are defined by $m \sinh\theta_{1,2} = p_{1,2}$, and

$$F(\theta) = \exp 2 \int_0^\infty \frac{d\xi}{\xi} \frac{e^{-\xi} - 1}{e^\xi + 1} \frac{\sin^2 \frac{\xi(\pi i - \theta)}{2\pi}}{\sinh \xi}$$

$$= \exp - \int_0^\infty \frac{d\xi}{\xi} \frac{e^{-\xi}}{\cosh^2 \frac{\xi}{2}} \sin^2 \frac{\xi(\pi i - \theta)}{2\pi}.$$

Note that the Kronecker deltas in (2.3) are automatically zero if an index takes the value 4. This expression is the result of Karowski and Weisz [13] for the $O(4) \simeq SU(2) \times SU(2)$ sigma-model form factors, after applying crossing [3].

The only difference between the free-field-theory matrix elements and (2.2) and (2.3) is the presence of the factor $F(\theta_1 - \theta_2 + i\pi)$. The physical interpretation of this factor is that the color of an FZ particle is not pointlike, but smeared over a region of size m^{-1}. This smearing will be made more explicit in the discussion below.

Since the mass of the FZ particles is assumed large compared to $(g_0')^2/a$, we assume that in the frame where the sources are static, these particles move slowly. We can therefore make the approximation that p_1 and p_2 in the Fourier transform in (2.2) are small compared to m. The result is

$$2^{-1/2}(p_1^2 + m^2)^{-1/4} 2^{-1/2}(p_2^2 + m^2)^{-1/4}$$
$$\times \langle p_1, k_1 | j_0^{L,R}(0) | p_2, k_2 \rangle$$
$$= \frac{i}{\sqrt{2}} (\delta_{k_1 4}\delta_{k_2 b} - \delta_{k_2 4}\delta_{k_1 b} \pm \epsilon_{bk_1 k_2}) \exp - \frac{A}{m^2}(p_1 - p_2)^2,$$

(2.4)

where the positive constant A is

$$A = \frac{1}{4\pi^2} \int_0^\infty d\xi \frac{\xi e^{-\xi}}{\cosh^2 \frac{\xi}{2}} = \frac{1}{12} - \frac{\ln 2}{\pi^2} = 0.1310284.$$

It is convenient that, to leading order, all the momentum dependence is in the exponent of (2.4). This result just means that the color distribution of an FZ particle is Gaussian. Inserting (2.4) into (2.2) yields

$$\langle z_1, k_1 | j_0^{L,R}(y) | z_2, k_2 \rangle = \sqrt{\frac{m^2}{4\pi A}} (\sigma_b^{L,R})_{kl}$$
$$\times \exp\left[-\frac{m^2}{4A}\left(\frac{z_1 + z_2}{2} - y\right)^2\right]$$
$$\times \delta(z_1 - z_2),$$

(2.5)

where the "spin" operators are

$$(\sigma_b^{L,R})_{kl} = i(\delta_{k4}\delta_{lb} - \delta_{l4}\delta_{kb} \pm \epsilon_{bkl}).$$

These operators are generators of independent spin-1/2 representations of color-SU(2). Specifically,

$$[\sigma_b^{L,R}, \sigma_c^{L,R}] = 2i\epsilon_{bcd}\sigma_d^{L,R}, \qquad [\sigma_b^L, \sigma_c^R] = 0,$$
$$\sum_b (\sigma_b^{L,R})^2 = 3.$$

III. THE STRING HAMILTONIAN

Next we use the smeared color-charge density (2.5) to write down the effective Hamiltonian of the string. We write $z = z(x^2)$ for each value of x^2. From the interaction Hamiltonian (1.2), and the kinetic term in (2.1), this is

$$H_{\text{string}} = \frac{m}{a}(v^2 - u^2) - \frac{1}{2m} \sum_{x^2 = u^2}^{v^2 - a} \frac{\partial^2}{\partial z(x^2)^2} + V_{\text{bulk}} + V_{\text{ends}},$$

where

$$V_{\text{bulk}} = -\frac{m^2}{4\pi A} \frac{(g_0')^2}{4g_0^4 a^2} \sum_{x^2 = u^2 + a}^{v^2 - a} \int dx^1 dy^1 |x^1 - y^1| \{e^{-(m^2/4A)[z(x^2) - x^1]^2} \sigma^L(x^2)_b - e^{-(m^2/4A)[z(x^2 - a) - x^1]^2} \sigma^R(x^2 - a)_b\}$$
$$\times \{e^{-(m^2/4A)[z(x^2) - y^1]^2} \sigma^L(x^2)_b - e^{-(m^2/4A)[z(x^2 - a) - y^1]^2} \sigma^R(x^2 - a)_b\},$$

(3.1)

and

$$V_{\text{ends}} = -\frac{(g_0')^2}{4g_0^4 a^2} \int dx^1 dy^1 |x^1 - y^1| \left\{\sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(u^2) - x^1]^2} \sigma^L(u^2)_b + \delta(x^1 - u^1)\bar{q}_b\right\}$$
$$\times \left\{\sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(u^2) - y^1]^2} \sigma^L(u^2)_b + \delta(y^1 - u^1)\bar{q}_b\right\} - \frac{(g_0')^2}{4g_0^4 a^2} \int dx^1 dy^1 |x^1 - y^1|$$
$$\times \left\{\sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(v^2) - x^1]^2} \sigma^R(v^2 - a)_b + \delta(x^1 - v^1)q_b\right\}$$
$$\times \left\{\sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(v^2) - y^1]^2} \sigma^R(v^2 - a)_b + \delta(y^1 - v^1)q_b\right\}.$$

(3.2)

We need to apply the constraint (1.3) to states. This becomes

$$\int dx^1 \left\{ -\sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(z^2)-x^1]^2} \sigma^L(x^2)_b + \sqrt{\frac{m^2}{2\pi A}} e^{-(m^2/4A)[z(z^2-a)-x^1]^2} \sigma^R(x^2-a)_b \right\} \Psi = 0,$$

for $x^2 = u^2 + a, \ldots, v^2 - a$, and

$$\int dx^1 \sqrt{\frac{m^2}{2\pi A}} \{ e^{-(m^2/4A)[z(u^2)-x^1]^2} \sigma^L(u^2)_b - \bar{q}_b \delta(x^1 - u^1) \} \Psi = 0,$$

$$\int dx^1 \sqrt{\frac{m^2}{2\pi A}} \{ e^{-(m^2/4A)[z(v^2-a)-x^1]^2} \sigma^L(v^2 - a)_b + q_b \delta(x^1 - v^1) \} \Psi = 0,$$

at the ends. These constraints simply reduce to the identification of $\sigma^L(x^2)_b$ with $\sigma^R(x^2 - a)_b$, for $x^2 = u^2 + a, \ldots, v^2 - a$, with $\sigma^L(u^2)_b/\sqrt{2}$ with $\bar{q}_b$ and $\sigma^R(v^2 - a)_b/\sqrt{2}$ with $-q_b$. In this way, the color degrees of freedom are completely eliminated from (3.1) and (3.2).

There are integrals remaining to be done in (3.1) and (3.2). One of these is straightforward:

$$\int dx^1 dy^1 |x^1 - y^1| e^{-(m^2/4A)[(x^1)^2+(y^1)^2]} = \frac{4\sqrt{2\pi}A^{3/2}}{m^3}.$$

We write another integral we need as

$$\int dx^1 dy^1 |x^1 - y^1| e^{-(m^2/4A)[(x^1+r)^2+(y^1)^2]}$$

$$= \frac{4\sqrt{2\pi}A^{3/2}}{m^3} P(r).$$

The third and final integral we need (simplifying the Hamiltonian near the endpoints of the string) is

$$\int dx^1 |x^1 - u^1| e^{-(m^2/4A)[x^1-z(u^2)]^2}$$

$$= \frac{2A}{m^2} P[\sqrt{2}z(u^2) - \sqrt{2}u^1].$$

The function $P(r)$ cannot be evaluated exactly, but for small or large r has the limiting forms

$$P(r) = \begin{cases} 1 + \frac{m^2 r^2}{4A}, & r \ll m^{-1} \\ \sqrt{\frac{\pi}{2A}} m|r|, & r \gg m^{-1}, \end{cases} \tag{3.3}$$

respectively. We note that the first of these forms can be derived from the power series:

$$P(r) = 1 + \frac{m^2 r^2}{4A} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!(2n+1)} \left(\frac{m^2 r^2}{A} \right)^n.$$

The small-r expression in (3.3) is due to the softening of the linear potential in the horizontal direction from color smearing. At large r, this smearing has no effect and the potential is linear.

Our result for the string Hamiltonian is

$$H_{\text{string}} = \frac{m}{a}(v^2 - u^2) - \frac{1}{2m} \sum_{x^2=u^2}^{v^2-a} \frac{\partial^2}{z(x^2)^2} - \frac{3(g_0')^2}{2g_0^4 ma^2} \sqrt{\frac{A}{2\pi}}$$

$$\times \sum_{x^2=u^2+a}^{v^2-a} \{1 - P[z(x^2) - z(x^2 - a)]\}$$

$$- \frac{3(g_0')^2}{2g_0^4 ma^2} \sqrt{\frac{A}{2\pi}} (1 + P\{\sqrt{2}[z(u^2) - u^1]\}$$

$$+ P\{\sqrt{2}[z(v^2 - a) - v^1]\}). \tag{3.4}$$

IV. THE STATIC POTENTIAL BETWEEN SOURCES

Our result (3.4) is simply a transversely oscillating discretized bosonic string. The only unusual feature is that the potential energy becomes linear for large transverse gradients. For small transverse gradients, however, the Hamiltonian (3.4) is quite conventional, since (3.3) yields a quadratic potential. We emphasize that this fortunate circumstance is due entirely to the smearing of color of the FZ particles. To determine the potential between static sources, we must find the ground-state energy of (3.4). This is feasible because of the quadratic nature of the potential for small gradients. In the small-gradient approximation that $|z(x^2) - z(x^2 - a)|$, for $u^2 < x^2 < v^2$, $|z(u^2) - u^1|$, and $|z(v^2) - v^1|$ are all much smaller than m^{-1}, the string Hamiltonian (3.4) becomes

$$H_{\text{string}} = \frac{3(g_0')^2}{2g_0^4 ma^2} \sqrt{\frac{A}{2\pi}} + \frac{m}{a}(v^2 - u^2) - \frac{1}{2m} \sum_{x^2=u^2}^{v^2-a} \frac{\partial^2}{\partial z(x^2)^2}$$

$$+ \frac{3(g_0')^2}{8g_0^4 ma^2} \sqrt{\frac{1}{2\pi A}} \sum_{x^2=u^2+a}^{v^2-a} [z(x^2) - z(x^2 - a)]^2$$

$$+ \frac{3(g_0')^2}{4g_0^4 ma^2} \sqrt{\frac{1}{2\pi A}} \{ [z(u^2) - u^1]^2$$

$$+ [z(v^2 - a) - v^1]^2 \}. \tag{4.1}$$

Let us now drop the first, constant term in (4.1) and denote $v^2 - u^2$ by L.

The analysis of (4.1) is straightforward. We drop the first term, which has no physical significance. The potential in (4.1) is diagonalized by means of normal modes w^q, which have components:

$$(w^q)_k = C^q \sin\left[\frac{\pi q}{Q}\left(k - \frac{1}{2}\right) + \frac{\pi}{2}\right],$$

where $k = (x^2 - u^2)/a$, $Q = (v^2 - u^2)/a = L/a$, k, $q = 1, 2, \ldots, Q$ and C^q is a constant of normalization. If we set $u^1 = v^1$, then the Hamiltonian becomes a set of Q simple harmonic oscillators. The ground-state energy of H_{string} is

$$E_0 = \frac{m}{a}L - \frac{\sqrt{3}g_0'}{g_0^2 a}\left(\frac{1}{2\pi A}\right)^{1/4}\sum_{q=0}^{Q}\sin\frac{\pi q}{2Q}, \qquad (4.2)$$

where all constant terms have been dropped. We apply the Euler summation formula

$$\sum_{q=0}^{Q}F\left(\frac{q}{Q}\right) = Q\int_0^1 dx F(x) - \frac{1}{2}[F(1) - F(0)]$$
$$+ \frac{1}{12Q}[F'(1) - F'(0)] + \mathcal{O}\left(\frac{1}{Q^2}\right),$$

to (4.2), and dropping constant terms once more, obtain the static quark-antiquark potential

$$V(L) = E_0$$
$$= \left[\frac{m}{a} - \frac{2\sqrt{3}}{\pi}\frac{g_0'}{g_0^2 a^2}\right]L - \frac{\pi\sqrt{3}}{24}\frac{g_0'}{g_0^2}\left(\frac{1}{2\pi A}\right)^{1/4}\frac{1}{L}$$
$$+ \mathcal{O}\left(\frac{1}{L^2}\right), \qquad (4.3)$$

which is our final result. Notice that in (4.3) there is a correction to the string tension of order g_0', namely,

$$\sigma_{\text{V}} = \frac{m}{a} - \frac{2\sqrt{3}}{\pi}\frac{g_0'}{g_0^2 a^2}.$$

There is also a new term present in the potential proportional to $1/L$. This term does not have the standard universal coefficient [14], but instead is proportional to g_0'.

V. SOME REMARKS ON THE ISOTROPIC CASE

The picture of confinement in the anisotropic theory is sufficiently compelling that we believe the behavior of the standard rotationally invariant theory is fundamentally similar. The necessity of the inequality (1.4) shows that the rotationally invariant theory is not easily accessible by the methods discussed in this section. We argued that applying an anisotropic renormalization group causes a theory for which $g_0' \approx g_0$ to flow to $g_0' \ll g_0$ in the infrared [4]. This infrared form of the theory is essentially just a nonrelativistic approximation for the isotropic theory. A theory with a mass gap has a nonrelativistic limit (the classical Yang-Mills theory, which is massless, has no

such limit). Consider the Yang-Mills action in $2 + 1$ dimensions with the speed of light included explicitly:

$$\mathcal{S} = \frac{1}{c}\int d^2x dt \, \text{Tr}\left[\frac{1}{2e^2}\sum_{i=1}^{2}(F_{0i})^2 - \frac{c^2}{2e^2}(F_{12})^2\right],$$

where e is the continuum coupling. Suppose we Wick rotate this action to Euclidean space by $x^0 \to ix^0$, rotate so that $F_{12} \to F_{01}$, and finally Wick rotate back. By identifying $g_0 = e/\sqrt{a}$ and $g_0' = e/(\sqrt{a}c^2)$, where a is a cutoff with units of centimeters, and taking $c \gg 1$, our naive result is just the anisotropic model discussed in this and previous papers. Certain observables in the anisotropic gauge theory can now be identified with observables in standard Yang-Mills theory, with a caveat. The caveat is that the mass scale is given by (1.5) rather than being proportional to the continuum coupling (this is because the justification for this procedure relies on the anisotropic renormalization-group argument given above).

After the rotations described above, the string tension would be given by the spacelike Wilson loop. By $(1 + 1)$-dimensional Lorentz invariance, that is exactly the vertical string tension, studied in this paper. Now the ratio of the string tension (which is σ_{V}) to the square of the mass gap M of the isotropic theory can be obtained by examining correlation functions

$$\langle j_\mu^{\text{L,R}}(x^1, x^2) j_\nu^{\text{L,R}}(x^1, x^2 + T)\rangle \sim \exp{-Mc^2 T},$$

for large T. This would be the first calculation of this ratio which is neither numerical, nor relying on strong-coupling expansions. If this idea can be made to work, the term proportional to $1/L$ in the potential (4.3) should have the universal coefficient of Ref. [14].

VI. CONCLUSIONS

To summarize, we have determined the potential between static sources, separated in the x^2-direction in $(2 + 1)$-dimensional SU(2) Yang-Mills theory with two couplings g_0 and g_0'. The calculation, like those in [2–4], is done entirely in a weak-coupling approximation, in which g_0' is smaller than any power of g_0. The nonpointlike nature of the color charge of the fundamental excitations of the principal-chiral sigma model is essential to understanding the result. The physical string states are color singlets by virtue of Gauss's law. This feature should also be the case for gauge group SU(N); unfortunately, nothing explicit can be done for $N > 2$, as the generalization of (2.3) is not known.

The composite-string Hamiltonian (4.1), describing the electric flux between a vertically separated quark-antiquark pair, can be studied by several techniques, among them numerical. One could eventually imagine real-space renormalization-group or numerical variational methods applied to this problem.

Using the exact S-matrix for FZ particles, the scattering problem of string states, either mesonic, such as those we have considered here, or purely gluonic, can be studied. In particular, amplitudes at large center of mass energies, *i.e.* Pomerons, in which gluonic processes dominate, are analytically accessible. Calculating the scattering amplitude in this asymptotically free version of $(2 + 1)$-dimensional Yang-Mills theory may give some general insight into large-s scattering.

In Sec. V, we conjectured that ratios of some quantities in the isotropic theory may be determined by those in our anisotropic model, through anisotropic renormalization flow. If this is the case, the string tension in the isotropic theory is proportional to the vertical string tension, *i.e.* that is studied in this paper. For $N > 2$, this would also mean that the k-string tensions in the isotropic theory should be proportional to $\sin \pi k/N$—for this is true of the vertical k-string tensions of our model [2]. This sine-law behavior was found in models of strong-coupling QCD, in particular, $\mathcal{N} = 2$ supersymmetric gauge theory softly broken to $\mathcal{N} = 1$ [15], in the M-theory 5-brane versions of QCD

[16], and in the AdS/QCD scheme [17]. The sine law was indicated in one calculation [18], but most simulations in four dimensions point to a result between the so-called Casimir law and the sine law [19–21]. In $(2 + 1)$ dimensions, Bringoltz and Teper's recent results indicate that the sine law does not hold [22]. This would bode ill for the conjecture of Sec. V, unless corrections to these string tensions of order g'_0 have significant $1/N$ dependence [thus far, we can find results like (4.3) only for $N = 2$]. We hope that behavior of k-string tensions will be settled soon, as more large-scale lattice simulations are carried out.

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[1] P. Orland, Phys. Rev. D **71**, 054503 (2005).
[2] P. Orland, Phys. Rev. D **75**, 025001 (2007).
[3] P. Orland, Phys. Rev. D **74**, 085001 (2006).
[4] P. Orland, Phys. Rev. D **75**, 101702(R) (2007).
[5] J. P. Greensite, Nucl. Phys. **B166**, 113 (1980); Q.-Z. Chen, X.-Q. Luo, and S.-H. Guo, Phys. Lett. B **341**, 349 (1995).
[6] D. Karabali and V. P. Nair, Nucl. Phys. **B464**, 135 (1996); Phys. Lett. B **379**, 141 (1996); D. Karabali, C. Kim, and V. P. Nair, Nucl. Phys. **B524**, 661 (1998); Phys. Lett. B **434**, 103 (1998); Nucl. Phys. **B566**, 331 (2000).Phys. Rev. D **64**, 025011 (2001).
[7] A. Agarwal, D. Karabali, and V. P. Nair, Nucl. Phys. **B790**, 216 (2008).
[8] J. Greensite and S. Olejnik, arXiv:0709.2370.
[9] D. Karabali and V. P. Nair, Phys. Lett. B **379**, 141 (1996).
[10] P. Orland and G. W. Semenoff, Nucl. Phys. **B576**, 627 (2000).
[11] P. Orland, in *Quark Confinement and the Hadron Spectrum VII* (Ponta Delgada, Azores, Portugal, 2006) [AIP Conf. Proc. 892, 206 (2007)].
[12] E. Abdalla, M. C. B. Abdalla, and A. Lima-Santos, Phys. Lett. **140B**, 71 (1984); P. B. Wiegmann, Phys. Lett. **142B**,

173 (1984); A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**, 121 (1983); P. B. Wiegmann, Phys. Lett. **141B**, 217 (1984).
[13] M. Karowski and P. Weisz, Nucl. Phys. **B139**, 455 (1978).
[14] M. Lüscher, K. Symanzik, and P. Weisz, Nucl. Phys. **B173**, 365 (1980); M. Lüscher, Nucl. Phys. **B180**, 317 (1981).
[15] M. R. Douglas and S. Shenker, Nucl. Phys. **B447**, 271 (1995).
[16] A. Hanany, M. J. Strassler, and A Zafforoni, Nucl. Phys. **B513**, 87 (1998).
[17] C. P. Herzog and I. R. Klebanov, Phys. Lett. B **526**, 388 (2002).
[18] L. Del Debbio, H. Panagopoulos, P. Rossi, and E. Vicari, J. High Energy Phys. 01 (2002) 009.
[19] L. Del Debbio, H. Panagopoulos, P. Rossi, and E. Vicari, Phys. Rev. D **65**, 021501(R) (2001).
[20] B. Lucini and M. J. Teper, Phys. Rev. D **64**, 105019 (2001).
[21] B. Lucini, M. J. Teper, and U. Wenger, J. High Energy Phys. 06 (2004) 012.
[22] B. Bringoltz and M. Teper, arXiv:0708.3447.